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# Boolean powers of groups

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## 1. *Introduction*

If $M$ is any algebraic structure, and $R$ is any Boolean ring, then a structure called the (bounded) Boolean power of $M$ by $R$, denoted $M^R$, can be defined. This construction, which is also called a bounded Boolean extension, is a sort of generalized direct power, and was introduced by Foster in the 1950's (as a refinement of his previous notion of a Boolean extension). In this paper we shall study isomorphism types and automorphisms of Boolean powers of groups, and obtain information about their characteristic subgroups: we shall be chiefly concerned with Boolean powers of finite groups.

Isomorphism types of Boolean powers of groups have been investigated by Jónsson (9) and Hickin and Plotkin (8), and our first theorem extends a result of (8):

Theorem A. *If $G$ is a group whose automorphisms are centrally indecomposable, then $G$ is Boolean separating (that is, non-isomorphic Boolean rings give rise to non-isomorphic Boolean powers of $G$).*

We remark that the hypothesis on $G$ (which we define in § 3) holds if $G$ is a finite non-abelian indecomposable group. We shall prove Theorem A by showing that $R$ can be recovered from the set of direct summands of $G^R$ (this method is also used in (9) and (8)).

For Boolean powers of finite groups, we can considerably strengthen Theorem A:

Theorem B. *If $\{G_i: 1 \leqslant i \leqslant m\}$ and $\{H_j: 1 \leqslant j \leqslant n\}$ are two families of non-isomorphic finite indecomposable groups, and $R_1, \ldots, R_m, S_1, \ldots, S_n$ are non-zero Boolean rings, with*

$$(G_1)^{R_1} \times \ldots \times (G_m)^{R_m} \cong (H_1)^{S_1} \times \ldots \times (H_n)^{S_n},$$

*then $m = n$, and for each $i$ we have (on re-ordering) $G_i \cong H_i$, and $|R_i| = |S_i|$, with $R_i \cong S_i$ if $G_i$ is non-abelian.*

This theorem (and its proof) can be thought of as a generalization of the Krull–Schmidt theorem for finite groups. Theorem B, in conjunction with a theorem of Ketonen (10), also enables us to give a necessary and sufficient condition for a finite group to be Boolean separating.

In § 4 we analyse the automorphisms of groups of form $(G_1)^{R_1} \times \ldots \times (G_m)^{R_m}$, with each $G_i$ finite and indecomposable; the following result indicates the pattern when $m = 1$ and $G_1$ is 'nice' (there are complications in the general case due to the existence of 'non-Boolean' automorphisms of $(G_i)^{R_i}$):

Corollary C1. *If $G$ is a finite non-abelian indecomposable group such that* Hom $[G, Z(G)] = \{e\}$, *and if $R$ is any Boolean ring with identity, then* $(\mathrm{Aut}\, G)^R$ *and* $\mathrm{Aut}\, R$ *act naturally on $G^R$, and* $\mathrm{Aut}\,(G^R)$ *is the split extension of* $(\mathrm{Aut}\, G)^R$ *by* $\mathrm{Aut}\, R$.

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For groups $H$ and $K$, we are writing $\mathrm{Hom}(H, K)$ for the set of homomorphisms from $H$ to $K$, $e: H \to K$ for the trivial homomorphism given by $he = 1 (h \in H)$, and $\mathrm{Aut}\, H$ for the group of automorphisms of $H$.

A substructure $N$ of a structure $M$ is said to be *characteristic* in $M$ if $N\theta = N$ for each $\theta \in \mathrm{Aut}\, M$; and a group (respectively ring) whose only characteristic subgroups (respectively characteristic ideals) are itself and the trivial subgroup (respectively zero ideal) is called *characteristically simple*. In section 5 we obtain an easy description of the characteristic subgroups of a Boolean power of a non-abelian simple group, and in section 6 we classify the countable characteristically simple Boolean rings: both these results are due to Hall, and were known to him at least as early as 1959. We use the work of these two sections in section 7 to construct further examples, following Wilson (17), of uncountable groups which are not characteristically simple, but in which each countable subgroup is contained in a countable characteristically simple subgroup.

Finally, we turn in section 8 to characteristic subgroups of Boolean powers of finite groups. Let $G$ be a finite non-abelian indecomposable group. We have:

THEOREM E($a$). *If $R$ is a non-zero Boolean ring, and $K$ is characteristic in $G$, then $K^R$ is characteristic in $G^R$.*

(Other results of a similar flavour are also obtained for less restricted $G$). Lastly, in Theorem F we classify the characteristic subgroups of $G^R$ when $G$ is as above and $R$ is a countably infinite characteristically simple Boolean ring (Theorem E($a$) does not give the whole picture in this case).

Our interest in Boolean powers stems largely from the theorem of Waszkiewicz and Weglorz (16) that if both $G$ and $R$ are $\aleph_0$-categorical, then so is $G^R$. We will use many of the results of this paper in some subsequent papers about $\aleph_0$-categorical groups.

Other papers treating algebraic aspects of Boolean powers are Hall (7), Hickin and Plotkin (8), Jónsson (9), and Neumann and Yamamuro (11). For a survey of both logical and algebraic aspects, see Burris (4).

*Notation.* If $G$ is a group, $g, h \in G$, and $H, K \leqslant G$ (this last signifying that $H$ and $K$ are subgroups of $G$), then we write

$g^h$ for $h^{-1}gh$;
$g^G$ for the normal subgroup $\langle g^k : k \in G \rangle$ generated by $g$;
$[g, h]$ for $g^{-1}h^{-1}gh = g^{-1}g^h$;
$[H, K]$ for $\langle [h', k'] : h' \in H, k' \in K \rangle$;
$G'$ for $[G, G]$, the derived subgroup of $G$;
$Z(G)$ for the centre of $G$;
and $G^\omega$ for the direct product of $\aleph_0$ copies of $G$.

We let $\mathrm{End}\, G$ denote the set of endomorphisms of $G$, writing $1_G$ for the identity automorphism of $G$, and $e$ for the trivial endomorphism of $G$ ($ge = 1$ for all $g \in G$). If $\alpha, \beta \in \mathrm{End}\, G$, and $[G\alpha, G\beta] = 1$, then we let $\alpha + \beta$ be the endomorphism of $G$ given by $g(\alpha + \beta) = (g\alpha)(g\beta)$ $(g \in G)$.

If $A$ and $B$ are sets, we write $A \subseteq B$ and $A \subset B$ for inclusion and strict inclusion respectively of $A$ in $B$, and $A - B$ for $\{x \in A : x \notin B\}$. Lastly, for any set $A$, we write $|A|$ for the cardinal number of $A$, and $2^A$ for the ring of subsets of $A$.

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## 2. *Definition and Basic Properties of Boolean Powers*

We recall that a Boolean ring $R$ is a commutative ring such that $x^2 = x$ for each $x \in R$ (we do not require $R$ to have a multiplicative identity; some authors require Boolean rings to possess an identity, and term those without identity 'generalized Boolean rings'). A Boolean space is a Hausdorff topological space in which the compact open sets form a base for the topology. By Stone's representation theorem, for any Boolean ring $R$ there is a Boolean space $X$, the Stone space of $R$, unique up to homeomorphism, such that $R$ is isomorphic to the ring of compact open subsets of $X$ (Stone's theorem is usually stated for rings with identity, but see Stone (15) or Balbes and Dwinger ((2); ch. IV, theorems 9, 12) for the more general form we require). Moreover, $X$ is locally compact, and is compact precisely when $R$ has an identity. We identify $R$ with the subring of $2^X$ comprising the compact open subsets of $X$, and freely use set theoretic notation $(\cap, \cup, -)$ in $R$, writing 0 for the empty subset of $X$. It is not difficult to see that automorphisms of $R$ correspond precisely to homeomorphisms of $X$ (see section 11 of Sikorski (14)).

Let $G$ be a group, and $R$ a Boolean ring with Stone space $X$. Following P. Hall, we write $G^R$ for the (bounded) *Boolean power of $G$ by $R$*, which is defined as follows:

$$G^R = \{\gamma : X \to G \mid \quad \gamma \text{ is continuous with compact support}\},$$

where $G$ is given the discrete topology, and the support of $\gamma$, written $\operatorname{supp}(\gamma)$, is given by

$$\operatorname{supp}(\gamma) = \{x \in X : x\gamma \neq 1\}.$$

For $g \in G$ and $A \in R$, we write $g_A$ for the element of $G^R$ given by $xg_A = g$ if $x \in A$, $xg_A = 1$ otherwise, and we let $G_A = \{g_A : g \in G\}$. Clearly $G_A \cong G$ if $A \neq 0$. We observe that if $\gamma \in G^R$, and $g \in G - 1$, then $\{x \in X : x\gamma = g\}$ is compact and open. Since $\operatorname{supp}(\gamma)$ is compact, $\{x \in X : x\gamma = g\}$ can be non-empty for only finitely many $g \neq 1$, and so each $\gamma \in G^R$ can be written uniquely (up to order) as a product

$$(g_1)_{A_1} \cdots (g_n)_{A_n},$$

where the $g_i$ are distinct non-trivial elements of $G$, and the $A_i$ are disjoint non-empty elements of $R$. We call $A_1, \ldots, A_n$ the *constancy sets* of $\gamma$.

This suggests an alternative way of defining $G^R$: namely, as the subset of the group ring $RG$ consisting of all elements of form $\Sigma_{g \in G-1} A_g g$, with $A_g \in R$, all but finitely many $A_g$ zero, and $A_g \cap A_h = 0$ if $g \neq h$, with

$$\Big(\sum_{g \neq 1} A_g g\Big)\Big(\sum_{h \neq 1} B_h h\Big) = \sum_{k \neq 1} C_k k, \quad \text{where} \quad C_k = \sum_{gh = k} A_g B_h$$

(and where $A_1 B_k = B_k - \bigcup_{g \neq 1} A_g$, and similarly for $A_k B_1$). This approach is adopted by Neumann and Yamamuro in (11), and indeed is akin to Foster's original definition in (5); it has the merit of not involving the Stone space of $R$, but it appears to be less suitable for calculations (as without the Stone space we do not have the natural projection maps $\pi_x$: see below).

Likewise, using the above form for elements of $G^R$, it is not difficult to see that $G^R$ has a presentation with generators $\{g_A : g \in G, A \in R\}$, and relations $g_{A+B} = g_A g_B$ and $[g_A, h_B] = 1$ if $AB = 0$, and $g_A h_A = (gh)_A$ $(g, h \in G; A, B \in R)$: cf. ((7), p. 465).



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We remark that if $X$ is any set, given the discrete topology, and $R$ is the ring of compact open (i.e. finite) subsets of $X$, then $G^R$ is just the direct product of $|X|$ copies of $G$. Thus Boolean powers can be viewed as generalized direct powers.

If $S$ is a subring of a Boolean ring $R$, then there is a natural map from the Stone space of $R$ to that of $S$ (see section 11 of Sikorski (14)), and so $G^S$ can be regarded as a subgroup of $G^R$ for any group $G$; in fact, $G^S \cong \langle g_A \in G^R : g \in G, A \in S \rangle$.

We record without proof the following elementary properties of Boolean powers:

PROPOSITION 2·1. *Suppose that $G$ and $H$ are groups, and that $R$ and $S$ are Boolean rings.*

(*a*) *If $T$ is an ideal of $R$, and $K$ is a normal subgroup of $G$, then $K^T$ is normal in $G^R$,*

(*b*) *$(G \times H)^R$ is naturally isomorphic to $G^R \times H^R$,*

(*c*) *$G^{R \times S}$ is naturally isomorphic to $G^R \times G^S$.* ∎

*Remark.* Burris (4) has shown that $(G^R)^S \cong G^{R*S}$, where $R * S$ is the Boolean product of $R$ and $S$. Iterated Boolean powers of $G$ are therefore still Boolean powers of $G$.

We conclude this section with an easy result on homomorphisms between Boolean powers of groups, which we shall use constantly in the sequel. If $\Gamma$ is the Boolean power of the group $G$ by the Boolean ring $R$, then we let $\pi_x : \Gamma \to G$ denote 'evaluation at $x$', for $x$ in the Stone space of $R$ (that is, $\gamma\pi_x = x\gamma$ if $\gamma \in \Gamma$).

PROPOSITION 2·2. *Let $R$ and $S$ be Boolean rings, and $G$ and $H$ groups. Let $Y$ be the Stone space of $S$. Suppose $\theta : G^R \to H^S$ is a homomorphism. For $A \in R - 0$, and $y \in Y$, define $\theta_{A,y} : G \to H$ by*

$$g\theta_{A,y} = (g_A\theta)\pi_y, \quad \text{for each} \quad g \in G.$$

*Then $\theta_{A,y}$ is a homomorphism. If in addition $G$ and $H$ are finite, then for each $A \in R - 0$, there are distinct homomorphisms $f_1, \ldots, f_n$ from $G$ to $H$ and disjoint non-empty sets $B_1, \ldots, B_n \in S$ such that*

$$g_A\theta = (gf_1)_{B_1} \ldots (gf_n)_{B_n}, \quad \text{for all} \quad g \in G.$$

*Proof.* It is immediate that $\theta_{A,y}$ is a homomorphism. Suppose now that $G$ and $H$ are finite, and $A \in R - 0$. If $f : G \to H$ is a homomorphism, let $C_f = \{y \in Y : \theta_{A,y} = f\} = \bigcap_{g \in G} D_g$, where $D_g = \{y \in Y : (g_A\theta)\pi_y = gf\}$. Now if $gf \neq 1$, then $D_g \in R$, as $g_A\theta$ is a continuous map from $Y$ to $H$, while if $gf = 1$, then $X - D_g \in R$ (for the same reason). So if $f$ is not the trivial homomorphism, then $D_g \in R$ for some $g$, and each $D_g$ is clopen. Hence $C_f$ is compact and open, and $C_f \in R$. Now let $f_1, \ldots, f_n$ be a list of those non-trivial $f$ for which $C_f \neq 0$, and let $B_i = C_{f_i}$ $(1 \leqslant i \leqslant n)$. By construction, we now have

$$g_A\theta = (gf_1)_{B_1} \ldots (gf_n)_{B_n} \quad \text{for each} \quad g \in G.$$ ∎

## 3. *Isomorphism Types of Boolean Powers*

Hickin and Plotkin (8) define a group $G$ to be *centrally indecomposable* if $G$ cannot be written as the join of two non-abelian subgroups $H$ and $K$ such that $[H, K] = 1$ (in other words, if $H, K \leqslant G$, $[H, K] = 1$, and $G = HK$, then $H \leqslant Z(G)$ or $K \leqslant Z(G)$). They show ((8); theorem 5) that any non-abelian centrally indecomposable group $G$ is Boolean separating (that is, if $R$ and $S$ are non-isomorphic Boolean rings, then $G^R$ and $G^S$ are non-isomorphic). This extends the earlier theorem of Jónsson (9) that indecomposable groups with trivial centre are Boolean separating.

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We shall say that an automorphism $\theta$ of $G$ is *centrally indecomposable* if whenever $\alpha, \beta \in \operatorname{End} G$, with $[G\alpha, G\beta] = 1$ and $\theta = \alpha + \beta$, then $G\alpha \leqslant Z(G)$ or $G\beta \leqslant Z(G)$. Since $\theta^{-1} \circ (\alpha + \beta) = \theta^{-1} \circ \alpha + \theta^{-1} \circ \beta$, we have that $\theta \in \operatorname{Aut} G$ is centrally indecomposable if and only if $1_G$ is centrally indecomposable. We note that if $G$ is centrally indecomposable, then so is $1_G$. We shall prove

THEOREM A. *If $G$ is a non-abelian group such that $1_G$ is centrally indecomposable, then $G$ is Boolean separating.*

We shall see shortly that finite non-abelian indecomposable groups satisfy the condition of Theorem A, and so are Boolean separating: indeed, this was our original version of Theorem A, and the present version (which was inspired by the result of Hickin and Plotkin) uses substantially the same proof. However, there are non-abelian indecomposable groups $G$ such that $1_G$ is not centrally indecomposable, and I am grateful to the referee for providing the following example. Let

$$G = \langle u, v, t \mid u^7 = v^3 = [u, v] = 1, t^{-1}ut = u^2, t^{-1}vt = v^2 \rangle.$$

Then $G$ is non-abelian, with $G'$ equal to $\langle u, v \rangle$ and cyclic of order 21, and $G$ is indecomposable (since $G$ is soluble and $G/G'$ is indecomposable). The assignments

$$u\alpha = u, \quad v\alpha = 1, \quad t\alpha = t^{-2}, \quad \text{and} \quad u\beta = 1, \quad v\beta = v, \quad t\beta = t^3,$$

determine endomorphisms $\alpha$ and $\beta$ of $G$, with $[G\alpha, G\beta] = 1$ and $1_G = \alpha + \beta$. However, neither $G\alpha$ nor $G\beta$ is obtained in $Z(G)$. We do not know whether every non-abelian indecomposable group is Boolean separating.

In Theorem A, the group $G$ is fixed, and $G^R$ is considered as $R$ varies. Our second result is more evenly balanced in this respect:

THEOREM B ('Krull–Schmidt theorem for Boolean powers'). *Suppose $\{G_i : 1 \leqslant i \leqslant m\}$ and $\{H_j : 1 \leqslant j \leqslant n\}$ are two families of finite indecomposable groups, with the groups in each family non-isomorphic. Suppose that $\{R_i : 1 \leqslant i \leqslant m\}$ and $\{S_j : 1 \leqslant j \leqslant n\}$ are two families of non-zero Boolean rings, and that*

$$G_1^{R_1} \times \ldots \times G_m^{R_m} \cong H_1^{S_1} \times \ldots \times H_n^{S_n}.$$

*Then $m = n$, and for each $i$ we have (on re-ordering) $G_i \cong H_i$, and $|R_i| = |S_i|$, with $R_i \cong S_i$ if $G_i$ is non-abelian.*

In the case when the $R_i$ and $S_j$ are finite, this is precisely the Krull–Schmidt theorem for finite groups (see, for example, section 4·6 of Scott (13)). We note that any finite direct product of Boolean powers of finite groups is isomorphic (by Proposition 2·1) to a group of form

$$(G_1)^{R_1} \times \ldots \times (G_m)^{R_m},$$

with $G_i$ and $R_i$ $(1 \leqslant i \leqslant m)$ as in Theorem B.

We shall prove these theorems by considering direct product decompositions, as in (8) and (9). We begin with some straightforward results on endomorphisms:

LEMMA 3·1. *Let $G$ be any group.*

(*a*) *Suppose $\alpha, \beta \in \operatorname{End} G$ are such that $[G\alpha, G\beta] = 1$, $\alpha + \beta = 1_G$, and $\beta$ is nilpotent. Then $\alpha$ is an automorphism of $G$.*

(*b*) *Suppose that $\theta \in \operatorname{Aut} G$, and that $\theta$ induces the trivial automorphism on $G/Z(G)$. Then $\theta = 1_G + \zeta$ for some $\zeta \in \operatorname{End} G$ such that $G\zeta \leqslant Z(G)$.*



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(c) *Suppose that $\zeta$ is a nilpotent endomorphism of $G$ such that $G\zeta \leqslant Z(G)$. Then $1_G + \zeta$ is an automorphism of $G$.*

*Proof.* (a) By hypothesis, we have $g = (g\alpha)(g\beta)$ for all $g \in G$. So if $g\alpha = 1$, then $g = g\beta$, and $g = 1$, as $\beta$ is nilpotent. Hence $\alpha$ is injective. Find $n$ such that $g\beta^n = 1$ for all $g \in G$. Then

$$[g(g\beta)\dots(g\beta^{n-1})]\alpha = [g(g\beta)^{-1}][(g\beta)(g\beta^2)^{-1}]\dots[(g\beta^{n-1})(g\beta^n)^{-1}] = g \quad (g \in G).$$

Hence $\alpha$ is surjective, and so is an automorphism of $G$.

(b) Now suppose that $\theta \in \operatorname{Aut} G$ and $g^{-1}(g\theta) \in Z(G)$ $(g \in G)$. Then we readily verify that the map $\zeta: G \to Z(G): g \mapsto g^{-1}(g\theta)$ is a homomorphism, and we have $\theta = 1_G + \zeta$, as required.

(c) Let $\eta: G \to G$ be given by $g\eta = (g\zeta)^{-1}$ $(g \in G)$. Then $\eta$ is a nilpotent endomorphism of $G$, $G\eta \leqslant Z(G)$, and $1_G = (1_G + \zeta) + \eta$. Hence $1_G + \zeta$ is an automorphism of $G$, by (a). ∎

LEMMA 3·2. *Suppose that $G$ is a finite non-abelian indecomposable group.*

(a) *If $\alpha \in \operatorname{End} G$, and $G\alpha^r \lhd G$ for all $r \geqslant 1$, then $\alpha$ is either an automorphism of $G$ or nilpotent. In particular, $\{\alpha \in \operatorname{End} G: G\alpha \leqslant Z(G)\}$ is a nilpotent ring.*

(b) *If $\alpha, \beta \in \operatorname{End} G$, $[G\alpha, G\beta] = 1$, and $1_G = \alpha + \beta$, then either $\alpha \in \operatorname{Aut} G$ and $G\beta \leqslant Z(G)$ or vice-versa. In particular, $1_G$ is centrally indecomposable.*

(c) *Let $R$ be any Boolean ring, and let $\Gamma = G^R$. Then $\{\theta \in \operatorname{End} \Gamma: \Gamma\theta \leqslant Z(\Gamma)\}$ is a nilpotent ring.*

*Proof.* (a) (cf. section 4·5 of Scott (13)). Find $r$ such that $G\alpha^r = G\alpha^{r+1}$. Since $G\alpha^r \lhd G$, we have $G = G\alpha^r \times \operatorname{Ker}(\alpha^r)$, and so $G\alpha^r = 1$ or $G\alpha^r = G$. Hence $\alpha \in \operatorname{Aut} G$ or $\alpha$ is nilpotent.

Now let $E = \{\alpha \in \operatorname{End} G: G\alpha \leqslant Z(G)\}$, which is a ring with operations $\alpha\beta = \alpha \circ \beta$, $\alpha + \beta: g \mapsto (g\alpha)(g\beta)$. This is a finite ring, each element of which is nilpotent, by the above. Hence $E$ is nilpotent.

(b) Suppose that $\alpha$ and $\beta$ satisfy the given conditions. Then $G\alpha^0 \lhd G$ (where $\alpha^0 = 1_G$), and if $r \geqslant 0$ and $G\alpha^r \lhd G$, then $G\alpha^{r+1}$ is normalized by $G\alpha$ and centralized by $G\beta$, and so $G\alpha^{r+1} \lhd G$. Hence $G\alpha^r \lhd G$ for each $r \geqslant 1$; and by (a), either $\alpha \in \operatorname{Aut} G$, when $G\beta \leqslant Z(G)$, or $\alpha$ is nilpotent, when $\beta \in \operatorname{Aut} G$ by Lemma 3·1, and so $G\alpha \leqslant Z(G)$.

(c) Let $E = \{\alpha \in \operatorname{End} G: G\alpha \leqslant Z(G)\}$, $F = \{\theta \in \operatorname{End} \Gamma: \Gamma\theta \leqslant Z(\Gamma)\}$. By (a), we can find $n$ such that $E^n = \{e\}$. Let $Z_0 = G$, and let $Z_i = \langle g\alpha: g \in G, \alpha \in E^i \rangle$ for $i \geqslant 1$. Then $G = Z_0 \geqslant Z_1 \geqslant \dots \geqslant Z_n = 1$, while if $i \geqslant 0$ and $\beta \in E$, then $Z_i\beta \leqslant Z_{i+1}$.

Now suppose $\theta \in F$, and let $z \in Z_i$, $A \in R - 0$. Since $G_A\theta \leqslant Z(\Gamma)$, we have $G\theta_{A,x} \leqslant Z(G)$ for each $x \in X$, where $X$ is the Stone space of $R$ (cf. Proposition 2·2). Hence $\theta_{A,x} \in E$, and $(z_A\theta)\pi_x \in Z_{i+1}$. Therefore $z_A\theta \in Z_{i+1}^R$ for all $A \in R$, $z \in Z_i$, and so $Z_i^R\theta \leqslant Z_{i+1}^R$. Likewise, if $\theta \in F$, then $\Gamma\theta \leqslant Z_1^R$. It follows that if $\theta \in F^n$, then $\Gamma\theta \leqslant Z_n^R = 1$, and so $F^n = \{e\}$, as required. ∎

We now show that the property of being non-abelian with centrally indecomposable identity map is partially preserved under the Boolean power operation:

LEMMA 3·3. *Suppose that $G$ is non-abelian and that $1_G$ is centrally indecomposable. Let $R$ be a Boolean ring, and let $\Gamma = G^R$. Suppose that $\alpha_i \in \operatorname{End} \Gamma (1 \leqslant i \leqslant n)$ are given such that $[\Gamma\alpha_i, \Gamma\alpha_j] = 1$ if $i \neq j$, and $1_\Gamma = \alpha_1 + \dots + \alpha_n$.*

*Then there are ideals $T_1, \dots, T_n$ of $R$ with $R = T_1 \oplus \dots \oplus T_n$ such that, writing $\Theta_i = G^{T_i}$, and $p_i: \Gamma \to \Theta_i$ for the $i$th projection, we have*

$$\Gamma\alpha_j p_i \leqslant Z(\Gamma) \quad \text{if} \quad j \neq i, \quad \text{and} \quad (\Gamma\alpha_j)Z(\Gamma) = G^{T_j}Z(\Gamma) \quad (1 \leqslant j \leqslant n).$$

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*If $G$ is finite, non-abelian, and indecomposable, then in addition $\alpha_j p_j : \Theta_j \to \Theta_j$ is an automorphism for each $j$.*

*If, further, $j \leqslant n$ is given such that $\alpha_j$ is a projection (so that $\alpha_j^2 = \alpha_j$), then $\Theta_j \alpha_j = \Gamma\alpha_j$, and $\Gamma\alpha_j \cong \Theta_j = G^{T_j}$.*

*Proof.* Let $X$ be the Stone space of $R$, and write $Z = Z(G)$. Suppose $x \in X$, and that $A \in R$ contains $x$. Now $g_A = (g_A\alpha_1) \dots (g_A\alpha_n)$ for all $g \in G$. Apply $\pi_x$ to obtain $g = [g(\alpha_1)_{A,x}] \dots [g(\alpha_n)_{A,x}]$ $(g \in G)$. Letting $\beta_i = (\alpha_i)_{A,x}$, we have $1_G = \beta_1 + \dots + \beta_n$. As $G$ is non-abelian, we can find $k \leqslant n$ such that $G\beta_k \nleqslant Z$. Let $\gamma = \Sigma_{j \neq k} \beta_j$. Since $1_G = \beta_k + \gamma$, we must have $G\gamma \leqslant Z$, and $(G\beta_k) Z = G$, as $1_G$ is centrally indecomposable.

Now suppose $B \in R$ and $j \neq k$. Then $[G_A\alpha_k, G_B\alpha_j] = 1$, and so $G(\alpha_j)_{B,x} \leqslant Z$. Hence $k$ depends only on $x$, and not on $A$. It follows that for each $x \in X$, there is a unique $k \leqslant n$, which we denote $k(x)$, such that

$$G(\alpha_{k(x)})_{A,x} \nleqslant Z \quad \text{if} \quad A \in R \quad \text{and} \quad x \in A,$$

and

$$G(\alpha_j)_{B,x} \leqslant Z \quad \text{if} \quad j \neq k(x) \quad \text{and} \quad B \in R.$$

We have defined a function $k : X \to \{1, \dots, n\}$, which we now claim to be continuous (where $\{1, \dots, n\}$ is given the discrete topology). For if $x \in X$ and $k(x) = i$, say, then there is some $g \in G$, and $A \in R$ containing $x$, such that $(g_A\alpha_i)\pi_x \notin Z$. But $g_A\alpha_i : X \to G$ is continuous, so we can find a neighbourhood $C$ of $x$ with $C \subseteq A$ such that $(g_A\alpha_i)\pi_y \notin Z$ $(y \in C)$. Hence for $y \in C$, we have $G(\alpha_i)_{A,y} \nleqslant Z$, and so $k(y) = i$. Thus $k$ is continuous.

Let $X_i = \{x \in X : k(x) = i\}$, and let $T_i = \{A \in R : A \subseteq X_i\}$ $(1 \leqslant i \leqslant n)$: $X_i$ is clopen, and so $R = T_1 \oplus \dots \oplus T_n$. Let $\Theta_i$ and $p_i$ be as in the statement of the proposition (note that $G^R = G^{T_1} \times \dots \times G^{T_n}$, by Proposition 2·1). By our choice of $X_j$, we have $\Gamma\alpha_j\pi_x \leqslant Z$ if $x \notin X_j$, and so $\Gamma\alpha_j p_i \leqslant Z(\Gamma)$ for all $i \neq j$. This is our first assertion. Hence also

$$\Gamma\alpha_j \leqslant \langle \Gamma\alpha_j p_i : 1 \leqslant i \leqslant n \rangle \leqslant \Theta_j Z(\Gamma).$$

Conversely,

$$\Theta_j \leqslant \langle \Gamma\alpha_i p_j : 1 \leqslant i \leqslant n \rangle \leqslant (\Gamma\alpha_j p_j) Z(\Gamma),$$

while $\Gamma\alpha_j p_j \leqslant \langle \Gamma\alpha_j, \Gamma\alpha_j p_i : i \neq j \rangle$ (since $1_\Gamma = p_1 + \dots + p_n$). Hence $\Gamma\alpha_j p_j \leqslant (\Gamma\alpha_j) Z(\Gamma)$, and $(\Gamma\alpha_j) Z(\Gamma) = \Theta_j Z(\Gamma)$, as asserted.

If now $G$ is finite, non-abelian, and indecomposable, then $\alpha_1 p_j + \dots + \alpha_n p_j$ is the identity on $\Theta_j$, while $\Gamma\alpha_i p_j \leqslant Z(\Gamma)$ if $i \neq j$. Let $\beta = \Sigma_{i \neq j} \alpha_i p_j$. We have $\Gamma\beta \leqslant Z(\Gamma)$, and so $\beta$ is nilpotent, by Lemma 3·2. Hence $\alpha_j p_j$ is an automorphism of $\Theta_j$, by Lemma 3·1.

Suppose lastly that $j$ is given such that $\alpha_j^2 = \alpha_j$. Then $\alpha_j p_1 \alpha_j + \dots + \alpha_j p_n \alpha_j$ is the identity on $\Gamma\alpha_j$. Let $\beta = \Sigma_{i \neq j} \alpha_j p_i \alpha_j$. Again, $\Gamma\beta \leqslant Z(\Gamma)$ (since $\Gamma\alpha_j p_i \leqslant Z(\Gamma)$ if $i \neq j$; and $Z(\Gamma)\alpha_j$ is centralized by $\Gamma\alpha_j$, and by $\Gamma\alpha_k$ if $k \neq j$; whence $Z(\Gamma)\alpha_j \leqslant Z(\Gamma)$). Hence $\beta$ is nilpotent, by Lemma 3·2, and so $\alpha_j p_j \alpha_j$ is an automorphism of $\Gamma\alpha_j$, using Lemma 3·1 again. Therefore $\Theta_j \alpha_j = \Gamma\alpha_j$, and $\alpha_j : \Theta_j \to \Gamma\alpha_j$ is an isomorphism. ∎

*Notation.* If $R$ is a Boolean ring, and $A \in R$, we write $(A)$ for the (principal) ideal of $R$ generated by $A$. We note that $(A)$ is a summand of $R$, with unique complement $\{B \in R : B \cap A = 0\}$.

*Proof of Theorem A.* Let $G$ be a non-abelian group such that $1_G$ is centrally indecomposable, and let $R$ and $S$ be Boolean rings such that $G^R \cong G^S$. We must show that $R \cong S$.

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Let $R'$ and $S'$ denote the rings of clopen subsets of the Stone spaces of $R$ and $S$ respectively. We identify $R'$ and $S'$ (via the Stone correspondence) with the sets of direct summands of $R$ and $S$ respectively. Clearly, $R$ is naturally embedded in $R'$ (as the set of compact open sets/principal ideals). Let $\Gamma = G^R$, $\Delta = G^S$, and suppose $\theta: \Gamma \to \Delta$ is some isomorphism.

If now $\Gamma = K_1 \times K_2$, let $\alpha_i: \Gamma \to K_i$ be the $i$th projection ($i = 1, 2$). Applying Lemma 3·3, we obtain ideals $T_1, T_2$ of $R$, with $R = T_1 \oplus T_2$, such that $K_i Z(\Gamma) = G^{T_i} Z(\Gamma)$ ($i = 1, 2$). Since $G$ is non-abelian, we obtain a bijection between elements of $R'$ and subgroups of $\Gamma$ of form $KZ(\Gamma)$, for $K$ a summand of $\Gamma$, given by $T \mapsto G^T Z(\Gamma)$ ($T \in R'$).

We now define a map $\theta_0: R' \to S'$ as follows: if $T \in R'$, there is a unique $U \in S'$ such that

$$[G^T Z(\Gamma)]\theta = G^U Z(\Delta),$$

since $G^T\theta$ is a summand of $G^S$ and $Z(\Gamma)\theta = Z(\Delta)$; let $U = T\theta_0$. Consideration of $\theta^{-1}$ yields the fact that $\theta_0$ is a bijection; also, if $T_1, T_2 \in R'$, with $T_1 \subseteq T_2$, then clearly $T_1\theta_0 \subseteq T_2\theta_0$. It follows easily that $\theta_0$ is a Boolean ring isomorphism.

But now, if $T \in R'$, then $T \in R$ if and only if there is some $\gamma \in \Gamma$ such that ($\gamma$ belongs to $G^U Z(\Gamma)$ precisely when $U \geqslant T$): for if $A \in R$, and $T = (A)$, then let $\gamma = g_A$ for some $g \in G - Z(G)$; conversely, if such a $\gamma$ exists, then $T$ is the ideal generated by $\{x \in X: x\gamma \notin Z(G)\}$. A similar statement holds for $S$. We deduce from our definition of $\theta_0$ that $R\theta_0 = S$, and that $R \cong S$. ∎

We now restrict our attention to finite groups. We are not certain whether any of the theorems below are true in greater generality, but certainly the Krull–Schmidt theorem breaks down for infinite groups (see Fuchs ((6); p. 154)).

Our next result strengthens Theorem A for finite non-abelian indecomposable groups:

THEOREM 3·4. *Suppose that $G$ and $H$ are finite non-abelian indecomposable groups, that $R$ and $S$ are non-zero Boolean rings, and that $G^R \cong H^S$. Then $G \cong H$, and $R \cong S$.*

*Proof.* Let $\theta: G^R \to H^S$ be the given isomorphism. Select $A \in R - 0$. By Proposition 2·2, there are disjoint sets $B_1, \ldots, B_n \in S$ and homomorphisms $f_1, \ldots, f_n: G \to H$ such that

$$g_A\theta = (gf_1)_{B_1} \ldots (gf_n)_{B_n}, \quad \text{for each} \quad g \in G.$$

Pick $x \in A$ (where $A$ is viewed as a subset of the Stone space of $R$, as usual). Then, writing $e_i = (\theta^{-1})_{B_i, x}: H \to G$, and observing that $g_A\theta\theta^{-1}\pi_x = g$, we obtain $g = (gf_1e_1)\ldots(gf_ne_n)$ for all $g \in G$. Furthermore, if $i \neq j$, then $[He_i, He_j] = 1$, since $B_i$ and $B_j$ are disjoint. Hence $1_G = f_1e_1 + \ldots + f_ne_n$.

Now $Gf_ie_i \not\leqslant Z(G)$ for some $i$, and Lemma 3·2 then gives $f_ie_i$ to be an automorphism of $G$. In particular, $f_i: G \to H$ is injective, and $|G| \leqslant |H|$. Reversing the roles of $G$ and $H$, we obtain $|H| \leqslant |G|$. Hence $|G| = |H|$, and $f_i: G \to H$ is an isomorphism. Theorem A applies, finally, to give $R \cong S$. ∎

*Proof of Theorem B.* Let $G_i$, $R_i$, $H_j$, and $S_j$ satisfy the given hypotheses. Let $\Gamma_i = G_i^{R_i}$, $\Delta_j = H_j^{S_j}$, $\Gamma = \Gamma_1 \times \ldots \times \Gamma_m$, and $\Delta = \Delta_1 \times \ldots \times \Delta_n$. Let $q_i: \Gamma \to \Gamma_i$ and $r_j: \Delta \to \Delta_j$ be the projection maps. Suppose $\theta: \Gamma \to \Delta$ is some isomorphism. We consider $G_1$; let $\alpha_j = [\theta r_j \theta^{-1} q_1]\big|_{\Gamma_1}$ for $j \leqslant n$.

*Case* 1. $G_1$ non-abelian.

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We have that $\Sigma_{1\leqslant j\leqslant n}\alpha_j$ is the identity on $\Gamma_1$. Applying Lemma 3·3, we obtain ideals $T_1, \ldots, T_n$ of $R_1$, with $R_1 = T_1 \oplus \ldots \oplus T_n$, such that for each $j$, the composite

$$\theta r_j \circ \theta^{-1} q_1 \circ p_j \colon G_1^{T_j} \to \Delta_j \to \Gamma_1 \to G_1^{T_j}$$

is an automorphism, where $p_j \colon \Gamma_1 \to G_1^{T_j}$ is the $j$th projection. But $G_1^{T_j} \lhd \Gamma$; hence $G_1^{T_j}\theta r_j \lhd \Delta_j$, and so $G_1^{T_j}\theta r_j$ is a summand of $\Delta_j$ (with complement $\operatorname{Ker}(\theta^{-1}q_1 p_j)$). Lemma 3·3 gives us a summand $U_j$ of $S_j$ such that

$$(G_1^{T_j}\theta r_j)\, Z(\Delta_j) = H_j^{U_j} Z(\Delta_j), \quad \text{and} \quad G_1^{T_j}\theta r_j \cong H_j^{U_j},$$

as $H_j$ is clearly non-abelian if $T_j \neq (0)$.

We now have $G_1^{T_j} \cong H_j^{U_j}$, since $\theta r_j$ is injective on $G_1^{T_j}$; so if $T_j \neq (0)$ then $G_1 \cong H_j$, by Theorem 3·4. But $R_1 \neq (0)$. Therefore there is a unique $j \leqslant n$ such that $T_j \neq (0)$; and $G_1 \cong H_j$ and $T_j = R_1$, with $\alpha_j$ an automorphism of $\Gamma_1$, for this $j$.

Similarly, considering $\Delta_j$, we obtain a unique $k \leqslant m$ such that $H_j \cong G_k$ (thus $k = 1$) and $\theta^{-1}q_1\theta r_j$ is an automorphism of $\Delta_j$. Hence $\theta r_j \colon \Gamma_1 \to \Delta_j$ is an isomorphism, and so $R_1 \cong S_j$, by Theorem A.

*Case* 2. $G_1$ abelian.

Suppose that $G_1$ is cyclic of order $p^s$, for some prime $p$ and some $s \geqslant 1$. Let $X_1$ be the Stone space of $R_1$. If $j \leqslant n$, and $A \in R_1 - 0$, find a finite subring $U$ of $S_j$ such that $(G_1)_A \theta r_j \leqslant H_j^U$. For each $x \in X_1$, consider the composite

$$(\alpha_j)_{A,x} = (\theta r_j)_A \circ \theta^{-1} q_1 \pi_x \colon G_1 \to H_j^U \to G_1,$$

where $g[(\theta r_j)_A] = g_A \theta r_j$ $(g \in G_1)$. If $(\alpha_j)_{A,x}$ is an isomorphism, then the above sequence splits (for $(G_1)_A \theta r_j \leqslant Z(\Delta)$, since $(G_1)^{R_1} \leqslant Z(\Gamma)$); and therefore $G_1$ is isomorphic to a summand of $H_j^U$. Hence $G_1 \cong H_j$, by the Krull–Schmidt theorem for finite groups (see section 4·6 of Scott (13)). It follows that if $G_1 \ncong H_j$, then for each $A \in R_1 - 0$ and each $x \in X_1$, the map $(\alpha_j)_{A,x} \colon G_1 \to G_1$ is not injective, and so $(G_1)_A \alpha_j \leqslant K_1^{R_1}$, where $K_1 = \{g^p : g \in G_1\}$ is the unique maximal subgroup of $G_1$. We therefore have $\Gamma_1 \alpha_j \leqslant K_1^{R_1}$ if $G_1 \ncong H_j$. But $\Sigma_{1\leqslant j\leqslant n}\alpha_j$ is the identity on $\Gamma_1$, while, writing $\beta = \Sigma\{\alpha_j : H_j \ncong G_1\}$, we have $\Gamma_1\beta \leqslant K_1^{R_1}$. Similarly $\Gamma_1\beta^s = 1$, and $\beta$ is nilpotent. There must therefore exist $j \leqslant n$ such that $H_j \cong G_1$, and now $\alpha_j \colon \Gamma_1 \to \Gamma_1$ is an automorphism, by Lemma 3·1. Reversing the argument, as in case 1, we obtain $\Gamma_1 \cong \Delta_j$, and so $|R_1| = |S_j|$.

Now consider each $G_i$ and $H_j$ in turn to see that $m = n$, and that (on re-ordering) $G_i \cong H_i$ and $|R_i| = |S_i|$, with $R_i \cong S_i$ if $G_i$ is non-abelian. ∎

COROLLARY B1. *Suppose that $G_1, \ldots, G_m$ are non-isomorphic finite indecomposable groups and that $R_1, \ldots, R_m$ are non-zero Boolean rings. Let $\Gamma_j = G_j^{R_j}$ for $j \leqslant m$, and let $\Gamma = \Gamma_1 \times \ldots \times \Gamma_m$. Then $\Gamma_j Z(\Gamma)$ and $\Gamma_j'$ are characteristic subgroups of $\Gamma$ for each $j \leqslant m$. If also* $\operatorname{Hom}[\Gamma, Z(\Gamma)] = \{e\}$, *then $\Gamma_j$ is characteristic in $\Gamma$ for each $j$.*

*Proof.* Let $\Delta = \Gamma$, and $\Delta_j = \Gamma_j$ for $j \leqslant m$. Suppose that $\theta$ is any automorphism of $\Gamma$. We view $\theta$ as an isomorphism from $\Gamma$ to $\Delta$, and apply case 1 of the proof of Theorem B to obtain that $\theta q_k \colon \Gamma_k \to \Gamma_k$ is an automorphism if $G_k$ is non-abelian, where $q_k \colon \Gamma \to \Gamma_k$ is the $k$th projection. So if $j \neq k$, and $G_k$ is non-abelian, then $[\Gamma_j\theta q_k, \Gamma_k \theta q_k] = [\Gamma_j, \Gamma_k]\,\theta q_k = 1$, and so $\Gamma_j \theta q_k \leqslant Z(\Gamma_k)$. This last holds trivially if $G_k$ is abelian.

Hence for each $j$, we have $\Gamma_j\theta \leqslant \Gamma_j Z(\Gamma)$, and so $(\Gamma_j Z(\Gamma))\,\theta = \Gamma_j Z(\Gamma)$ (considering $\theta^{-1}$). Hence $\Gamma_j Z(\Gamma)$ and $\Gamma_j' = [\Gamma_j Z(\Gamma)]'$ are characteristic in $\Gamma$.

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If in addition $\mathrm{Hom}[\Gamma, Z(\Gamma)] = \{e\}$, $j \leqslant m$, and $\theta \in \mathrm{Aut}\,\Gamma$, then $\Gamma_j \theta q_k = 1$ for each $k \neq j$, and so $\Gamma_j \theta \leqslant \Gamma_j$. Hence $\Gamma_j$ is characteristic in $\Gamma$. ∎

COROLLARY B2. *Suppose that $G$ is a finite group, with $G \cong (H_1)^{t_1} \times \ldots \times (H_m)^{t_m}$, where $H_1, \ldots, H_m$ are non-isomorphic finite indecomposable groups, and $t_i \geqslant 1$ for each $i$. Then $G$ is Boolean separating if and only if $H_i$ is non-abelian and $t_i = 1$ for some $i \leqslant m$.*

Our proof of Corollary B2 uses the existence of a countable Boolean ring $R$ such that $R^m \cong R^n$ for all $m, n \geqslant 2$, but $R \ncong R^2$. I am very grateful to J. Plotkin and to the referee for independently pointing out that the existence of such an $R$ follows from the work of Ketonen (10) (for by ((10), theorem 7·1), the semigroup $\{1, a, a^2\}$, with relation $a^2 = a^3$, can be embedded in the semigroup of isomorphism types of countable Boolean algebras under direct product).

*Proof of Corollary* B2. 'If' is immediate from Theorem B and Proposition 2·1. Conversely, suppose that for each $i$, either $H_i$ is abelian or $t_i \geqslant 2$. Let $R$ be as above, and take $S = R^2$, so that $R \ncong S$. Now if $H$ is a cyclic $p$-group for some prime $p$, and $T$ is any countably infinite Boolean ring, then $H^T$ is isomorphic to $H^\omega$: for $H^T$ is a direct product of cyclic groups, and all elements of order $p$ in $H^T$ have height $s-1$, where $|H| = p^s$. So for any finite abelian group $H$, we have $H^R \cong H^S$. If, on the other hand, $H$ is non-abelian and $t \geqslant 2$, then $(H^t)^R \cong (H^t)^S$, since $R^t \cong S^t$. Thus $G^R \cong G^S$, and so $G$ is not Boolean separating. ∎

## 4. *The Automorphism Group*

Let $G$ be any group, $R$ a Boolean ring with Stone space $X$, and let $\Gamma = G^R$. Let $R' = \{A \subseteq X : A \text{ is clopen}\}$, which we regard as a Boolean ring containing $R$, with $R' = R$ precisely when $R$ has an identity. As one might expect, automorphisms of $G$ or $R$ yield automorphisms of $\Gamma$:

(1) If $\alpha \in \mathrm{Aut}\,R$, we define $\alpha^\Gamma : \Gamma \to \Gamma$ by

$$\gamma\alpha^\Gamma = \alpha^{-1} \circ \gamma : X \to G \quad (\gamma \in \Gamma)$$

(here, we are identifying automorphisms of $R$ and homeomorphisms of $X$); equivalently, we have

$$(g_A)\,\alpha^\Gamma = g_{A\alpha} \quad \text{if} \quad g \in G \quad \text{and} \quad A \in R.$$

We write $(\mathrm{Aut}\,R)^\Gamma = \{\alpha^\Gamma : \alpha \in \mathrm{Aut}\,R\}$, which is a subgroup of $\mathrm{Aut}\,\Gamma$ isomorphic to $\mathrm{Aut}\,R$.

(2) If $f \in \mathrm{Aut}\,G$, we define $f_X : \Gamma \to \Gamma$ by

$$\gamma f_X = \gamma \circ f : X \to G \quad (\gamma \in \Gamma);$$

equivalently,

$$(g_A) f_X = (gf)_A \quad (g \in G, A \in R).$$

More generally, if $f_1, \ldots, f_n \in \mathrm{Aut}\,G$, and $\{X_1, \ldots, X_n\}$ is a partition of $X$ into clopen subsets, we have an automorphism $\phi$ of $\Gamma$ given by

$$x(\gamma\phi) = (x\gamma) f_i \quad \text{if} \quad \gamma \in \Gamma \quad \text{and} \quad x \in X_i;$$

or

$$g_A \phi = (g f_i)_A \quad \text{if} \quad A \in R \quad \text{and} \quad A \subseteq X_i$$

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(and for any $A \in R$, we have $A = \bigcup_{i \leqslant n} (A \cap X_i)$, and $A \cap X_i \in R$). We denote the automorphism $\phi$ by $(f_1)_{X_1} \dots (f_n)_{X_n}$, and the group of such automorphisms (which is canonically isomorphic to the Boolean power of $\operatorname{Aut} G$ by $R'$) by $(\operatorname{Aut} G)^{R'}$.

These are the two most important types of automorphisms of $\Gamma$, as they respect the Boolean structure of $\Gamma$. There is a third, less 'pleasant', type of automorphism, which does not in general respect the Boolean structure:

(3) Suppose $\nu: \Gamma \to Z(\Gamma)$ is a nilpotent endomorphism. Let $\theta: \Gamma \to \Gamma$ be given by $\gamma\theta = \gamma . \gamma\nu$ $(\gamma \in \Gamma)$, so that $\theta = 1_\Gamma + \nu$: then $\theta$ is an automorphism of $\Gamma$, by Lemma 3·1. We note that $\nu$ corresponds to a homomorphism from $\Gamma/\Gamma'$ to $Z(\Gamma)$.

In this section we shall only be concerned with Boolean powers of finite groups. We show first that the three kinds of automorphisms above generate $\operatorname{Aut}(G^R)$ when $G$ is non-abelian and indecomposable:

THEOREM C. *Let $G$ be a finite non-abelian indecomposable group, and $R$ a Boolean ring. Let $\Gamma = G^R$, and $\Delta = \Gamma/Z(\Gamma)$, which is isomorphic to $[G/Z(G)]^R$. Suppose that $\theta \in \operatorname{Aut} \Gamma$, and let $\theta^*$ be the induced automorphism of $\Delta$.*

*Then $\theta$ induces an automorphism $\theta_R$ of $R$ such that*

$$\operatorname{supp}(\delta\theta^*) = \operatorname{supp}(\delta)\theta_R \quad \textit{for all} \quad \delta \in \Delta,$$

*and*

$$\operatorname{supp}(\gamma\theta) = \operatorname{supp}(\gamma)\theta_R \quad \textit{for all} \quad \gamma \in \Gamma';$$

*and the map $\theta \mapsto \theta_R$ from $\operatorname{Aut} \Gamma$ to $\operatorname{Aut} R$ is a homomorphism.*

*Moreover, there is some $\psi \in (\operatorname{Aut} G)^{R'}$ such that*

$$\theta^* = \psi^* \circ (\theta_R)^\Delta \quad \textit{on} \quad \Delta,$$

*and*

$$\theta = \psi \circ (\theta_R)^\Gamma \quad \textit{on} \quad \Gamma'.$$

*Finally, $\theta = (1_\Gamma + \nu) \circ \psi \circ (\theta_R)^\Gamma$ for some nilpotent $\nu \in \operatorname{Hom}[\Gamma, Z(\Gamma)]$.*

*Proof.* (*a*) We obtain $\theta_R$. The proof of Theorem A shows that there is some $\theta_0 \in \operatorname{Aut}(R')$ corresponding to the automorphism $\theta$ of $\Gamma$ such that $[G^T Z(\Gamma)]\theta = G^{T\theta_0} Z(\Gamma)$, and that $R\theta_0 = R$, where we identify elements of $R'$ with summands of $R$. Let $\theta_R = \theta_0|_R$. Then $\theta_R \in \operatorname{Aut} R$, and

$$[G^{(A)} Z(\Gamma)]\theta = G^{(A\theta_R)} Z(\Gamma) \quad \text{for each} \quad A \in R.$$

It follows, writing $G^* = G/Z(G)$, that $G^{*(A)}\theta^* = G^{*(A\theta_R)}$, and that $G'^{(A)}\theta = G'^{(A\theta_R)}$, since $G'^{(A)} = [G^{(A)} Z(\Gamma)]'$.

If now $\delta \in \Delta$ and $A = \operatorname{supp}(\delta)$, then $\delta \in G^{*(A)}$, and so $\delta\theta^* \in G^{*(A\theta_R)}$, whence $\operatorname{supp}(\delta\theta^*) \subseteq \operatorname{supp}(\delta)\theta_R$. Considering $\theta^{-1}$, and noting that $(\theta^{-1})_R = (\theta_R)^{-1}$, we obtain

$$\operatorname{supp}(\delta\theta^*) = \operatorname{supp}(\delta)\theta_R \quad \text{for all} \quad \delta \in \Delta.$$

Similarly, $\operatorname{supp}(\gamma\theta) = \operatorname{supp}(\gamma)\theta_R$ for all $\gamma \in \Gamma'$. Also, the definition of $\theta_R$ immediately yields the fact that the map $\theta \mapsto \theta_R$ from $\operatorname{Aut} \Gamma$ to $\operatorname{Aut} R$ is a homomorphism.

(*b*) We construct $\psi$. Let $\phi = \theta \circ [(\theta_R)^\Gamma]^{-1}$. Then $\phi_R = 1_R$, and we have $\operatorname{supp}(\delta\phi^*) = \operatorname{supp}(\delta)$, and $\operatorname{supp}(\gamma\phi) = \operatorname{supp}(\gamma)$, for all $\delta \in \Delta$ and $\gamma \in \Gamma'$. We first claim that if $A \in R$ and $x \in A$, then $\phi_{A,x} \in \operatorname{Aut} G$.

For by Proposition 2·2 we can find $C \in R$ such that $x \in C \subseteq A$ and $\phi_{A,y} = \phi_{A,x}$ for each $y \in C$. Let $f = \phi_{A,x}$. Then, writing $R = (C) \oplus T$, $p: \Gamma \to G^T$ for the projection onto

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$G^T$, and $\beta: G \to \Gamma$ for the map $g \mapsto (g_A\phi)p$, we have $g_A\phi = (gf)_C \cdot (g\beta)\,(g \in G)$; and $\operatorname{supp}(g\beta) \cap C = 0$ for each $g \in G$. Applying $\phi^{-1} \circ \pi_x$, we obtain

$$g = [gf(\phi^{-1})_{C,x}] \cdot [g\beta\phi^{-1}\pi_x] \quad (g \in G),$$

and we note that $G\beta\phi^{-1}\pi_x \leqslant Z(G)$, since $\operatorname{supp}[(g\beta\phi^{-1})^*] = \operatorname{supp}[(g\beta)^*]$ and $x \notin \operatorname{supp}(g\beta)$ for each $g \in G$. Hence $f \circ (\phi^{-1})_{C,x} \in \operatorname{Aut} G$, by Lemma 3·2. Therefore $f = \phi_{A,x} \in \operatorname{Aut} G$, as claimed.

Now if $x \in X$, pick $A \in R$ such that $x \in A$, and let $\phi_x = \phi_{A,x}$, which is an automorphism of $G$, by the above. Suppose also that $C \in R$, with $x \in C \subseteq A$, and let $D = A - C$. Then $g_A\phi = (g_C\phi)\,(g_D\phi)$, and $\operatorname{supp}[(g_D\phi)^*] \subseteq D\,(g \in G)$. Hence $(g_C\phi)^*\pi_x = (g_A\phi)^*\pi_x = (g\phi_x)^*$ for each $g \in G$. We therefore obtain that $(g_B\phi)^*\pi_x = (g\phi_x)^* = g^*\phi_x^*$ for each $B \in R$ such that $x \in B$ (by considering $C = A \cap B$), and so $\phi_x^*$ is uniquely determined by $x$.

Select $f_1, \ldots, f_n \in \operatorname{Aut} G$ such that $f_1^*, \ldots, f_n^*$ are distinct with $\{f_1^*, \ldots, f_n^*\} = \{\phi_x^* : x \in X\}$. Let $X_i = \{x \in X : \phi_x^* = f_i^*\}$. Clearly, $\{X_1, \ldots, X_n\}$ is a partition of $X$. Also, if $x \in X_i$ and $A \in R$ with $x \in A$, then we can find $B \in R$ such that $x \in B \subseteq A$ and $\phi_{A,y} = \phi_{A,x}$ for each $y \in B$ (Proposition 2·2); hence $\phi_y^* = (\phi_{A,y})^* = \phi_x^* = f_i^*\,(y \in B)$, and so $B \subseteq X_i$. It follows that $X_i$ is clopen.

By construction, we now have

$$(g^*)_A\phi^*\pi_x = g^*f_i^* \quad \text{if} \quad x \in A \cap X_i,$$
$$= 1 \quad \text{if} \quad x \notin A.$$

Therefore for each $\delta \in \Delta$, we have $x(\delta\phi^*) = (x\delta)f_i^*$ if $x \in X_i$. Letting $\psi = (f_1)_{X_1} \ldots (f_n)_{X_n}$, we obtain $\phi^* = \psi^*$, and so $\theta^* = \psi^* \circ (\theta_R)^\Delta$, with $\psi \in (\operatorname{Aut} G)^{R'}$.

Now commutation induces a well-defined map from $G/Z(G) \times G/Z(G)$ to $G'$, which we denote $[\ ,\ ]$, and clearly

$$[\delta_1, \delta_2]\theta = [\delta_1\theta^*, \delta_2\theta^*] = [\delta_1, \delta_2]\psi(\theta_R)^\Gamma \quad \text{if} \quad \delta_1, \delta_2 \in G/Z(G).$$

Hence $\theta = \psi \circ (\theta_R)^\Gamma$ on $\Gamma'$, as claimed.

The final assertion comes by applying Lemma 3·1(*b*) to the automorphism $\theta \circ [\psi \circ (\theta_R)^\Gamma]^{-1}$ of $\Gamma$, and recalling Lemma 3·2(*c*). ∎

COROLLARY C1. *If $G$, $R$, and $\Gamma$ satisfy the conditions of theorem C and, in addition,* $\operatorname{Hom}[G, Z(G)] = \{e\}$, *then automorphisms of $\Gamma$ have form*

$$\psi \circ \alpha^\Gamma, \quad \textit{with} \quad \psi \in (\operatorname{Aut} G)^{R'}, \quad \textit{and} \quad \alpha \in \operatorname{Aut} R.$$

*Furthermore,* $\operatorname{Aut}\Gamma$ *is the split extension of* $(\operatorname{Aut} G)^{R'}$ *by* $(\operatorname{Aut} R)^\Gamma$.

*Proof.* By Lemma 1 of (8), or by Proposition 2·2, we have $\operatorname{Hom}[\Gamma, Z(\Gamma)] = \{e\}$. Hence, by Theorem C, each automorphism $\theta$ of $\Gamma$ has form $\theta = \psi \circ (\theta_R)^\Gamma$, with $\psi \in (\operatorname{Aut} G)^{R'}$. This is the first assertion. Now the map $\theta \mapsto \theta_R$ from $\operatorname{Aut}\Gamma$ to $\operatorname{Aut} R$ is a homomorphism, and $\theta_R = 1_R$ if and only if $\theta \in (\operatorname{Aut} G)^{R'}$ ('only if' follows from the above form for $\theta$; conversely, if $\theta \in (\operatorname{Aut} G)^{R'}$, then it is easily seen that $\operatorname{supp}(\gamma\theta) = \operatorname{supp}(\gamma)$, for all $\gamma \in \Gamma$, and so $\theta_R = 1_R$). This also shows that $(\operatorname{Aut} R)^\Gamma \cap (\operatorname{Aut} G)^{R'} = \{1_\Gamma\}$, since $(\alpha^\Gamma)_R = \alpha$ for each $\alpha \in \operatorname{Aut} R$. Hence $\operatorname{Aut}\Gamma$ is the split extension of $(\operatorname{Aut} G)^{R'}$ by $(\operatorname{Aut} R)^\Gamma$: the action of $(\operatorname{Aut} R)^\Gamma$ on $(\operatorname{Aut} G)^{R'}$ is obtained by identifying $(\operatorname{Aut} R)^\Gamma$ with $\operatorname{Aut} R$, and noting that automorphisms of $R$ induce automorphisms of $R'$, and so of $(\operatorname{Aut} G)^{R'}$. ∎

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For a general finite non-abelian indecomposable group $G$, the automorphism group of $\Gamma = G^R$ has a more complicated structure: Theorem C gives us a series

$$1 < N \lhd \Psi \lhd \operatorname{Aut}\Gamma,$$

with $\Psi = \{\theta \in \operatorname{Aut}\Gamma : \theta_R = 1_R\}$, $(\operatorname{Aut}\Gamma)/\Psi \cong \operatorname{Aut}R$, and $\operatorname{Aut}\Gamma$ the split extension of $\Psi$ by $(\operatorname{Aut}R)^\Gamma$. Also, $N = \{\theta \in \operatorname{Aut}\Gamma : \theta^* = 1\}$, with $\Psi/N \cong [(\operatorname{Aut}G)^*]^{R'}$, where $(\operatorname{Aut}G)^* = \{f^* \in \operatorname{Aut}(G^*) : f \in \operatorname{Aut}G\}$ (writing asterisks modulo $Z(G)$ and $Z(G)^R$). There is a bijection between $\operatorname{Hom}[\Gamma, Z(\Gamma)]$ and $N$ given by

$$\nu \mapsto 1_\Gamma + \nu, \quad \text{for} \quad \nu \in \operatorname{Hom}[\Gamma, Z(\Gamma)],$$

and multiplication in $N$ is given by

$$(1_\Gamma + \mu)(1_\Gamma + \nu) = 1_\Gamma + (\mu + \nu + \mu\nu), \quad \text{for} \quad \mu, \nu \in \operatorname{Hom}[\Gamma, Z(\Gamma)]$$

(this is just the normal way of constructing a nilpotent group out of a nilpotent ring).

Lastly in this section, we draw out the implications of Theorems B and C for automorphisms of a product of Boolean powers of finite groups:

COROLLARY C2. *Suppose that $G_1, \ldots, G_m$ are non-isomorphic finite indecomposable groups, and that $R_1, \ldots, R_m$ are non-zero Boolean rings. Let $\Gamma_i = G_i^{R_i}$ $(1 \leqslant i \leqslant m)$, and let $\Gamma = \Gamma_1 \times \ldots \times \Gamma_m$. Suppose that $\theta \in \operatorname{Aut}\Gamma$. Then $\theta$ induces an automorphism $\theta_i^*$ of $\Delta_i = \Gamma_i/Z(\Gamma_i)$ such that*

$$\theta_i^* = \psi_i^* \circ (\alpha_i)^{\Delta_i} \quad \textit{for some} \quad \psi_i \in (\operatorname{Aut}(G_i))^{R_i} \quad \textit{and} \quad \alpha_i \in \operatorname{Aut}(R_i);$$
$$\textit{further,} \quad \theta = \psi_i \circ (\alpha_i)^{\Gamma_i} \quad \textit{on} \quad \Gamma_i'.$$

*Finally, there is some $\zeta \in \operatorname{Hom}[\Gamma, Z(\Gamma)]$ (not necessarily nilpotent) such that*

$$\theta = (1_\Gamma + \zeta) \circ \prod_{i \leqslant n} [\psi_i \circ (\alpha_i)^{\Gamma_i}].$$

*Proof.* Suppose that $i \leqslant n$. By Corollary B1, $\theta$ fixes $\Gamma_i Z(\Gamma)$ and $\Gamma_i'$, and so induces an automorphism of $\Gamma_i Z(\Gamma)/Z(\Gamma)$ – which is isomorphic to $\Gamma_i/Z(\Gamma_i)$ – say $\theta$ induces $\theta_i^* \in \operatorname{Aut}(\Delta_i)$. Let $q_i : \Gamma \to \Gamma_i$ be the $i$th projection. If $G_i$ is abelian, there is nothing to prove (take $\psi_i = 1, \alpha_i = 1$); otherwise, $\theta q_i : \Gamma_i \to \Gamma_i$ is an automorphism (cf. the proof of Corollary B1), and $(\theta q_i)^* = \theta_i^*$. Theorem C applied to $\theta q_i$ now gives the required form for $\theta_i^*$ and for $\theta|_{\Gamma_i}$ (since $\theta$ and $\theta q_i$ have identical action on $\Gamma_i'$).

Let $\eta = \Pi_{i \leqslant n}(\psi_i \circ (\alpha_i)^{\Gamma_i})$: this is an automorphism of $\Gamma = \Pi_{i \leqslant n} \Gamma_i$. We have $[\theta \circ \eta^{-1}]_i^* = 1_{\Delta_i}$ for each $i$, and so $\theta \circ \eta^{-1}$ induces the trivial automorphism of $\Gamma/Z(\Gamma)$. Hence $\theta \circ \eta^{-1} = 1_\Gamma + \zeta$ for some $\zeta \in \operatorname{Hom}[\Gamma, Z(\Gamma)]$, by Lemma 3·1, and the assertion follows. ∎

## 5. *Boolean Powers of Simple Groups*

Let $G$ be a non-abelian simple group, $R$ a Boolean ring, and let $\Gamma = G^R$. We obtain a complete description of the normal and characteristic subgroups of $\Gamma$, and then show in section 6 that there are just three countably infinite characteristically simple Boolean rings. These results were all included in a seminar given by Hall in 1959.

Our first result, about normal subgroups of $\Gamma$, is contained in Hall ((7); lemma 13·2) and in Neumann and Yamamuro ((11); theorem 5).

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PROPOSITION 5·1. *For each $\gamma \in \Gamma$, we have $\gamma^\Gamma = G^{(\mathrm{supp}\,\gamma)}$. Furthermore, the normal subgroups of $\Gamma$ are just those of form $G^S$, for $S$ an ideal of $R$.*

*Proof.* Suppose that $\gamma \in \Gamma$, and write $\gamma = (g_1)_{A_1} \dots (g_n)_{A_n}$, with each $g_i \in G - 1$, and $A_1, \dots, A_n$ disjoint. Suppose that $i \leqslant n$, $B \in R$, and $B \subseteq A_i$. We can find $h \in G$ with $[g_i, h] \neq 1$. Now $[\gamma, h_B] = [g_i, h]_B \in \gamma^\Gamma$. Write $k = [g_i, h]$; then $k_B \in \gamma^\Gamma$, and so $g_B^{-1} k_B g_B = (k^g)_B \in \gamma^\Gamma$ $(g \in G)$. Since $G$ is simple, we see that $f_B \in \gamma^\Gamma$ for all $f \in G$.

It follows that $f_B \in \gamma^\Gamma$ for all $f \in G$ and all $B \in R$ such that $B \subseteq A_1 \cup \dots \cup A_n = \mathrm{supp}\,(\gamma)$; hence $G^{(\mathrm{supp}\,\gamma)} \leqslant \gamma^\Gamma$. Conversely, $\gamma \in G^{(\mathrm{supp}\,\gamma)} \lhd G^R$ (using Proposition 2·1), and so $\gamma^\Gamma \leqslant G^{(\mathrm{supp}\,\gamma)}$, and equality follows.

Suppose finally that $\Delta \lhd \Gamma$. Clearly $\Delta = \langle \delta^\Gamma : \delta \in \Delta \rangle$, and so $\Delta = G^S$, where $S$ is the ideal of $R$ generated by $\{\mathrm{supp}\,(\delta) : \delta \in \Delta\}$. Proposition 2·1 $(a)$ completes the proof. ∎

PROPOSITION 5·2 (Hall, unpublished). *The characteristic subgroups of $G^R$ are just those of form $G^S$, with $S$ a characteristic ideal of $R$. Moreover, $G^R$ is characteristically simple if and only if $R$ is characteristically simple.*

*Proof.* Let $\Gamma = G^R$, and suppose $\theta \in \mathrm{Aut}\,\Gamma$. Then $\theta$ permutes $\{\gamma^\Gamma : \gamma \in \Gamma\}$; but $\{\gamma^\Gamma : \gamma \in \Gamma\} = \{G^{(A)} : A \in R\}$, by Proposition 5·1. Hence $\theta$ induces $\theta_R \in \mathrm{Aut}\,R$ such that $G^{(A)}\theta = G^{(A\theta R)}$ for all $A \in R$; and so we also have $G^S\theta = G^{S\theta_R}$ for each ideal $S$ of $R$. The assertions now follow, noting that characteristic subgroups are normal, and that the map $\theta \mapsto \theta_R$ from $\mathrm{Aut}\,\Gamma$ to $\mathrm{Aut}\,R$ is onto. ∎

*Remarks.* (1) We could have used the proof of Theorem A to obtain $\theta_R$, but the above approach is more elementary.

(2) By some results of Hall, and Wilson (see ((1); theorems B and C)), any characteristically simple group satisfying certain finiteness conditions is a Boolean power of a finite simple non-abelian group (by a characteristically simple Boolean ring).

## 6. *Countable Characteristically Simple Boolean Rings*

Proposition 5·2 leads us to ask which Boolean rings are characteristically simple. The answer for countable Boolean rings is contained in Theorem 6·3 below; this result is stated (without proof) in Hall ((7); p. 468).

We recall that an element $A$ of a Boolean ring $R$ is said to be an *atom* of $R$ if $A \neq 0$, and for each $B \in R$, either $AB = 0$ or $AB = A$; and that a non-zero Boolean ring with no atoms is termed *atomless*.

PROPOSITION 6·1. *Up to isomorphism, there is just one countable atomless Boolean ring with identity, and just one without identity.*

This result is well known for rings with identity (see Sikorski ((14); section 9 C), or Barwise ((3); p. 8)), but we include a proof of this proposition as we have been unable to find a reference for the other case.

*Proof.* For the existence of such rings, consider

$$R_1 = \langle m + 2^n\mathbb{Z} : m \in \mathbb{Z}, n \geqslant 0 \rangle, \quad \text{and} \quad R_0 = \{A \in R_1 : 0 \notin A\},$$

viewed as subrings of $2^{\mathbb{Z}}$ (cf. ((7); pp. 468, 469)). It is readily verified that $R_1$ and $R_0$ are countable atomless Boolean rings with identity and without identity respectively.

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For uniqueness, suppose that $R$ and $S$ are two countable atomless Boolean rings without identity (respectively with identity). We claim that if $J$ and $K$ are finite subrings (respectively finite subrings containing the ring identity) of $R$ and $S$ respectively, $\theta_0\colon J \to K$ is any isomorphism, and $A \in R$, then there is some $B \in S$ and an isomorphism $\theta_1\colon \langle J, A\rangle \to \langle K, B\rangle$ extending $\theta_0$ such that $A\theta_1 = B$.

For let $C_1, \ldots, C_n$ be the atoms of $J$, and let $D_i = C_i\theta_0 (1 \leqslant i \leqslant n)$, so that $D_1, \ldots, D_n$ are the atoms of $K$. For each $i$, let $A_i = A \cap C_i$, and pick $B_i \in S$ such that $B_i \subseteq D_i$, and such that $B_i = 0$, $B_i = D_i$, or $0 \subset B_i \subset D_i$ according as $A_i = 0$, $A_i = C_i$, or $0 \subset A_i \subset C_i$. If $R$ and $S$ are without identities, let $C = \bigcup_{i=1}^n C_i$, $D = \bigcup_{i=1}^n D_i$, and $A_0 = A - C$; now pick $B_0 \in S$ such that $B_0 \cap D = 0$, with $B_0 = 0$ or $B_0 \neq 0$ according as $A_0 = 0$ or $A_0 \neq 0$. Take $B = \bigcup_{i=0}^n B_i$ (respectively $\bigcup_{i=1}^n B_i$), and let the map $\theta_1\colon \langle J, A\rangle \to \langle K, B\rangle$ be determined by $A_i\theta_1 = B_i$, $C_i\theta_1 = D_i$ for each $i$; $\theta_1$ is an isomorphism, and $A\theta_1 = B$.

The claim is established, and an application of the 'back and forth' argument (cf. Barwise ((3); pp. 5–9)) shows that $R$ and $S$ are isomorphic. ∎

COROLLARY 6·2. *If $R$ is a countable atomless Boolean ring, and $A$ and $B$ are non-zero, non-identity elements of $R$, then there is an automorphism $\theta$ of $R$ such that $A\theta = B$.*

*Proof.* This is immediate from the proof of Proposition 6·1, taking $J$ to be $\{0, A, X - A, X\}$ (where $X$ is the Stone space of $R$) or $\{0, A\}$ according as $R$ has an identity or not (and similarly for $K$). ∎

We denote the countable atomless Boolean ring without identity (respectively with identity) by $V_0$ (respectively $V_1$); these have Stone spaces $2^\omega - \{x\}$ (any $x \in 2^\omega$) and $2^\omega$ respectively (cf. section 9 C of (14)). Also, $V_0$ embeds in $V_1$ as an ideal of index 2, while $V_0 \cong V_1^\omega$. We can think of $V_0$ (respectively $V_1$) as being the free object on countably many generators in the category of Boolean rings (respectively Boolean rings with identity). We note in passing that Proposition 6·1 above shows that $V_0$ and $V_1$ are $\aleph_0$-categorical: for the non-existence of atoms, and the existence or non-existence of an identity, are easily seen to be first-order properties in the theory of Boolean rings. This fact, which appears to have acquired the status of 'folklore', was, I believe, first noticed by Waszkiewicz and Weglorz (16).

We also write $V_a$ for the ring of finite subsets of $\mathbb{Z}$; $V_a$ is generated by its atoms, and lies at the opposite end of the atomic spectrum to $V_0$ and $V_1$.

We now have the following classification:

THEOREM 6·3 (Hall). *Up to isomorphism, the only countable characteristically simple Boolean rings are the finite Boolean rings, and $V_0$, $V_1$, and $V_a$.*

*Proof.* Suppose first that $R$ is a countably infinite characteristically simple Boolean ring. If $R$ is atomless, then $R \cong V_0$ or $R \cong V_1$, by Proposition 6·1. Otherwise, $L = \{A \in R : A \text{ is an atom}\}$ is non-empty, and so $\langle L\rangle$, being a non-zero characteristic ideal of $R$, must equal $R$. Hence $R$ is isomorphic to the ring of finite subsets of $L$; that is, $R \cong V_a$.

Conversely, $V_0$ and $V_1$ are characteristically simple by Corollary 6·2. Suppose, lastly, that $R$ is finite or $R \cong V_a$, so that $R$ is generated by its atoms. If $S$ is a non-zero characteristic ideal of $R$, then $S$ contains an atom of $R$, and so contains all the atoms of $R$, whence $S = R$. Hence $R$ is characteristically simple. ∎

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### 7. *An Application*

It is known (see, for example, Phillips(12)) that if a group $G$ is characteristically simple, then each countable subgroup of $G$ is contained in some countable characteristically simple subgroup of $G$. This suggests the following question:

If $G$ is a group such that every countable subgroup of $G$ is contained in a countable characteristically simple subgroup of $G$, is $G$ necessarily characteristically simple?

Wilson constructed a counter-example in (17), using McLain groups. We show below that certain Boolean powers of any countable non-abelian simple group $G$ also serve as counter-examples; and if we take this group $G$ to be finite, then we obtain counter-examples which are locally finite of bounded exponent (cf. ((7); p. 465)). We shall use the fact that any countable atomless Boolean ring is characteristically simple in conjunction with:

LEMMA 7·1. *Let $R$ be an atomless Boolean ring. Then every countable subring of $R$ is contained in a countable atomless subring of $R$.*

*Proof.* Let $L$ be a countable subring of $R$. We successively construct countable subrings $L = M_0 \leqslant M_1 \leqslant M_2 \leqslant \ldots$ of $R$ such that no element of $M_i$ is an atom of $M_{i+1}$. Let $M_0 = L$. Suppose that $i \geqslant 0$ and that we have $M_i$. Since $R$ is atomless, we can find a map (of sets) $f: M_i \to R$ such that $0f = 0$ and $0 \subset Af \subset A$ for each $A \in M_i - 0$. Let $M_{i+1} = \langle M_i, M_i f \rangle$: this clearly has the required properties. Now set $M = \bigcup_{i \geqslant 0} M_i$: this is a countable subring of $R$ containing $L$, and is easily seen to be atomless. ∎

LEMMA 7·2. *There is an uncountable atomless Boolean ring $R$ with a non-zero countable characteristic ideal $S$.*

*Proof.* Let $R$ be the cartesian product of $\aleph_0$ copies of $V_1$, and let $S = \{A \in R: (A)$ is countable$\}$. Then $R$ is uncountable and atomless, while $S$ is (naturally isomorphic to) the direct product of $\aleph_0$ copies of $V_1$. Hence $S$ is a countable ideal of $R$ which, by its definition, is clearly characteristic in $R$. ∎

THEOREM D. *There is an uncountable Boolean ring $R$ such that if $G$ is any countable simple non-abelian group, then $G^R$ is uncountable and has a non-trivial countable characteristic subgroup; however, each countable subgroup of $G^R$ is contained in a countable characteristically simple subgroup of $G^R$.*

*Proof.* Let $R$ and $S$ be as in Lemma 7·2. Suppose that $G$ is any countable non-abelian simple group, and let $\Gamma = G^R$. Then $G^S$ is countable, and is characteristic in $\Gamma$ by Proposition 5·2. Let $\Delta$ be any countable subgroup of $\Gamma$, and let $T$ be the subring of $R$ generated by the constancy sets of elements of $\Delta$: $T$ is a countable subring of $R$, and $\Delta \leqslant G^T$. By Lemma 7·1, we can find a countable atomless subring $U$ of $R$ containing $T$. Now $\Delta \leqslant G^U$, and $G^U$ is characteristically simple by Proposition 5·2 and Theorem 6·3. ∎

### 8. CHARACTERISTIC SUBGROUPS

Let $G$ be a finite non-abelian group and $R$ a Boolean ring. We investigate which subgroups of $G^R$ are characteristic, using the results of section 4. The obvious candidates are subgroups of form $K^S$, with $K$ a characteristic subgroup of $G$, and $S$ a characteristic ideal of $R$. In fact, we have:

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THEOREM E. (*a*) *Let $G$ be a finite non-abelian indecomposable group, and $R$ a non-zero Boolean ring. Suppose that $K$ is a characteristic subgroup of $G$. Then $K^R$ is a characteristic subgroup of $G^R$. If also* $\mathrm{Hom}[G, Z(G)] = \{e\}$, *and $S$ is a characteristic ideal of $R$, then $K^S$ is characteristic in $G^R$.*

(*b*) *Let $G_1, \ldots, G_n$ be non-isomorphic finite indecomposable groups, and $R_1, \ldots, R_n$ non-zero Boolean rings. Let $\Gamma_i = (G_i)^{R_i} (1 \leqslant i \leqslant n)$, and let $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$. Suppose that $i \leqslant n$ is given, that $K_i$ is a characteristic subgroup of $G_i$, and that $S_i$ is a characteristic ideal of $R_i$. Then $(K_i)^{S_i} Z(\Gamma)$ is characteristic in $\Gamma$. If also $K_i \leqslant G_i'$, then $(K_i)^{S_i}$ is characteristic in $\Gamma$.*

*Proof.* (*a*) Let $\Gamma = G^R$. By Theorem C, each automorphism of $\Gamma$ has form $(1_\Gamma + \nu) \circ \psi \circ \alpha^\Gamma$, with $\nu \in \mathrm{Hom}[\Gamma, Z(\Gamma)]$, $\psi \in (\mathrm{Aut}\, G)^{R'}$, and $\alpha \in \mathrm{Aut}\, R$. Let $X$ be the Stone space of $R$; in the notation of Proposition 2·2, we have that $\nu_{A,x}$ is an endomorphism of $G$ belonging to $\mathrm{Hom}[G, Z(G)]$ for each $A \in R - 0$ and $x \in X$.

But if $\delta \in \mathrm{Hom}[G, Z(G)]$, then $\delta$ is nilpotent (Lemma 3·2), and so $1_G + \delta$ is an automorphism of $G$, by Lemma 3·1. Since $K$ is characteristic in $G$, we have $g(g\delta) \in K$ for each $g \in K$, and so $K\delta \leqslant K$.

In particular, $K\nu_{A,x} \leqslant K$ for each $A \in R - 0$ and $x \in X$. Hence $K^R\nu \leqslant K^R$, and $K^R(1_\Gamma + \nu) \leqslant K^R$. Moreover, $K^R$ is clearly invariant under $\psi$ and $\alpha^\Gamma$. Therefore $K^R$ is characteristic in $\Gamma$.

If in addition $\mathrm{Hom}[G, Z(G)] = \{e\}$, then $\mathrm{Hom}[\Gamma, Z(\Gamma)] = \{e\}$; and if $S$ is a characteristic ideal of $R$, then $K^S$ is invariant under elements of $(\mathrm{Aut}\, G)^{R'}$ (since $K$ is characteristic in $G$ and $S$ is an ideal of $R$), and under elements of $(\mathrm{Aut}\, R)^\Gamma$ (since $S$ is characteristic in $R$). The result follows from Corollary C1.

(*b*) As in (*a*), $(K_i)^{S_i}$ is invariant under elements of $(\mathrm{Aut}\, G_i)^{R_i}$ and of $(\mathrm{Aut}\, R_i)^{\Gamma_i}$; the assertions follow from Corollary C2. ∎

If $G$ is any finite group, $R$ is any Boolean ring, $K$ is a characteristic subgroup of $G$, and $S$ is a characteristic ideal of $R$, then there are two reasons why $K^S$ may not be characteristic in $G^R$, which explain the gaps in the statement of Theorem E.

First, if $G$ is indecomposable and $\mathrm{Hom}[G, Z(G)] \neq \{e\}$, then $G^S$ will not be characteristic in $G^R$ unless $S = 0$ or $S = R$. For let $f$ be a non-trivial element of $\mathrm{Hom}[G, Z(G)]$; suppose that $0 < S < R$, and pick non-zero elements $A, B$ of $R$ with $A \in S, B \notin S$. Write $Z = Z(G)$ and $\Gamma = G^R$. Then $Z_A$ and $Z_B$ are both summands of $Z^R$ (for they are easily seen to be pure in $Z^R$: cf. corollary 24·6 of (6)), and so there is some homomorphism $\theta$ from $Z^R$ to $Z^R$ such that $z_A\theta = z_B (z \in Z)$. Let $\nu = f_X \circ \theta : \Gamma \to Z(\Gamma)$, where $f_X : \Gamma \to Z(\Gamma)$ is given by $\gamma f_X = \gamma \circ f (\gamma \in \Gamma)$. Now $\nu$ is nilpotent and $\phi = 1_\Gamma + \nu$ is an automorphism of $\Gamma$, by Lemmas 3·1 and 3·2. However, $G^S\nu \geqslant (G_A)\nu = (Gf)_A\theta = (Gf)_B$, and so $G^S\phi \neq G^S$. Thus $G^S$ is not characteristic in $G^R$ (this illustrates how an automorphism of $\Gamma$ need not respect the Boolean structure of $\Gamma$).

Second, if $G$ is decomposable, then $K^R$ need not be characteristic in $G^R$. For example, take $G$ to be $Q_8 \times D_8$, while $Q_8$ and $D_8$ denote the quaternion and dihedral groups respectively of order 8. Let $q$ and $d$ be the central involutions of $Q_8$ and $D_8$ respectively. Each element of $Z(G)$ is fixed by each automorphism of $G$ (as $qf = 1$ for any homomorphism $f$ from $Q_8$ to $D_8$, and similarly for $d$); hence $K = \{1, (q, d)\}$ is a characteristic subgroup of $G$. Take $R$ to be the finite Boolean ring with two atoms. Then $G^R \cong G \times G = (Q_1 \times D_1) \times (Q_2 \times D_2)$, say, with $Q_i \cong Q_8$ and $D_i \cong D_8 (i = 1, 2)$. However, $K \times K$ is not characteristic in $G \times G$: for let $\theta$ be an automorphism of $G \times G$ such that

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$Q_1\theta = Q_2, Q_2\theta = Q_1$, and $D_i\theta = D_i$ $(i = 1, 2)$; then $(q, d, 1, 1) \in K \times K$, but $(q, d, 1, 1)\theta = (1, d, q, 1) \notin K \times K$.

The characteristic subgroups of $G^R$ considered in Theorem E had form $K^S$ (with $K$ a characteristic subgroup of $G$, and $S$ a characteristic ideal of $R$), or were the join of finitely many such subgroups (since $Z(G^R) = Z(G)^R$). There is an obvious converse question: do all characteristic subgroups of $G^R$ have this form? In general, this appears to be a difficult question, even if $R$ is finite. In section 5 we considered this question for a simple group $G$. We now turn to the case when $R$ is countably infinite and characteristically simple (so that $R \cong V_0$, $V_1$, or $V_a$ by Theorem 6·3): we obtain partial results on the characteristic subgroups of $G^R$ for any group $G$, and give a complete description of them when $G$ is finite, non-abelian, and indecomposable.

We recall from section 4 that any automorphism $\theta$ of $R$ induces an automorphism $\theta^\Gamma$ of $\Gamma = G^R$. We therefore view $\mathrm{Aut}\, R$ as acting on $\Gamma$, and write $\gamma\theta$ for $\gamma(\theta^\Gamma)$ when $\gamma \in \Gamma$ and $\theta \in \mathrm{Aut}\, R$. We begin with the case when $R$ is atomless:

PROPOSITION 8·1. *Let $G$ be any group. Let $R = V_1$ (respectively $V_0$), and let $X$ be the Stone space of $R$. Then the subgroups of $G^R$ invariant under the action of* $\mathrm{Aut}\, R$ *are precisely those of form $H^R K_X$, with $H \lhd K \leqslant G$, where $K_X = \{k_X : k \in K\}$ (respectively $H^R$, with $H \leqslant G$).*

*Proof.* Let $G$, $R$, and $X$ be as given; let $\Gamma = G^R$. Subgroups of the given form are readily verified to be invariant under $\mathrm{Aut}\, R$. Suppose, conversely, that $\Delta$ is a subgroup of $\Gamma$ invariant under $\mathrm{Aut}\, R$. Let $H = \{\delta\pi_x : x \in X, \delta \in \Delta, \mathrm{supp}\,(\delta) \neq X\}$, and let $K = \{k \in G : k_X \in \Delta\}$. Clearly, $K$ is a subgroup of $G$. We show that $H$ is a normal subgroup of $K$ and that $H^R \leqslant \Delta$.

Suppose, then, that $h \in H - 1$. Select $\delta \in \Delta$ and $x \in X$ such that $\delta\pi_x = h$ and $\mathrm{supp}\,(\delta) \neq X$. Find $B \in R$ containing $x$ such that $\delta\pi_y = h$ for each $y \in B$. Let $C = \mathrm{supp}\,(\delta) - B$, and let $I = \{A \in R : A \cap C = 0\}$. Now $R = (C) \oplus I$, and $B$ is not the identity of $I$, as $\mathrm{supp}\,(\delta) \neq X$. Pick $D \in I$ such that $0 \subset D \subset B$. By Corollary 6·2, there is some $\alpha \in \mathrm{Aut}\, I$ such that $B\alpha = D$ (for $I$ is atomless); and $\alpha$ extends to an automorphism $\beta$ of $R$ such that $A\beta = A$ for each $A \subseteq C$. We can write $\delta = h_B\delta_1$ for some $\delta_1 \in \Gamma$ such that $\mathrm{supp}\,(\delta_1) = C$, and we now have $\delta\beta = h_D\delta_1$. Hence $h_{B-D} = \delta(\delta\beta)^{-1} \in \Delta$. Using Corollary 6·2 a second time, we have $h_A \in \Delta$ for each $A \in R$ such that $0 \subset A \subset X$. Hence $H$ is a subgroup of $G$; and $H^R$, being generated by elements of form $h_A$, with $h \in H - 1$ and $0 \subset A \subset X$, is contained in $\Delta$. Also, if $h \in H$, $k \in K$, and $0 \subset A \subset X$, then $(k^{-1}hk)_A = (k_X)^{-1} h_A k_X \in \Delta$, and so $k^{-1}hk \in H$; thus $H$ is normal in $K$.

If now $R = V_0$, then $\mathrm{supp}\,(\delta) \neq X$ for each $\delta \in \Delta$. Therefore $\Delta \leqslant H^R$, and equality follows.

Suppose, therefore, that $R = V_1$. We have seen that $H^R$, and so $H^R K_X$, is contained in $\Delta$; we prove the converse. If $\delta \in \Delta$ and $\mathrm{supp}\,(\delta) \neq X$, then $\delta \in H^R$. Suppose $\delta \in \Delta$ and $\mathrm{supp}\,(\delta) = X$. If $\delta\pi_x = \delta\pi_y$ for all $x, y \in X$, then $\delta = g_X$ for some $g \in G$, and so $\delta \in K_X$. Otherwise, let $g$ and $h$ be any two distinct values taken by $\delta$ (regarded as a function from $X$ to $G$). As above, we can find $A, B \in R - 0$, and $\delta_1 \in \Gamma$, with $A, B$, and $\mathrm{supp}\,(\delta_1)$ disjoint, such that $\delta = g_A h_B \delta_1$. Find $C \in R$ such that $0 \subset C \subset A$ and let $D = A - C$; then there is some $\alpha \in \mathrm{Aut}\, R$ such that $A\alpha = C$, $(A + B)\alpha = A + B$, and $\alpha$ is the identity on $(\mathrm{supp}\,\delta_1)$ (apply Corollary 6·2 in the atomless ring $(A + B)$). We have $\delta\alpha = g_C h_{D+B}\delta_1$; so $(gh^{-1})_D = \delta(\delta\alpha)^{-1} \in \Delta$, and $gh^{-1} \in H$.

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Now fix some value $h$ taken by $\delta$, and let $\delta_2 = \delta(h_X)^{-1}$. Then $\delta_2\pi_x \in H$ for all $x \in X$, and so $\delta_2 \in H^R \leqslant \Delta$. Hence $h_X \in \Delta$; thus $h \in K$, and $\delta = \delta_2 h_X \in H^R K_X$. Therefore $\Delta \leqslant H^R K_X$, and equality follows, as required. ∎

Now let $R = V_a$, and let $X$ be the Stone space of $R$, which we may take to be the set of atoms of $R$ (with the discrete topology); we identify $R$ with the ring of finite subsets of $X$. For any group $K$, we define a map

$$\det_K : K^R \to K/K' : \gamma \mapsto \prod_{x \in X} K'(\gamma\pi_x) \quad (\gamma \in K^R);$$

$\det_K$ is well defined (since $\gamma\pi_x = 1$ for all but finitely many $x \in X$ if $\gamma \in K^R$), and is clearly a homomorphism.

PROPOSITION 8·2. *Let $G$ be any group. Then the subgroups of $G^R$ invariant under* Aut $R$ *are precisely those of shape*

$$\{\gamma \in K^R : \det_K \gamma \in H/K'\},$$

*where $H$ and $K$ are subgroups of $G$ such that $K' \leqslant H \leqslant K$.*

*Proof.* We observe that Aut $R$ is essentially the group of permutations of $X$. Thus subgroups of the given form are invariant under Aut $R$.

Conversely, suppose that $\Delta$ is a subgroup of $\Gamma$ invariant under the action of Aut $R$, and let $K = \{\delta\pi_x : \delta \in \Delta, x \in X\}$. We show first that if $k \in K - 1$, and $C, D$ are distinct atoms of $R$, then

$$k_C(k^{-1})_D \in \Delta. \tag{1}$$

For select $\delta \in \Delta$ and $x \in X$ such that $\delta\pi_x = k$. Let $A = \{x\}$ and $B = \{y\}$, where $y$ is any element of $X - \mathrm{supp}\,(\delta)$. Find $\theta \in \mathrm{Aut}\, R$ such that $A\theta = B$ and $\theta$ fixes $\mathrm{supp}\,(\delta) - A$. Then $k_A(k^{-1})_B = \delta(\delta\theta)^{-1} \in \Delta$. Hence $k_C(k^{-1})_D \in \Delta$ for every pair $(C, D)$ of distinct atoms of $R$, as claimed.

It follows that $K$ is a subgroup of $G$; and clearly $\Delta \leqslant K^R$. Let $A, B, C$ be distinct atoms of $R$, and suppose that $h, k \in K$: then $[h, k]_A = [h_A(h^{-1})_B, k_A(k^{-1})_C] \in \Delta$. But $R$ is generated by its atoms, and so $(K')^R \leqslant \Delta \leqslant K^R$.

Now let the subgroup $H$ of $G$ be given by $K' \leqslant H \leqslant K$ and $H/K' = \det_K(\Delta)$. We obtain immediately that $\Delta \leqslant \{\gamma \in K^R : \det_K \gamma \in H/K'\}$; we prove the reverse inclusion. Suppose that $\gamma \in K^R$ and $\det_K \gamma \in H/K'$. Select $\delta \in \Delta$ such that $\det_K \delta = \det_K \gamma$; let $\gamma_1 = \gamma\delta^{-1}$, so that $\gamma_1 \in K^R$ and $\det_K \gamma_1 = K'$. Write asterisks modulo $K'$ and $(K')^R$; then $\gamma_1^*$ can be written as a product $(k_1^*)_{A_1} \dots (k_n^*)_{A_n}$, with $k_i \in K$ and $A_i$ $(1 \leqslant i \leqslant n)$ distinct atoms of $R$. So $k_1^* \dots k_n^* = 1$ in $K^*$, and thus $\gamma_1^* = \prod_{i=2}^n [(k_i)_{A_i}(k_i^{-1})_{A_1}]^*$. Hence $\gamma_1^* \in \Delta^*$, by (1). But $\Delta \geqslant (K')^R$, and therefore $\gamma \in \Delta$. This completes the proof. ∎

THEOREM F. *Let $G$ be a finite non-abelian indecomposable group, and let $R$ be isomorphic to $V_0$ or $V_1$ or $V_a$. Then the characteristic subgroups of $G^R$ are precisely those of form*

(1) *$H^R K_X$, with $H \leqslant K \leqslant G$ and $k^{-1}(kf) \in H$ for all $k \in K$, $f \in \mathrm{Aut}\, G$, if $R = V_1$;*

(2) *$H^R$, with $H$ characteristic in $G$, if $R = V_0$;*

(3) *$\{\gamma \in K^R : \det_K \gamma \in H/K'\}$, with $K' \leqslant H \leqslant K$ and $k^{-1}(kf) \in H$ for all $k \in K, f \in \mathrm{Aut}\, G$, if $R = V_a$.*

*Note.* The conditions on $H$ and $K$ in cases (1) and (3) imply that both $H$ and $K$ are characteristic in $G$.

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*Proof.* Let $\Gamma = G^R$, and let $X$ be the Stone space of $R$.

*Case* 1. $R = V_1$. Suppose first that $\Delta$ is a characteristic subgroup of $\Gamma$. Then $\Delta$, being invariant under $\mathrm{Aut}\,R$, has form $H^R K_X$ for some subgroups $H, K$ of $G$ with $H \lhd K$, by Proposition 8·1. Suppose that $k \in K$ and $f \in \mathrm{Aut}\,G$. Pick $A \in R$ such that $0 \subset A \subset X$, and let $\theta = f_A(1_G)_{X-A}$. We have $(kf)_A k_{X-A} = k_X \theta \in \Delta$, and so

$$[k^{-1}(kf)]_A = (k_X)^{-1}(k_X\theta) \in \Delta.$$

Hence $k^{-1}(kf) \in H$ (by our definition of $H$ in Proposition 8·1), as required.

Conversely, if $H$ and $K$ satisfy the given conditions, then $H^R K_X$ is invariant under $(\mathrm{Aut}\,R)^\Gamma$, by Proposition 8·1. Also, we see easily that

$$\gamma^{-1}(\gamma\theta) \in H^R \quad \text{for all} \quad \gamma \in K^R \tag{2}$$

if $\theta \in (\mathrm{Aut}\,G)^R$ (using our conditions on $H$ and $K$). Suppose that $\theta = 1_\Gamma + \nu$ for some $\nu \in \mathrm{Hom}\,[\Gamma, Z(\Gamma)]$: we show that (2) still holds. For if $A \in R - 0$ and $x \in X$, then $\nu_{A,x}$ belongs to $\mathrm{Hom}\,[G, Z(G)]$ and $1_G + \nu_{A,x}$ is an automorphism of $G$ (by Lemmas 3·1 and 3·2); thus $K\nu_{A,x} \leqslant H$, by our assumptions on $H$ and $K$. Hence $K^R\nu \leqslant H^R$, and $\gamma^{-1}(\gamma\theta) = \gamma\nu \in H^R$ for all $\gamma \in K^R$, as claimed.

It now follows easily from Theorem $C$ and the above that $H^R K_X$ is characteristic in $\Gamma$.

*Case* 2. $R = V_0$. This proceeds as case 1, without the complication of the subgroup $K$. To show that if $H^R$ is characteristic in $G^R$ then $H$ is characteristic in $G$, we choose $A \in R - 0$ and apply the automorphism $f_X$ of $\Gamma$ to $h_A$ for each $h \in H$ and $f \in \mathrm{Aut}\,G$.

*Case* 3. $R = V_a$. Suppose first that $\Delta$ is characteristic in $\Gamma$. By Proposition 8·2, we have subgroups $H, K$ of $G$ with $K' \leqslant H \leqslant K$ and $\Delta = \{\gamma \in K^R : \det_K \gamma \in H/K'\}$. Suppose that $k \in K$ and $f \in \mathrm{Aut}\,G$. Let $A$ and $B$ be distinct atoms of $R$; take $\delta = k_A(k^{-1})_B \in \Delta$. Let $\theta$ be the automorphism $f_A(1_G)_{X-A}$ of $\Gamma$: we have $\delta\theta = (kf)_A(k^{-1})_B \in \Delta$. Let $\gamma = [k^{-1}(kf)]_A$; then $\gamma = \delta^{-1}(\delta\theta) \in \Delta$, and so $\det_K \gamma = K'[k^{-1}(kf)]$ belongs to $H/K'$. Hence $k^{-1}(kf) \in H$, as required.

Conversely, suppose that $H$ and $K$ satisfy the given conditions, and let $\Delta = \{\gamma \in K^R : \det_K \gamma \in H/K'\}$. Then $\Delta$ is invariant under $(\mathrm{Aut}\,R)^\Gamma$, by Proposition 8·2; and (2) holds if $\theta$ belongs to $(\mathrm{Aut}\,G)^{R'}$ or has form $1_\Gamma + \nu$, with $\nu \in \mathrm{Hom}\,[\Gamma, Z(\Gamma)]$, as in case 1. Hence $\Delta$ is characteristic in $\Gamma$, by Theorem C. ∎

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