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Complexity for partial computable functions over computable Polish spaces

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In the framework of effectively enumerable topological spaces, we introduce the notion of a partial computable function. We show that the class of partial computable functions is closed under composition, and the real-valued partial computable functions defined on a computable Polish space have a principal computable numbering. With respect to the principal computable numbering of the real-valued partial computable functions, we investigate complexity of important problems such as totality and root verification. It turns out that for some problems the corresponding complexity does not depend on the choice of a computable Polish space, whereas for other ones the corresponding choice plays a crucial role.

1. Introduction

In numerical computations, many important problems such as real number equality, set equality and etc., are undecidable (non-computable). Therefore, studying degrees of non-computability became recently an active research area in computable analysis (see among others Brattka and Gherardi (2009), Cenzer and Remmel (1998), Brodhead and Cenzer (2008) and Selivanov and Schroeder (2014)).

We use numbering theory and index sets as promising techniques merging classical recursion theory and computable analysis in topological spaces. There are several reasons for doing this. One of them is that the theory of index sets provides methods for encoding problems in an effective way by natural numbers, i.e., generate the corresponding index sets, which can be used for the analysis of their complexity in the settings of arithmetical and analytical hierarchies. Another reason is that the theory of index sets has been already successfully employed in many areas in mathematics and computer science. In recursion theory, index sets have been applied to obtain both new results and new elegant proofs of classical theorems such as the Post's theorem and the density theorem (Ershov 1977; Rogers 1967; Shoenfield 1971). See also observations in Selivanov (1988). Many recent advancements in computable model theory are also closely related to index sets (Calvert *et al.* 2007, 2006; Downey *et al.* 2003; Downey and Montalban 2008). In computer science, the Rice–Shapiro theorem provides a simple description of effectively enumerable properties of program languages, and the complexity of some decision problems in programming has been studied in Spreen (1995) using index sets. While some important results have been obtained in Cenzer and Remmel (1998, 1999), Brodhead and Cenzer

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(2008), Spreen (1984, 1995, 1998) and Weihrauch and Deil (1980), many methods and techniques of numbering theory have not been fully employed in computable analysis. In this paper, we work with an effectivisation of Polish spaces (see Moschovakis (2009), Kechris (1995), Selivanov (2015) among others), which is compatible with the notion of computable (recursive) metric spaces (Moschovakis 1964).

We assume that a computable Polish space is a complete computable metric space without isolated points. It is worth noting that the class of computable Polish spaces is one of the main object for investigation in the Effective Descriptive Set Theory (Selivanov 2015).

The main contributions of this paper are as follows. We introduce the notion of partial computability over effectively enumerable spaces. After constructing principal computable numberings for real-valued partial computable functions defined on a computable Polish space, we calculate the complexity of index sets for important problems such as totality, somewhere definability, root verification and function equality. It is remarkable that for some problems m-degrees of the corresponding index sets do not depend on the choice of a Polish space, whereas for other ones the corresponding choice plays a crucial role.

2. Preliminaries

We refer the reader to Rogers (1967) and Soare (1987) for basic definitions and fundamental concepts of recursion theory. We recall that, in particular, φ_e denotes the partial computable (recursive) function with an index e in the Kleene numbering, φ_e^s denotes the computation of φ_e for s steps such that the function φ_e^s is uniformly primitive recursive. In this paper, we also use notations $W_e = \mathrm{dom}(\varphi_{\mathrm{e}})$, $W_e^s = \mathrm{dom}(\varphi_{\mathrm{e}}^s)$ and $\pi_e = \mathrm{im}(\varphi_{\mathrm{e}})$. A sequence $\{V_i\}_{i\in\omega}$ of computably enumerable (c.e.) sets is *computable* (or uniformly computably enumerable), if $\{(n,i)|\, n \in V_i\}$ is computably enumerable. It is worth noting that this is equivalent to existence of a computable function $f : \omega \to \omega$ such that $V_i = W_{f(i)}$.

In the major part of our paper, we work with the following notion of a computable Polish space. A computable Polish space is a complete separable metric space X without isolated points and with metric d such that there is a countable dense set $\mathcal{B} = \{b_1, b_2, \dots\}$ called a *basis of* X that makes the following two relations $\{(n,m,i) \mid d(b_n, b_m) < q_i,\ q_i \in \mathbb{Q}\}$ $\{(n,m,i) \mid d(b_n, b_m) > q_i,\ q_i \in \mathbb{Q}\}$ computably enumerable (c.f. Moschovakis (1964)). The standard notations $B(x,y)$ and $\overline{B}(x,y)$ are used for open and closed balls with the centre x and the radius y. We consider this concept in the framework of effectively enumerable spaces (see Section 3.2). We work with the real numbers $(\mathbb{R}, \tau_{\mathbb{R}}, \beta_{\mathbb{R}})$, where $\beta_{\mathbb{R}}$ is a computable numbering of the open intervals with rational endpoints, $\varnothing$ and $\mathbb{R}$. We use the Baire space $\mathcal{N} = (\omega^\omega, \tau_{\mathcal{N}})$ defined as follows.

$$\omega^0 = \{\bot\},\ \text{where } \bot \text{ is the empty word},$$
$$\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n,$$
$$\omega^\omega = \{f \mid f : \omega \to \omega\}$$
(informally, the set of all paths in the tree $\omega^{<\omega}$).

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The standard topology $\tau_{\mathcal{N}}$ on ω^ω is generated by the base that contains all clopen sets of the type

$$\mathfrak{A}_w = \{f \in \mathcal{N} \mid f[s] = w,\ s = \text{length}(w)\},$$

where $w \in \omega^{<\omega}$ and the interpretation of $f[s]$ is as follows:

$$f[0] = \bot,$$
$$f[s] = \langle f(0), \ldots, f(s-1)\rangle .$$

We take a standard agreement (Rogers 1967) that a downward closed nonempty subset $T \subseteq \omega^{<\omega}$ is a tree, and $[T]$ denotes the set $\{f \in \mathcal{N} \mid (\forall s \in \omega) f[s] \in T\}$.

3. Basic background

3.1. *Logic for the reals*

In this paper, we work with the ordered structure of the real numbers in the finite predicate language without equality, $(\mathbb{R}, \sigma_P) = \left(\mathbb{R}, \mathcal{M}_E^*, \mathcal{M}_H^*, \mathcal{P}_E^+, \mathcal{P}_H^+, <\right)$, where $\mathcal{M}_E^*, \mathcal{M}_H^*$ are interpreted as the open epigraph and the open hypograph of multiplication, respectively, and $\mathcal{P}_E^+, \mathcal{P}_H^+$ are interpreted as the open epigraph and the open hypograph of addition, respectively, e.g., $\mathcal{P}_H^+(x,y,z) \rightleftharpoons x + y < z$ and $\mathcal{P}_E^+(x,y,z) \rightleftharpoons x + y > z$. For details, we refer to Korovina and Kudinov (2009). In this section, we consider all formulas in the language σ_P.

The *atomic formulas* are the atomic formulas in the predicate language σ_P. For simplicity, we use standard notations, e.g., $x + y < z$, $x + y > z$, $x \cdot y < z$, $x \cdot y > z$.

The set of $\exists$-*formulas* is the closure of the set of atomic formulas under $\wedge, \vee$ and existential quantifiers. It is worth noting that in $\exists$-formulas, we can use the expressions $x > 0$ and $y < 1$ (and similar) as notations for the formulas $\exists y\,(x > y \cdot y)$, $(\exists x > 0)\,(x \cdot y < x)$, respectively.

In the paper, we use the following standard model-theoretical agreement see, e.g., Hodges (1993): the notation $\mathbb{M} \models \varphi(\bar{a})$ means the validity of the formula $\varphi(\bar{x})$ after the assignment of the variables $\bar{x}$ to elements $\bar{a}$ of $\mathbb{M}$.

Definition 3.1. A set $B \subseteq \mathbb{R}^n$ is $\exists$*–definable* if there exists a $\exists$-formula φ such that $\bar{x} \in B \leftrightarrow \mathbb{R} \models \varphi(\bar{x})$.

Remark 3.1. We recall that all predicates $Q_i \in \sigma_P$ occur only positively in $\exists$-formulas. Hence, all $\exists$-definable subsets of $\mathbb{R}^n$ are open sets and equality on the reals is not representable by a $\exists$-formula.

Proposition 3.1. Let η be a Gödel numbering of $\exists$-formulas and Sent denote the set of all $\exists$-sentences. Then, the following statements hold.

1. The set $\{n \mid \eta(n) \in \text{Sent}\}$ is computable.
2. The set $\{n \mid \eta(n) \in \text{Sent, and } \mathbb{R} \models \eta(n)\}$ is computably enumerable.

Proof. The first claim follows from properties of a Gödel numbering. The second claim follows from the density of $\mathbb{Q}$ in $\mathbb{R}$ and Remark 3.1. It is also can be deduced from quantifier elimination in the strictly ordered field of the reals (Delzell 1982). □

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A sequence of $\exists$-formulas $\{\varphi_i\}_{i\in\omega}$ is called *computable* if there exists a computable function $g : \omega \to \omega$ such that $\varphi_i = \eta(g(i))$. For the computable sequence of formulas $\{\varphi_i\}_{i\in\omega}$, the notation $\mathbb{M} \models \bigvee_{i\in\omega} \varphi_i(\bar{a})$ means that there exists $i \in \omega$ such that $\mathbb{M} \models \varphi_i(\bar{a})$ (Ash and Knight 2000).

Proposition 3.2 (Korovina and Kudinov 2009). Let $\bar{x}, a$ and b range over the real numbers. There is an algorithm which by a computable sequence of $\exists$–formulas $\{\psi_i\}_{i\in\omega}$ produces a computable sequence of $\exists$–formulas $\{\varphi_i\}_{i\in\omega}$ such that

$$\mathbb{R} \models (\forall y \in [a, b]) \bigvee_{i\in\omega} \psi_i(y, \bar{x}) \leftrightarrow \mathbb{R} \models \bigvee_{i\in\omega} \varphi_i(a, b, \bar{x}).$$

Proposition 3.3 (Korovina and Kudinov 2009). There is an algorithm which by a computable sequence of $\exists$–formulas $\{\psi_i\}_{i\in\omega}$, computable real functions $\bar{f}$, computable real numbers $\bar{r}$ produces a computable sequence of $\exists$–formulas $\{\varphi_i\}_{i\in\omega}$ such that

$$\mathbb{R} \models \bigvee_{i\in\omega} \psi_i(\bar{f}(\bar{y}), \bar{r}, \bar{x}) \leftrightarrow \mathbb{R} \models \bigvee_{i\in\omega} \varphi_i(\bar{y}, \bar{x}).$$

It is worth noting that in this paper along with the logic recalled above, we widely use the set theoretic language and semantics that are common in effective descriptive set theory (Moschovakis 2009).

3.2. *Effectively enumerable topological spaces*

Now, we recall the notion of effectively enumerable topological space. Let (X, τ, α) be a topological space, where X is a non-empty set, $B_\tau \subseteq 2^X$ is a base of the topology τ and $\alpha : \omega \to B_\tau$ is a numbering.

Definition 3.2 (Korovina and Kudinov 2008). A topological space (X, τ, α) is *effectively enumerable* if the following conditions hold.

1. There exists a computable function $g : \omega \times \omega \times \omega \to \omega$ such that
$$\alpha(i) \cap \alpha(j) = \bigcup_{n\in\omega} \alpha(g(i, j, n)).$$
2. The set $\{i | \alpha(i) \neq \varnothing\}$ is computably enumerable.

For a computable Polish space $(X, \mathcal{B}, d)$ in a natural way, we define the numbering of the base of the standard topology as follows.

$$\alpha(0) = \varnothing,$$
$$\alpha(i) = B(b_{u(i)}, r_i) \text{ for } i > 0, \text{where}$$

$u : \omega \to \omega$ is a suitable computable function, $\{r_i\}_{i\in\omega}$ is a computable sequence of all positive rational numbers such that $\{\langle u(i), r_i\rangle \mid i > 0\} = \big(\omega \setminus \{0\}\big) \times \mathbb{Q}^+$. It is easy to see that (X, τ, α) is an effectively enumerable topological space. Therefore, we consider the computable Polish spaces as a proper subclass of the effectively enumerable topological spaces. For details, we refer to Korovina and Kudinov (2008). Below, we use the irreflexive

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and transitive relation defined as follows: $i \prec j \leftrightharpoons d(b_{u(i)}, b_{u(j)}) + r_i < r_j$ (for details see, e.g., Spreen (1998)). It is easy to see that this relation is computably enumerable on the indices of basic balls, and if $i \prec j$, then $\mathrm{cl}(\alpha(i)) \subseteq \alpha(j)$.

We recall the notion of an effectively open set.

Definition 3.3 (Korovina and Kudinov 2008). Let (X, τ, α) be an effectively enumerable topological space. A set $A \subseteq X$ is *effectively open* if there exists a computably enumerable set V such that

$$A = \bigcup_{n \in V} \alpha(n).$$

Let $\mathcal{O}_X$ denote the set of all open subsets of X, and $\mathcal{O}^e_X$ denote the set of all effectively open subsets of X. It is worth noting that $\mathcal{O}^e_X$ is closed under intersection and union since the class of effectively enumerable sets is a lattice. The following propositions are natural corollaries of the definition.

Proposition 3.4 (Korovina and Kudinov 2008). A set $A \subseteq \mathbb{R}^n$ is effectively open if and only if there exists a computable sequence of $\exists$–formulas $\{\varphi_i\}_{i \in \omega}$ such that $\bar{a} \in A \leftrightarrow \mathbb{R} \models \bigvee_{i \in \omega} \varphi_i(\bar{a})$.

Remark 3.2. It is straightforward that for effectively enumerable topological spaces (X, τ, α) and $(\mathbb{R}, \tau_{\mathbb{R}}, \beta_{\mathbb{R}})$, an effectively open set $U \subseteq X \times \mathbb{R}$ can be represented as $U = \bigcup_{(i,j) \in I} \alpha(i) \times \beta_{\mathbb{R}}(j)$, where $I \subseteq \omega^2$ is computably enumerable. It is well known, see, e.g., Rogers (1967), Moschovakis (2009), the formulas $(\exists r \in \mathbb{Q}) U(x, r)$ and $\bigvee_{r \in \mathbb{Q}} U(x, r)$ define the same open set which is the projection of U over X. Moreover, this set is effectively open since this is exactly the set $\{x \mid \bigcup_{i \in J} x \in \alpha(i)\}$, where $J = \{i \mid (\exists j \in \omega)(\exists r \in \mathbb{Q})((i, j) \in I \wedge r \in \beta_{\mathbb{R}}(j))\}$ is computably enumerable. More generally, by a similar use of first order formulas, one can deal with Borel sets and we will use this technique throughout the paper.

Proposition 3.5 (Korovina and Kudinov 2015b). For every effectively enumerable topological space $\mathcal{X}$, there exists a principal computable numbering α^e_X of $\mathcal{O}^e_X$.

4. Partial computable functions over effectively enumerable topological spaces

In this section, we introduce the notion of a partial computable function $f : X \to Y$, where $\mathcal{X} = (X, \tau_X, \alpha)$ is an effectively enumerable topological space and $\mathcal{Y} = (Y, \tau_Y, \beta)$ is an effectively enumerable T_0–space. For a motivation of our definition, let us first consider $\mathcal{Y} = \mathbb{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta_{\mathbb{R}})$.

Proposition 4.1. For a partial function $f : X \to \mathbb{R}$, the following conditions are equivalent.

1. There exist two effectively open sets $U, V \subseteq X \times \mathbb{R}$ such that
   a. for all $x \in X$, $U(x)$ is closed downward, and $V(x)$ is closed upward;
   b. if $x \in \mathrm{dom}(f)$, then $f(x) = y \leftrightarrow \{y\} = \mathbb{R} \setminus (U(x) \cup V(x))$;
   c. if $x \notin \mathrm{dom}(f)$, then the set $\mathbb{R} \setminus (U(x) \cup V(x))$ is not a singleton.

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2. There exist a computable sequence of effectively open sets $\{A_n\}_{n\in\omega}$ and a computable function $H : \omega^2 \to \omega$ such that
   a. $\mathrm{dom}(f) = \bigcap_{n\in\omega} A_n$ and
   b. $f^{-1}(\beta_{\mathbb{R}}(m)) = \bigcup_{i\in\omega} \alpha(H(m,i)) \cap \mathrm{dom}(f)$.

Proof.
$1 \to 2$. The first property follows from the observation that

$$x \in \mathrm{dom}(f) \leftrightarrow (\forall y \in \mathbb{Q})(\forall z \in \mathbb{Q})\Big(\big(U(x,y) \wedge V(x,z)\big) \to y < z\Big) \wedge$$
$$(\forall n \in \mathbb{N})(\exists y \in \mathbb{Q})(\exists z \in \mathbb{Q})\Big(U(x,y) \wedge V(x,z) \wedge |z-y| < \frac{1}{n+1}\Big).$$

Therefore, $\mathrm{dom}(f) \in \Pi^0_2$ in the effective Borel hierarchy on $\mathcal{X}$, i.e. it is the intersection of a computably enumerable sequence of effectively open sets (Moschovakis 2009).

The second property holds since, for $x \in \mathrm{dom}(f)$,

$$x \in f^{-1}(\beta_{\mathbb{R}}(m)) \leftrightarrow \big(\exists y \in \mathbb{Q} \cap \beta_{\mathbb{R}}(m)\big)\big(\exists z \in \mathbb{Q} \cap \beta_{\mathbb{R}}(m)\big)\Big((x,y) \in U \wedge (x,z) \in V\Big)$$
$$\leftrightarrow x \in \bigcup_{i\in\omega} \alpha(H(m,i)),$$

for a suitable computable function $H : \omega^2 \to \omega$.

$2 \to 1$. For the function f, let us define the corresponding U and V. Without loss of generality, we assume that $A_{n+1} \subseteq A_n$. Put

$$(x,r) \in U \leftrightarrow (\exists n \in \omega)\Big(x \in A_n \wedge (\exists m \in \omega)(\exists a \in \mathbb{Q})(\exists b \in \mathbb{Q})\Big(\beta_{\mathbb{R}}(m) = (a,b) \wedge$$
$$a > r + \frac{1}{n+1} \wedge x \in \bigcup_{i\in\omega} \alpha(H(m,i))\Big)\Big) \vee$$
$$(\exists m \in \omega)(\exists s \in \omega)\Big(\beta_{\mathbb{R}}(m) \cap \beta_{\mathbb{R}}(s) = \varnothing \wedge x \in \bigcup_{i\in\omega} \alpha(H(m,i)) \wedge$$
$$x \in \bigcup_{i\in\omega} \alpha(H(s,i))\Big).$$

$$(x,r) \in V \leftrightarrow (\exists n \in \omega)\Big(x \in A_n \wedge (\exists m \in \omega)(\exists a \in \mathbb{Q})(\exists b \in \mathbb{Q})\Big(\beta_{\mathbb{R}}(m) = (a,b) \wedge$$
$$b < r - \frac{1}{n+1} \wedge x \in \bigcup_{i\in\omega} \alpha(H(m,i))\Big)\Big) \vee$$
$$(\exists m \in \omega)(\exists s \in \omega)\Big(\beta_{\mathbb{R}}(m) \cap \beta_{\mathbb{R}}(s) = \varnothing \wedge x \in \bigcup_{i\in\omega} \alpha(H(m,i)) \wedge$$
$$x \in \bigcup_{i\in\omega} \alpha(H(s,i))\Big).$$

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Let us verify the required properties of U and V. The first property is straightforward from the definitions. For the second property, it is worth noting that if $x \in \mathrm{dom}(f)$ then $(x,z) \in U \leftrightarrow z < f(x)$ and $(x,z) \in V \leftrightarrow z > f(x)$. So, $f(x) = y \leftrightarrow \{y\} = \mathbb{R} \setminus (U(x) \cup V(x))$.

To verify the third property, we assume $x \notin \mathrm{dom}(f)$. Then there exists $n \in \omega$ that $(\forall k \geqslant n) x \notin A_k$. We are going to show that $\mathbb{R} \setminus (U(x) \cup V(x))$ is not a singleton.

Suppose the opposite that $\mathbb{R} \setminus (U(x) \cup V(x)) = \{r\}$ for some $r \in \mathbb{R}$. So, $r' \in U(x)$ and $r'' \in V(x)$, where $r' = r - \frac{1}{n+1}$ and $r'' = r + \frac{1}{n+1}$. If the second disjunct in the definitions of U and V holds, then $U(x) = V(x) = \mathbb{R}$; this contradicts to the assumption. If the first disjuncts hold, then there exists $n' \in \omega$ and $n'' \in \omega$ such that $r' \in A_{n'}$, $r'' \in A_{n''}$,

$$(\exists m \in \omega)(\exists a \in \mathbb{Q})(\exists b \in \mathbb{Q})\,(\beta_{\mathbb{R}}(m) = (a,b) \wedge$$
$$a > r' + \frac{1}{n'+1} \wedge x \in \bigcup_{i \in \omega} \alpha(H(m,i))\Bigg)$$

and

$$(\exists s \in \omega)(\exists c \in \mathbb{Q})(\exists d \in \mathbb{Q})\,(\beta_{\mathbb{R}}(s) = (c,d) \wedge$$
$$d < r'' - \frac{1}{n''+1} \wedge x \in \bigcup_{i \in \omega} \alpha(H(s,i))\Bigg).$$

Since $(\forall k \geqslant n) x \notin A_k$, we have $n' < n$, $n'' < n$ and, as corollary, $d < r < a$. So the intervals (a,b) and (c,d) are disjoint; therefore, the second disjunct holds. This means that $U(x) = V(x) = \mathbb{R}$, this contradicts to the assumption. So, $\mathbb{R} \setminus (U(x) \cup V(x))$ is not a singleton. □

The following definition is motivated by the proposition above.

Definition 4.1. Let $\mathcal{X} = (X, \tau_X, \alpha)$ be an effectively enumerable topological space, and $\mathcal{Y} = (Y, \tau_Y, \beta)$ be an effectively enumerable T_0-space. A partial function $f : X \to Y$ is called *partial computable* if the following properties hold. There exists a computable sequence of effectively open sets $\{A_n\}_{n \in \omega}$ and a computable function $H : \omega^2 \to \omega$ such that

1. $\mathrm{dom}(f) = \bigcap_{n \in \omega} A_n$ and
2. $f^{-1}(\beta(m)) = \bigcup_{i \in \omega} \alpha(H(m,i)) \cap \mathrm{dom}(f)$.

In the following, if a partial computable function f is everywhere defined, we say f is a *total computable function.*

Remark 4.1. In computable Polish spaces, it is clear that $B = \bigcap_{n \in \omega} A_n$ for a computable sequence of effectively open sets $\{A_n\}_{n \in \omega}$ if and only if B is a Π^0_2-set in the effective Borel hierarchy on $\mathcal{X}$ (see, e.g., Moschovakis (2009)). Further, we show that such sets are precisely the domains of partial computable functions.

The following theorem shows that, for effectively enumerable T_0-spaces, the set of partial computable functions is closed under composition.

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Theorem 4.1. Let $\mathcal{X} = (X, \tau_X, \alpha)$, $\mathcal{Y} = (Y, \tau_Y, \beta)$ and $\mathcal{Z} = (Y, \tau_Z, \varrho)$ be effectively enumerable T_0-spaces. If partial functions $f : X \to Y$ and $g : Y \to Z$ are partial computable, then $h = g \circ f$ is a partial computable function.

Proof. In the following, we check (1) and (2) properties of partial computability.

1.

$$\begin{aligned}\mathrm{dom}(h) &= \mathrm{dom}(f) \cap f^{-1}(\mathrm{dom}(g)) \\ &= \mathrm{dom}(f) \cap f^{-1}\left(\bigcap_{m\in\omega} B_m\right) \\ &= \mathrm{dom}(f) \cap \left(\bigcap_{m\in\omega} f^{-1}(B_m)\right) \\ &= \mathrm{dom}(f) \cap \left(\bigcap_{m\in\omega} O_m\right),\end{aligned}$$

where $\{O_m\}_{m\in\omega}$ is a computable sequence of effectively open subsets of X and $\{B_m\}_{m\in\omega}$ is a computable sequence of effectively open subsets of Y. So h satisfies the property (1).

2 It is easy to see that

$$h^{-1}(\varrho(n)) = f^{-1}(g^{-1}(\varrho(n))) = f^{-1}(\mathrm{dom}(g) \cap B_n) = \mathrm{dom}(h) \cap f^{-1}(B_n),$$

where B_n is an effectively open subset of Y. Therefore, h satisfies the property (2).

□

For an effectively enumerable space $\mathcal{X}$, we denote the set of partial computable function $f : X \to \mathbb{R}$ as $\mathcal{PCF}_X$ and nowhere defined function as $\perp$. It is natural to tell that a sequence $\{f_n\}_{n\in\omega}$ of functions from $\mathcal{PCF}_X$ is *c*omputable if the corresponding function $F : \omega \times X \to \mathbb{R}$ is partial computable, where $F(n, x) = f_n(x)$. From Definition 4.1 and Proposition 4.1, it follows that this is equivalent to the existence of computable sequences $\{U_n\}_{n\in\omega}$ and $\{U_n\}_{n\in\omega}$ of effectively open subsets of $X \times \mathbb{R}$ that, for every $n \in \omega$, U_n and V_n define f_n i.e. satisfy the first condition of Proposition 4.1.

A function $\gamma : \omega \to \mathcal{PCF}_X$ is called a *numbering* of $\mathcal{PCF}_X$ if $\{\gamma(n) \mid n \in \omega\} = \mathcal{PCF}_X$. A numbering $\gamma : \omega \to \mathcal{PCF}_X$ is called *computable* if $\{\gamma(n)\}_{n\in\omega}$ is a computable sequence. The numbering γ is called *principal computable* if it is computable and every computable numbering ξ is computably reducible to γ, i.e., there exists a computable function $f : \omega \to \omega$ such that $\xi(i) = \gamma(f(i))$. In particular, for any computable sequence of partial computable functions $\{f_n\}_{n\in\omega}$, there exists a computable function $g : \omega \to \omega$ such that $f_n = \gamma(g(n))$. It is worth noting that these definitions agree with the notions of computable and principal computable numberings of $A \subseteq \mathcal{P}(\omega)$ (Ershov 1999).

The following proposition is a straightforward corollary of Proposition 4.1, Definition 4.1 and the results from Korovina and Kudinov (2015b).

Proposition 4.2. For every effectively enumerable space $\mathcal{X}$, there exists a principal computable numbering γ of the set of partial computable functions $f : X \to \mathbb{R}$.

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In the following for the partial computable function $\gamma(n)$, (U_n, V_n) denotes the pair of effectively open sets which define $\gamma(n)$, i.e. satisfy the first condition of Proposition 4.1, and H_n denotes the function that satisfies the second condition of Proposition 4.1, i.e. $\gamma_n^{-1}(\beta_{\mathbb{R}}(m)) = \bigcup_{i\in\omega} \alpha(H_n(m,i)) \cap \mathrm{dom}(\gamma(n))$. For convenience, in some places, we use the notation γ_n for $\gamma(n)$.

5. Complexity of index sets for partial computable functions

In this section, we work with the subclass of effectively enumerable topological space, which is the set of computable Polish spaces, the real numbers $\mathbb{R}$ and the Baire space $\mathcal{N}$. Now, we focus on calculating the complexity of index sets for classes of partial computable functions $f : X \to \mathbb{R}$. We consider $(\mathcal{PCF}_X, \gamma)$, where γ is a principal computable numbering of the real-valued partial computable functions over X, where $\mathcal{X}$ is a computable Polish space.

Definition 5.1 (Rogers 1967; Shoenfield 1971). For the class K, the set $Ix(K) = \{n|\gamma(n) \in K\}$ is called an index set for the class $K \subseteq \mathcal{PCF}_X$.

In order to calculate complexity of index sets, we use arithmetical and analytical hierarchies, which classifies certain sets of the natural numbers based on the complexity of formulas defining them (Rogers 1967; Soare 1987).

Below, we recall the notion of multiple m-reducibility that is one of the central concepts in the classical theory of computability.

Definition 5.2. Let $A, B, C, D \subseteq \omega$.
The set A is *m-reducible* to D, denoted by $A \leqslant_m D$, if there exists a computable function $f : \omega \to \omega$ such that

$$x \in A \leftrightarrow f(x) \in D.$$

The pair (A, B) is *multiple m-reducible* to (C, D), denoted by $(A, B) \leqslant_m (C, D)$, if there exists a computable function $f : \omega \to \omega$ such that

$$x \in A \leftrightarrow f(x) \in C \text{ and } x \in B \leftrightarrow f(x) \in D.$$

The function f is called a reduction function.

Definition 5.3. Let Γ be a class of arithmetical (analytical) complexity. A set $A \subseteq \omega$ is called Γ-*hard* if $B \leqslant_m A$ for any $B \in \Gamma$. A set A is called Γ-*complete* if $A \in \Gamma$ and A is Γ-hard.

The following theorem summarises results on complexity of well-known problems in the classical computability theory (see, e.g., Rogers (1967) and Soare (1987)), which we will use further on.

Proposition 5.1.

- $\mathrm{Inf} = \{e \mid W_e \text{ is infinite}\}$ is Π_2^0-complete.
- $\mathrm{Fin} = \{e \mid W_e \text{ is finite}\}$ is Σ_2^0-complete.

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- Cofin $= \{e \mid W_e$ is cofinite$\}$ is Σ^0_3-complete.
- $R = \{e \mid W_e$ is computable$\}$ is Σ^0_3-complete.
- There exists a computable sequence of computable trees $\{T_m\}_{m\in\omega}$ such that the set $I = \{m \mid T_m$ has an infinite path$\}$ is Σ^1_1-complete.

In this paper, we investigate complexity of the following problems for a computable Polish space $\mathcal{X}$:

Problem 1: $\{f : X \to \mathbb{R} \mid f$ is a partial computable function and $f \neq \perp\}$.

Problem 2: $\{f : X \to \mathbb{R} \mid f$ is a partial computable function and $\exists x\, f(x) = 0\}$.

Problem 3: $\{f : X \to \mathbb{R} \mid f$ is a total computable function$\}$.

Problem 4: $\{f : X \to \mathbb{R} \mid f$ is a total computable function and $\exists x\, f(x) = 0\}$.

Theorem 5.1. For any computable Polish space $\mathcal{X} = (X, \tau_X, \alpha)$, the index set $Ix(\{f \in \mathcal{PCF}_X | f \neq \perp\})$ is Σ^1_1-complete.

Our proof is based on the following theorem and lemmas.

Theorem 5.2 (Nested Sphere principle; Kolmogorov 1999). A necessary and sufficient condition that the metric space X be complete is that every sequence of closed nested spheres in X with radii tending to zero have nonvoid intersection; moreover, the intersection is a one point set.

Lemma 5.1. There exists an effective procedure which given any computable sequence $\{A_n\}_{n\in\omega}$ of effectively open subsets of $\mathcal{N}$ produces a computable sequence $\{E_n\}_{n\in\omega}$ of effectively open subsets of X such that $\bigcap_{n\in\omega} A_n = \varnothing \leftrightarrow \bigcap_{n\in\omega} E_n = \varnothing$.

Proof. First, we construct a homeomorphism $F : \mathcal{N} \to X$ such that $F^{-1} : X \to \mathcal{N}$ is a partial computable function. It is worth noting that for any ball $B(x, r)$ one can choose two nonempty balls $\overline{B}(x_1, r_1)$ and $\overline{B}(x_2, r_2)$, which are inside of $B(x, r)$ and do not intersect each other. Continuing the process, one can effectively generate infinitely many disjoint nonempty balls which are inside of $B(x, r)$; therefore, one can produce a computable infinite sequence of such balls $\{B(x_n, r_n)\}_{n\in\omega}$. Without loss of generality, we could assume that for all $n \in \omega$, $r_n < 1$. Now, we construct F by steps.

Step 1. Put $B(x^1_n, r^1_n) = B(x_n, r_n)$ and make correspondence between each $\langle n\rangle \in \omega^{<\omega}$ and the ball $B(x^1_n, r^1_n)$.

Step s+1. Assume that on the step s, we already constructed balls $B(x^s_w, r^s_w)$, where $w \in \omega^{<\omega}$ and length$(w) = s$, and made correspondence between each w and the ball $B(x^s_w, r^s_w)$. Next we proceed as follows. Inside of every ball $B(x^s_w, r^s_w)$, we construct an effective sequence of disjoint nonempty balls $B(x^{s+1}_{w\sqcup\langle n\rangle}, r^{s+1}_{w\sqcup\langle n\rangle})$ such that for all $n \in \omega$, $r^{s+1}_{w\sqcup\langle n\rangle} \leqslant \frac{1}{2} r^s_w$. Then, we make correspondence between each $w \sqcup \langle n\rangle$ and the ball $B(x^{s+1}_{w\sqcup\langle n\rangle}, r^{s+1}_{w\sqcup\langle n\rangle})$.

Finally, we define

$$\{F(v)\} = \bigcap_{s>0} \overline{B}(x^s_{v[s]}, r^s_{v[s]}).$$

By the Nested Sphere principle (see Theorem 5.2), F is correctly defined on $\mathcal{N}$ since $r^s_{v[s]} \to 0$. Moreover, $\{F(v)\} = \bigcap_{s>0} B(x^s_{v[s]}, r^s_{v[s]})$ since, by construction, $B(x^s_{v[s]}, r^s_{v[s]}) \supset$

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$\overline{B}(x^{s+1}_{v[s]\sqcup\langle n\rangle}, r^{s+1}_{v[s]\sqcup\langle n\rangle})$. Using this strict inclusion, we show that $\mathrm{im}(F)$ is Π^0_2-set in the effective Borel hierarchy on $\mathcal{X}$.

Indeed, assume for all $s \in \omega$, there exists a word w of the length s, $x \in B(x^s_w, r^s_w)$. By assumption,

$$x \in \bigcup_{w:\mathrm{length}(w)=s+1} B(x^{s+1}_w, x^{s+1}_w)$$

and, by construction, $B(x^s_u, r^s_u) \cap B(x^s_v, r^s_v) = \varnothing$ for any different words u and v of the length s. Therefore, if $x \in B(x^s_v, r^s_v)$, then there exists n and a word $w = v \sqcup \langle n\rangle$ of the length $s+1$ such that $x \in B(x^{s+1}_w, r^{s+1}_w)$. So we construct a chain of words $w_1 \sqsubseteq w_2 \sqsubseteq \ldots$ such that $v[s] = w_s$ for some $v \in \mathcal{N}$ and $x = F(v)$. Therefore, $x \in \mathrm{im}(F)$ if and only if for all $s \in \omega$ there exists words w of the length s such that $x \in B(x^s_w, r^s_w)$. This is a Π^0_2-condition. Therefore, $\mathrm{im}(F)$ is a Π^0_2-set in the effective Borel hierarchy on $\mathcal{X}$. By construction, $F(\mathfrak{A}_w) = B(x^s_w, r^s_w) \bigcap \mathrm{im}(F)$, where $s = \mathrm{length}(w)$. Since $\mathcal{X}$ is a computable Polish space, F^{-1} is a partial computable function.

To finish the proof of the lemma using partial computability of F^{-1}, let us define $\widetilde{E}_n$ as an effectively open set such that $\widetilde{E}_n \cap \mathrm{im}(F) = F(A_n)$. It is easy to see that if $\bigcap_{n\in\omega} A_n = \varnothing$, then $\bigcap_{n\in\omega} F(A_n) = \varnothing$. Suppose $\mathrm{im}(F) = \bigcap_{n\in\omega} C_n$, where C_n is an effectively open set. Without loss of generality, we can assume that $C_{n+1} \subseteq C_n$.

Finally, put $E_n = \widetilde{E}_n \cap C_n$. Then, if $\bigcap_{n\in\omega} A_n = \varnothing$, then $\bigcap_{n\in\omega} (F(A_n) \cap C_n) = \varnothing$. So $\bigcap_{n\in\omega} E_n = \varnothing$. If $\bigcap_{n\in\omega} E_n = \varnothing$, then $\bigcap_{n\in\omega} F(A_n) = \varnothing$. So, $\bigcap_{n\in\omega} A_n = \varnothing$. Therefore, $\{E_n\}_{n\in\omega}$ is required. □

Lemma 5.2. Let $\{E_n\}_{n\in\omega}$ be a computable sequence of effectively open subsets of an effectively enumerable topological space $\mathcal{X} = (X, \tau, \alpha)$. There is an algorithm producing a partial computable function $f : X \to \mathbb{R}$ such that $\mathrm{dom}(f) = \bigcap_{n\in\omega} E_n$.

Proof. Put

$$U = \bigcup_{n\in\omega} \left(\bigcap_{i\leqslant n} E_i \right) \times \left(-\infty, -\frac{1}{n+1} \right),$$

$$V = \bigcup_{n\in\omega} \left(\bigcap_{i\leqslant n} E_i \right) \times \left(\frac{1}{n+1}, +\infty \right).$$

It is clear that U and V define a partial computable function such that if $x \in \bigcap_{n\in\omega} E_n$ then $f(x) = 0$, otherwise $f(x)\uparrow$. □

Below, we use the result from Proposition 5.1, which states that there exists a computable sequence of computable trees $\{T_m\}_{m\in\omega}$ such that the set $I = \{m | T_m \text{ has an infinite path}\}$ is Σ^1_1-complete.

Lemma 5.3. There exists an effective procedure that generates a computable sequence $\{A^m_n\}_{m,n\in\omega}$ of effectively open subsets of $\mathcal{N}$ such that $\bigcap_{n\in\omega} A^m_n \neq \varnothing \leftrightarrow m \in I$.

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Proof. Let $T_m \subseteq \mathcal{N}$ such that $m \in I$. Define

$$\{A_n^m\}_{m,n\in\omega} = \{f \in \mathcal{N} \mid \langle f(0), \ldots, f(n)\rangle \in T_m\}.$$

It is easy to see that $f \in \bigcap_{n\in\omega} A_n^m$ iff f is an infinite path in T_m iff $m \in I$. □

Proof. (Theorem 5.1) First, let us note that the standard Kleene–Addison algorithm (Rogers 1967) shows that $Ix(\{f|f \neq \bot\} \in \Sigma_1^1$. Our goal is to show that $I \leqslant_m Ix(\{f|f \neq \bot\})$. For that, we construct a computable sequence $\mathcal{F} = \{f_m\}_{m\in\omega}$ of partial computable functions such that $m \in I \leftrightarrow f_m \neq \bot$. Using Lemmas 5.3 and 5.1, we construct a computable sequence $\{A_n^m\}_{m,n\in\omega}$ of effectively open subsets of $\mathcal{N}$ and a computable sequence $\{E_n^m\}_{m,n\in\omega}$ of effectively open subsets of X such that $\bigcap_{n\in\omega} A_n^m \neq \varnothing \leftrightarrow m \in I$ and $\bigcap_{n\in\omega} A_n^m = \varnothing \leftrightarrow \bigcap_{n\in\omega} E_n^m = \varnothing$. Using Lemma 5.2, we can effectively construct a partial computable function f_m such that $\mathrm{dom}(f_m) = \bigcap_{n\in\omega} E_n^m$. So, there exists a computable function $\chi : \omega \to \omega$ such that $f_m = \gamma(\chi(m))$. The function χ is a reduction, since $m \in I \leftrightarrow \gamma(\chi(m)) \neq \bot$. □

Corollary 5.1. Let $\mathcal{X} = (X, \tau_X, \alpha)$ be any computable Polish space. The index set $Ix(\bot) = \{n \mid \gamma(n) = \bot\}$ is Π_1^1-complete.

Proof. This follows immediately. □

Corollary 5.2. Let $\mathcal{X} = (X, \tau_X, \alpha)$ be any computable Polish space. For a partial computable functions $f : X \to \mathbb{R}$, the problem of equality is Π_1^1-complete.

Proof. This follows immediately. □

Theorem 5.3. Let $\mathcal{X} = (X, \tau_X, \alpha)$ be any computable Polish space. The index set $Z = Ix(\{f \in \mathcal{PCF}_X \mid \exists x\, f(x) = 0\})$ is Σ_1^1-complete.

Our proof is based on the following lemma. Let $\mathcal{N}_0 \subset \mathcal{N}$ be the set of all functions that represent fast Cauchy sequences, i.e. $h \in \mathcal{N}_0$ iff $(\forall l \in \omega)(\forall m > l)\, d(b_{h(m)}, b_{h(l)}) < \frac{1}{2^l}$.

Lemma 5.4. For any effectively open set $A \subset X$, there exists an effectively open set $C \subset \mathcal{N}$ such that for all $h \in \mathcal{N}_0$,

$$\lim_{n\to\infty} b_{h(n)} = x \in A \leftrightarrow h \in C.$$

Proof. To approach our proof, we make several observations. The first one is that if $\lim_{i\to\infty} b_i = x$, $r_i \to 0$ and $\alpha(t_i) = B(b_i, r_i) \supseteq \overline{B}(b_{i+1}, r_{i+1}) = \bar{\alpha}(t_{i+1})$ for any $i \in \omega$, then from $x \in B(a,r) = \alpha(c)$ it follows that there exists $N \in \omega$ such that for all $i \geqslant N$, $t_i \prec c$, where the relation $\prec$ is defined in Section 3.2. The second one is that since $\mathcal{X}$ is a computable Polish space, there exists a computable function $F : \omega^2 \to \omega$ such that $B(b_{h(n)}, \frac{1}{2^{n-1}}) = \alpha(F(h(n), n))$, for $n > 0$.

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Let $A = \bigcup_{i\in T} \alpha(i)$, where T is a computably enumerable set. Now we show that for $x \in X$ and $h \in \mathcal{N}_0$, such that $\lim_{n\to\infty} b_{h(n)} = x$, the following equivalence holds:

$$x \in A \leftrightarrow \bigvee_{i\in T} (\exists N \in \omega) F(h(N), N) \prec i.$$

Assume $\lim_{n\to\infty} b_{h(n)} = x$ and $h \in \mathcal{N}_0$. Then, there exists $N_0 \in \omega$ such that for all $n \geqslant N_0$, $x \in \alpha(F(h(n), n))$.

If $x \in A$, then $x \in \alpha(j)$ for some $j \in T$. By the choice of h, we have $t_i = F(h(i), i)$. Since $\overline{B}(b_{h(n+1)}, \frac{1}{2^n}) \subseteq B(b_{h(n)}, \frac{1}{2^{n-1}})$, we can use the first observation above to find $N_1 \in \omega$ such that for all $i \geqslant N_1$, $t_i \prec j$. Taking $N = N_1$, we get $F(h(N), N) \prec j$.

If $F(h(N), N) \prec i$ for some $N \in \omega$ and $i \in T$, then, for all $n \geqslant N$, $F(h(n), n) \prec i$ by transitivity of the relation $\prec$ and the choice of h. Put $n = \max(N_0, N)$. Then, $x \in \alpha(F(h(n), n))$ and $F(h(n), n) \prec i$. Therefore, $x \in \alpha(i)$ by the property of the relation $\prec$. So, $x \in A$.

Put

$$C_i = \{h \in \mathcal{N} \mid (\exists N \in \omega) i \succ F(h(N), N)\},$$
$$C = \bigcup_{i\in T} C_i.$$

Every C_i is an effectively open set since this is the projection of the effectively open set $\{(h, i, N) \mid i \succ F(h(N), N)\} = \bigcup_{(k,i,N)\in B} \{(h, i, N) \mid h(N) = k\}$, where $B = \{(k, i, N) \mid i \succ F(k, N)\}$. Therefore, C is required by the characterisation of A given above. □

It is worth noting that the construction in the lemma above is uniform that means that the index of C is computed by an index of A.

Proof. (Theorem 5.3) The Σ^1_1-hardness follows from Theorems 5.1 and 4.1. Indeed, let $H(x) = x \cdot 0 \equiv 0$. Then, for $F_n = H \circ \gamma_n$, $\{F_n\}_{n\in\omega}$ is a computable sequence of partial computable functions (see the proof of Theorem 4.1). It is easy to see that the following equivalences hold:

$$\exists x\, F_n(x) = 0 \leftrightarrow \exists x\, \gamma(n)(x) \downarrow \leftrightarrow n \notin Ix(\bot).$$

So, $\omega \setminus Ix(\bot) \leqslant_m \{n \mid \exists x\, F_n(x) = 0\}$. By Theorem 5.1, we get Σ^1_1-hardness.

Now, we show that $Z \in \Sigma^1_1$. Since $n \in Z$ could we written as follows:

$$(\exists g \in \mathcal{N}_0) \left(\lim_{k\to\infty} b_{g(k)} \in \mathrm{dom}(\gamma(n)) \wedge \gamma_n(\lim_{k\to\infty} b_{g(k)}) = 0 \right) \leftrightharpoons$$
$$(\exists g \in \mathcal{N}_0) \left(\varphi_1(g, n) \wedge \varphi_2(g, n)\right),$$

it is sufficient to show that φ_1 and φ_2 are Π^0_2–conditions on $\mathcal{N} \times \omega$.

By definition, $\mathrm{dom}(\gamma(n)) = \bigcap_{i\in\omega} A^n_i$ for some computable sequence of effectively open sets $\{A^n_i\}_{i\in\omega}$. Using Lemma 5.4, for $i \in \omega$, we construct the corresponding $C^n_i \subset \mathcal{N}$ such that $\lim_{k\to\infty} b_{g(k)} \in A^n_i \leftrightarrow g \in C^n_i$ for any $g \in \mathcal{N}_0$. From uniformity of Lemma 5.4, it follows that the sequence $\{C^n_i\}_{n,i\in\omega}$ is computable. Therefore, $\varphi_1(g, n) \leftrightharpoons (\forall i \in \omega) g \in C^n_i$ is a Π^0_2–condition on $\mathcal{N} \times \omega$. It is easy to see that the condition φ_2 is equivalent to

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$(\forall m \in \omega) \lim_{k\to\infty} b_{g(k)} \in \gamma_n^{-1}((-\frac{1}{2^m}, +\frac{1}{2^m}))$. In other words, $(\forall m \in \omega) \lim_{k\to\infty} b_{g(k)} \in A_{nm} = \alpha(H_n(\chi(m)))$, where H_n is a function corresponding to H from Definition 4.1 for $\gamma(n)$ and $\beta_{\mathbb{R}}(\chi(m)) = (-\frac{1}{2^m}, +\frac{1}{2^m})$. Since A_{nm} are uniformly effectively open, we can use Lemma 5.4 to construct the corresponding sets $C_{nm} \subset \mathcal{N}$ such that $\lim_{k\to\infty} b_{g(k)} \in A_{nm} \leftrightarrow g \in C_{nm}$ for any $g \in \mathcal{N}_0$. From uniformity of Lemma 5.4, it follows that the sequence $\{C_{nm}\}_{n,m\in\omega}$ is computable. Therefore, $\varphi_2(g, n) \leftrightharpoons (\forall m \in \omega) g \in C_{nm}$ is a Π^0_2–condition on $\mathcal{N} \times \omega$. □

Theorem 5.4 (Korovina and Kudinov 2015b). For partial computable real functions $f : \mathbb{R} \to \mathbb{R}$, totality is Π^0_2-complete.

Theorem 5.5. For partial computable functions $f : \mathcal{N} \to \mathbb{R}$, totality is Π^1_1-complete.

Proof. It is clear that $\{n \mid \exists x\, \gamma(n)(x) \uparrow\} \in \Sigma^1_1$. So $\{n \mid \forall x\, \gamma(n)(x) \downarrow\} \in \Pi^1_1$.

Now, we show that totality is Π^1_1-hard. From Lemma 5.2, it follows that every effectively open set $\mathcal{O}_n \subseteq \mathcal{N}$ is the domain of an appropriate partial computable function $f : \mathcal{N} \to \mathbb{R}$. Moreover, its index can be constructed effectively i.e. $\mathcal{O}_n = \mathrm{dom}(\gamma(h(n)))$ for a suitable computable function $h : \omega \to \omega$. Therefore, it is sufficient to show that the set $\{n \mid \mathcal{O}_n = \mathcal{N}\}$ is Π^1_1-hard. In order to do that with every computable tree $T_n \subseteq \omega^{<\omega}$, we uniformly associate by computable g the effectively open set

$$\mathcal{O}_{g(n)} = \mathcal{N} \setminus [T_n] = \{f \in \mathcal{N} \mid \exists m\, f[m] \notin T_n\}.$$

After that, it is straightforward that $\mathcal{O}_{g(n)} = \mathcal{N}$ if and only if T_n has no infinite path. Therefore, $\{n \mid \mathcal{O}_n = \mathcal{N}\}$ is Π^1_1-hard. □

Proposition 5.2. Let $\mathcal{X} = \mathcal{N}$, $K_1 = \{n \mid \gamma(n) \text{ is total}\}$ and $K_2 = \{m \mid \neg\, \exists x\, \gamma(m)(x) \downarrow = 0\}$. The pair (K_1, K_2) is m-complete in the class of the pairs of Π^1_1-sets.

The proof is based on the following lemmas.

Lemma 5.5. For any Π^1_1-set A, there exists a computable sequence $\{g_n\}_{n\in\omega}$ of partial computable functions $g_i : \mathcal{N} \to \mathbb{R}$ such that for all $n \in \omega$

1. g_n is total;
2. $n \notin A \leftrightarrow (\exists x \in \mathcal{N})\, g_n(x) = 0$;
3. $(\forall x \in \mathcal{N})(g_n(x) = 0 \to x \in \mathfrak{A}_{\langle 2n\rangle})$.

Proof. Let us choose a computable sequence $\{T_n\}_{n\in\omega}$ of computable trees $T_n \in \omega^{<\omega}$ such that

1. $(\exists f \in \mathcal{N})(\forall s \in \omega) f[s] \in T_n \leftrightarrow n \notin A$;
2. $(\forall h \in T_n) h(0) = 2n$.

Then by every tree T_n, we construct a function $g_n : \mathcal{N} \to \mathbb{R}$ by the rules:

1. For $f \in \mathcal{N}$, if $\exists k\, f[k] \notin T_n$, then we define $g_n(f) = 2^{-k}$.
2. For $f \in \mathcal{N}$, if $\forall k\, f[k] \in T_n$, then we define $g_n(f) = 0$.

By construction, $\{g_n\}$ is a required computable sequence of partial computable functions. □

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Let $U(x) < V(x)$ denote that, for all $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}$, if $z_1 \in U(x)$ and $z_2 \in V(x)$ then $z_1 < z_2$.

Lemma 5.6. For any Π^1_1-set B, there exists a computable sequence $\{f_n\}_{n\in\omega}$ of partial computable functions $f_i : \mathcal{N} \to \mathbb{R}$ such that for all $n \in \omega$

1. $n \in B \leftrightarrow f_n$ is total;
2. $(\forall x \in \mathcal{N})(f_n(x) \uparrow \to x \in \mathfrak{A}_{\langle 2n+1\rangle})$;
3. $\mathcal{N} \times \{0\} \subseteq U_n$, where (U_n, V_n) defines f_n such that, $U_n(x) < V_n(x)$.

Proof. In the proof of Theorem 5.5, we already established that $\{n \mid \mathcal{O}_n = \mathcal{N}\}$ is Π^1_1-complete, where $\mathcal{O}_n$ is nth effective open set of $\mathcal{N}$. So $n \in B \leftrightarrow \mathcal{O}_{h(n)} = \mathcal{N}$ for a suitable computable function $h : \omega \to \omega$. Let $\mathcal{O}_{h(n)} = A_n$. It is worth noting that $A_n \in \Pi^0_2$, and that there is an algorithm producing a partial computable function $f : \mathcal{N} \to \mathbb{R}$ such that $\mathrm{dom}(f) = A_n$ (see, e.g., Lemma 5.2). Therefore, we can effectively construct computable sequence of $\{f_n\}_{n\in\omega}$ of partial computable functions $f_i : \mathcal{N} \to \mathbb{R}$ such that for all $n \in \omega$

1. $\mathrm{dom}(f_n) = A_n$;
2. $f_n(x) \downarrow \to f_n(x) = 1$;
3. $V_n(x, y) = \{(x, y) \mid (x \in A_n \wedge y > 1) \vee (y > \frac{3}{2})\}$;
4. $U_n(x, y) = \{(x, y) \mid (x \in A_n \wedge y < 1) \vee (y < \frac{1}{2})\}$.

Moreover, we can satisfy the condition for all $n \in \omega$ $A_n \cup \mathfrak{A}_{\langle 2n+1\rangle} = \mathcal{N}$, e.g., redefining $A'_n = \{f \in \mathcal{N} \mid f(0) = 2n + 1 \wedge \mathrm{delay}(f) \in A_n\} \cup (\mathcal{N} \setminus \mathfrak{A}_{\langle 2n+1\rangle})$, where for all $m \in \omega$ $\mathrm{delay}(f)(m) = f(m + 1)$. Then, $A'_n \cup \mathfrak{A}_{\langle 2n+1\rangle} = \mathcal{N}$.

After all, we get for all $n \in \omega$

1. f_n is total $\leftrightarrow A_n = \mathcal{N} \leftrightarrow n \in B$;
2. $f_n(x) \uparrow \to x \in \mathfrak{A}_{\langle 2n+1\rangle}$;
3. $\mathcal{N} \times \{0\} \subseteq U_n$.

Hence, $\{f_n\}_{n\in\omega}$ is a required sequence. □

Proof. (Proposition 5.2)

Let (A, B) be a pair of Π^1_1-sets. Using g_n from Lemma 5.5 and f_n from Lemma 5.6, we put $h_n = f_n \cdot g_n$. By definition,

1. h_n is total $\leftrightarrow f_n$ is total $\leftrightarrow n \in B$;
2. $(\exists x \in \mathcal{N})h_n(x) \downarrow = 0 \leftrightarrow g_n(x) = 0 \leftrightarrow n \notin A$.

Since h_n is uniformly constructed, there is a computable function $\lambda : \omega \to \omega$ such that $h_n = \gamma(\lambda(n))$. The function λ m-reduces the pair (B, A) to the pair (K_1, K_2). By Theorems 5.5 and 5.3, the last pair is m-complete in the class of the pairs of Π^1_1-sets. □

Theorem 5.6. Let $\mathcal{X} = \mathcal{N}$. The index set

$$K = \{f : \mathcal{N} \to \mathbb{R} \mid f \text{ is a total computable function and } \exists x\, f(x) = 0\}$$

is m-complete in the class of the differences of Π^1_1-sets.

Proof. The claim is straightforward form the previous proposition since $K = K_1 \setminus K_2$. □

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Proposition 5.3. Let $X = \mathbb{R}$, $K_1 = \{n \mid \gamma(n) \text{ is not total}\}$ and $K_2 = \{n \mid \mathbb{R} \times \{0\} \not\subseteq U_n \cup V_n\}$. The pair (K_1, K_2) is m-complete in the class of the pairs of Σ^0_2-sets.

The proof is based on the following lemmas.

Lemma 5.7. Let A be Σ^0_2-complete. There exists a computable sequence $\{g_n\}_{n\in\omega}$ of partial computable functions $g_i : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \omega$

1. g_n is total and $g_n \geqslant 0$;
2. $n \in A \leftrightarrow (\exists x \in \mathbb{R})\, g_n(x) = 0$;
3. $g_n(x) = 0 \to x \in \bigcup_{m\in\mathbb{Z}}(2m, 2m+1)$.

Proof. It is sufficient to prove for $A = \text{Fin} = \{n \mid W_n \text{ is finite}\}$. Let $K_w \subset \omega$ be m-complete c.e. set and $\{a_n\}_{n\in\omega}$ be a computable sequence of computable non-negative real numbers such that $n \in K_w \leftrightarrow a_n > 0$. For example, $\{a_n\}_{n\in\omega}$ can be defined as

$$a_n = 1 - \sum_{m>0 : n \notin K^m_w} \frac{1}{2^m}.$$

Indeed, $\{a_n\}_{n\in\omega}$ is a computable sequence since for every $n \in \omega$

$$q < a_n \leftrightarrow q < 0 \vee \exists k \left(n \in K^{k+1}_w \setminus K^k_w \wedge q < 1 - \sum_{0<i\leqslant k} \frac{1}{2^i} \right),$$

$$q > a_n \leftrightarrow \exists k \left(n \notin K^k_w \wedge q > 1 - \sum_{0<i\leqslant k} \frac{1}{2^i} \right),$$

where $q \in \mathbb{Q}$. Using $\{a_n\}_{n\in\omega}$, we construct $\{g_n\}_{n\in\omega}$ as follows. Let $C_k = \{n \mid |W_n| \geqslant k\}$. Since $C_k \leqslant_m K_w$ uniformly on k, there exists a computable sequence $\{f_n : \omega \to \omega\}_{n\in\omega}$ such that $C_k = f_k^{-1}(K_w)$. So, $n \in C_k$ if and only if $f_k(n) \in K_w$ and if and only if $a_{f_k(n)} > 0$. Therefore, we can define $b_{n,k} = a_{f_k(n)}$. Then, g_n is the merging of partial functions $H_{n,k}$, $k \geqslant 0$, the function G and the function F such that

1. $\text{dom}(H_{n,k}) = [2k, 2k+1]$ and $H_{n,k}(x) = b_{n,k} + |x - (2k + \frac{1}{2})| \cdot (1 - b_{n,k}) \cdot 2$;
2. $\text{dom}(G) = \bigcup_{k\geqslant 0}[2k+1, 2k+2]$ and $G(x) \equiv 1$ on the domain;
3. $\text{dom}(F) = (-\infty, 0]$ and $F(x) \equiv 1$ on the domain.

By construction, g_n is total, W_n is infinite if and only if for all $k \in \omega$, $b_{n,k} > 0$ and if and only if g_n does not have roots. Moreover, if $g_n(x) = 0$, then $x \in (2m, 2m+1)$ for some $m \in \omega$. Hence, $\{g_n\}_{n\in\omega}$ is required. □

Lemma 5.8. Let B be Σ^0_2-complete. There exists a computable sequence $\{h_n\}_{n\in\omega}$ of partial computable functions $h_i : \mathbb{R} \to \mathbb{R}$, i.e. $h_n = \gamma(s(n))$ for a suitable computable function $s : \omega \to \omega$, such that for all $n \in \omega$.

1. $\mathbb{R} \times \{0\} \subseteq U_{s(n)}$, where $(U_{s(n)}, V_{s(n)})$ defines h_n such that $U_{s(n)}(x) < V_{s(n)}(x)$;
2. if $h_n(x) \uparrow$ then $x \in \bigcup_{m\in\mathbb{Z}}(2m-1, 2m)$;
3. $n \in B \leftrightarrow$ h_n is not total.

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Proof. It is sufficient to prove for $B = \mathrm{Fin} = \{n \mid W_n \text{ is finite}\}$. Let us construct effectively open sets U^n and V^n as follows.

$$U^n = \mathbb{R} \times (-\infty, 1),$$
$$V^n = \bigcup_{m \geqslant 0} \left[m, m + \frac{1}{3}\right] \times (1, +\infty) \cup (-\infty, 0] \times (1, +\infty) \cup \bigcup_{m \geqslant 0} \left[m + \frac{2}{3}, m + 1\right] \times (1, +\infty) \cup$$
$$\bigcup_{k \geqslant 0} [2k, 2k + 1] \times (1, +\infty) \cup \bigcup_{k \geqslant 0 : |W_n| \geqslant k} \left[2k - 1 + \frac{1}{3}, 2k - \frac{1}{3}\right] \times (1, +\infty).$$

Put $U_{s(n)} = U^n$ and $V_{s(n)} = V^n$. By construction, the pair $(U_{s(n)}, V_{s(n)})$ defines h_n such that

1. h_n is total if and only if W_n is infinite;
2. if W_n is finite then $\mathbb{R} \setminus \mathrm{dom}(h_n) = \bigcup_{k : |W_n| < k} [2k - 1 + \frac{1}{3}, 2k - \frac{1}{3}]$;
3. if $h_n(x) \downarrow$ then $h_n(x) = 1$;
4. if $h_n(x) \uparrow$ then $x \in (2k - 1, 2k)$ for some $k \in \omega$.

Hence, $\{h_n\}_{n \in \omega}$ is required. □

Proof. (Proposition 5.3) From Proposition 4.1 and Theorem 5.4, it follows that $K_1 \in \Sigma^0_2$. Since $n \in K_2 \leftrightarrow (\exists a \in \mathbb{Q})(\exists b \in \mathbb{Q})(a < b \wedge \neg([a, b] \times \{0\} \subseteq U_n \bigcup V_n))$, $K_2 \in \Sigma^0_2$ by Proposition 3.2. Let (A, B) be a pair of Σ^0_2-sets. Further, we use g_n from Lemma 5.7 and h_n from Lemma 5.8. Let $g_n = \gamma(G(n))$ and $h_n = \gamma(H(n))$ for suitable computable functions $G : \omega \to \omega$ and $H : \omega \to \omega$. Now, we construct (U^n, V^n) which will define the partial computable function $f_n = g_n \cdot h_n$ as follows.

$$U^n(x, y) \rightleftharpoons y < 0 \vee (\exists a > 0)(\exists c > 0)\big(U_{G(n)}(x, a) \wedge U_{H(n)}(x, c) \wedge a \cdot c > y\big),$$
$$V^n(x, y) \rightleftharpoons (\exists b > 0)(\exists d > 0)\big(V_{G(n)}(x, b) \wedge V_{H(n)}(x, d) \wedge b \cdot d < y\big).$$

It is worth noting that $U^n < V^n$ and $V^n \cap (\mathbb{R} \times \{0\}) = \varnothing$. By construction, there is a computable function $F : \omega \to \omega$ such that $U_{F(n)} = U^n$ and $V_{F(n)} = V^n$, in particular, $f_n = \gamma(F(n))$. Therefore,

$$\exists x\, f_n(x) \uparrow \leftrightarrow n \in B \leftrightarrow F(n) \in K_1.$$

By construction,

$$U^n \supseteq (\mathbb{R} \times \{0\}) \leftrightarrow U_{G(n)} \supseteq (\mathbb{R} \times \{0\}) \leftrightarrow \neg\, \exists x\, g_n(x) = 0 \leftrightarrow n \notin A.$$

Therefore, $n \in A \leftrightarrow F(n) \in K_2$. Hence, F m-reduces (B, A) to (K_1, K_2). So, (K_1, K_2) is m-complete in the class of the pairs of Σ^0_2-sets. □

Theorem 5.7. Let $\mathcal{X} = \mathbb{R}$. The index set

$$K = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a total computable function and } \exists x\, f(x) = 0\}$$

is m-complete in the class of differences of Σ^0_2-sets.

Proof. The claim is straightforward form the previous proposition since $K = K_2 \setminus K_1$. □

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6. Conclusion and related work

Recently, several approaches to measure complexity of undecidable problems in computable analysis have been intensively developed. Our work is closely related to the approach that originated from the research on computing index sets of closed subsets of classical Polish spaces and computably continuous functions (Cenzer and Remmel 1998, 1999).

Our main results extend this approach as follows. In the framework of effectively enumerable topological spaces, we have introduced the notion of a partial computable function and constructed principal computable numberings of the partial computable functions $f : X \to \mathbb{R}$. With respect to these principal computable numberings, we have shown complexity of problems over partial computable functions from a computable Polish space to $\mathbb{R}$. It will be challenging to extend this approach to the class of partial computable functions $f : X \to Y$ for computable Polish spaces X and Y.

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