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# Volume integral means over spherical shell

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*Abstract.* We investigate integral means over spherical shell of holomorphic functions in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ with respect to the weighted volume measures and their relation with the weighted Hadamard product. The main result of this paper has many consequences which improve some well-known estimates related to the Hadamard product in Hardy spaces and weighted Bergman spaces.

## 1 Introduction

The notion of the coefficient-wise product of two power series was first considered in the classical paper [3] by Hadamard. Since then a numerous problems related to the Hadamard product of holomorphic functions in the open unit disc in the complex plane have been considered. Of particular importance is the study of the behavior of the Hadamard product on various standard spaces of holomorphic functions on the unit disc. The effect of the Hadamard product on the classical Hardy spaces on the unit disc is described in [7]. In the case of the several complex variables, the weighted Hadamard product of holomorphic functions in the unit ball has been studied in [1] in the context of the weighted Bergman spaces. Some estimates for the Hadamard product on the classical Hardy and Bergman spaces on the unit disc in the complex plane are obtained in [10], where a list of references with various interesting applications of the Hadamard product is given. Some results related to the weighted Hadamard product and various weighted area integral means have been obtained very recently in [4, 5].

In this paper, we explore the weighted volume integral means over spherical shell of holomorphic functions in the unit ball, which can be seen as a generalization of the weighted volume integral means introduced in [12]. Our primary goal is to examine the mentioned volume integral means together with the appropriate weighted Hadamard product, which is a generalization of the weighted Hadamard products presented in [1] and [5]. The main result of this paper includes a number of special cases that improve many well-known existing results. Before describing the main result and its special cases we need to set up some standard notation, which will be used throughout the paper.

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## 1.1 Basic notation

Throughout the text, we fix a positive integer $n$, noting that in some situations we will consider the obtained results in a special case when $n = 1$. Let $\mathbb{C}^n$ be the Euclidean space of the complex dimension $n$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, $c \in \mathbb{C}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we write $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$, $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$, $z \cdot w = (z_1 w_1, \ldots, z_n w_n)$, $cz = (cz_1, \ldots, cz_n)$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$. Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$ and let also $\mathbb{S}_n = \partial\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| = 1\}$ be the unit sphere in $\mathbb{C}^n$. With this notation, $\mathbb{D} = \mathbb{B}_1$ is the open unit disc in $\mathbb{C}$ and $\mathbb{T} = \mathbb{S}_1$ is the unit circle in $\mathbb{C}$. For $0 \le r \le 1$, we write $\mathbb{B}_{n,r} = r\mathbb{B}_n$ and $\mathbb{S}_{n,r} = r\mathbb{S}_n$. On the other hand, for $0 \le r < R \le 1$, we also write $\mathbb{A}_{n,r,R} = \mathbb{B}_{n,R} \setminus \overline{\mathbb{B}}_{n,r} = \{z \in \mathbb{C}^n : r < |z| < R\}$. The set $\mathbb{A}_{n,r,R}$ is called *spherical shell* in the open unit ball $\mathbb{B}_n$. It is a generalization of the annulus to higher dimensions. We mention that spherical shells will play an important role in the results that follow. In the case of one complex dimension $n = 1$, we simply write $\mathbb{D}_r = \mathbb{B}_{1,r}$, $\mathbb{T}_r = \mathbb{S}_{1,r}$ and $\mathbb{A}_{r,R} = \mathbb{A}_{1,r,R}$ is an annulus in the open unit disc $\mathbb{D}$.

We let dV denote the volume measure on the open unit ball $\mathbb{B}_n$ and the surface measure on the unit sphere $\mathbb{S}_n$ will be denoted by dS. Of course, measures dV and dS are rotation invariant. It is well-known that $\mathrm{V}(\mathbb{B}_n) = \pi^n/n!$ and $\mathrm{S}(\mathbb{S}_n) = 2\pi^n/(n-1)!$. If $z \in \mathbb{B}_n$, then $z = \rho\zeta$ for some $0 \le \rho < 1$, $\zeta \in \mathbb{S}^n$ and $\mathrm{dV}(z) = \frac{\mathrm{V}(\mathbb{B}_n)}{\mathrm{S}(\mathbb{S}_n)} 2n\rho^{2n-1}\,\mathrm{d}\rho\,\mathrm{dS}(\zeta)$, which implies integration in polar coordinates

$$\frac{1}{\mathrm{V}(\mathbb{B}_n)} \int_{\mathbb{B}_n} f(z)\,\mathrm{dV}(z) = \int_0^1 \frac{1}{\mathrm{S}(\mathbb{S}_n)} \int_{\mathbb{S}_n} f(\rho\zeta)\,\mathrm{dS}(\zeta)\, 2n\rho^{2n-1}\,\mathrm{d}\rho,$$

for a given function $f : \mathbb{B}_n \to \mathbb{C}$. On the other hand, by $\mathrm{Hol}(\mathbb{B}_n)$, we denote the space of all holomorphic functions on the open unit ball $\mathbb{B}_n$, that is $\mathrm{Hol}(\mathbb{B}_n) = \{f : \mathbb{B}_n \to \mathbb{C} : f \text{ is holomorphic function}\}$. If $f \in \mathrm{Hol}(\mathbb{B}_n)$, then there exists a unique power series representing a holomorphic function $f$, i.e., $f(z) = \sum_\alpha a_\alpha z^\alpha$, $z \in \mathbb{B}_n$, where the summation is over all multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Also, for each $0 \le s \le 1$, the notation $f_s$ will refer to the dilatation of the function $f$, that is $f_s(z) = f(sz)$, $z \in \mathbb{B}_n$. For further information related to the theory of holomorphic functions in the unit ball we refer to the classical monographs [2, 6, 8, 9, 14].

## 1.2 Weighted volume measures

We define the weighted volume measures on the unit ball $\mathbb{B}_n$ in the following way $\mathrm{dV}_\gamma(z) = \left(1 - |z|^2\right)^\gamma \mathrm{dV}(z)$ for $z \in \mathbb{B}_n$, where $\gamma > -1$. Then integration in polar coordinates shows

$$\mathrm{V}_\gamma(\mathbb{B}_n) = \int_{\mathbb{B}_n} \left(1 - |z|^2\right)^\gamma \mathrm{dV}(z) = \frac{\mathrm{V}(\mathbb{B}_n)}{\mathrm{S}(\mathbb{S}_n)} \int_0^1 \int_{\mathbb{S}_n} \left(1 - |\rho\zeta|^2\right)^\gamma \mathrm{dS}(\zeta)\, 2n\rho^{2n-1}\,\mathrm{d}\rho,$$

or equivalently

$$\mathrm{V}_\gamma(\mathbb{B}_n) = n\mathrm{V}(\mathbb{B}_n) \int_0^1 2\rho^{2n-1} \left(1 - \rho^2\right)^\gamma \mathrm{d}\rho = n\mathrm{V}(\mathbb{B}_n)\, \mathrm{B}(n, \gamma + 1),$$

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i.e.,

$$\mathrm{V}_\gamma(\mathbb{B}_n) = \frac{n!\Gamma(\gamma+1)}{\Gamma(n+\gamma+1)}\mathrm{V}(\mathbb{B}_n), \tag{1.1}$$

where $\Gamma$ and B are the classical Gamma and Beta functions, respectively, that is, where $\Gamma(x) = \int_0^\infty \rho^{x-1}e^{-\rho}\,\mathrm{d}\rho$ and $\mathrm{B}(x,y) = \int_0^1 \rho^{x-1}(1-\rho)^{y-1}\,\mathrm{d}\rho$ for $x, y > 0$ and where it is valid well-known formula $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Also, if $\alpha, \beta \in \mathbb{N}_0^n$ are multi-indexes of non-negative integers, based on [14], we can write

$$\frac{1}{\mathrm{S}(\mathbb{S}_n)}\int_{\mathbb{S}_n} \zeta^\alpha \overline{\zeta}^\beta\,\mathrm{dS}(\zeta) = \begin{cases} 0, & \alpha \neq \beta \\ \dfrac{1}{\mathrm{S}(\mathbb{S}_n)}\displaystyle\int_{\mathbb{S}_n} |\zeta^\alpha|^2\,\mathrm{dS}(\zeta) = \frac{\alpha!\Gamma(n)}{\Gamma(n+|\alpha|)}, & \alpha = \beta \end{cases}. \tag{1.2}$$

## 1.3 Weighted Bergman spaces

For $0 < p < \infty$ and $\gamma > -1$ the weighted Bergman space on the unit ball is defined as follows

$$A^p_\gamma(\mathbb{B}_n) = \left\{ f \in \mathrm{Hol}(\mathbb{B}_n) : \|f\|_{A^p_\gamma(\mathbb{B}_n)} = \left( \frac{1}{\mathrm{V}_\gamma(\mathbb{B}_n)} \int_{\mathbb{B}_n} |f|^p\,\mathrm{dV}_\gamma \right)^{\frac{1}{p}} < \infty \right\}.$$

If $f \in A^p_\gamma(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$ we have the following sharp pointwise estimate

$$|f(z)| \leq \left(1-|z|^2\right)^{-\frac{n+\gamma+1}{p}} \|f\|_{A^p_\gamma(\mathbb{B}_n)}, \tag{1.3}$$

which was first obtained in [11]. We note that if $\gamma = 0$ then $A^p(\mathbb{B}_n) = A^p_0(\mathbb{B}_n)$ are standard unweighted Bergman spaces. Also, if $0 \leq r < 1$, $0 < p < \infty$ and $f \in \mathrm{Hol}(\mathbb{B}_n)$ we consider the following surface area integral means

$$\mathrm{M}_p(r,f) = \left( \frac{1}{\mathrm{S}(\mathbb{S}_n)} \int_{\mathbb{S}_n} |f_r|^p\,\mathrm{dS} \right)^{\frac{1}{p}} = \left( \frac{1}{\mathrm{S}(\mathbb{S}_n)} \int_{\mathbb{S}_n} |f(r\zeta)|^p\,\mathrm{dS}(\zeta) \right)^{\frac{1}{p}}.$$

It is well-known that $r \mapsto \mathrm{M}_p(r,f)$ is an increasing function and $\log \mathrm{M}_p(r,f)$ is a convex function of $\log r$. See [12, Lemma 1]. The Hardy space on the unit ball is defined as follows

$$H^p(\mathbb{B}_n) = \{ f \in \mathrm{Hol}(\mathbb{B}_n) : \|f\|_{H^p(\mathbb{B}_n)} = \sup_{0 \leq r < 1} \mathrm{M}_p(r,f) < \infty \}.$$

It is a well-known fact that in the sense of the boundary values of the function $f \in H^p(\mathbb{B}_n)$, we may write $\|f\|_{H^p(\mathbb{B}_n)} = \mathrm{M}_p(1,f)$. See [9, 14]. Therefore, if $f \in H^p(\mathbb{B}_n)$, $0 < p < \infty$ and $0 \leq s \leq 1$, we can write $\|f_s\|_{H^p(\mathbb{B}_n)} = \mathrm{M}_p(s,f)$. On the other hand, if $f \in \mathrm{Hol}(\mathbb{B}_n)$, then based on [12, 13] we can also write

$$\lim_{\gamma \to -1^+} \|f\|_{A^p_\gamma(\mathbb{B}_n)} = \|f\|_{H^p(\mathbb{B}_n)}, \quad 0 < p < \infty. \tag{1.4}$$

Therefore, the Hardy space $H^p(\mathbb{B}_n)$ is the limit case of the weighted Bergman spaces $A^p_\gamma(\mathbb{B}_n)$ when $\gamma \to -1^+$. By using integration in polar coordinates, we easily obtain $\|f\|^p_{A^p(\mathbb{B}_n)} = \int_0^1 \mathrm{M}^p_p(\rho,f)\,2n\rho^{2n-1}\,\mathrm{d}\rho$. Hence, if $f \in H^p(\mathbb{B}_n)$ then $\|f\|^p_{A^p(\mathbb{B}_n)} \leq$

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$\int_0^1 \|f\|^p_{H^p(\mathbb{B}_n)} 2n\rho^{2n-1}\,\mathrm{d}\rho = \|f\|^p_{H^p(\mathbb{B}_n)}$, that is $\|f\|_{A^p(\mathbb{B}_n)} \le \|f\|_{H^p(\mathbb{B}_n)}$. Actually, we have the following interesting result, which will be useful later.

***Lemma 1.1*** *Let $0 < p < \infty$ and $f \in H^p(\mathbb{B}_n)$. Then*

$$\left\| f_{e^{-\frac{1}{2n}}} \right\|_{H^p(\mathbb{B}_n)} \le \|f\|_{A^p(\mathbb{B}_n)} \le \|f\|_{H^p(\mathbb{B}_n)}.$$

**Proof** We have already shown $\|f\|_{A^p(\mathbb{B}_n)} \le \|f\|_{H^p(\mathbb{B}_n)}$. On the other hand, we denote $\mathrm{I}_p[f](r) = \mathrm{M}_p^p(r, f)$. Then $\mathrm{I}_p[f] \circ \exp$ is a convex function. By using Jensen inequality and integration by parts, we conclude

$$\begin{aligned}
\|f\|^p_{A^p(\mathbb{B}_n)} &= \int_0^1 \big(\mathrm{I}_p[f] \circ \exp\big)(\log\rho)\; 2n\rho^{2n-1}\,\mathrm{d}\rho \\
&\ge \big(\mathrm{I}_p[f] \circ \exp\big)\left(\int_0^1 2n\rho^{2n-1}\log\rho\,\mathrm{d}\rho\right) \\
&= \big(\mathrm{I}_p[f] \circ \exp\big)\left(\rho^{2n}\log\rho\,\Big|_0^1 - \int_0^1 \rho^{2n-1}\,\mathrm{d}\rho\right) \\
&= \big(\mathrm{I}_p[f] \circ \exp\big)\left(-\frac{1}{2n}\right) = \mathrm{M}_p^p\left(e^{-\frac{1}{2n}}, f\right) = \left\| f_{e^{-\frac{1}{2n}}} \right\|^p_{H^p(\mathbb{B}_n)}.
\end{aligned}$$

This finishes the proof. ∎

## 1.4 Volume integral means

Let $f$ be a holomorphic function in the unit ball $\mathbb{B}_n$, $0 < p < \infty$, $\gamma > -1$, $0 \le r < R \le 1$ and $0 \le s \le 1$. We consider the following volume integral means over spherical shell

$$\mathcal{M}_{p,\gamma}[f](r, R, s) = \left(\frac{1}{\mathrm{V}_\gamma(\mathbb{A}_{n,r,R})}\int_{\mathbb{A}_{n,r,R}} |f_s|^p\,\mathrm{dV}_\gamma\right)^{\frac{1}{p}}. \tag{1.5}$$

Note that

$$\mathcal{M}_{p,\gamma}[f](0, 1, s) = \left(\frac{1}{\mathrm{V}_\gamma(\mathbb{B}_n)}\int_{\mathbb{B}_n} |f_s|^p\,\mathrm{dV}_\gamma\right)^{\frac{1}{p}} = \|f_s\|_{A^p_\gamma(\mathbb{B}_n)}.$$

Also, in the special case when $r = 0$ and $s = 1$, we obtain the volume integral means

$$\mathcal{M}_{p,\gamma}[f](0, R, 1) = \left(\frac{1}{\mathrm{V}_\gamma(\mathbb{B}_{n,R})}\int_{\mathbb{B}_{n,R}} |f|^p\,\mathrm{dV}_\gamma\right)^{\frac{1}{p}},$$

which are considered in [12], where, among other things, it is shown that $R \mapsto \mathcal{M}_{p,\gamma}[f](0, R, 1)$ is an increasing function. In fact, it is much more true, since $\mathcal{M}_{p,\gamma}[f](r, R, s)$ is an increasing function in each of the arguments $r$, $R$ and $s$. Namely, we have the following result.

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***Lemma 1.2*** *Let $f \in \mathrm{Hol}(\mathbb{B}_n)$, $0 < p < \infty$, $\gamma > -1$, $0 \leq r < R \leq 1$, and $0 \leq s \leq 1$. Then the following functions*

$$r \mapsto \mathcal{M}_{p,\gamma}[f](r,R,s), \quad R \mapsto \mathcal{M}_{p,\gamma}[f](r,R,s) \text{ and } s \mapsto \mathcal{M}_{p,\gamma}[f](r,R,s),$$

*are increasing.*

**Proof** Integration in polar coordinates gives

$$\begin{aligned}\int_{\mathbb{A}_{n,r,R}} |f_s|^p \, \mathrm{dV}_\gamma &= \mathrm{V}(\mathbb{B}_n) \int_r^R \frac{1}{\mathrm{S}(\mathbb{S}_n)} \int_{\mathbb{S}_n} |f_s(\rho\zeta)|^p \, \mathrm{dS}(\zeta) \, 2n\rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho \\ &= \mathrm{V}(\mathbb{B}_n) \int_r^R \mathrm{M}_p^p(\rho, f_s) \, 2n\rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho,\end{aligned}$$

and

$$\mathrm{V}_\gamma(\mathbb{A}_{n,r,R}) = \mathrm{V}(\mathbb{B}_n) \int_r^R 2n\rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho,$$

which implies

$$\mathcal{M}_{p,\gamma}^p[f](r,R,s) = \frac{\displaystyle\int_r^R \mathrm{M}_p^p(\rho, f_s) \, \rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho}{\displaystyle\int_r^R \rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho}. \tag{1.6}$$

By using formula (1.6) and the fact that $s \mapsto \mathrm{M}_p^p(\rho, f_s) = \mathrm{M}_p^p(s\rho, f)$ is an increasing function, we directly obtain that $s \mapsto \mathcal{M}_{p,\gamma}[f](r,R,s)$ is an increasing function too. On the other hand, by using again relation (1.6) and the fact that $\rho \mapsto \mathrm{M}_p^p(\rho, f_s)$ is an increasing function, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}R} \mathcal{M}_{p,\gamma}^p[f](r,R,s) \\ &= \frac{R^{2n-1}\left(1-R^2\right)^\gamma \displaystyle\int_r^R \overbrace{\left(\mathrm{M}_p^p(R, f_s) - \mathrm{M}_p^p(\rho, f_s)\right)}^{\geq 0} \rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho}{\left(\displaystyle\int_r^R \rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho\right)^2} \geq 0.\end{aligned}$$

Hence, $R \mapsto \mathcal{M}_{p,\gamma}[f](r,R,s)$ is an increasing function. Similarly, we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r} \mathcal{M}_{p,\gamma}^p[f](r,R,s) \\ &= \frac{r^{2n-1}\left(1-r^2\right)^\gamma \displaystyle\int_r^R \overbrace{\left(\mathrm{M}_p^p(\rho, f_s) - \mathrm{M}_p^p(r, f_s)\right)}^{\geq 0} \rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho}{\left(\displaystyle\int_r^R \rho^{2n-1} \left(1-\rho^2\right)^\gamma \mathrm{d}\rho\right)^2} \geq 0.\end{aligned}$$

Thus, $r \mapsto \mathcal{M}_{p,\gamma}[f](r,R,s)$ is an increasing function. This completes the proof. ∎

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In the case $n = 1$, the integral means given by the formula (1.5) become

$$\mathcal{M}_{p,\gamma}[f](r,R,s) = \left(\frac{1}{\mathrm{m}_\gamma\left(\mathbb{A}_{r,R}\right)}\int_{\mathbb{A}_{r,R}} |f_s|^p \,\mathrm{dm}_\gamma\right)^{\frac{1}{p}},$$

where $\mathrm{dm}_\gamma(z) = \left(1-|z|^2\right)^\gamma \mathrm{dm}(z)$, $z \in \mathbb{B}_1 = \mathbb{D}$ and dm is area measure on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Specially, we get

$$\mathcal{M}_{p,\gamma}[f](0,1,s) = \left(\frac{1}{\mathrm{m}_\gamma\left(\mathbb{D}\right)}\int_{\mathbb{D}} |f_s|^p \,\mathrm{dm}_\gamma\right)^{\frac{1}{p}}, \tag{1.7}$$

and

$$\mathcal{M}_{p,\gamma}[f](0,R,1) = \left(\frac{1}{\mathrm{m}_\gamma\left(\mathbb{D}_R\right)}\int_{\mathbb{D}_R} |f|^p \,\mathrm{dm}_\gamma\right)^{\frac{1}{p}}, \tag{1.8}$$

noting that means given by (1.7) and (1.8) play a major role in the paper [5] and in the case when $\gamma = 0$ we obtain the integral means considered in the paper [4]. See also [10].

## 1.5 Weighted Hadamard product

Let $f$ and $g$ be functions in $\mathrm{Hol}\left(\mathbb{B}_n\right)$ with the following power series representations

$$f(z) = \sum_\alpha a_\alpha z^\alpha \quad \text{and} \quad g(z) = \sum_\alpha b_\alpha z^\alpha, \quad z \in \mathbb{B}_n.$$

Then their classical *Hadamard product* is defined by

$$(f * g)(z) = \sum_\alpha a_\alpha b_\alpha z^\alpha, \quad z \in \mathbb{B}_n.$$

On the other hand, their *weighted Hadamard product*, with parameters $\gamma > -1$ and $0 \le r < R \le 1$, is defined in the following way

$$\begin{aligned}&(f * g)_{\gamma,r,R}(z)\\ &\quad= \sum_\alpha \frac{\alpha!}{\Gamma(n+|\alpha|)}\frac{\pi^n}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)}\left(\mathrm{B}_R(n+|\alpha|,\gamma+1) - \mathrm{B}_r(n+|\alpha|,\gamma+1)\right) a_\alpha b_\alpha z^\alpha,\end{aligned}$$

for $z \in \mathbb{B}_n$. Note that $\mathrm{B}_t(x,y)$ is the standard incomplete Beta function, that is $\mathrm{B}_t(x,y) = \int_0^t \rho^{x-1}(1-\rho)^{y-1}\,\mathrm{d}\rho$, where $0 \le t \le 1$, $x > 0$ and $y > 0$. In the special case, when $r = 0$ and $R = 1$, we obtain

$$\begin{aligned}&(f * g)_{\gamma,0,1}(z)\\ &\quad= \sum_\alpha \frac{\alpha!}{\Gamma(n+|\alpha|)}\frac{\pi^n}{\mathrm{V}_\gamma\left(\mathbb{B}_n\right)}\mathrm{B}(n+|\alpha|,\gamma+1)\, a_\alpha b_\alpha z^\alpha\\ &\quad\overset{(1.1)}{=} \sum_\alpha \frac{\alpha!}{\Gamma(n+|\alpha|)}\frac{n!\Gamma(n+\gamma+1)\mathrm{V}\left(\mathbb{B}_n\right)}{n!\Gamma(\gamma+1)\mathrm{V}\left(\mathbb{B}_n\right)}\frac{\Gamma(n+|\alpha|)\Gamma(\gamma+1)}{\Gamma(n+|\alpha|+\gamma+1)}\, a_\alpha b_\alpha z^\alpha,\end{aligned}$$

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or equivalently

$$(f*g)_{\gamma,0,1}(z)=\sum_{\alpha}\frac{\alpha!\Gamma(n+\gamma+1)}{\Gamma(n+|\alpha|+\gamma+1)}a_\alpha b_\alpha z^\alpha. \tag{1.9}$$

Note that our weighted Hadamard product $(f*g)_{\gamma,r,R}$ in this special case when $r=0$ and $R=1$, that is $(f*g)_{\gamma,0,1}$, plays an important role in the paper [1]. We also mention that integral representation of the weighted Hadamard product $(f*g)_{\gamma,r,R}$ will be obtained in Section 2 (actually, see formula (2.2) in the proof of Lemma 2.6), which will guarantee that the function $(f*g)_{\gamma,r,R}$ is holomorphic in the open unit ball $\mathbb{B}_n$, under the initial condition that $f,g\in \mathrm{Hol}(\mathbb{B}_n)$. Finally, in the case of one complex dimension $n=1$, the weighted Hadamard product $(f*g)_{\gamma,r,R}$ with $r=0$, that is $(f*g)_{\gamma,0,R}$, was studied in [5].

On the other hand, letting $\gamma\to -1^+$ in (1.9), we also can write

$$(f*g)_{-1,0,1}(z)=\sum_{\alpha}\frac{\alpha!\Gamma(n)}{\Gamma(n+|\alpha|)}a_\alpha b_\alpha z^\alpha=\sum_{\alpha}\frac{\alpha!(n-1)!}{(n-1+|\alpha|)!}a_\alpha b_\alpha z^\alpha. \tag{1.10}$$

We observe that, from the above formula (1.10), in the case $n=1$, we have

$$(f*g)_{-1,0,1}(z)=\sum_{\alpha=0}^{\infty}\frac{\alpha!}{|\alpha|!}a_\alpha b_\alpha z^\alpha=\sum_{\alpha=0}^{\infty}\frac{\alpha!}{\alpha!}a_\alpha b_\alpha z^\alpha=\sum_{\alpha=0}^{\infty}a_\alpha b_\alpha z^\alpha=(f*g)(z),$$

for $z\in\mathbb{B}_1=\mathbb{D}$, that is $(f*g)_{-1,0,1}=f*g$, which means that in the case $n=1$ our weighted Hadamard product $(f*g)_{-1,0,1}$ becomes the standard Hadamard product $f*g$.

## 1.6 The main result

Based on the above notations, we are ready to express our main result.

***Theorem 1.3*** *Let $f\in A^p_\gamma(\mathbb{B}_n)$ and $g\in A^q_\gamma(\mathbb{B}_n)$, with $0<p\le 1$, $p\le q<\infty$, $\gamma>-1$, and $0\le r<R\le 1$, $0\le s\le 1$, $0\le a<b\le 1$. Then*

$$\mathcal{M}_{q,\gamma}\left[(f*g)_{\gamma,r,R}\right](a,b,s)\le\frac{\|f\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)}\mathcal{M}^p_{p,\gamma}[f]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right)\mathcal{M}^{1-p}_{q,\gamma}[g](a,b,1)\mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}\sqrt{R}\right)}{(1-sR)^{(n+\gamma+1)\left(\frac{1}{p}-1\right)}}.$$

This result allows us several interesting special cases. We list some of them below.

***Theorem 1.4*** *Let $f\in A^p_\gamma(\mathbb{B}_n)$ and $g\in A^q_\gamma(\mathbb{B}_n)$, with $0<p\le 1$, $p\le q<\infty$, $\gamma>-1$ and $0\le r<R\le 1$, $0\le s\le 1$. Then*

$$\left\|\left((f*g)_{\gamma,r,R}\right)_s\right\|_{A^q_\gamma(\mathbb{B}_n)}\le\frac{\|f\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)}\mathcal{M}^p_{p,\gamma}[f]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right)\|g\|^{1-p}_{A^q_\gamma(\mathbb{B}_n)}\left\|g_{\sqrt{s}\sqrt{R}}\right\|^p_{A^q_\gamma(\mathbb{B}_n)}}{(1-sR)^{(n+\gamma+1)\left(\frac{1}{p}-1\right)}}.$$

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**Proof** Put $a = 0$ and $b = 1$ in Theorem 1.3. We note also that $\left((f * g)_{\gamma,r,R}\right)_s$ is dilatation of the weighted Hadamard product $(f * g)_{\gamma,r,R}$, that is $\left((f * g)_{\gamma,r,R}\right)_s(z) = (f * g)_{\gamma,r,R}(sz)$ for each $z \in \mathbb{B}_n$. ∎

By using Theorem 1.4 in the case when $r = 0$ and $R = 1$, we obtain the following interesting result.

***Corollary 1.5*** *Let $f \in A^p_\gamma(\mathbb{B}_n)$ and $g \in A^q_\gamma(\mathbb{B}_n)$, with $0 < p \le 1$, $p \le q < \infty$, $\gamma > -1$ and $0 \le s \le 1$. Then*

$$\left\|\left((f * g)_{\gamma,0,1}\right)_s\right\|_{A^q_\gamma(\mathbb{B}_n)} \le (1-s)^{(n+\gamma+1)\left(1-\frac{1}{p}\right)} \|f\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)} \left\|f_{\sqrt{s}}\right\|^p_{A^p_\gamma(\mathbb{B}_n)} \|g\|^{1-p}_{A^q_\gamma(\mathbb{B}_n)} \left\|g_{\sqrt{s}}\right\|^p_{A^q_\gamma(\mathbb{B}_n)}.$$

***Remark 1.6*** Since

$$\left\|f_{\sqrt{s}}\right\|_{A^p_\gamma(\mathbb{B}_n)} = \mathcal{M}_{p,\gamma}[f]\left(0,1,\sqrt{s}\right) \le \mathcal{M}_{p,\gamma}[f](0,1,1) = \|f\|_{A^p_\gamma(\mathbb{B}_n)},$$

and

$$\left\|g_{\sqrt{s}}\right\|_{A^q_\gamma(\mathbb{B}_n)} = \mathcal{M}_{q,\gamma}[g]\left(0,1,\sqrt{s}\right) \le \mathcal{M}_{q,\gamma}[g](0,1,1) = \|g\|_{A^q_\gamma(\mathbb{B}_n)},$$

as a direct consequence of Corollary 1.5, we obtain the following inequality

$$\left\|\left((f * g)_{\gamma,0,1}\right)_s\right\|_{A^q_\gamma(\mathbb{B}_n)} \le (1-s)^{(n+\gamma+1)\left(1-\frac{1}{p}\right)} \|f\|_{A^p_\gamma(\mathbb{B}_n)} \|g\|_{A^q_\gamma(\mathbb{B}_n)}, \tag{1.11}$$

which was originally proved in [1, Proposition 3.10]. Therefore, our result presented in Corollary 1.5 can be viewed as an improvement of [1, Proposition 3.10]. Note also that inequality (1.11) in the case of one complex dimension $n = 1$, i.e.,

$$\left\|\left((f * g)_{\gamma,0,1}\right)_s\right\|_{A^q_\gamma(\mathbb{D})} \le (1-s)^{(\gamma+2)\left(1-\frac{1}{p}\right)} \|f\|_{A^p_\gamma(\mathbb{D})} \|g\|_{A^q_\gamma(\mathbb{D})},$$

was also obtained in [5, Remark 3.5], where the functions $f$ and $g$ are replaced by their appropriate fractional derivatives of Hardy–Littlewood. For more information see the recent paper [5]. ◇

Letting $\gamma \to -1^+$ in Corollary 1.5 and by using the fact (1.4), we also get the following result.

***Corollary 1.7*** *Let $f \in H^p(\mathbb{B}_n)$ and $g \in H^q(\mathbb{B}_n)$, with $0 < p \le 1$, $p \le q < \infty$, and $0 \le s \le 1$. Then*

$$\left\|\left((f * g)_{-1,0,1}\right)_s\right\|_{H^q(\mathbb{B}_n)} \le (1-s)^{n\left(1-\frac{1}{p}\right)} \|f\|^{1-p}_{H^p(\mathbb{B}_n)} \left\|f_{\sqrt{s}}\right\|^p_{H^p(\mathbb{B}_n)} \|g\|^{1-p}_{H^q(\mathbb{B}_n)} \left\|g_{\sqrt{s}}\right\|^p_{H^q(\mathbb{B}_n)},$$

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*or equivalently*

$$\begin{aligned}&\mathrm{M}_q\left(s,(f*g)_{-1,0,1}\right)\\&\quad\le(1-s)^{n\left(1-\frac{1}{p}\right)}\|f\|_{H^p(\mathbb{B}_n)}^{1-p}\left\|f_{\sqrt{s}}\right\|_{H^p(\mathbb{B}_n)}^{p}\|g\|_{H^q(\mathbb{B}_n)}^{1-p}\left\|g_{\sqrt{s}}\right\|_{H^q(\mathbb{B}_n)}^{p}.\end{aligned}$$

In the case of one complex dimension $n=1$, our Corollary 1.7 gives the next result.

***Corollary 1.8*** *Let $f\in H^p(\mathbb{D})$ and $g\in H^q(\mathbb{D})$, with $0<p\le 1$, $p\le q<\infty$ and $0\le s\le 1$. Then*

$$\mathrm{M}_q(s,f*g)\le(1-s)^{1-\frac{1}{p}}\|f\|_{H^p(\mathbb{D})}^{1-p}\left\|f_{\sqrt{s}}\right\|_{H^p(\mathbb{D})}^{p}\|g\|_{H^q(\mathbb{D})}^{1-p}\left\|g_{\sqrt{s}}\right\|_{H^q(\mathbb{D})}^{p}.$$

***Remark 1.9*** Since

$$\left\|f_{\sqrt{s}}\right\|_{H^p(\mathbb{D})}=\mathrm{M}_p\left(\sqrt{s},f\right)\le\|f\|_{H^p(\mathbb{D})},$$

and

$$\left\|g_{\sqrt{s}}\right\|_{H^q(\mathbb{D})}=\mathrm{M}_q\left(\sqrt{s},g\right)\le\|g\|_{H^q(\mathbb{D})},$$

as a direct consequence of Corollary 1.8, we derive the following inequality

$$\mathrm{M}_q(s,f*g)\le(1-s)^{1-\frac{1}{p}}\|f\|_{H^p(\mathbb{D})}\|g\|_{H^q(\mathbb{D})},\tag{1.12}$$

which was originally proved in [7]. Hence our Corollary 1.8 can be viewed as an improvement of the main result of [7]. ◇

***Corollary 1.10*** *Let $f\in H^p(\mathbb{B}_n)$ and $g\in H^q(\mathbb{B}_n)$, with $0<p\le 1$, $p\le q<\infty$ and $0\le s\le e^{-\frac{1}{n}}$. Then*

$$\begin{aligned}&\left\|\left((f*g)_{-1,0,1}\right)_s\right\|_{H^q(\mathbb{B}_n)}\\&\quad\le(1-s)^{n\left(1-\frac{1}{p}\right)}\left(\|f\|_{H^p(\mathbb{B}_n)}\|g\|_{H^q(\mathbb{B}_n)}\right)^{1-p}\left(\|f\|_{A^p(\mathbb{B}_n)}\|g\|_{A^q(\mathbb{B}_n)}\right)^{p},\end{aligned}$$

*or equivalently*

$$\begin{aligned}&\mathrm{M}_q\left(s,(f*g)_{-1,0,1}\right)\\&\quad\le(1-s)^{n\left(1-\frac{1}{p}\right)}\left(\|f\|_{H^p(\mathbb{B}_n)}\|g\|_{H^q(\mathbb{B}_n)}\right)^{1-p}\left(\|f\|_{A^p(\mathbb{B}_n)}\|g\|_{A^q(\mathbb{B}_n)}\right)^{p}.\end{aligned}$$

**Proof** Since $0\le s\le e^{-\frac{1}{n}}$, by using Lemma 1.1, we find

$$\left\|f_{\sqrt{s}}\right\|_{H^p(\mathbb{B}_n)}=\mathrm{M}_p\left(\sqrt{s},f\right)\le\mathrm{M}_p\left(e^{-\frac{1}{2n}},f\right)=\left\|f_{e^{-\frac{1}{2n}}}\right\|_{H^p(\mathbb{B}_n)}\le\|f\|_{A^p(\mathbb{B}_n)},$$

and

$$\left\|g_{\sqrt{s}}\right\|_{H^q(\mathbb{B}_n)}=\mathrm{M}_q\left(\sqrt{s},g\right)\le\mathrm{M}_q\left(e^{-\frac{1}{2n}},g\right)=\left\|g_{e^{-\frac{1}{2n}}}\right\|_{H^q(\mathbb{B}_n)}\le\|g\|_{A^q(\mathbb{B}_n)},$$

whence follows the desired conclusion by direct application of Corollary 1.7. ■

In the case of one complex dimension $n=1$, Corollary 1.10 produces the next result.

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***Corollary 1.11*** *Let $f \in H^p(\mathbb{D})$ and $g \in H^q(\mathbb{D})$, with $0 < p \le 1$, $p \le q < \infty$ and $0 \le s \le e^{-1}$. Then*

$$\mathrm{M}_q(s, f * g) \le (1-s)^{1-\frac{1}{p}} \left(\|f\|_{H^p(\mathbb{D})} \|g\|_{H^q(\mathbb{D})}\right)^{1-p} \left(\|f\|_{A^p(\mathbb{D})} \|g\|_{A^q(\mathbb{D})}\right)^p.$$

***Remark 1.12*** Note that $\|f\|_{A^p(\mathbb{D})} \le \|f\|_{H^p(\mathbb{D})}$ and $\|g\|_{A^q(\mathbb{D})} \le \|g\|_{H^q(\mathbb{D})}$. Thus, Corollary 1.11 represents an interesting refinement of the inequality (1.12), which is the main result of the paper [7], but only in the case when $0 \le s \le e^{-1}$. ◇

By using Theorem 1.4 in the special case when $r = 0$ and $s = 1$, we obtain the next result.

***Corollary 1.13*** *Let $f \in A^p_\gamma(\mathbb{B}_n)$ and $g \in A^q_\gamma(\mathbb{B}_n)$, where $0 < p \le 1$, $p \le q < \infty$, $\gamma > -1$ and let also $0 < R \le 1$. Then*

$$\begin{aligned}&\left\|(f * g)_{\gamma,0,R}\right\|_{A^q_\gamma(\mathbb{B}_n)}\\ &\quad\le (1-R)^{(n+\gamma+1)\left(1-\frac{1}{p}\right)} \|f\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)} \mathcal{M}^p_{p,\gamma}[f]\left(0, \sqrt{R}, 1\right) \|g\|^{1-p}_{A^q_\gamma(\mathbb{B}_n)} \left\|g_{\sqrt{R}}\right\|^p_{A^q_\gamma(\mathbb{B}_n)}.\end{aligned}$$

***Remark 1.14*** In the case of one complex dimension $n = 1$, our Corollary 1.13 produces the following inequality

$$\begin{aligned}&\left\|(f * g)_{\gamma,0,R}\right\|_{A^q_\gamma(\mathbb{D})}\\ &\quad\le (1-R)^{(\gamma+2)\left(1-\frac{1}{p}\right)} \|f\|^{1-p}_{A^p_\gamma(\mathbb{D})} \mathcal{M}^p_{p,\gamma}[f]\left(0, \sqrt{R}, 1\right) \|g\|^{1-p}_{A^q_\gamma(\mathbb{D})} \left\|g_{\sqrt{R}}\right\|^p_{A^q_\gamma(\mathbb{D})}.\end{aligned} \tag{1.13}$$

Note that

$$\mathcal{M}^p_{p,\gamma}[f]\left(0, \sqrt{R}, 1\right) \le \mathcal{M}^p_{p,\gamma}[f](0,1,1) = \|f\|^p_{A^p_\gamma(\mathbb{D})},$$

and

$$\left\|g_{\sqrt{R}}\right\|^p_{A^q_\gamma(\mathbb{D})} = \mathcal{M}^p_{q,\gamma}[g]\left(0, 1, \sqrt{R}\right) \le \mathcal{M}^p_{q,\gamma}[g](0,1,1) = \|g\|^p_{A^q_\gamma(\mathbb{D})}.$$

Therefore,

$$\left\|(f * g)_{\gamma,0,R}\right\|_{A^q_\gamma(\mathbb{D})} \le (1-R)^{(\gamma+2)\left(1-\frac{1}{p}\right)} \|f\|_{A^p_\gamma(\mathbb{D})} \|g\|_{A^q_\gamma(\mathbb{D})}. \tag{1.14}$$

As we noted earlier, in the case of one complex dimension $n = 1$, the weighted Hadamard product $(f * g)_{\gamma,0,R}$ was studied in [5]. Actually, the inequality (1.14) is the main result of the paper [5], where the holomorphic functions $f$ and $g$ are replaced by their appropriate fractional derivatives of Hardy–Littlewood. See [5]. Thus, the inequality (1.13) can be viewed as an improvement of [5, Theorem 2.1]. ◇

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## 2 Preliminaries

If $f \in \mathrm{Hol}(\mathbb{B}_n)$ and $0 < p < \infty$, then

$$r \mapsto \mathrm{I}_p[f](r) = \mathrm{M}_p^p(r, f) = \frac{1}{\mathrm{S}(\mathbb{S}_n)} \int_{\mathbb{S}_n} |f(r\zeta_1, \ldots, r\zeta_n)|^p \, \mathrm{dS}(\zeta_1, \ldots, \zeta_n),$$

is an increasing function, where $0 \leq r < 1$. In fact, the following stronger property holds.

***Lemma 2.1*** *Let $f \in \mathrm{Hol}(\mathbb{B}_n)$, $0 < p < \infty$ and*

$$\mathrm{L}_p[f](r_1, \ldots, r_n) = \int_{\mathbb{S}_n} |f(r_1\zeta_1, \ldots, r_n\zeta_n)|^p \, \mathrm{dS}(\zeta_1, \ldots, \zeta_n),$$

*where $0 \leq r_1 < 1, \ldots, 0 \leq r_n < 1$. Then, for each $j \in \{1, \ldots, n\}$,*

$$r_j \mapsto \mathrm{L}_p[f](r_1, \ldots, r_j, \ldots, r_n),$$

*is an increasing function.*

**Proof** Let $j \in \{1, \ldots, n\}$ and $r_k \in [0, 1)$, where $k \neq j$, be fixed. Since dS is a rotation invariant measure, for each $\theta \in [0, 2\pi)$, we have

$$\begin{aligned}
&\mathrm{L}_p[f](r_1, \ldots, r_j, \ldots, r_n) \\
&\quad = \int_{\mathbb{S}_n} \left|f\left(r_1\zeta_1, \ldots, r_j\zeta_j, \ldots, r_n\zeta_n\right)\right|^p \mathrm{dS}\left(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_n\right) \\
&\quad = \int_{\mathbb{S}_n} \left|f\left(r_1\zeta_1, \ldots, r_j e^{i\theta}\zeta_j, \ldots, r_n\zeta_n\right)\right|^p \mathrm{dS}\left(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_n\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{L}_p[f](r_1, \ldots, r_j, \ldots, r_n) \\
&\quad = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{S}_n} \left|f\left(r_1\zeta_1, \ldots, r_j e^{i\theta}\zeta_j, \ldots, r_n\zeta_n\right)\right|^p \mathrm{dS}\left(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_n\right) \mathrm{d}\theta,
\end{aligned}$$

and by using Fubini theorem, we obtain

$$\begin{aligned}
&\mathrm{L}_p[f](r_1, \ldots, r_j, \ldots, r_n) \\
&\quad = \int_{\mathbb{S}_n} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(r_1\zeta_1, \ldots, r_j e^{i\theta}\zeta_j, \ldots, r_n\zeta_n\right)\right|^p \mathrm{d}\theta \, \mathrm{dS}\left(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_n\right).
\end{aligned}$$

For each $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{S}_n$, we consider the following function

$$F_\zeta(z) = f\left(r_1\zeta_1, \ldots, z\zeta_j, \ldots, r_n\zeta_n\right), \quad z \in \mathbb{D}.$$

Then $F_\zeta \in \mathrm{Hol}(\mathbb{D})$, which implies that

$$r_j \mapsto \frac{1}{2\pi} \int_0^{2\pi} \left|F_\zeta\left(r_j e^{i\theta}\right)\right|^p \mathrm{d}\theta,$$

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is an increasing function, where $0 \leq r_j < 1$. Hence

$$r_j \mapsto \mathrm{L}_p[f]\left(r_1, \ldots, r_j, \ldots, r_n\right) = \int_{\mathbb{S}_n} \frac{1}{2\pi} \int_0^{2\pi} \left|F_\zeta\left(r_j \mathrm{e}^{\mathrm{i}\theta}\right)\right|^p \mathrm{d}\theta \, \mathrm{dS}(\zeta),$$

is an increasing function too. This completes the proof. ∎

Based on the previous lemma, we have the following useful statement.

***Lemma 2.2*** *Let $f \in \mathrm{Hol}\left(\mathbb{B}_n\right)$, $0 < p < \infty$ and $w \in \mathbb{B}_n$. Then*

$$\int_{\mathbb{S}_n} |f(w \cdot \zeta)|^p \, \mathrm{dS}(\zeta) \leq \int_{\mathbb{S}_n} |f(|w|\zeta)|^p \, \mathrm{dS}(\zeta).$$

**Proof** First of all, we can write $w = (w_1, \ldots, w_n) = \left(r_1 \mathrm{e}^{\mathrm{i}\theta_1}, \ldots, r_n \mathrm{e}^{\mathrm{i}\theta_n}\right) \in \mathbb{B}_n$, where $r_1, \ldots, r_n \geq 0$ and $\theta_1, \ldots, \theta_n \in [0, 2\pi)$. Note that $r_j \leq \sqrt{r_1^2 + \cdots + r_n^2} = |w| < 1$, for all $j = 1, \ldots, n$, which implies that $r_1, \ldots, r_n \in [0, 1)$. By using the fact that dS is a rotation invariant measure and Lemma 2.1, we get

$$\begin{aligned}
\int_{\mathbb{S}_n} |f\left(w \cdot \zeta\right)|^p \, \mathrm{dS}(\zeta) &= \int_{\mathbb{S}_n} \left|f\left(r_1 \mathrm{e}^{\mathrm{i}\theta_1} \zeta_1, \ldots, r_n \mathrm{e}^{\mathrm{i}\theta_n} \zeta_n\right)\right|^p \mathrm{dS}\left(\zeta_1, \ldots, \zeta_n\right) \\
&= \int_{\mathbb{S}_n} |f\left(r_1 \zeta_1, \ldots, r_n \zeta_n\right)|^p \, \mathrm{dS}\left(\zeta_1, \ldots, \zeta_n\right) = \mathrm{L}_p[f]\left(r_1, \ldots, r_n\right) \\
&\leq \mathrm{L}_p[f]\left(|w|, \ldots, |w|\right) = \int_{\mathbb{S}_n} |f\left(|w|\zeta_1, \ldots, |w|\zeta_n\right)|^p \, \mathrm{dS}\left(\zeta_1, \ldots, \zeta_n\right) \\
&= \int_{\mathbb{S}_n} |f(|w|\zeta)|^p \, \mathrm{dS}(\zeta).
\end{aligned}$$

This finishes the proof. ∎

Continuing in a similar way, we obtain the following result, where the main ingredient is the previous lemma.

***Lemma 2.3*** *Let $f \in \mathrm{Hol}\left(\mathbb{B}_n\right)$, $0 < p < \infty$, $\gamma > -1$, $0 \leq a < b \leq 1$ and $w \in \mathbb{B}_n$. Then*

$$\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} |f\left(w \cdot z\right)|^p \, \mathrm{dV}_\gamma(z) \leq \mathcal{M}_{p,\gamma}^p[f](a, b, |w|).$$

**Proof** By using integration in polar coordinates and Lemma 2.2, we conclude

$$\begin{aligned}
&\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} |f\left(w \cdot z\right)|^p \, \mathrm{dV}_\gamma(z) \\
&= \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \frac{\mathrm{V}\left(\mathbb{B}_n\right)}{\mathrm{S}\left(\mathbb{S}_n\right)} \int_a^b \int_{\mathbb{S}_n} |f\left(w \cdot \rho\zeta\right)|^p \, \mathrm{dS}(\zeta) \, 2n\rho^{2n-1} \left(1 - \rho^2\right)^\gamma \mathrm{d}\rho \\
&= \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \frac{\mathrm{V}\left(\mathbb{B}_n\right)}{\mathrm{S}\left(\mathbb{S}_n\right)} \int_a^b \int_{\mathbb{S}_n} |f\left(\rho w \cdot \zeta\right)|^p \, \mathrm{dS}(\zeta) \, 2n\rho^{2n-1} \left(1 - \rho^2\right)^\gamma \mathrm{d}\rho \\
&\leq \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \frac{\mathrm{V}\left(\mathbb{B}_n\right)}{\mathrm{S}\left(\mathbb{S}_n\right)} \int_a^b \int_{\mathbb{S}_n} |f\left(\rho|w|\zeta\right)|^p \, \mathrm{dS}(\zeta) \, 2n\rho^{2n-1} \left(1 - \rho^2\right)^\gamma \mathrm{d}\rho
\end{aligned}$$

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$$= \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} |f\left(|w|z\right)|^p \, \mathrm{dV}_\gamma(z)$$
$$= \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} \left|f_{|w|}\left(z\right)\right|^p \, \mathrm{dV}_\gamma(z) = \mathcal{M}_{p,\gamma}^p[f](a,b,|w|),$$

which actually needed to be proven. ∎

We continue with another auxiliary result, which will be useful below and which may be of independent interest.

***Lemma 2.4*** *Let $f \in A_\gamma^p\left(\mathbb{B}_n\right)$, $0 < p \le q < \infty$, $\gamma > -1$ and $0 \le r < R \le 1$, $0 \le s \le 1$. Then*

$$\mathcal{M}_{q,\gamma}[f](r,R,s) \le \left(1 - s^2R^2\right)^{(n+\gamma+1)\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{A_\gamma^p(\mathbb{B}_n)}^{1-\frac{p}{q}} \mathcal{M}_{p,\gamma}^{\frac{p}{q}}[f](r,R,s).$$

**Proof** If $z \in \mathbb{A}_{n,r,R}$ then $|sz| = s|z| \le sR$ and by using sharp pointwise estimate (1.3), we obtain

$$|f_s(z)| = |f(sz)| \le \left(1 - |sz|^2\right)^{-\frac{n+\gamma+1}{p}} \|f\|_{A_\gamma^p(\mathbb{B}_n)} \le \left(1 - s^2R^2\right)^{-\frac{n+\gamma+1}{p}} \|f\|_{A_\gamma^p(\mathbb{B}_n)}.$$

Since $q - p \ge 0$, we conclude

$$|f_s(z)|^{q-p} \le \left(1 - s^2R^2\right)^{(n+\gamma+1)\left(1-\frac{q}{p}\right)} \|f\|_{A_\gamma^p(\mathbb{B}_n)}^{q-p}, \tag{2.1}$$

for each $z \in \mathbb{A}_{n,r,R}$. Therefore, we may write

$$\mathcal{M}_{q,\gamma}[f](r,R,s) = \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,r,R}\right)} \int_{\mathbb{A}_{n,r,R}} |f_s|^{q-p} |f_s|^p \, \mathrm{dV}_\gamma\right)^{\frac{1}{q}}$$
$$\overset{(2.1)}{\le} \left(\left(1 - s^2R^2\right)^{(n+\gamma+1)\left(1-\frac{q}{p}\right)} \|f\|_{A_\gamma^p(\mathbb{B}_n)}^{q-p} \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,r,R}\right)} \int_{\mathbb{A}_{n,r,R}} |f_s|^p \, \mathrm{dV}_\gamma\right)^{\frac{1}{q}}$$
$$= \left(\left(1 - s^2R^2\right)^{(n+\gamma+1)\left(1-\frac{q}{p}\right)} \|f\|_{A_\gamma^p(\mathbb{B}_n)}^{q-p} \mathcal{M}_{p,\gamma}^p[f](r,R,s)\right)^{\frac{1}{q}}$$
$$= \left(1 - s^2R^2\right)^{(n+\gamma+1)\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{A_\gamma^p(\mathbb{B}_n)}^{1-\frac{p}{q}} \mathcal{M}_{p,\gamma}^{\frac{p}{q}}[f](r,R,s).$$

This finishes the proof. ∎

***Remark 2.5*** By Lemma 2.4, in the special case when $f \in A_\gamma^p\left(\mathbb{B}_n\right)$, $0 < p \le 1 = q$, $\gamma > -1$ and $0 \le r < R \le 1$, $0 \le s \le 1$, we obtain

$$\mathcal{M}_{1,\gamma}[f](r,R,s) \le \left(1 - s^2R^2\right)^{(n+\gamma+1)\left(1-\frac{1}{p}\right)} \|f\|_{A_\gamma^p(\mathbb{B}_n)}^{1-p} \mathcal{M}_{p,\gamma}^p[f](r,R,s),$$

which will be useful below. ◇

We continue with the following result, which will play an important role in the proof of the main result of this paper.

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***Lemma 2.6*** *Let $f, g \in \mathrm{Hol}(\mathbb{B}_n)$, $0 < p \le 1$, $\gamma > -1$, $0 \le r < R \le 1$, $0 \le s \le 1$ and let*

$$\mathrm{K}(z,w) = \overline{f(\overline{w})}\, g(z \cdot w), \quad \text{where} \quad z, w \in \mathbb{B}_n.$$

*Then*

$$\begin{aligned}&\left|\left((f * g)_{\gamma,r,R}\right)_s(z)\right| \\ &\quad \le (1-sR)^{(n+\gamma+1)\left(1-\frac{1}{p}\right)} \|\mathrm{K}(z,\cdot)\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)}\, \mathcal{M}^p_{p,\gamma}[\mathrm{K}(z,\cdot)]\left(\sqrt{r}, \sqrt{R}, \sqrt{s}\right),\end{aligned}$$

*for all $z \in \mathbb{B}_n$.*

**Proof** Let

$$f(z) = \sum_\alpha a_\alpha z^\alpha \ \text{ and } \ g(z) = \sum_\alpha b_\alpha z^\alpha, \ \ z \in \mathbb{B}_n.$$

Then, for $z \in \mathbb{B}_n$, we have

$$\begin{aligned}&\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} f\left(\sqrt{s}\overline{w}\right) g\left(\sqrt{s}z \cdot w\right) \mathrm{dV}_\gamma(w) \\ &\quad = \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \sum_\alpha a_\alpha \left(\sqrt{s}\overline{w}\right)^\alpha \sum_\beta b_\beta \left(\sqrt{s}z \cdot w\right)^\beta \left(1-|w|^2\right)^\gamma \mathrm{dV}(w) \\ &\quad = \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \sum_\alpha a_\alpha \sqrt{s}^{|\alpha|} \overline{w}^\alpha \sum_\beta b_\beta \sqrt{s}^{|\beta|} z^\beta w^\beta \left(1-|w|^2\right)^\gamma \mathrm{dV}(w),\end{aligned}$$

and by using integration in polar coordinates, we obtain

$$\begin{aligned}&\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} f\left(\sqrt{s}\overline{w}\right) g\left(\sqrt{s}z \cdot w\right) \mathrm{dV}_\gamma(w) \\ &\quad = \frac{\mathrm{V}(\mathbb{B}_n) \int_{\sqrt{r}}^{\sqrt{R}} \frac{\int_{\mathbb{S}_n} \sum_{\alpha,\beta} a_\alpha b_\beta \sqrt{s}^{|\alpha|+|\beta|} \rho^{|\alpha|+|\beta|} z^\beta \overline{\zeta}^\alpha \zeta^\beta \mathrm{dS}(\zeta)}{\mathrm{S}(\mathbb{S}_n)} 2n\rho^{2n-1}\left(1-\rho^2\right)^\gamma \mathrm{d}\rho}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \\ &\quad = \frac{\mathrm{V}(\mathbb{B}_n) \int_{\sqrt{r}}^{\sqrt{R}} \sum_{\alpha,\beta} a_\alpha b_\beta \sqrt{s}^{|\alpha|+|\beta|} \rho^{|\alpha|+|\beta|} z^\beta \frac{\int_{\mathbb{S}_n} \overline{\zeta}^\alpha \zeta^\beta \mathrm{dS}(\zeta)}{\mathrm{S}(\mathbb{S}_n)} 2n\rho^{2n-1}\left(1-\rho^2\right)^\gamma \mathrm{d}\rho}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)}.\end{aligned}$$

Also, by using (1.2), we find

$$\begin{aligned}&\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} f\left(\sqrt{s}\overline{w}\right) g\left(\sqrt{s}z \cdot w\right) \mathrm{dV}_\gamma(w) \\ &\quad = \frac{\mathrm{V}(\mathbb{B}_n)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\sqrt{r}}^{\sqrt{R}} \sum_\alpha a_\alpha b_\alpha \sqrt{s}^{2|\alpha|} \rho^{2|\alpha|} z^\alpha \frac{\alpha!\Gamma(n)}{\Gamma(n+|\alpha|)} 2n\rho^{2n-1}\left(1-\rho^2\right)^\gamma \mathrm{d}\rho \\ &\quad = \frac{\mathrm{V}(\mathbb{B}_n)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \sum_\alpha a_\alpha b_\alpha \frac{\alpha!\Gamma(n+1)}{\Gamma(n+|\alpha|)} \int_{\sqrt{r}}^{\sqrt{R}} \rho^{2(n+|\alpha|-1)}\left(1-\rho^2\right)^\gamma 2\rho\, \mathrm{d}\rho\, (sz)^\alpha\end{aligned}$$

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$$
\begin{aligned}
&= \frac{\mathrm{V}(\mathbb{B}_n)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \sum_\alpha a_\alpha b_\alpha \frac{\alpha!\Gamma(n+1)}{\Gamma(n+|\alpha|)} \int_r^R \rho^{n+|\alpha|-1}(1-\rho)^\gamma \,\mathrm{d}\rho\,(sz)^\alpha \\
&= \sum_\alpha \frac{\alpha!\pi^n a_\alpha b_\alpha (sz)^\alpha}{\Gamma(n+|\alpha|)\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \left(\mathrm{B}_R(n+|\alpha|,\gamma+1) - \mathrm{B}_r(n+|\alpha|,\gamma+1)\right) \\
&= (f*g)_{\gamma,r,R}(sz),
\end{aligned}
$$

that is

$$
\left((f*g)_{\gamma,r,R}\right)_s(z) = \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} f\left(\sqrt{s}\overline{w}\right) g\left(\sqrt{s}z\cdot w\right) \mathrm{dV}_\gamma(w). \tag{2.2}
$$

Therefore, from the formula (2.2), we derive

$$
\begin{aligned}
\left|\left((f*g)_{\gamma,r,R}\right)_s(z)\right| &\le \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left|f\left(\sqrt{s}\overline{w}\right) g\left(\sqrt{s}z\cdot w\right)\right| \mathrm{dV}_\gamma(w) \\
&= \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left|\mathrm{K}\left(z,\sqrt{s}w\right)\right| \mathrm{dV}_\gamma(w) \\
&= \mathcal{M}_{1,\gamma}\left[\mathrm{K}(z,\cdot)\right]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right) \\
&\le (1-sR)^{(n+\gamma+1)\left(1-\frac{1}{p}\right)} \left\|\mathrm{K}(z,\cdot)\right\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)} \mathcal{M}^p_{p,\gamma}\left[\mathrm{K}(z,\cdot)\right]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right).
\end{aligned}
$$

where we used Lemma 2.4, that is Remark 2.5. This completes the proof. ∎

## 3 The proof of Theorem 1.3

In this final section, we are ready to provide the proof of the main theorem of this paper.

**Proof of Theorem 1.3** Let

$$
\mathrm{K}(z,w) = \overline{f\left(\overline{w}\right)} g\left(z\cdot w\right),
$$

for $z, w \in \mathbb{B}_n$. By using Lemma 2.6, we find

$$
\begin{aligned}
\mathcal{M}_{q,\gamma}\left[(f*g)_{\gamma,r,R}\right](a,b,s) &= \left(\frac{\int_{\mathbb{A}_{n,a,b}} \left|\left((f*g)_{\gamma,r,R}\right)_s(z)\right|^q \mathrm{dV}_\gamma(z)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)}\right)^{\frac{1}{q}} \\
&\le \frac{\left(\frac{\int_{\mathbb{A}_{n,a,b}} \left\|\mathrm{K}(z,\cdot)\right\|^{q(1-p)}_{A^p_\gamma(\mathbb{B}_n)} \mathcal{M}^{qp}_{p,\gamma}\left[\mathrm{K}(z,\cdot)\right]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right) \mathrm{dV}_\gamma(z)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)}\right)^{\frac{1}{q}}}{(1-sR)^{(n+\gamma+1)\left(\frac{1}{p}-1\right)}}.
\end{aligned}
$$

On the other hand, note that

$$
\frac{\int_{\mathbb{A}_{n,a,b}} \left\|\mathrm{K}(z,\cdot)\right\|^{q(1-p)}_{A^p_\gamma(\mathbb{B}_n)} \mathcal{M}^{qp}_{p,\gamma}\left[\mathrm{K}(z,\cdot)\right]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right) \mathrm{dV}_\gamma(z)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \le I^{1-p} J^p, \tag{3.1}
$$

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where

$$I = \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} \left\|\mathrm{K}(z,\cdot)\right\|^q_{A^p_\gamma(\mathbb{B}_n)} \,\mathrm{dV}_\gamma(z),$$

and

$$J = \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} \mathcal{M}^q_{p,\gamma}\left[\mathrm{K}(z,\cdot)\right]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right) \mathrm{dV}_\gamma(z).$$

Namely, in the case when $0 < p < 1$, the previous inequality (3.1) is a simple consequence of the Hölder inequality with the conjugate exponents $1/(1-p)$ and $1/p$. In the case when $p = 1$, the above inequality (3.1) is trivial. Actually, in that case we have the equality between the left and right hand side in the (3.1). Therefore, we get

$$\mathcal{M}_{q,\gamma}\left[(f * g)_{\gamma,r,R}\right](a,b,s) \le (1-sR)^{(n+\gamma+1)\left(1-\frac{1}{p}\right)} I^{\frac{1-p}{q}} J^{\frac{p}{q}}. \tag{3.2}$$

Then, by using the integral Minkowski inequality, Lemmas 2.3 and 1.2, we obtain

$$\begin{aligned}
I^{\frac{1-p}{q}} &= \left(\left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} \left(\frac{\int_{\mathbb{B}_n} |\mathrm{K}(z,w)|^p \,\mathrm{dV}_\gamma(w)}{\mathrm{V}_\gamma\left(\mathbb{B}_n\right)}\right)^{\frac{q}{p}} \mathrm{dV}_\gamma(z)\right)^{\frac{p}{q}}\right)^{\frac{1-p}{p}} \\
&\le \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{B}_n\right)} \int_{\mathbb{B}_n} \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} |\mathrm{K}(z,w)|^q \,\mathrm{dV}_\gamma(z)\right)^{\frac{p}{q}} \mathrm{dV}_\gamma(w)\right)^{\frac{1-p}{p}} \\
&= \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{B}_n\right)} \int_{\mathbb{B}_n} |f(\overline{w})|^p \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} |g(z\cdot w)|^q \,\mathrm{dV}_\gamma(z)\right)^{\frac{p}{q}} \mathrm{dV}_\gamma(w)\right)^{\frac{1-p}{p}} \\
&\le \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{B}_n\right)} \int_{\mathbb{B}_n} |f(\overline{w})|^p \,\mathcal{M}^p_{q,\gamma}[g](a,b,|w|) \,\mathrm{dV}_\gamma(w)\right)^{\frac{1-p}{p}} \\
&\le \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{B}_n\right)} \int_{\mathbb{B}_n} |f(\overline{w})|^p \,\mathcal{M}^p_{q,\gamma}[g](a,b,1) \,\mathrm{dV}_\gamma(w)\right)^{\frac{1-p}{p}} \\
&= \left(\frac{1}{\mathrm{V}_\gamma\left(\mathbb{B}_n\right)} \int_{\mathbb{B}_n} |f(\overline{w})|^p \,\mathrm{dV}_\gamma(w)\right)^{\frac{1-p}{p}} \mathcal{M}^{1-p}_{q,\gamma}[g](a,b,1) \\
&= \|f\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)} \,\mathcal{M}^{1-p}_{q,\gamma}[g](a,b,1),
\end{aligned}$$

that is

$$I^{\frac{1-p}{q}} \le \|f\|^{1-p}_{A^p_\gamma(\mathbb{B}_n)} \,\mathcal{M}^{1-p}_{q,\gamma}[g](a,b,1). \tag{3.3}$$

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Similarly, by using again integral Minkowski inequality and Lemma 2.3, we derive

$$
\begin{aligned}
J^{\frac{p}{q}} &= \left( \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} \left( \frac{\int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left|\mathrm{K}\left(z,\sqrt{s}w\right)\right|^p \mathrm{dV}_\gamma(w)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \right)^{\frac{q}{p}} \mathrm{dV}_\gamma(z) \right)^{\frac{p}{q}} \\
&\leq \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left( \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} \left|\mathrm{K}\left(z,\sqrt{s}w\right)\right|^q \mathrm{dV}_\gamma(z) \right)^{\frac{p}{q}} \mathrm{dV}_\gamma(w) \\
&= \frac{\int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left|f\left(\sqrt{s}\overline{w}\right)\right|^p \left( \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,a,b}\right)} \int_{\mathbb{A}_{n,a,b}} \left|g\left(z\cdot\sqrt{s}w\right)\right|^q \mathrm{dV}_\gamma(z) \right)^{\frac{p}{q}} \mathrm{dV}_\gamma(w)}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \\
&\leq \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left|f\left(\sqrt{s}\overline{w}\right)\right|^p \mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}|w|\right) \mathrm{dV}_\gamma(w).
\end{aligned}
$$

Based on Lemma 1.2, for $w \in \mathbb{A}_{n,\sqrt{r},\sqrt{R}}$, we have

$$
\mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}|w|\right) \leq \mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}\sqrt{R}\right),
$$

which implies

$$
\begin{aligned}
J^{\frac{p}{q}} &\leq \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left|f\left(\sqrt{s}\overline{w}\right)\right|^p \mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}\sqrt{R}\right) \mathrm{dV}_\gamma(w) \\
&= \frac{1}{\mathrm{V}_\gamma\left(\mathbb{A}_{n,\sqrt{r},\sqrt{R}}\right)} \int_{\mathbb{A}_{n,\sqrt{r},\sqrt{R}}} \left|f_{\sqrt{s}}\left(\overline{w}\right)\right|^p \mathrm{dV}_\gamma\left(w\right) \mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}\sqrt{R}\right) \\
&= \mathcal{M}^p_{p,\gamma}[f]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right) \mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}\sqrt{R}\right),
\end{aligned}
$$

that is

$$
J^{\frac{p}{q}} \leq \mathcal{M}^p_{p,\gamma}[f]\left(\sqrt{r},\sqrt{R},\sqrt{s}\right) \mathcal{M}^p_{q,\gamma}[g]\left(a,b,\sqrt{s}\sqrt{R}\right). \tag{3.4}
$$

Finally, from (3.2)–(3.4), we obtain the required conclusion. ∎

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