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ROUGH PATHS *VIA* SEWING LEMMA

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Abstract. We present the rough path theory introduced by Lyons, using the swewing lemma of Feyel and de Lapradelle.

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1. Introduction

The main purpose of this work is to explain how define and solve integral equations like

$$y(t) = y_0 + \int_0^t f(y(s))\mathrm{d}x(s), \quad t \in [0,T]; \tag{1.1}$$

where $m, n \in \mathbb{N}^*$, $x : [0,1] \to \mathbb{R}^n$ is x is a irregular path and f is a regular enough function from $\mathbb{R}^n$ into the set of the linear applications from $\mathbb{R}^m$ into $\mathbb{R}^d$, $T > 0$.

The first step is to explain how define

$$\int f(x_s)\mathrm{d}x_s, \tag{1.2}$$

where f is a regular enough function from $\mathbb{R}^m$ into the set of the linear applications from $\mathbb{R}^m$ into $\mathbb{R}^d$; and x is a irregular path.

In [22], Lyons has introduced natural familly of metrics on the space of paths with finite variation such that the integral (1.2) and the Itô map $x \mapsto y$, y solution of (1.1) are uniformelly continuous with respect to these metrics if f is regular enough. Then, the Itô map is extended to the completion of the space of paths with bounded variation with respect to this metrics. This family relies on the notion of p variation of a function. A path x is said to be of finite p variation on $[S,T]$ if and only if

$$\sum_{t_i \in \mathcal{D}} |x(t_{i+1}) - x(t_i)|^p < \infty,$$

where the suppremum runs over all subdivions of $[S,T]$.

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The case $p < 2$ was the first solved. The definition of the integral was achieved for the first time by Young in 1936 [31], extended by Bertoin for processes with p finite variation, [2]. Lyons, in [21], defines and solves differential equations like (1.1), when x is of finite p variation using fix point argument, see also Lejay [20].

The case $p \geq 2$ was studied in 1998, by Lyons, in [22], where was introduced the concept of Rough paths. Rough paths theory is now an active field of research as it allows to integrate differential forms along irregular path, and to solve controlled differential equations against irregular paths, see the book of Lyons and Qian, [23], or the book of Friz and Victoir,[13], the introductions of Lejay [18, 19].

Some alternative views on the theory are developped by Gubinelli, [15], Feyel and de La Pradelle [12] or Hu and Nualart [16].

The rough path theory is a way to extend the notion of stochastic differential equations to many other processes such as fractional Brownian motion [7] and general Gaussian processes [14]. It allow ro recover some results on asymptotic expansion of stochastic flows or the Laplace approximation [17].

The object of this work is to present some results on Rough path theory. A path of finite p variation is, up to a change of variable, Hölder continuous. For simplicity, we restrict ourself to α Hölder paths. We consider only the case of α Hölder continuous driving process with $\alpha \in]1/3; 1/2]$. Then, using ideas introduced by Feyel and De La Pradelle in [12], we construct $\int f(x)\mathrm{d}x$ and solve differential equations *via* and some fix point argument. Unfortunately, the fix point argument seems not work in the case $\alpha \leq 1/3$.

Some usefull notations are introduced by Section 2. In Section 3, following the works of Chen, we explain how sequences of iterated integrals of the underlying process x appears in solution of linear differential equations and in a theory of integration with respect to α Hölder continuous paths. In Section 4, we present the sewing lemma of Feyel and De La Pradelle [12]. This lemma is a toolbox which allows to construct some additive functional from almost additive functional. Lyons has extracted the algebraic properties of sequences of iterated integrals required for the definition of an integral (Chen rules since we restrict ourself to $\alpha > \frac{1}{3}$). Functionals fulfilling the Chen rule are called multiplicative functionals and are presented in Section 5.1. Using the sewing lemma, we recover some results on "almost multiplicative functionals" proved by Lyons and Qian in [23], for all α. When $\alpha > \frac{1}{2}$, the integral (1.2) is the Young integral, studied in Section 6. In the remaind of this work, we assume that x is a α Hölder continuous path with $\alpha \in]\frac{1}{3}, \frac{1}{2}]$ and its increments define the first level of a multiplicative functional. In Section 7, we define an integral (1.2) wich is an extension of the Rieman-Stieljes or the Young integral. In the last Section, 8, using the same lines as in [12], we prove a result of existence an uniqness for equation (1.1).

2. Notations

The notations given in this section are used in the sequel.

- For $s \in \mathbb{R}$, $[s]$ is the interger part of s.
- The space $(\mathbb{B}, \|.\|)$ is a Banach space. The closed ball around 0 with radius K is denoted by $\overline{B}_{\mathbb{B}}(0, K)$.
- Let $T \in \mathbb{R}_+^*$ and denotes

$$\Delta_T^2 = \{(a, b), \quad 0 \leq a \leq b \leq T\},$$
$$\Delta_T^3 = \{(a, c, b), \quad 0 \leq a \leq c \leq b \leq T\}.$$

- Let $\alpha \in (0, 1]$. A function $x : [S, T] \to \mathbb{B}$ is said to be α Hölder continuous on $[S, T]$ if and only if

$$\|x\|_{\alpha,S,T} := \sup_{S \leq s < t \leq T} \frac{\|x(t) - x(s)\|}{|t - s|^\alpha} < +\infty.$$

  Then, $N_{\alpha,\infty,S,T}(.) = \|.\|_{\infty,S,T} + \|.\|_{\alpha,S,T}$ is a norm on the set of real functions α Hölder continuous denoted by $C^\alpha([S, T], \mathbb{B})$. Here $\|.\|_{\infty,S,T}$ is the suppremum norm on $C([S, T], \mathbb{B})$.

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- Let $\alpha \in]0,1]$. A functional $\mu : \Delta_T^2 \to \mathbb{B}$ is said to be α Hölder continuous on $\{S \leq s \leq t \leq T\}$ if and only if

$$\|\mu\|_{\alpha,S,T} = \sup_{S\leq a<b\leq T} \frac{\|\mu(a,b)\|}{|b-a|^\alpha} < \infty.$$

Let $C^\alpha(\Delta_T^2, \mathbb{B})$ be the set of α Hölder continuous functionals on Δ_T^2.

3. Motivations

In order to motivate the introduction of sequences of iterated integrals and the tensor algebra , we introduce the formal Chen developpement of solutions of ordinary differential equations (see the work of Chen [3] or the book of Baudoin [1]). Second, we construct a naive theory of integration for exact differential form. We conclude this section with some few words on the rough paths theory.

3.1. Formal Chen developpement of linear ordinary differential equations

Let $x = (x^1, \ldots, x^m) : [0,T] \to \mathbb{R}^m$ be a path with bounded variation, $A_1, \ldots, A_m$ be square real $d \times d$ matrices and y be the solution of the linear differential equation

$$\mathrm{d}y_t = \sum_{i=1}^m A_i y_t \mathrm{d}x_t^i, \quad y_0 \in \mathbb{R}^d, \quad t \in [0,T]. \tag{3.3}$$

Its integral form is

$$y_t = y_0 + \sum_{i=1}^m \int_0^t A_i y_s \mathrm{d}x_s^i, \quad t \in [0,T].$$

Since y_s has also an integral expression we obtain

$$y_t = y_0 + \sum_{i=1}^m A_i y_0 [x_t^i - x_0^i] + \sum_{i_1,i_2=1}^m \int_{0<s_2<s_1<t} A_{i_1} A_{i_2} y_s \mathrm{d}x_{s_2}^{i_2} \mathrm{d}x_{s_1}^{i_1}, \quad t \in [0,T].$$

By induction, we derive for $n \in \mathbb{N}^*$

$$\begin{aligned} y_t = y_0 &+ \sum_{k=1}^n \sum_{i_1,\ldots,i_k=1}^m A_{i_1} \ldots A_{i_k} y_0 \int_{0<s_k<\ldots<s_1<t} \mathrm{d}x_{s_k}^{i_k} \ldots \mathrm{d}x_{s_1}^{i_1} \\ &+ \sum_{i_1,\ldots,i_{n+1}=1}^m \int_{0<s_{n+1}<\ldots<s_1<t} A_{i_1} \ldots A_{i_{n+1}} y_{s_{n+1}} \mathrm{d}x_{s_{n+1}}^{i_{n+1}} \ldots \mathrm{d}x_{s_1}^{i_1}, \quad t \in [0,T]. \end{aligned}$$

Formally, we derive the Chen developpement of y,

$$y_t = y_0 + \sum_{k=1}^\infty \sum_{i_1,\ldots,i_k=1}^m A_{i_1} \ldots A_{i_k} y_0 X_{0,t}^{(k),i_k,\ldots,i_1}, \quad t \in [0,T]. \tag{3.4}$$

where

$$X_{0,t}^{(n),i_n,\ldots,i_1} = \int_{0<s_n<\ldots<s_1<t} \mathrm{d}x_{s_n}^{i_n} \ldots \mathrm{d}x_{s_1}^{i_1}, \quad t \in [0,T], \quad i_1, \ldots, i_n = 1, \ldots, m, \quad n \in \mathbb{N}.$$

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Then, we address ourself the following questions:

- is the expansion in the right member of (3.4) be correct?
- is the solution of the linear differential equation (3.3) be a continuous function of $X = (X_{0,t}^{(n),i_n,\dots,i_1}, (i_1,\dots,i_n) \in \{1,\dots,m\}^n, n \in \mathbb{N}^*)$? For which norm?
- which algebraic properties of sequence of iterated integrals are used?

In the next subsection, we will see that sequence of iterated integrals X can play a role in some integration theory.

3.2. Naive theory of integration of exact differential forms

Let $x = (x^1,\dots,x^m) \in C^\alpha([0,T],\mathbb{R}^m)$ and $F : \mathbb{R}^m \to \mathbb{R}$ be a $C^{[\frac{1}{\alpha}]+1}$ continuous function. From for $n \in \mathbb{N}^*$

$$F(x_T) - F(x_0) = \sum_{i=0}^{n-1} \left[F(x_{\frac{i+1}{n}}) - F(x_{\frac{i}{n}}) \right].$$

Using Taylor expansion we derive

$$F(x_T) - F(x_0) = \sum_{i=0}^{n-1} \left\{ \sum_{k=1}^{[\frac{1}{\alpha}]} \frac{1}{k!} D^k F\left(x_{\frac{i}{n}}\right) \cdot \left[x_{\frac{i+1}{n}} - x_{\frac{i}{n}} \right]^{\otimes k} + R\left(\frac{i}{n}, \frac{i+1}{n}\right) \right\}$$

where for $(s,t) \in \Delta_T^2$

$$R(s,t) = \int_0^1 \frac{(1-u)^{[\frac{1}{\alpha}]+1}}{[\frac{1}{\alpha}]!} D^{[\frac{1}{\alpha}]+1} F(x_s + u(x_t - x_s)) \mathrm{d}u \cdot [x_t - x_s]^{\otimes [\frac{1}{\alpha}]+1}.$$

First, $D^{[\alpha]+1}F$ is continuous and x is α Hölder continuous, then

$$\left\| \sum_{i=0}^{n-1} R\left(\frac{i}{n}, \frac{i+1}{n}\right) \right\| = O\left(n^{1-\alpha([\frac{1}{\alpha}]+1)}\right).$$

Second, the differentials $D^k F(a)$ are symetric multilinear functions from $(\mathbb{R}^m)^{\otimes k}$ into $\mathbb{R}$. Let $X^{(k)} : \Delta_T^2 \to (\mathbb{R}^m)^{\otimes k}$ be such that the symetric part of $X_{s,t}^{(k)}$ is $\frac{(x_t - x_s)^{\otimes k}}{k!}$ for $(s,t) \in \Delta_T^2$ and $k = 1,\dots,[\frac{1}{\alpha}]$ then

$$F(x_T) - F(x_0) = \int_0^T DF(x_s) \mathrm{d}x_s := \lim_{n\to\infty} \sum_{i=0}^{n-1} \sum_{k=1}^{[\frac{1}{\alpha}]} \frac{1}{k!} D^k F(x_{\frac{i}{n}}) \cdot X_{\frac{i}{n},\frac{i+1}{n}}^{(k)}. \tag{3.5}$$

If x has finite variation, then a possible choice for $X^{(k)}$ is $\int_{s \le u_1 \le \dots \le u_k \le t} \mathrm{d}x_{u_1} \otimes \dots \otimes \mathrm{d}x_{u_k}$ for $(s,t) \in \Delta_T^2$, $k = 1,\dots,[\frac{1}{\alpha}]$. Again, we can address ourself the following questions.

- has the definition (3.5) of the integral of $DF(x)$ with respect to x an extension to non exact linear form?
- is this extension a continuous function of $X = (X_{0,t}^{(n),i_n,\dots,i_1}, \quad (i_1,\dots,i_n) \in \{1,\dots,m\}^n, \quad n \le [\frac{1}{[\alpha]}])$? For which norm?

In the next subsection, we propose an extension to integration of general differential forms.

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3.3. Rough integration of one form and differential equations

Now, we are in position to give some few elements of answer using the deeper results of [22].
Let $\alpha \in]\frac{1}{3}, 1]$. Let $F : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^m)$ be a $C^{[\frac{1}{\alpha}]+1}$ continuous function where $L(\mathbb{R}^m, \mathbb{R}^m)$ is the set of linear functions from $\mathbb{R}^m$ into $\mathbb{R}^m$. In general, $D^k F$ are not symetric multilinear functions from $(\mathbb{R}^m)^{\otimes k+1}$ into $\mathbb{R}^m$.
Let $x = (x^1, \ldots, x^m) \in C^\alpha([0,T], \mathbb{R}^m)$ and assume that there exists a collection of tensors $X^{(k)} : \Delta_T^2 \to (\mathbb{R}^m)^{\otimes k}$, $k \leq [\frac{1}{\alpha}]$ such

that the symetric part of $X_{s,t}^{(m)}$ is $\frac{(x_t - x_s)^{\otimes k}}{k!}$ for $(s,t) \in \Delta_T^2$ and $k = 1, \ldots, [\frac{1}{\alpha}]$;
and they fulfill some algebraic consistency relation (see Chen formula or a multiplicative functional below)

$$X_{s,t}^{(k)} = \sum_{i=0}^{k} X_{s,u}^{(i)} \otimes X_{u,t}^{(k-i)}, \quad (s,u,t) \in \Delta_T^3, \quad k \leq \left[\frac{1}{\alpha}\right],$$

and

$$\sup_{(s,t)\in\Delta_T^2} \sup_{k=1,\ldots,[\frac{1}{\alpha}]} \frac{\|X_{s,t}^{(k)}\|}{|t-s|^{k\alpha}} < +\infty.$$

Then, Lyons in [22] has proved first that the functional

$$\hat{Y}_{s,t}^{(1)} = \sum_{k=1}^{[\frac{1}{\alpha}]} \frac{1}{k!} D^k F(x_s) \cdot X_{s,t}^{(k)} \quad (s,t) \in \Delta_T^2$$

is an almost additive one, that is there exists $C > 0$ and $\varepsilon > 0$ such that

$$\left\| \hat{Y}_{s,t}^{(1)} - \hat{Y}_{s,u}^{(1)} - \hat{Y}_{u,t}^{(1)} \right\| \leq C|t-s|^{1+\varepsilon}, \quad (s,u,t) \in \Delta_T^3.$$

Second, he proves that

$$Y_{s,t} := \lim_{n\to\infty} \sum_{i=0}^{n-1} \sum_{k=1}^{[\frac{1}{\alpha}]} \frac{1}{k!} D^k F(x_{\frac{i}{n}}) \cdot X_{\frac{i}{n},\frac{i+1}{n}}^{(k)}, \quad (s,t) \in \Delta_T^2$$

exists. The limit is denoted by $\int F(x_s) \mathrm{d}x_s$.

As, we will see below, this notion of integral also allow to solve differential equations driven by x (and the associated collection of tensors)

$$dy_t = f(y_t)dx_t, \quad y(0) = y_0, \quad t \in [0,T].$$

Moreover, the solution is continuous with respect to the underlying path x (and the associated collection of tensors) for the Hölder distance. This last strong result is very usefull for the applications.

4. The sewing lemma

In some proofs in the original paper of Lyons [22] or see for instance [23] Theorems 3.1.2 and 3.2.1, the following fact is hidden and used several times:

To any almost additive functional (see definition below) a unique additive functional is associated.

This fact is pointed out by Feyel and De la Pradelle and proved in [12]. Indeed, they have extract the analytical part of the original proof of Lyons, and postpone the proof of the algebraic part. They also obtain some continuity or Fréchet differentiablity results more easily. In this section, we give the proof of [12].

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Notation 4.1. For any $\mu : \Delta_T^2 \to \mathbb{B}$ and $(a,c,b) \in \Delta_T^3$

$$\delta\mu(a,c,b) = \mu(a,b) - \mu(a,c) - \mu(c,b).$$

Definition 4.1. A continuous function μ from Δ_T^2 into $\mathbb{B}$ is said to be an almost additive functional if and only if such there exists two constants K and $\varepsilon > 0$

$$\|\delta\mu(a,c,b)\| \leq K|b-a|^{1+\varepsilon}, \quad \forall \quad (a,c,b) \in \Delta_T^3. \tag{4.6}$$

With this definition, the sewing lemma, Lemma 2.1 of [12], is the following.

Lemma 4.1 (sewing lemma)**.**
Let μ be an almost additive functional fulfilling (4.6), there exists a unique functional $u : \Delta_T^2 \to \mathbb{B}$ such that

$$\|(u-\mu)(a,b)\| \leq cte|b-a|^{1+\varepsilon}, \quad \forall (a,b) \in \Delta_T^2. \tag{4.7}$$

Moreover, the least constant is at most $K\theta(\varepsilon)$ with $\theta(\varepsilon) = (1-2^{-\varepsilon})^{-1}$.

For sake of completness, the proof is given.

Proof of Lemma 4.1. For any integer n, let μ_n be the continuous function from Δ_T^2 into $\mathbb{B}$ defined by

$$\mu_n(a,b) = \sum_{i=0}^{2^n-1} \mu(t_i^n(a,b), t_{i+1}^n(a,b));$$

where for $i \in \{0,\dots,2^n\}$

$$t_i^n(a,b) = a + (b-a)i2^{-n}.$$

Then the context is clear, we will omitt (a,b) in $t_i^n(a,b)$.

The sketch of proof is the following:

- first, we prove that $(\mu_n)_n$ is a Cauchy sequence and converges to a continuous function denoted by u;
- second, we prove that u is the unique continuous, semi-additive function (see Eq. (4.9) below for a definition) such that there exists a constant $\tilde{K}$ such that

$$\|u(a,b) - \mu(a,b)\| \leq \tilde{K}|b-a|^{1+\varepsilon}, \quad \forall (a,b) \in \Delta_T^2; \tag{4.8}$$

- third, we will prove that u is additive.

(1) Note that for $(a,b) \in \Delta_T^2$, $n \in \mathbb{N}$, $i \in \{0,\dots,2^n\}$:

$$t_{2i}^{n+1} = t_i^n \qquad \text{and} \qquad t_{2i+1}^{n+1} = \frac{t_i^n + t_{i+1}^n}{2},$$

and,

$$\mu_{n+1}(a,b) - \mu_n(a,b) = -\sum_{i=0}^{2^n-1} \delta\mu\left(t_i^n, t_{2i+1}^{n+1}, t_{i+1}^n\right).$$

Using (4.6) on μ, we conclude that

$$\|\mu_{n+1}(a,b) - \mu_n(a,b)\| \leq K|b-a|^{1+\varepsilon}2^{-n\varepsilon}.$$

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Hence, the sequence $(\mu_n)_n$ is , uniformly in (a,b), a Cauchy sequence which converges to a continuous function denoted by u, fulfilling (4.7).
Note that for $n \in \mathbb{N}$,

$$\mu_{n+1}(a,b) = \mu_n\left(a, \frac{b+a}{2}\right) + \mu_n\left(\frac{b+a}{2}, b\right), \quad \forall (a,b) \in \Delta_T^2.$$

When n goes to infinity, we derive that u is semi-additive, that means

$$u(a,b) = u\left(a, \frac{b+a}{2}\right) + u\left(\frac{b+a}{2}, b\right), \quad \forall (a,b) \in \Delta_T^2. \tag{4.9}$$

(2) Let v be a semi-additive function such that there exists a constant $\tilde{K}$

$$\|v(a,b) - \mu(a,b)\| \leq \tilde{K}|b-a|^{1+\varepsilon} \quad \forall (a,b) \in \Delta_T^2.$$

Then, the difference $w = u - v$ is semi-additive and satisfies

$$\|w(a,b)\| \leq (K\theta(\varepsilon) + \tilde{K})|b-a|^{1+\varepsilon}, \quad \forall (a,b) \in \Delta_T^2.$$

For $(a,b) \in \Delta_T^2$, introducing the point $(a+b)2^{-1}$, w fulfills

$$\|w(a,b)\| \leq \left\|w\left(a, \frac{a+b}{2}\right)\right\| + \left\|w\left(\frac{a+b}{2}, b\right)\right\| \leq 2^{-\varepsilon}(K\theta(\varepsilon) + \tilde{K})|b-a|^{1+\varepsilon}.$$

By induction on n, we derive that

$$\|w(a,b)\| \leq 2^{-\varepsilon n}(K\theta(\varepsilon) + \tilde{K})|b-a|^{1+\varepsilon}, \quad \forall (a,b) \in \Delta_T^2;$$

and then $w = 0$.

(3) For $k \in \mathbb{N}^*$, let v_k be the continuous function defined by

$$v_k(a,b) = \sum_{i=0}^{k-1} u\left(a + \frac{b-a}{k}i, a + \frac{b-a}{k}(i+1)\right), \quad \forall (a,b) \in \Delta_T^2.$$

Let $c = (a+b)2^{-1}$, then for $i \in \{0, \ldots, k\}$ then

$$\begin{aligned} a + \frac{b-a}{k}i &= a + \frac{c-a}{k}(2i) \qquad \text{for} \quad i \leq \frac{k}{2}, \\ a + \frac{b-a}{k}i &= c + \frac{b-c}{k}(2i-k) \qquad \text{for} \quad i \geq \frac{k}{2}, \end{aligned}$$

and the mid point of $[a + (b-a)k^{-1}i; a + (b-a)k^{-1}(i+1)]$ is

$$\begin{aligned} a + \frac{c-a}{k}(2i+1) \qquad &\text{for} \quad i \leq \frac{k-1}{2}, \\ c + \frac{b-c}{k}(2i+1-k) \qquad &\text{for} \quad i \geq \frac{k-1}{2}. \end{aligned}$$

Combining with the fact that u is a semi-additive function, we derive that v_k is a continuous, semi-additive function.

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Note that $v_1 = u$.
Assume for $k \geq 2$ that $v_{k-1} = u$.
By construction and hypothesis of induction, we have

$$\begin{aligned} v_k(a,b) &= v_{k-1}(a, a + \frac{(b-a)(k-1)}{k}) + u(a + \frac{(b-a)(k-1)}{k}, b) \\ &= u(a, a + \frac{(b-a)(k-1)}{k}) + u(a + \frac{(b-a)(k-1)}{k}, b), \quad \forall (a,b) \in \Delta_T^2. \end{aligned}$$

Then,

$$\|(v_k - \mu)(a,b)\| \leq K[2\theta(\varepsilon) + 1]|b-a|^{1+\varepsilon} \quad \forall (a,b) \in \Delta_T^2.$$

Hence, by uniqueness, $v_k = u$ for $k \in \mathbb{N}^*$.
Note that for $k \in \mathbb{N}^*$, $i \in \{0, \ldots, k-1\}$,

$$v_k(a,b) = v_i\left(a, a + \frac{b-a}{k}i\right) + v_{k-i}\left(a + \frac{b-a}{k}i, b\right)$$

then for all $0 \leq a \leq b \leq T$ and all rational barycenter c of $[a,b]$,

$$u(a,b) = u(a,c) + u(c,b).$$

Since u is continuous, u is additive. □

Let $\mathcal{A}a$ be the subspace of $C^\alpha(\Delta_T^2, \mathbb{B})$ of functions μ fulfilling inegality (4.6) endowed with $N_{\alpha,a,\varepsilon}(\mu) = N_\alpha(\mu) + \inf\{K,\ \mu \text{ satisfies inegality (4.6)}\}$. For $\mu \in \mathcal{A}a$, $\mathcal{S}(\mu)$ is defined by $\mathcal{S}(\mu) = u$ where u is given by Lemma 4.1. Then, we obtain the following Corollary

Corollary 4.2. The map $\mathcal{S}$ is a linear continuous map from $(\mathcal{A}a, N_{\alpha,a,\varepsilon})$ into $C^\alpha(\Delta_T^2, \mathbb{B})$ with norm at most $(1 + T^{1+\varepsilon-\alpha}\theta(\varepsilon))$.

In order to identify $\mathcal{S}(\mu)$ with some limit of Riemann sums, we need the following corollary. Let $\mathcal{D} = \{t_1, \ldots, t_k\}$ be an arbitray subdivision of $[a,b]$ with mesh $|\mathcal{D}| := \sup_{i=1,\ldots,k-1} |t_{i+1} - t_i|$. Define the Riemann sum of μ along $\mathcal{D}$ as

$$J_{\mathcal{D}}(a,b,\mu) := \sum_{i=1}^{k-1} \mu(t_i, t_{i+1}).$$

Let $\mathcal{D} = \{t_1, \ldots, t_k\}$ be an arbitray subdivision of $[0,T]$. Then, for all $(a,b) \in \Delta_T^2$, $\mathcal{D}_{|[a,b]} := \{t_i$, such that $a \leq t_i \leq b\} \cup \{a,b\}$ defines a subdivision of $[a,b]$ and $|\mathcal{D}_{|[a,b]}| \leq |\mathcal{D}|$.

Corollary 4.3 (Cor. 2.4 of [12])**.** Let μ fulfilling hypothesis of Lemma 4.1 and $(\mathcal{D}^n)_n$ be a sequence of subdivisions of $[0,T]$ with mesh converging to 0. Then $J_{\mathcal{D}^n_{|[a,b]}}(a,b,\mu)$ converges to $\mathcal{S}(\mu)(a,b)$ as $|\mathcal{D}_n|$ shrinks to 0, uniformely on Δ_T^2.

Proof of Corollary 4.3. Note that

$$\|\mathcal{S}(\mu)(a,b) - J_{\mathcal{D}}(a,b,\mu)\| \leq \sum_{i=1}^{k-1} \|\mathcal{S}(\mu)(t_i,t_{i+1}) - \mu(t_i,t_{i+1})\| \leq (b-a)K\theta(\varepsilon)|\mathcal{D}|^\varepsilon$$

wich converges to 0 as $|\mathcal{D}|$ schrinks to 0. □

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Using uniqueness in Lemma 4.1, we derive the following Lemma.

Lemma 4.4. Let $\mu : [0,1] \times \Delta_T^2 \to \mathbb{R}$ be continuous such that

$$\|\delta\mu(t;a,c,b)\| \leq C|b-a|^{1+\varepsilon}, \quad \forall t \in [0,1], \quad \forall (a,c,b) \in \Delta_T^3.$$

Then, the functional $\nu : [0,1] \times \Delta_T^2 \to \mathbb{R}$ $\nu(t;a,b) = \int_0^t \mu(s;a,b)ds, \quad \forall (a,b) \in \Delta_T^2$ is almost additive and

$$\mathcal{S}(\nu(t;.,.)) = \int_0^t \mathcal{S}(\mu(t;,.,.))ds, \quad \forall t \in [0,1].$$

In the original work of Lyons [22] or [23], Corollary 4.3 is hidden in several results, see for instance Theorem 3.1.2 of [23] or Theorem 1.16 of [24]. A proof can be sumarized in the following way.

Let us fix a partition $\mathcal{D} = \{0 = t_0 < \ldots < t_k = 1\}$. For any partition $\mathcal{D}'$, we have

$$|J_{\mathcal{D}}(a,b,\mu)| \leq |J_{\mathcal{D}'}(a,b,\mu)| + |J_{\mathcal{D}}(a,b,\mu) - J_{\mathcal{D}'}(a,b,\mu)|, \quad (a,b) \in \Delta_T^2.$$

Let $\mathcal{D}' = \mathcal{D} \setminus \{t_j\}$ where t_j is t_1 if $r = 2$ and otherwise chosen such that

$$|t_{j-1} - t_{j+1}| \leq \frac{2}{r-1}|b-a|.$$

In particular,

$$|J_{\mathcal{D}}(a,b,\mu) - J_{\mathcal{D}'}(a,b,\mu)| \leq C\left[\frac{2}{r-1}|b-a|\right]^{1+\varepsilon}, \quad (a,b) \in \Delta_T^2.$$

By successively dropping point of $\mathcal{D}$ until $\mathcal{D} = \{s,t\}$ we get

$$|\mu(a,b) - J_{\mathcal{D}}(a,b,\mu)| \leq C2^{1+\varepsilon}\sum_{j=2}^{r-1}\frac{1}{j^{1+\varepsilon}}|b-a|^{1+\varepsilon}, \quad (a,b) \in \Delta_T^2.$$

If $(\mathcal{D}_n)_n$ is a sequence of subdivisions such that $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ with mesh converging to 0, $(J_{\mathcal{D}}(a,b,\mu))_n$ converge to a functional $\tilde{\mu}$ fulfilling inequality (4.8). It remainds to prove that the limit is additive an does not depend on the chosen sequence of subdivisions.

5. Tensor algebra, multiplicative and almost multiplicative functionals

We recall some definitions and prove results on rough path theory stated in [23], using the tools and proofs of [12].

5.1. Tensor algebra

Let $(V, |\ |_1)$ be a d dimensional Euclidian space, $d \geq 2$. The tensor product is $V^{\otimes k} = V \otimes \ldots \otimes V$ (of k copies of V) endowed with a norm $|.|_k$ compatible with the tensor product that is for $l \geq 1$, $k \geq 1$,

$$|\xi \otimes \eta|_{k+l} \leq |\xi|_k|\eta|_l, \quad \forall \xi \in V^{\otimes k}, \quad \forall \eta \in V^{\otimes l}.$$

For each $n \in \mathbb{N}$, the truncated tensor algebra $T^{(n)}(V)$ is

$$T^{(n)}(V) := \oplus_{k=0}^n V^{\otimes k}, \quad V^{\otimes 0} = \mathbb{R}.$$

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The addition on $T^{(n)}(V)$ is the usual one:

$$\xi + \eta := (\xi^{(i)} + \eta^{(i)})_{i=0}^{n}, \quad \xi = (\xi^{(i)})_{i=0}^{n}, \quad \eta = (\eta^{(i)})_{i=0}^{n} \in T^{(n)}(V).$$

The multiplication on $T^{(n)}(V)$ is defined as

$$(\xi \otimes \eta)^{(k)} := \sum_{j=0}^{k} \xi^{(j)} \otimes \eta^{(k-j)}, \quad \forall \xi,\ \eta \in T^{(n)}(V), \quad k \leq n.$$

The norm $|.|_n$ on $T^{(n)}(V)$ is defined by

$$|\xi|_{(n)} := \sum_{i=0}^{n} |\xi^{(i)}|_i, \qquad \text{if } \xi = (\xi^{(0)}, \ldots, \xi^{(n)}).$$

The space $(T^{(n)}(V),\ |.|_{(n)}, \oplus, \otimes)$ is a tensor algebra with identity element for the multiplication $(1, 0, \ldots, 0)$ and for $\xi, \eta \in T^{(n)}(V), \quad |\xi \otimes \eta|_{(n)} \leq |\xi|_{(n)} |\eta|_{(n)}$.

Example 5.1. If $V = \mathbb{R}^d$, then $V^{\otimes 2}$ is the set of d square matrices and for $\xi = (\xi^{(0)}, \xi^{(1)}, \xi^{(2)}), \quad \eta = (\eta^{(0)}, \eta^{(1)}, \eta^{(2)}) \in T^{(2)}(V)$,

$$\begin{aligned}
(\xi \otimes \eta)^{(0)} &= \xi^{(0)} \cdot \eta^{(0)}, \\
(\xi \otimes \eta)^{(1)} &= \xi^{(0)} \cdot \eta^{(1)} + \eta^{(0)} . \xi^{(1)}, \\
(\xi \otimes \eta)^{(2)} &= \xi^{(0)} \cdot \eta^{(2)} + \xi^{(1)} \otimes \eta^{(1)} + \eta^{(0)} . \xi^{(2)}.
\end{aligned}$$

The tensor algebra $T^{(\infty)}(V)$ is

$$T^{(\infty)}(V) =: \oplus_{k=0}^{\infty} V^{\otimes k}, \quad V^{\otimes 0} = \mathbb{R}.$$

Remark 5.1. When $d = 1$, the definition is slightly modified. Indeed,

$$T^{(n)}(V) = \mathbb{R} \oplus V \oplus \ldots \oplus V.$$

Definition 5.1. A map $X : \Delta_T^2 \to T^{(n)}(V)$ is said to be α Hölder continuous if for $k = 1, \ldots, n$, $X^{(i)}$ belongs to $C^{\alpha i}(\Delta_T^2, V^{\otimes i})$, and $N_{\alpha,S,T}(X) = \max(\|X^{(i)}\|_{i\alpha,S,T}^{1/i}, \quad i = 1, \ldots, n)$.

5.2. Multiplicative functional

Let $n \in \mathbb{N}^*$ and $T \in \mathbb{R}_+^*$.

Definition 5.2. A map $X : \Delta_T^2 \to T^{(n)}(V)$, $X = (X^{(0)}, \ldots, X^{(n)})$ with components $X_{s,t}^{(k)} \in V^{\otimes k}$ for any $(s,t) \in \Delta_T^2, \quad k = 0, \ldots, n$, is called a multiplicative functional of degree n if

$$\begin{aligned}
X_{s,t}^{(0)} &= 1, \\
X_{s,t} \otimes X_{t,u} &= X_{s,u}, \quad \forall (s,t,u) \in \Delta_T^3,
\end{aligned} \tag{5.10}$$

where the tensor product $\otimes$ is taken in $T^{(n)}(V)$.

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Let $\mathcal{M}^{(n)}(V)$ be the set of multplicative functionals from Δ_T^2 into $T^{(n)}(V)$.

Equality (5.10) is called Chen identity, although it appears long before Chen's fundamental works in which a connection is made from iterated path integrals along smooth paths to a class of differential forms on a space of loops on manifold, see [3–5]. At the order k, equality (5.10) means

$$X_{s,t}^{(k)} = \sum_{i=0}^{k} X_{s,u}^{(i)} \otimes X_{u,t}^{(k-i)}, \quad \forall (s,u,t) \in \Delta_T^3. \tag{5.11}$$

That means for $n \geq 2$

$$\begin{aligned} X_{s,t}^{(0)} &= 1, \\ X_{s,t}^{(1)} &= X_{s,u}^{(1)} + X_{u,t}^{(1)}, \\ X_{s,t}^{(2)} &= X_{s,u}^{(2)} + X_{s,u}^{(1)} \otimes X_{u,t}^{(1)} + X_{u,t}^{(2)}, \quad (s,u,t) \in \Delta_T^3. \end{aligned}$$

Sequel of iterated integrals of smooth paths are the generic example multiplicative functionals.

Proposition 5.2. Let $d \geq 2$. Let $x : [0,T] \to V$ be a Lipschitz path and define $X = (1, X^{(1)}, \ldots, X^{(n)})$ where $X_{s,t}^{(k)} = \int_{s \leq t_1 \leq \ldots \leq t_k \leq t} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_k}$, $k = 1, \ldots, n$, $(s,t) \in \Delta_T^2$. Then, X is a multiplicative functional of degree n.

In this case, identity (5.10) is equivalent to the additive property of iterated path integrals over differents domains.

Proof of Proposition 5.2. Let $X = (1, X^{(1)}, \ldots, X^{(n)})$ be as in statement of Proposition 5.2. Equality (5.11) when $k = 1$ is the Chalse relation for Rieman Stieljes integral.

Assume that equality (5.11) holds for $k \leq n-1$ and note that

$$X_{s,t}^{(k+1)} = \int_s^t \mathrm{d}x_{t_1} \otimes X_{t_1,t}^{(k)}, \quad (s,t) \in \Delta_T^2. \tag{5.12}$$

Using the the Chalse relation for Rieman Stieljes integral

$$X_{s,t}^{(k+1)} = \int_s^u \mathrm{d}x_{t_1} \otimes X_{t_1,t}^{(k)} + X_{u,t}^{(k+1)}, \quad (s,u,t) \in \Delta_T^3.$$

According to induction hypothesis on $X^{(k)}$, we replace $X_{t_1,t}^{(k)}$ by $\sum_{j=0}^{k} X_{t_1,u}^{(j)} \otimes X_{u,t}^{(k-j)}$ and

$$X_{s,t}^{(k+1)} = \sum_{j=0}^{k} \int_s^u \mathrm{d}x_{t_1} \otimes X_{t_1,u}^{(j)} \otimes X_{u,t}^{(k-j)} + X_{u,t}^{(k+1)}, \quad (s,u,t) \in \Delta_T^3.$$

Using identity (5.12) and a change of index in the sum, we recognise equality (5.11) for $k+1$. □

When $d = 1$, using the binomial formula of Newton, we have the following exemple of multiplicative functional.

Example 5.3. If $V = \mathbb{R}$, or $d = 1$, then for any path $x : [0,T] \to \mathbb{R}$, the functional defined by for $k = 0, \ldots, n$, $(s,t) \in \Delta_T^2$

$$X_{s,t} = (X_{s,t}^{(0)}, \ldots, X_{s,t}^{(n)}) \qquad \text{with} \qquad X_{s,t}^{(k)} = \frac{[x(t) - x(s)]^k}{k!}, \quad k \leq n,$$

is a multiplicative functional of order n.

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Definition 5.3. Let $x : [0,T] \to V$ be continuous. A multiplicative functional X is a multiplicative functional over x if and only if

$$X^{(1)}_{s,t} = x(t) - x(s), \quad \forall (s,t) \in \Delta^2_T.$$

A functional over a path, when it exits is not unique in general.

Lemma 5.4 ([23])**.** Let $X = (1, X^{(1)}, X^{(2)})$ be a multiplicative functional, then $Y = (1, X^{(1)}, X^{(2)} + \Phi)$ is multiplicative if and only if Φ is additive.

Proof of Lemma 5.4. According to the Chen Rule, equation (5.10), Y is multiplicative if and only if

$$Y^{(2)}_{s,t} = Y^{(2)}_{s,u} + Y^{(2)}_{u,t} + Y^{(1)}_{s,u} \otimes Y^{(1)}_{u,t}, \quad (s,u,t) \in \Delta^3_T.$$

Since X is multiplicative we obtain for all $0 \le s \le u \le t$,

$$\Phi_{s,t} = \Phi_{s,u} + \Phi_{u,t}.$$

□

Remark 5.2. For $n \ge 2$, the set $\mathcal{M}^{(n)}(V)$ of is not a convex set.

Indeed, if X and Y belong to $\mathcal{M}^{(n)}(V)$ and $\theta \in (0,1)$ then for instance for $(a,c,b) \in \Delta^3_T$,

$$\begin{aligned}&[(\theta X_{a,b} + (1-\theta)Y_{a,b}) - (\theta X_{a,c} + (1-\theta)Y_{a,c}) \otimes (\theta X_{c,b} + (1-\theta)Y_{c,b})]^{(2)} \\ &= -\theta(1-\theta)\left(X^{(1)}_{a,c} \otimes Y^{(1)}_{c,b} + Y^{(1)}_{a,c} \otimes X^{(1)}_{c,b}\right)\end{aligned}$$

which is non null in general and then $\theta X + (1-\theta)Y$ is not a multiplicative functional.

Definition 5.4. A multiplicative functional X of order n, α-Hölder continuous is called a α Hölder continuous n rough path. The set of all α-Hölder continuous n rough paths is denoted by $\Omega H^{(n)}_{\alpha,T}(V)$.

Note that $d_{\alpha,T}$ is a distance on $\Omega H^{(n)}_{\alpha,T}(V)$, where $d_{\alpha,T}(X,\tilde{X}) = N_{\alpha,0,T}(\tilde{X} - X)$. When the context is clear, we omitt the subscript T.

Definition 5.5. A smooth rough path X is an element of $\Omega H^{(n)}_{\alpha,T}(V)$ with $n \ge [\frac{1}{\alpha}]$ such that there exists a Lipschitz function $x : [0,T] \to V$ and

$$X^{(k)}_{s,t} = \int_{s<t_1<\ldots<t_k<t} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_k}, \quad k = 1,\ldots,n, \quad \forall (s,t) \in \Delta^2_T.$$

(When $d = 1$ the tensor product is indeed the product in $\mathbb{R}$.)

The set of the geometric rough paths α Hölder continuous is the closure of the set of smooth rough paths of order $[\frac{1}{\alpha}]$ for $d_{\alpha,T}$. It is denoted by $G\Omega H_{\alpha,T}(V)$. It is important to note that when $\alpha \le \frac{1}{2}$, neither $G\Omega H_{\alpha,T}(V)$ or $\Omega H^{(n)}_{\alpha,T}(V)$ are vector spaces.

5.3. Exemples: rough paths over Gaussian processes

Existence of rough paths and geometrics rough paths over Gaussian processes are deeply studied by Friz and Victoir in [14], see also [8].

Here, we present a partial result. Let $G = (G^1,\ldots,G^d)$ be a centered Gaussian process, $\mathbb{R}^d$ valued, with independent components. Let $R^i(t,s) = \mathbb{E}(G^i_s G^i_t)$ for $i = 1,\ldots,d$, $(s,t) \in \Delta^2_T$ be their convariance function.

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Assume that their exist $H > 1/4$ and a constant C such that for $(s,t) \in [0,T]^2$, such that

$$|R^i(t,t) + R^i(s,s) - 2R^i(s,t)| \leq C|t-s|^{2H} \tag{5.13}$$
$$\left|R^i(t,t+\tau) + R^i(s,s+\tau) - R^i(s,t+\tau) - R^i(t,s+\tau)\right| \leq C\tau^{2H}\left|\frac{t-s}{\tau}\right|^2,$$

provided $|(t-s)/\tau| \leq 1$. Let G^n be the linear interpolation of G along the dyadic subdivisions of mesh 2^{-n} of $[0,T]$. Let $\mathbf{G}^n$ be the smooth multiplicative functional built on G^n. Then, from Theorem 4.5.1 of [23], the sequence $(\mathbf{G}^n)_n$ converges almost surely and in all L^p, $p \geq 1$ in $G\Omega H_\alpha(V)$ for any $\alpha < H$ to a geometric rough path over G: that means

$$\lim_{n\to\infty} \sup_{(s,t)\in\Delta_T^2} \sum_{i=1}^{[1/\alpha]} \frac{|\mathbf{G}_n^{(i)}(s,t) - \mathbf{G}^{(i)}(s,t)|}{|t-s|^{i\alpha}} = 0 \tag{5.14}$$

converges almost surely and in L^p for any $p \geq 1$.

Example 5.5. Let $B = (B^1, \ldots, B^d)$ be a d dimensional Brownian motion then its covariance is given by $R^i(s,t) = \min(s,t)$, $i = 1, \ldots, d$ and $(s,t) \in \Delta_T^2$ and fulfills condition (5.13) with $C = 1$. Moreover, the geometric functional over B is $\mathbf{B} = (1, \mathbf{B}^{(1)}, \mathbf{B}^{(2)})$ where

$$\mathbf{B}_{s,t}^{(1)\,,i} = B^i(t) - B^i(s), \quad \mathbf{B}_{s,t}^{(2)\,,i,j} = \int_0^t [B^i(u) - B^i(s)] \circ \mathrm{d}B^j(u), \quad i,j = 1, \ldots, \mathrm{d}, \quad (s,t) \in \Delta_T^2,$$

where $\circ d$ means Stratonovitch integral.

Example 5.6. The d dimensional fractional Brownian motion with Hurst parameter H is the unique centered Gaussian process $B_H = (B_H^1, \ldots, B_H^d)$ with independant components and covariance function

$$R_H^i(s,t) = \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right], \quad (s,t) \in \Delta_T^2, \quad i = 1, \ldots, d.$$

In [7], it is proved that the condition (5.13) is fulfilled for $H > \frac{1}{4}$. Moreover, the convergence (5.14) does not hold when $H \leq 1/4$.

5.4. Almost to multiplicative functionals *via* sewing lemma

The "fundamental blocks" in Rieman sums will be almost multiplicative functionals. In this section we give a method to construct multiplicative functionals from almost multiplicative functionals using the sewing lemma. This lemma deeply simplifies the proofs of the results of the section called Almost Rough paths in the book [23]. The steps of proof are the following:

- **Step 1.** To an almost multiplicative functionals of degree n, we associate n almost additive functionals.
- **Step 2.** The sewing Lemma yields n additive functionals.
- **Step 3.** The inverse of step 1 applying to additive functionals given in step 2 provide a multiplicative functional.

Since every step are continuous (in a mean we precise below), we derive without extra effort the continuity of the procedure.

Definition 5.6. A functional $Y : \Delta_T^2 \to T^{(n)}(V)$ is called an almost multiplicative functional if $Y_{s,t}^{(0)} = 1$ and for some constants $C > 0$ and $\varepsilon > 0$,

$$|(Y_{s,t} \otimes Y_{t,u})^{(i)} - Y_{s,u}^{(i)}|_i \leq C^i|u-s|^{1+\varepsilon}, \quad \forall (s,t,u) \in \Delta_T^3, \quad i = 1, \ldots, n. \tag{5.15}$$

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In the sequel, we will used

$$\delta_{T^{(n)}(V)}Y(s,t,u) := (Y_{s,t} \otimes Y_{t,u}) - Y_{s,u}, \quad \forall (s,t,u) \in \Delta_T^3.$$

Let us denote by $\mathcal{A}m_{C,\varepsilon}^{(n)}(V)$ the set of almost multiplicative fonctional of degree n fufilling inequality (5.15).

Example 5.7. Let x and y be α Hölder continuous paths in V with $\alpha > 1/2$, and μ given by

$$\mu(a,b) = [x(a) - x(0)] \otimes [y(b) - y(a)], \quad (a,b) \in \Delta_T^2.$$

Then, $\delta\mu(a,c,b) = -[x(c) - x(a)] \otimes [y(b) - y(c)], \quad (a,c,b) \in \Delta_T^3$. The functional $(1,\mu)$ is an almost multiplicative functional.

The following proposition, due to [22], justifies the name of almost rough path. The proof given here is the proof of [12], where the original proof of Lyons is splitten into three parts. The first step (or the analytical one) is the Sewing Lemma, the second one given in Proposition 5.9 allows us to pass to an almost multiplicative functional of order $n+1$ such that its restriction to $T^{(n)}(V)$ is multiplicative to a multiplicative functional, see Lemma 5.8 and Proposition 5.9. The third step is a recursevely one, see Theorem 5.11.

Lemma 5.8. Let n, X, $Y^{(n+1)}$ be as in Proposition 5.9, and $\mu_Y^{(n+1)} : \Delta_T^2 \to V^{\otimes(n+1)}$ be

$$\mu_Y^{(n+1)}(a,b) := Y_{a,b}^{(n+1)} + \sum_{k=1}^{n} X_{0,a}^{(k)} \otimes X_{a,b}^{(n+1-k)}. \tag{5.16}$$

then $\mu_Y^{(n+1)}$ is an almost additive functional.

Proof of Lemma 5.8. First, using Chen rule applying to X, identity (5.10) we obtain

$$\sum_{k=1}^{n} X_{a,c}^{(k)} \otimes X_{c,b}^{(n+1-k)} = \sum_{k=1}^{n} X_{0,a}^{(k)} \otimes X_{a,b}^{(n+1-k)} - \sum_{k=1}^{n} X_{0,a}^{(k)} \otimes X_{a,c}^{(n+1-k)} - \sum_{k=1}^{n} X_{0,c}^{(k)} \otimes X_{c,b}^{(n+1-k)}. \tag{5.17}$$

Indeed,

$$\begin{aligned} &\sum_{k=1}^{n} X_{0,a}^{(k)} \otimes X_{a,b}^{(n+1-k)} - \sum_{k=1}^{n} X_{0,a}^{(k)} \otimes X_{a,c}^{(n+1-k)} - \sum_{k=1}^{n} X_{0,c}^{(k)} \otimes X_{c,b}^{(n+1-k)} \\ &= \sum_{k=1}^{n} X_{0,a}^{(k)} \otimes [X_{a,b}^{(n+1-k)} - X_{a,c}^{(n+1-k)} - X_{c,b}^{(n+1-k)}] \\ &\quad + \sum_{k=1}^{n} [X_{0,a}^{(k)} - X_{0,c}^{(k)} - X_{a,c}^{(k)}] \otimes X_{c,b}^{(n+1-k)} \\ &\quad + \sum_{k=1}^{n} X_{a,c}^{(k)} \otimes X_{c,b}^{(n+1-k)}. \end{aligned} \tag{5.18}$$

Note that

$$X_{a,b}^{(n+1-k)} - X_{a,c}^{(n+1-k)} - X_{c,b}^{(n+1-k)} = \sum_{j=1}^{n-k} X_{a,c}^{(j)} \otimes X_{c,b}^{(n+1-k-j)} \tag{5.19}$$

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and

$$X^{(k)}_{0,a} - X^{(k)}_{0,c} - X^{(k)}_{a,c} = \sum_{j=1}^{k-1} X^{(j)}_{0,a} \otimes X^{(k-j)}_{a,c} \tag{5.20}$$

where the sum on an empty set of indices is null. Plugging equalities (5.19) and (5.20) into (5.18) and using a change variable and permutation of the order of summation, we prove that

$$\sum_{k=1}^{n} X^{(k)}_{0,a} \otimes \left[X^{(n+1-k)}_{a,b} - X^{(n+1-k)}_{a,c} - X^{(n+1-k)}_{c,b}\right] + \sum_{k=1}^{n} [X^{(k)}_{0,a} - X^{(k)}_{0,c} - X^{(k)}_{a,c}] \otimes X^{(n+1-k)}_{c,b} = 0,$$

and derive (5.17) from (5.18).

From equality (5.17) we obtain,

$$\begin{aligned} \delta\mu_Y^{(n+1)}(a,c,b) &= Y^{(n+1)}_{a,b} - Y^{(n+1)}_{a,c} - Y^{(n+1)}_{c,b} - \sum_{k=1}^{n} X^{(k)}_{a,c} \otimes X^{(n+1-k)}_{c,b} \\ &= [Y_{a,b} - Y_{a,c} \otimes Y_{c,b}]^{(n+1)} \end{aligned} \tag{5.21}$$

hence $|\delta\mu_Y^{(n+1)}(a,c,b)|_{n+1} \leq C^{n+1}|b-a|^{1+\varepsilon}$. That means $\mu_Y^{(n+1)}$ is an almost additive functional. □

Proposition 5.9. Let X be a continuous multiplicative functional of degree n, and let $Y^{(n+1)} : \Delta_T^2 \to V^{\otimes(n+1)}$ be continuous and such that

$$Y = \left(1, X^{(1)}, X^{(2)}, \ldots, X^{(n)}, Y^{(n+1)}\right)$$

is an $n+1$ almost multiplicative functional fulfilling (5.15). Then, $X^{(n+1)} = \mathcal{S}(\mu_Y^{(n+1)}) - \mu_Y^{(n+1)} + Y^{(n+1)}$ is the unique fonctional from Δ_T^2 info $V^{\otimes(n+1)}$ such that

$$Z = \left(1, X^{(1)}, X^{(2)}, \ldots, X^{(n)},\ X^{(n+1)}\right)$$

is an $n+1$ multiplicative functional with the condition

$$|X^{(n+1)}_{s,t} - Y^{(n+1)}_{s,t}|_{n+1} \leq Cst \times C^{n+1}|t-s|^{1+\varepsilon}, \quad (s,t) \in \Delta_T^2. \tag{5.22}$$

The least constant is at most $\theta(\varepsilon) = (1-2^{-\varepsilon})^{-1}$ and C is given in (5.15).

Proposition 5.10. Let n and α such that $(n+1)\alpha \leq 1$. The map $\mathcal{E}$ is continuous from $\{Y \in \mathcal{A}m_{C,\varepsilon}^{(n+1)}(V),\ \ (1, Y^{(1)}, \ldots, Y^{(n)}) \in \mathcal{M}^{(n)}(V)\} \cap C^{\alpha}(\Delta_T^2, T^{(n+1)}(V))$ into $\Omega H_{\alpha,T}^{(n+1)}(V)$, where $\mathcal{E}^{(n+1)}(Y) = \mathcal{S}(\mu_Y^{(n+1)}) - \mu_Y^{(n+1)} + Y^{(n+1)}$, $\varepsilon^{(i)}(Y) = Y^{(i)}, \quad i = 1, ..., n$.

Proof of Proposition 5.9. By the Lemma 5.8 and the sewing lemma, Lemma 4.1, we get an additive functional $\mathcal{S}(\mu_Y^{(n+1)})$ having its values in $V^{\otimes(n+1)}$ such that

$$|\mathcal{S}(\mu_Y^{(n+1)})(a,b) - \mu_Y^{(n+1)}(a,b)|_{n+1} \leq \theta(\varepsilon) C^{n+1}|b-a|^{1+\varepsilon}.$$

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Put for $(a,b) \in \Delta_T^2$

$$X_{a,b}^{(n+1)} := \mathcal{S}(\mu_Y^{(n+1)})(a,b) - \sum_{k=1}^{n} X_{0,a}^{(k)} \otimes X_{a,b}^{(n+1-k)}$$

we then get from equality (5.17)

$$X_{a,b}^{(n+1)} - X_{a,c}^{(n+1)} - X_{c,b}^{(n+1)} = \sum_{k=1}^{n} X_{a,c}^{(k)} \otimes X_{c,b}^{(n+1-k)}$$

and

$$|X_{a,b}^{(n+1)} - Y_{a,b}^{(n+1)}|_{n+1} = |\mathcal{S}(\mu_Y^{(n+1)})(a,b) - \mu_Y^{(n+1)}(a,b)|_{n+1} \leq \theta(\varepsilon) C^{n+1} |b-a|^{1+\varepsilon}.$$

Uniqueness follows by the routine argument. □

Proof of Proposition 5.10. Note that the projection of the map $\mathcal{E}$ on the tensor algebra $T^{(n)}(V)$ is equal to the identity $Id_{T^{(n)}(V)}$ then it remaindes to prove the continuity for the $(n+1)$ level.

Since $\mathcal{E}^{(n+1)}(Y) = \mathcal{S}(\mu_Y^{(n+1)}) - \mu_Y^{(n+1)} + Y^{(n+1)}$, and $\mathcal{S} - Id$ is a linear continuous map, we only have to study the continuity of the map $Y \mapsto \mu_Y^{(n+1)}$.

Let $\eta < \frac{\varepsilon}{1+\varepsilon-(n+1)\alpha}$, we will prove that $Y \mapsto \mu_Y^{(n+1)}$ is locally η Hölder continuous from $\{Y \in \mathcal{A}m_{C,\varepsilon}^{(n+1)}(V), (1, Y^{(1)}, \ldots, Y^{(n)}) \in \mathcal{M}^{(n)}(V)\} \cap C^{\alpha}(\Delta_T^2, T^{(n+1)}(V))$ into $(\mathcal{A}a, N_{\alpha,a,\tilde{\varepsilon}})$ where $\tilde{\varepsilon} = \eta(n+1)\alpha + (1-\eta)(1+\varepsilon) - 1 > 0$.

Let $(X, \tilde{X}) \in \Omega H_{\alpha}^{(n)}(V)^2$ and $(Y^{(n+1)}, \tilde{Y}^{(n+1)}) \in C^{\alpha}(\Delta_T^2, V^{\otimes(n+1)})$ such that $Y = (X, Y^{(n+1)})$ and $(\tilde{X}, \tilde{Y}^{(n+1)})$ belong to $\mathcal{A}m_{C,\varepsilon}^{(n+1)}$. From identity (5.21), in one hand for all $(a,c,b) \in \Delta_T^3$

$$|\delta(\mu_Y^{(n+1)} - \mu_{\tilde{Y}}^{(n+1)})(a,c,b)|_{n+1} \leq 2C^{n+1}|b-a|^{1+\varepsilon};$$

in the other hand for all $(a,c,b) \in \Delta_T^3$

$$|\delta(\mu_Y^{(n+1)} - \mu_{\tilde{Y}}^{(n+1)})(a,c,b)|_{n+1} \leq n N_{\alpha,0,T}(Y - \tilde{Y})[1 + N_{\alpha,0,T}^{n}(Y) + N_{\alpha,0,T}^{n}(\tilde{Y})]|b-a|^{(n+1)\alpha}.$$

Then, for all $(a,c,b) \in \Delta_T^3$

$$|\delta(\mu_Y^{(n+1)} - \mu_{\tilde{Y}}^{(n+1)})(a,c,b)|_{n+1} \leq K N_{\alpha,0,T}(Y - \tilde{Y})^{\eta} |b-a|^{1+\tilde{\varepsilon}},$$

where $\tilde{\varepsilon} = \eta(n+1)\alpha + (1-\eta)(1+\varepsilon) - 1 > 0$ and $K = [n(1 + N_{\alpha,0,T}^{n}(Y) + N_{\alpha,0,T}^{n}(\tilde{Y}))]^{\eta} 2^{1-\eta} C^{(1-\eta)(n+1)}$. That means, $Y \mapsto \mu_Y^{(n+1)}$ is locally η Hölder continuous as $\mathcal{E}^{(n+1)}$. □

The functional $X^{(n+1)}$ is indeed limit of kind of Riemann sums.

Remark 5.3. Using the notations and assumptions of Proposition 5.9 and Corollary 4.3, in the case $n = 0$ we have for $(a,b) \in \Delta_T^2$

$$X_{a,b}^{(1)} = \lim_{\mathcal{D}_m} \sum_{i=1}^{k_m - 1} Y_{t_i^m, t_{i+1}^m}^{(1)};$$

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and in the case $n = 1$, for $(a, b) \in \Delta_T^2$ we have

$$X_{a,b}^{(2)} = \lim_{\mathcal{D}_m} \sum_{i=1}^{k_m - 1} \left[Y_{t_i^m, t_{i+1}^m}^{(2)} + X_{a,t_i^m}^{(1)} \otimes X_{t_i^m, t_{i+1}^m}^{(1)} \right];$$

where $\mathcal{D}_m = \{t_1^m = a < t_2^m < \ldots < t_{k_n}^m = b\}$ is a sequence of subdivisions such that $\lim_{m\to\infty} |\mathcal{D}_m|$.

Theorem 5.11. Let Y be a α Hölder continuous almost multiplicative functional of order $[\frac{1}{\alpha}]$such that there exist some constants C, $\varepsilon > 0$,

$$|(Y_{s,t} - Y_{s,u} \otimes Y_{u,t})^{(i)}|_i \leq C|t - s|^{1+\varepsilon}, \quad (s, u, t) \in \Delta_T^3.$$

Then, there exists a unique α Hölder continuous rough path X such that

$$|Y_{s,t}^{(i)} - X_{s,t}^{(i)}|_i \leq CCst(\alpha, \varepsilon, T, N_{\alpha,0,T}(Y))|t - s|^{1+\varepsilon}, \quad \forall i = 1, \ldots, [1/\alpha], \quad \forall (s, t) \in \Delta_T^2,$$

where for $\alpha \in]1/3, 1/2]$ $Cste(C, \alpha, \varepsilon, T, Y) = [1 + 2N_{\alpha,0,T}(Y)T^\alpha\theta(\varepsilon) + \theta(\varepsilon)^2 CT^{1+\varepsilon}]\theta(\varepsilon)$.

Let be $\mathcal{E} : \{Y \in \mathcal{A}m_{C,\varepsilon}^{([\frac{1}{\alpha}])}(V), \quad \text{continuous}\} \to \mathcal{M}^{(n)}(V)$ defined by Theorem 5.11.

Corollary 5.12. The map $\mathcal{E}$ is continuous from $\mathcal{A}m_{C,\varepsilon}^{([\frac{1}{\alpha}])}(V) \cap C^\alpha(\Delta_T^2, T^{([\frac{1}{\alpha}])}(V))$ into $\Omega H_{\alpha,T}^{([\frac{1}{\alpha}])}(V)$.

Proof of Theorem 5.11 and Corollary 5.12. We give only the proof for $\alpha \in]1/3, 1]$.

For the first level, we apply Proposition 5.9 to $n = 0$, the multiplicative functional of degree 0 *i.e.* 1 and $Y^{(1)}$. There exists $X^{(1)} : \Delta_T^2 \to V$ such that $(1, X^{(1)})$ is a multiplicative functional and

$$|X_{s,t}^{(1)} - Y_{s,t}^{(1)}|_1 \leq \theta(\varepsilon)C|t - s|^{1+\varepsilon}, \quad \forall (s, t) \in \Delta_T^2.$$

For the second level, we have to prove that the functional $(1, X^{(1)}, Y^{(2)})$ is almost multiplicative and conclude with Proposition 5.9. Indeed, for $(s, u, t) \in \Delta_T^3$

$$\begin{aligned} Y_{s,t}^{(2)} - Y_{s,u}^{(2)} - Y_{u,t}^{(2)} - X_{s,u}^{(1)} \otimes X_{u,t}^{(1)} = (Y_{s,t} - Y_{s,u} \otimes Y_{u,t})^{(2)} + Y_{s,u}^{(1)} \otimes [Y_{u,t}^{(1)} - X_{u,t}^{(1)}] \\ + [Y_{s,u}^{(1)} - X_{s,u}^{(1)}] \otimes X_{u,t}^{(1)} \end{aligned}$$

and

$$|Y_{s,t}^{(2)} - Y_{s,u}^{(2)} - Y_{u,t}^{(2)} - X_{s,u}^{(1)} \otimes X_{u,t}^{(1)}|_2 \leq C[1 + 2N_{\alpha,0,T}(Y)T^\alpha\theta(\varepsilon) + \theta(\varepsilon)^2 CT^{1+\varepsilon}]|t - s|^{1+\varepsilon}.$$

According to Proposition 5.9, there exists $X^{(2)} : \Delta_T^2 \to V^{\otimes 2}$ such that $(1, X^{(1)}, X^{(2)})$ is a multiplicative functional and

$$|X_{s,t}^{(2)} - Y_{s,t}^{(2)}|_2 \leq \theta(\varepsilon)C[1 + 2T^\alpha\theta(\varepsilon)N_{\alpha,0,T}(Y) + \theta(\varepsilon)^2 T^{1+\varepsilon}]|t - s|^{1+\varepsilon}, \quad \forall (s, t) \in \Delta_T^2.$$

The continuity of the map $\mathcal{E}$ is a consequence of the proof of Theorem 5.11 and Proposition 5.10. □

In the case $\alpha \in]\frac{1}{3}; \frac{1}{2}[$, we have the following kind of differentiablity property.

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Corollary 5.13. Let $\hat{Y} : [0,1] \times \Delta_T^2 \to T^{(2)}(W)$ be continuous and continuously differentiable with respect to θ such that $\hat{Y}(\theta,.,.) = (1,0,0)$

$$\left|\delta d_\theta \hat{Y}^{(1)}(a,c,b)\right| \leq C|b-a|^{1+\varepsilon}, \quad \forall (a,c,b) \in \Delta_T^3 \tag{5.23}$$
$$\left|d_\theta \hat{Y}^{(2)}(a,b) - d_\theta \hat{Y}^{(2)}(a,c) - d_\theta \hat{Y}^{(2)}(c,b) - \hat{Y}^{(1)}_{a,c} \otimes d_\theta \hat{Y}^{(1)}(c,b) - d_\theta \hat{Y}^{(1)}(a,c) \otimes \hat{Y}^{(1)}_{c,b}\right| \leq C|b-a|^{1+\varepsilon}.$$

Then, for all $\theta \in [0,1]$, $\hat{Y}(\theta;.,.)$ is an almost multiplicative functional and $\mathcal{E}(\hat{Y}(\theta;.,.))$ is continuously differentiable with respect to θ, and

$$|\mathcal{E}(\hat{Y})^{(i)}_{\theta;s,t} - \hat{Y}^{(i)}_{\theta;s,t}|_i \leq CCst(\alpha,\varepsilon,T)|t-s|^{1+\varepsilon}, \quad \forall i = 1,\dots,[1/\alpha], \quad \forall (s,t) \in \Delta_T^2,$$

where for $\alpha \in]1/3, 1/2]$ $Cste(\alpha,\varepsilon,T) = [1 + 2N_\alpha(Y)T^\alpha\theta(\varepsilon) + \theta(\varepsilon)^2 CT^{1+\varepsilon}]\theta(\varepsilon)$. Moreover, if in addition $\hat{Y}_{\theta;.,.}$ belongs to $C^\alpha(\Delta_T^2, W) \oplus C^{2\alpha}(\Delta_T^2, W^{\otimes 2})$, then

$$\|d_\theta \mathcal{E}(\hat{Y})^{(i)}_{\theta;.,.}\|_{i\alpha,0,T} \leq [\|\hat{Y}^{(i)}_{\theta;.,.}\|_{i\alpha,0,T} + CCst(\alpha,\varepsilon,T)]|T|^{1+\varepsilon-\alpha i}, \quad i = 1,2. \tag{5.24}$$

Proof of Corollary 5.13.

- Integrating with respect to θ inequalities (5.23) yiedls the fact that $\hat{Y}(\theta;.,.)$ is an almost multiplicative functional.
- Let $\mu^{(1)}_{\hat{Y}(\theta;.,.)} = \hat{Y}^{(1)}(\theta;.,.)$, then $\mu^{(1)}_{\hat{Y}}(\theta;.,.)$ is differentiable with respect to θ, $d_\theta \mu^{(1)}_{\hat{Y}}(\theta;.,.)$ fulfills assumption of Lemma 4.4 and

$$\mathcal{E}(\hat{Y}(\Theta;.,.)^{(1)}) = \mathcal{S}\left(\int_0^\Theta d_\theta \mu^{(1)}_{\hat{Y}}(\theta;.,.)\mathrm{d}\theta\right) = \int_0^\Theta \mathcal{E}(d_\theta \hat{Y}(\theta;.,.))^{(1)}\mathrm{d}\theta.$$

  Inequality (5.24) for $i = 1$ follows from

$$d_\theta \mathcal{E}(\hat{Y})^{(1)}_{\theta;.,.} = d_\theta \mathcal{E}(\hat{Y})^{(1)}_{\theta;.,.} - d_\theta \hat{Y}^{(1)}_{\theta;.,.} + d_\theta \hat{Y}^{(1)}_{\theta;.,.}.$$

- Let $\mu^{(2)}_{\hat{Y}(\theta;.,.)}(\theta;a,b) = \hat{Y}^{(2)}(\theta;a,b) + \hat{Y}(\theta;0,a)^{(1)} \otimes \hat{Y}(\theta;a,b)^{(1)}$ then $\mu^{(2)}_{\hat{Y}(\theta;.,.)}$ is differentiable with respect to θ, $d_\theta \mu^{(2)}_{\hat{Y}(\theta;.,.)}$ fulfills assumption of Lemma 4.4 and

$$\mathcal{S}\left(\int_0^\Theta d_\theta \mu^{(2)}_{\hat{Y}}(\theta;.,.)\mathrm{d}\theta\right) = \int_0^\Theta \mathcal{S}(d_\theta \mu^{(2)}_{\hat{Y}}(\theta;.,.))\mathrm{d}\theta.$$

  Since $\mathcal{E}(\hat{Y}(\theta;.,.))^{(2)}_{a,b} = \mathcal{S}(\mu^{(2)}_{\hat{Y}}(\theta;.,.))_{a,b} - Y^{(1)}_{\theta;0,a} \otimes Y^{(1)}_{\theta;a,b}$, then $\mathcal{E}(\hat{Y}(\theta;.,.))^{(2)}_{a,b}$ is differentiable with respect to θ.

Inequality (5.24) for $i = 2$ follows from

$$d_\theta \mathcal{E}(\hat{Y})^{(2)}_{\theta;.,.} = d_\theta \mathcal{S}(\mu^{(2)}_{\hat{Y}}(\theta;.,.)) - d_\theta \mu^{(2)}_{\hat{Y}}(\theta;.,.) + d_\theta \hat{Y}^{(2)}_{\theta;.,.}.$$ □

We also deduce the following Corollary (see Rem. 2.6 of [12] and Thm. 3.1.3 of [23]), which can be seen as a generalization of the following inequality on sequences of iterated integrals of smooth paths. For $x \in C^1([0,T],V)$:

$$\left|\int_{a \leq u_1 \leq \dots \leq u_n \leq b} \mathrm{d}x(u_1) \otimes \dots \otimes \mathrm{d}x(u_n)\right|_n \leq \frac{\|\dot{x}\|^n |b-a|^n}{n!}, \quad n \in \mathbb{N}^*, \quad (a,b) \in \Delta_T^2.$$

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It relies on the following binomial inequality (Thm 3.1.1 of [23]): for $a, b \geq 0$,

$$\sum_{i=0}^{n} \frac{a^{i\alpha} b^{(n-i)\alpha}}{(i\alpha)!((n-i)\alpha)!} \leq \frac{1}{\alpha^2} \frac{(a+b)^{n\alpha}}{(n\alpha)!}. \tag{5.25}$$

Corollary 5.14. Let $0 < \alpha \leq 1$, $n \geq [1/\alpha]$ and $X = (1, X^{(1)}, \ldots, X^{(n)})$ be in $\mathcal{M}^{(n)}(V)$ such that

$$|X^{(i)}_{s,t}|_i \leq \frac{\alpha^2(1 - 2^{-\frac{\alpha}{[\alpha]}})C^i|t-s|^{\alpha i}}{(i\alpha)!} \quad (s,t) \in \Delta_T^2, \quad 1 \leq i \leq n. \tag{5.26}$$

Then, $(1, \ldots, X^{(n+1)})$ is a multiplicative functional and inequality (5.26) is true for $i = n+1$, where $X^{(n+1)} = \mathcal{E}^{(n+1)}(1, \ldots, X^{(n)}, 0)$.

If $\tilde{X} = (1, \tilde{X}^{(1)}, \ldots, \tilde{X}^{(n)})$ is a multiplicative functional fulfilling inequality (5.26) for $i = 1, \ldots, n$ and there exists $0 < \varepsilon < 1$ such that

$$|X^{(i)}_{s,t} - \tilde{X}^{(i)}_{s,t}|_i \leq \varepsilon \frac{\alpha^2(1 - 2^{-\frac{\alpha}{[\alpha]}})2^i C^i|t-s|^{\alpha i}}{2(i\alpha)!} \quad (s,t) \in \Delta_T^2, \quad 1 \leq i \leq n. \tag{5.27}$$

Then, inequality (5.27) is true for $i = n+1$.

Note that this Corollary allows to define a Lipschitz map from $(\mathcal{M}^{([1/\alpha])}(V), \|.\|_\alpha)$ into $(\mathcal{M}^{(n)}(V), \|.\|_\alpha)$ for all $n > [1/\alpha]$.

Proof of Corollary 5.14. Let $Y^{(n+1)}_{s,t} = 0, \quad (s,t) \in \Delta_T^2$. Then, $Y = (1, X^{(1)}, \ldots, X^{(n)}, 0)$ is an almost multiplicative functional. Indeed, using the definition of the product in $T^{(n+1)}(V)$,

$$Y^{(n+1)}_{a,b} - (Y_{a,c} \otimes Y_{c,b})^{(n+1)} = 0 - \sum_{k=1}^{n} X^{(k)}_{a,c} \otimes X^{(n+1-k)}_{c,b} + 0,$$

and using hypothesis on X (inequality (5.26)) we have

$$\left| Y^{(n+1)}_{a,b} - (Y_{a,c} \otimes Y_{c,b})^{(n+1)} \right|_{n+1} \leq (1 - 2^{-\frac{\alpha}{[\alpha]}})^2 \alpha^4 C^{n+1} \sum_{k=1}^{n} \frac{(c-a)^{k\alpha}(b-c)^{(n+1-k)\alpha}}{(k\alpha)!((n-k)\alpha)!}.$$

Then, from the binomial inequality (5.25), we obtain

$$\left| Y^{(n+1)}_{a,b} - (Y_{a,c} \otimes Y_{c,b})^{(n+1)} \right|_{n+1} \leq \alpha^2 (1 - 2^{-\frac{\alpha}{[\alpha]}})^2 C^{n+1} \frac{(b-a)^{(n+1)\alpha}}{((n+1)\alpha)!}.$$

The existence and uniqueness follow from Proposition 5.10, and

$$X^{(n+1)} := \mathcal{S}(\mu_Y^{(n+1)}) - \mu_Y^{(n+1)}.$$

Now, for $0 \leq a \leq c \leq b \leq T$

$$\delta\mu_Y^{(n+1)}(a,b,c) - \delta\mu_{\tilde{Y}}^{(n+1)}(a,b,c) = \sum_{k=1}^{n} \tilde{X}^{(k)}_{a,c} \otimes \tilde{X}^{(n+1-k)}_{c,b} - \sum_{k=1}^{n} X^{(k)}_{a,c} \otimes X^{(n+1-k)}_{c,b}$$

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and using inequality (5.27),

$$\left|\delta\mu_Y^{(n+1)}(a,c,b)-\delta\mu_{\tilde{Y}}^{(n+1)}(a,c,b)\right|_{n+1}\leq 2^n\varepsilon\alpha^2(1-2^{-\frac{\alpha}{[\alpha]}})^2C^{n+1}\frac{(b-a)^{(n+1)\alpha}}{((n+1)\alpha)!}.$$

By the sewing lemma, Lemma 4.1, and the fact that $X^{(n+1)}-\tilde{X}^{(n+1)})=\mathcal{S}(\mu_Y^{(n+1)}-\mu_{\tilde{Y}}^{(n+1)})-\mu_Y^{(n+1)}+\mu_{\tilde{Y}}^{(n+1)}$ we obtain

$$\left|X^{(n+1)}(a,b)-\tilde{X}^{(n+1)}(a,b)\right|_{n+1}\leq\varepsilon(1-2^{-\frac{\alpha}{[\alpha]}})\alpha^2C^{n+1}2^n\frac{|b-a|^{(n+1)\alpha}}{((n+1)\alpha)!}.$$ □

5.4.1. Concatenation of (almost) multiplicative functionals

Let $N\in\mathbb{N}^*$, $S_0=0$, $(S_i,\ \ i=1,\dots,N)\in[0,+\infty[^N$ and $T_i=\sum_{j=0}^i S_j$.
For $a\in]0,T_N[$, we define $i_{a-}:=\max\{i,\ \ T_i\leq a\}$, $i_{a+}:=\min\{i,\ \ a<T_i\}$, and $i_{0-}=i_{0+}=0$, $i_{T-}=I_{T+}=N$.

Lemma 5.15.

1. Let $X:\Delta^2_{T_N}\to T^{(n)}(V)$ be a multiplicative (resp. an almost multiplicative) functional and define
$$^iX_{a,b}=X_{a+T_i,b+T_i},\quad (a,b)\in\Delta^2_{S_{i+1}},\quad i=0,\dots,N-1.$$
Then, for $i=0,\dots,N-1$, $^iX:\Delta^2_{S_{i+1}}\to T^{(n)}(V)$ are multiplicative (resp. almost multiplicative) functionals.
2. On the converse, if for $i=0,\dots,N-1$, $^iX:\Delta^2_{S_{i+1}}\to T^{(n)}(V)$ are bounded multiplicative (resp. almost multiplicative) functionals, then $^0X\otimes\ldots\otimes\ ^{N-1}X$ given by for $(a,b)\in\Delta^2_{T_N}$
$$(^0X\otimes\ldots\otimes\ ^{N-1}X)_{a,b}=^{i_{a-}}X_{a-T_{i_{a-}};S_{i_{a-}+1}}\otimes^{i_{a+}}X_{0,S_{i_{a+}+1}}\otimes\ldots\otimes\ ^{i_{b-}-1}X_{0,S_{i_{b-}}}\otimes\ ^{i_{b-}}X_{0,b-S_{i_{b-}}},$$
is a multiplicative (resp. an almost multiplicative) functional.
3. Moreover, if for $i=0,\dots,N-1$, $^iX:\Delta^2_{S_{i+1}}\to T^{(n)}(V)$ are bounded almost multiplicative functionals, then $\mathcal{E}(^0X)\otimes\ldots\otimes\mathcal{E}(^{N-1}X)=\mathcal{E}(^0X\otimes\ldots\otimes\ ^{N-1}X)$.

Proof of Lemma 5.15. This proposition is proved recursevely on N and it is enough to give the proof for $N=2$.

1. Let $i\in\{0,1\}$ and $(a,c,b)\in\Delta^3_{S_{i+1}}$ then
$$\delta_{T^{(n)}(V)}\ ^iX(a,c,b)=\delta_{T^{(n)}(V)}X(a+T_i,c+T_i,b+T_i).$$
Then, ^iX has the same functional property as X.
2. Let $(a,c,b)\in\Delta^3_{T_2}$, then
$$\begin{aligned}\delta_{T^{(n)}(V)}(\ ^0X\otimes\ ^1X)(a,c,b)&=\delta_{T^{(n)}(V)}\ ^iX(a,c,b)\ \text{ if }\ (a-T_i,c-T_i,b-T_i)\in\Delta^3_{S_{i+1}},\\&=\delta_{T^{(n)}(V)}\ ^0X(a,c,S_1)\otimes\ ^1X_{0,b-S_1}\ \text{ if }\ (a,c)\in\Delta^2_{S_1},\quad b\in[S_1,T_2],\\&=\ ^0X_{a,S_1}\otimes\delta_{T^{(n)}(V)}\ ^1X_{0,c-S_1,b-S_1}\ \text{ elswhere.}\end{aligned}$$
Then $^0X\otimes\ ^1X$ has the same functional property as ^iX, $\ i=1,2$.
3. Note that
$$\begin{aligned}[\mathcal{E}(^0X)\otimes\mathcal{E}(^1X)]_{a,b}-(^0X\otimes\ ^1X)_{a,b}&=\mathcal{E}(^iX)_{a,b}-\ ^iX_{a,b}\ \text{ if }\ (a-S_i,b-S_i)\in\Delta^2_{S_{i+1}}\\&=\mathcal{E}(^0X_{a,S_1}-\ ^0X_{a,S_1})\otimes\mathcal{E}(^1X)_{0,b-S_1}\\&\quad+\ ^0X_{a,S_1}\otimes(\mathcal{E}(^1X)_{0,b-S_1}-\ ^1X_{0,b-S_1}),\quad\text{elswhere}\end{aligned}$$
and conclude with the uniqueness result in Theorem 5.11. □

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5.5. (Almost) multiplicative functionals on product spaces

In this section, we assume that $U = V \oplus W$, where V and W are two finite dimensional Banach spaces. Then, $Z : \Delta_T^2 \to T^{(2)}(V \oplus W)$ has the following decomposition

$$Z^{(1)} = (X^{(1)}, Y^{(1)}), \quad Z^{(2)} = \begin{pmatrix} X^{(2)} & \tilde{R} \\ R & Y^{(2)} \end{pmatrix}$$

where $X : \Delta_T^2 \to T^{(2)}(V)$, $Y : \Delta_T^2 \to T^{(2)}(W)$, $R : \Delta_T^2 \to W \otimes V$ and $\tilde{R} : \Delta_T^2 \to V \otimes W$.

A map $Z : \Delta_T^2 \to T^{(2)}(V \oplus W)$ is a multiplicative functional if and only if X and Y are two multiplicative functionals and R, $\tilde{R}$, fulfill

$$\begin{aligned} R_{s,t} &= R_{s,u} + R_{u,t} + Y_{s,u}^{(1)} \otimes X_{u,t}^{(1)}, \quad \forall (s,u,t) \in \Delta_T^3, \\ \tilde{R}_{s,t} &= \tilde{R}_{s,u} + \tilde{R}_{u,t} + X_{s,u}^{(1)} \otimes Y_{u,t}^{(1)}. \end{aligned} \tag{5.28}$$

A map $Z : \Delta_T^2 \to T^{(2)}(V \oplus W)$ belongs to $\mathcal{A}m_{C,\varepsilon}(U \oplus W)$ if and only if X and Y are two almost multiplicative functionals associated to C, ε , and there exists two constants $C > 0, \varepsilon > 0$ such that R, $\tilde{R}$ fulfill $\forall (s,u,t) \in \Delta_T^3$,

$$\begin{aligned} \left| R_{s,t} - R_{s,u} - R_{u,t} - Y_{s,u}^{(1)} \otimes X_{u,t}^{(1)} \right| &\leq C|t-s|^{1+\varepsilon} \\ \left| \tilde{R}_{s,t} - \tilde{R}_{s,u} - \tilde{R}_{u,t} - X_{s,u}^{(1)} \otimes Y_{u,t}^{(1)} \right| &\leq C|t-s|^{1+\varepsilon}. \end{aligned} \tag{5.29}$$

This leads us to introduce for $X \in \mathcal{M}^{(2)}(V)$, the following restriction of the spaces $\mathcal{M}(V \oplus W)$, $\mathcal{A}m(V \oplus W)$, and operator $\delta_{T^{(2)}(V \oplus W)}$:

$$\begin{aligned} \mathcal{C}_{X,T} &:= \{Y = (Y_W^{(1)}, Y_{W \otimes V}^{(2)}) : \Delta_T^2 \to W \oplus (W \otimes V), \quad Y_V^{(1)} \text{ is additive}, \quad Y_{W \otimes V}^{(2)} \quad \text{fulfills (5.28)}\}, \\ \mathcal{AC}_{X,C,\varepsilon,T} &:= \{(Y_W^{(1)}, Y_{W \otimes V}^{(2)}) : \Delta_T^2 \to W \oplus (W \otimes V), \quad Y^{(1)} \in \mathcal{A}a_C, \quad Y_{W \otimes V}^{(2)} \quad \text{fulfills (5.29)}\}. \end{aligned}$$

Let us denote for $Y = (Y_W^{(1)}, Y_{W \otimes V}^{(2)}) : \Delta_T^2 \to W \oplus (W \otimes V)$, for all $(a,c,b) \in \Delta_T^3$,

$$\begin{aligned} \delta_{\mathcal{C}_X}^{(1)} Y(a,c,b) &:= [Y_{W \otimes V}^{(1)}(a,b) - Y_{W \otimes V}^{(1)}(a,c) - Y_{W \otimes V}^{(1)}(c,b)] \\ \delta_{\mathcal{C}_X}^{(2)} Y(a,c,b) &:= [Y_{W \otimes V}^{(2)}(a,b) - Y_{W \otimes V}^{(2)}(a,c) - Y_{W \otimes V}^{(2)}(c,b)] - Y_W^{(1)}(a,c) \otimes X_{c,b}^{(1)}. \end{aligned}$$

With the same lines as Theorems 5.11 and Corollary 5.12, on can prove

Theorem 5.16. Let $\hat{Y} \in \mathcal{AC}_{X,C,\varepsilon}$. There exists a unique $Y \in \mathcal{C}_X$ such that

$$|Y_{s,t}^{(i)} - \hat{Y}_{s,t}^{(1)}| + |Y_{s,t}^{(2)} - \hat{Y}_{s,t}^{(2)}| \leq Cst(C, \alpha, \varepsilon, T, N_{\alpha,0,T}(\hat{Y}))|t-s|^{1+\varepsilon}, \quad \forall i = 1, \ldots, [1/\alpha], \quad \forall (s,t) \in \Delta_T^2, \tag{5.30}$$

where for $\alpha \in]1/3, 1/2]$ $Cste(C, \alpha, \varepsilon, T) = C[1 + 2N_{\alpha,0,T}(\hat{Y})T^{\alpha}\theta(\varepsilon) + \theta(\varepsilon)^2 CT^{1+\varepsilon}]\theta(\varepsilon)$.

Proof of Theorem 5.16. We omitt the subscripts W and $W \otimes V$ in $\hat{Y}$ and Y.
Firstly, let $\mu_{X,\hat{Y}}^1$ be defined by $\mu_{X,\hat{Y}}^1(a,b) = \hat{Y}^{(1)}(a,b), \quad (a,b) \in \Delta_T^2$. Then, $\mu_{X,\hat{Y}}^1$ is almost additive and set $Y^{(1)} := \mathcal{S}(\mu_{X,\hat{Y}})$.

Secondly, let $\mu_{X,\hat{Y}}^2$ be defined by $\mu_{X,\hat{Y}}^2(a,b) = \hat{Y}^{(2)}(a,b) + Y_{0,a}^{(1)} \otimes X_{a,b}, \quad (a,b) \in \Delta_T^2$. Then, $\mu_{X,\hat{Y}}^2$ is almost additive and set $Y_{a,b}^{(2)} := \mathcal{S}(\mu_{X,\hat{Y}})(a,b) - Y_{0,a}^{(1)} \otimes X_{a,b}, \quad (a,b) \in \Delta_T^2$.

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If $Y := (Y^{(1)}, Y^{(2)})$, the first component of the functional Y is additive and for all $(a,c,b) \in \Delta_T^3$,

$$\begin{aligned}\delta^{(2)}_{\mathcal{C}_X} Y(a,c,b) &= \delta\mathcal{S}(\mu^2_{X,\hat{Y}})(a,c,b) - Y^{(1)}_{0,a} \otimes X_{a,b} + Y^{(1)}_{0,a} \otimes X_{a,c} + Y^{(1)}_{0,c} \otimes X_{c,b} - Y^{(1)}_{a,c} \otimes X^{(1)}_{c,b} \\ &= 0.\end{aligned}$$

That means Y belongs to $\mathcal{C}_X$. Inequality (5.30) is a consequence of the Sewing lemma. □

For all $\hat{Y} \in \mathcal{AC}_{X,C,\varepsilon}$ let us denote by $\mathcal{E}(\hat{Y}) := Y$ where Y is given in Theorem 5.16.

We also introduce, for $Y : \Delta_T^2 \to W \oplus (W \otimes V)$, $N_{\alpha,0,T}(Y) := \|Y^{(1)}_W\|_{\alpha,0,T} + \|Y^{(2)}_{W\otimes V}\|_{2\alpha,0,T}$, and the spaces of functionals

$$\begin{aligned}\mathcal{C}^{\alpha}_{X,T} &:= \{Y \in \mathcal{C}_{X,T}, \quad N_{\alpha,0,T}(Y) < +\infty\}, \\ \mathcal{AC}^{\alpha}_{X,C,\varepsilon,T} &:= \{Y \in \mathcal{AC}_{X,T}, \quad N_{\alpha,0,T}(Y) < +\infty\};\end{aligned}$$

Corollary 5.17. The map $\mathcal{E}$ is continuous from $\mathcal{AC}^{\alpha}_{X,C,\varepsilon,T}$ into $\mathcal{C}^{\alpha}_{X,T}$ endowed with $N_{\alpha,0,T}$.

The map $\mathcal{E}$ is also differentiable.

Corollary 5.18. Let $(\hat{Y}(\theta;.,.))_{\theta\in[0,1]} \in \mathcal{AC}^{\alpha}_{X,C,\varepsilon,T}$ be continuously differentiable with respect to θ with derivative $d_\theta\hat{Y}(\theta;.,.))_{\theta\in[0,1]} \in \mathcal{AC}^{\alpha}_{X,C,\varepsilon,T}$ starting from (0,0). Then, $(\mathcal{E}(\hat{Y}(\theta;.,.))_{\theta\in[0,1])}$ is differentiable with respect to θ and

$$d_\theta\mathcal{E}(\hat{Y}(\theta;.,.)) = \mathcal{E}(d_\theta\hat{Y}(\theta;.,.)).$$

Moreover,

$$\begin{aligned}\|d_\theta\mathcal{E}(\hat{Y})\|_{\alpha,0,T} &\le Cste(\alpha,\varepsilon,T)T^{2\alpha} + \|\hat{Y}^{(1)}_W\|_{\alpha,0,T}, \\ \|\Pi_{W\otimes V}(d_\theta\mathcal{E}(\hat{Y})\|_{2\alpha,0,T} &\le Cste(\alpha,\varepsilon,T)T^{\alpha} + \|\hat{Y}^{(2)}_{W\otimes V}\|_{2\alpha,0,T},\end{aligned} \tag{5.31}$$

where for $\alpha \in]1/3, 1/2]$ $Cste(\alpha,\varepsilon,T) = [1 + 2N_{\alpha,0,T}(\hat{Y})T^{\alpha}\theta(\varepsilon) + \theta(\varepsilon)^2 CT^{1+\varepsilon}]\theta(\varepsilon)$.

Proof of Corollary 5.18. We omitt the subscripts W and $W \otimes V$ in $\hat{Y}$ and Y.

- For the first level, Corollary 5.18 is a consequence of Lemma 4.4. Indeed, let $\mu^1_{X,\hat{Y}_\theta} = d_\theta\hat{Y}^{(1)}_\theta$, then it fulfills assumption of Lemma 4.4, and

$$\mathcal{S}\left(\int_0^{\Theta} \mu^1_{X,d_\theta\hat{Y}_\theta}(.,.)\mathrm{d}\theta\right) = \int_0^{\Theta} \mathcal{S}(\mu^1_{X,d_\theta\hat{Y}_\theta}(.,.))\mathrm{d}\theta;$$

  and then

$$\Pi_W[\mathcal{E}(\hat{Y}_\Theta)(.,.)] = \int_0^{\Theta} \Pi_W[\mathcal{E}(d_\theta\hat{Y}_\theta(.,.))]\mathrm{d}\theta.$$

- For the second level, we introduce for $(a,b) \in \Delta_T^2$,

$$\mu^2_{X,d_\theta\hat{Y}_\theta}(a,b) = d_\theta\hat{Y}^{(2)}_\theta(a,b) + \Pi_W[\mathcal{E}(d_\theta\hat{Y}_\theta(.,.))]_{0,a} \otimes X^{(1)}_{a,b}$$

  which fulfills assumption of Lemma 4.4, and

$$\mathcal{S}\left(\int_0^{\Theta} \mu^2_{X,d_\theta\hat{Y}_\theta}(.,.)\mathrm{d}\theta\right) = \int_0^{\Theta} \mathcal{S}(\mu^2_{X,d_\theta\hat{Y}_\theta}(.,.))\mathrm{d}\theta.$$

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But, we identify

$$\mathcal{S}(\mu^2_{X,d_\theta\hat{Y}_\theta})_{a,b} = \Pi_{W\otimes V}\mathcal{E}(d_\theta\hat{Y}_\theta)_{a,b} + \Pi_W\mathcal{E}(d_\theta\hat{Y}_\theta)_{0,a}\otimes X^{(1)}_{a,b};$$
$$\int_0^\Theta \mu^2_{X,d_\theta\hat{Y}_\theta}(.,.)\mathrm{d}\theta = \mu^2_{X,Y_\Theta}.$$

and then

$$\Pi_{W\otimes V}[\mathcal{E}(\hat{Y}_\Theta)(.,.)] = \int_0^\Theta \Pi_{W\otimes V}[\mathcal{E}(d_\theta\hat{Y}_\theta(.,.))]\mathrm{d}\theta.$$

- Inequalities (5.31) are obtained by integrating each terms of (5.30) with respect to θ. □

6. Young integrals

In this section, we first define and study an extension of the Rieman integral: the Young integral. The definition of this integral was achieved for the first time by Young in 1936 [31] extended by Bertoin for processes with p finite variation in [2]. It was aslo studied by Ciesielski *et al.* [6] using properties of Besov-Orlicz type space, or by Zähle in [32] using Liouville integral (see [27] for a definition of Liouville integral). The proof given here is based on the sewing lemma and is short.

The second purpose of this section is to define and solve integral equations like

$$y^j(t) = y_0^j + \sum_{i=1}^m \int_0^t V_j^i(y(s))\mathrm{d}x^i(s), \quad t\in[0,T], \quad j\in\{1,\dots,d\}; \tag{6.32}$$

where $m,d\in\mathbb{N}^*$, $x:[0,1]\to\mathbb{R}^m$ is "regular enough" and $V_j^i:\mathbb{R}^d\to\mathbb{R}$ is "regular enough" for $(i,j)\in\{1,\dots,m\}\times\{1,\dots,d\}$, $T>0$; and $\int_0^t V_j^i(y(s))\mathrm{d}x^j(s)$ is defined as a Young integral. Here, we follow the proof of Lyons [24], or Lejay, [19].

Existence and uniqueness of solution of (6.32) was obtained by Ciesielski, Kerkyacharian and Roynette , in [6] when x is a fractional Brownian motion with Hurst parameter greater than 1/2. In 1994, using the Young integral and the norm of p variation, Lyons, in [21] prove the same result when x is with finite p variation with $p<2$. The sharpness of the conditions on V_i have been deeply studied by Davie in [9]. These results can be also find in [19, 24]. Zähle, in [32] has a similar result in the frame work of Liouville integrals. They are well described by Nualart and Rascanu in [26].

Lyons and Qian obtained some results on the flow of solutions, see [23].

For the particular case $d=m=1$, Doss [11] and Susmann [29] proved that when x is a Brownian motion the solution of (6.32) is given by solving an ordinary differential equation and using a change of time in Brownian motion. The same ideas are used by Nourdin [25] for fractional Brownian motion with any Hurst parameter.

The framework using the distance in finite p variation is well adapted to the construction of integral, apply ies to a large field of processes, *e.g.* in Bertoin [2]. But for simplicity, we work in the framework of Hölder continuous functions.

6.1. Young integrals *via* sewing lemma

The results contained in this section are well known and their proof is simplify by using sewing lemma.

Let x and y be real continuous functions on $[0,T]$. For $(a,b)\in\Delta_T^2$, and $\mathcal{D}=\{t_1,\dots,t_k\}$ a subdivion of $[a,b]$, the associated Riemann sum is given by

$$J_{\mathcal{D}}(a,b,x,y) = \sum_{i=1}^{k-1} x(t_i)\left[y(t_{i+1})-y(t_i)\right].$$

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Proposition 6.1 (Young [31])**.** Let x and y be respectively α and β Hölder continuous functions on $[0,T]$ with $\alpha+\beta>1$. Let $(a,b)\in\Delta_T^2$. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[a,b]$, such that the sequence of their mesh $(|\mathcal{D}_n|)_n$ converges to 0. Then the sequence $(J_{\mathcal{D}_n}(a,b,x,y))_n$ converges, the limit is independant of the sequence of subdivisions and is denoted by $J(a,b,x,y)=\int_a^b x(s)\mathrm{d}y(s)$. Moreover, for $(a,b)\in\Delta_T^2$ and $n\in\mathbb{N}$,

$$\left|\int_a^b x(s)\mathrm{d}y(s)\right| \leq \left[\|x\|_{\infty,0,T}+\theta(\alpha+\beta-1)T^\alpha\|x\|_{\alpha,0,T}\right]\|y\|_{\beta,0,T}|b-a|^\beta; \tag{6.33}$$

$$\left|\int_a^b x(s)\mathrm{d}y(s)-J_{\mathcal{D}_n}(x,y,a,b)\right| \leq \|x\|_{\alpha,0,T}\|y\|_{\beta,0,T}|\mathcal{D}_n|^{\alpha+\beta-1}(b-a). \tag{6.34}$$

Proof of Proposition 6.1. We do not follow the original proof of [31] but the proof given in [12].
If $\mu(a,b)=x(a)[y(b)-y(a)]$ for $(a,b)\in\Delta_T^2$ then for $(a,c,b)\in\Delta_T^3$

$$|\delta\mu(a,c,b)|=|-[x(c)-x(a)][y(b)-y(c)]|\leq\|x\|_{\alpha,0,T}\|y\|_{\beta,0,T}|b-a|^{\alpha+\beta}.$$

Since $\alpha+\beta>1$, μ is almost additive and the Lemma 4.1 yields a functionnal $\mathcal{S}(\mu)$ such that

$$|\mathcal{S}(\mu)(a,b)-\mu(a,b)|\leq\|x\|_{\alpha,0,T}\|y\|_{\beta,0,T}\theta(\alpha+\beta-1)|b-a|^{\alpha+\beta},\quad\forall(a,b)\in\Delta_T^2.$$

Then, from Corollary 4.3, the sequence $(J_{\mathcal{D}_n}(a,b,x,y))_n$ converges, the limit is independant of the choosen sequence of partitions, and fulfills inequality (6.34). Inequality (6.33) is a consequence of $|\mu(a,b)|\leq\|x\|_{\infty,0,T}\|y\|_{\beta,0,T}|b-a|^\beta$, $\forall(a,b)\in\Delta_T^2$. □

The Young integral is an extension of the Stieljes integral.

Corollary 6.2. Let x and y be two real functions piecewise continuously differentiable then the associated Young integral coincides with the Riemann Stieltjes integral.

Proof of Corollary 6.2. If $\psi(a)=\int_0^a x(u)\mathrm{d}y(u)$ denotes the Riemann Stieljes integral, then for all $(a,b)\in\Delta_T^2$,

$$\begin{aligned}|\psi(b)-\psi(a)-x(a)[y(b)-y(a)]| &\leq \left|\int_a^b[x(s)-x(a)]\mathrm{d}y(s)\right| \\ &\leq \frac{1}{2}\|x^\cdot\|_\infty\|y^\cdot\|_\infty|b-a|^2,\end{aligned} \tag{6.35}$$

where $x^\cdot$ is the derivative of x. Since $\mathcal{S}(\mu)$ is the unique additive functionnal fulfilling (6.35), $\psi=\mathcal{S}(\mu)(0,.)$. □

Using the fact that the functional $\int x(s)\mathrm{d}y(s)$ does not depend on the sequence of subdivisions we derive the following change of variable.

Corollary 6.3. Let x and y be an α and β Hölder continuous functions on $[0,T]$ with $\alpha+\beta>1$. Let f be a increasing, γ Hölder continuous function from $[0,T]$ into $[0,T]$, with $(\alpha+\beta)\gamma>1$, then for $0\leq a\leq b\leq T$

$$\int_a^b(x\circ f)(u)\mathrm{d}(y\circ f)(u)=\int_{f(a)}^{f(b)}x(u)\mathrm{d}y(u).$$

Proof of Corollary 6.3. Since f is a γ Hölder continuous function, then $x\circ f$ is $\alpha\gamma$ Hölder continuous and $y\circ f$ is $\gamma\beta$ Hölder continuous. Moreover, if $(\mathcal{D}_n)_n$, where $\mathcal{D}_n=\{t_1^n,\ldots,t_{k_n}^n\}$, is a sequence of subdivisions of $[a,b]$

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such that $(|\mathcal{D}_n|)_n$ converges to 0, then for $n \in \mathbb{N}$, $\tilde{\mathcal{D}}_n = \{f(t_1^n), \dots, f(t_{k_n}^n)\}$ define a sequence of subdivisions of $[f(a), f(b)]$ such that $|\tilde{\mathcal{D}}_n| \leq |\mathcal{D}_n|^\gamma$. According to Proposition 6.1,

$$\begin{aligned} J(a,b,x\circ f, y\circ f) &= \lim_{n\to\infty} \sum_{i=1}^{k_n-1} x\circ f(t_i^n)[y\circ f(t_{i+1}^n) - y\circ f(t_i^n)] \\ &= \lim_{n\to\infty} J_{\tilde{\mathcal{D}}_n}(f(a), f(b), x, y) \\ &= J(f(a), f(b), x, y). \end{aligned}$$

□

As a consequence of the uniqueness in the Lemma 4.1, we obtain

Corollary 6.4. Let f, g, and h be real, α Hölder continous functions on $[0,T]$ and denote by φ the function given by

$$\varphi(t) = \int_0^t g(u)\mathrm{d}h(u), \quad t \in [0,T].$$

Then,

$$\int_a^b f(u)d\varphi(u) = \int_a^b f(u)g(u)\mathrm{d}h(u), \quad \forall (a,b) \in \Delta_T^2.$$

The following change of variable formula is a consequence of uniqueness in Lemma 4.1.

Corollary 6.5. Let $x = (x_i)_{i=1}^d \in C^\alpha([0,T], \mathbb{R}^d)$ for $\alpha > 1/2$ and $\Psi : \mathbb{R}^d \to \mathbb{R}^m$ be differentiable, with bounded differential $d\Psi$, γ Hölder continuous with $(\gamma+1)\alpha > 1$. Then, for all $t \in [0,T]$,

$$\Psi(x(t)) = \Psi(x(0)) + \int_0^t \sum_{i=1}^d d_i\Psi(x(u))\mathrm{d}x^i(u),$$

where $d_i\Psi$ is the partial derivative of Ψ with respect to x_i.

Proof of Corollary 6.5. Let us introduce $\mu(a,b) = \sum_{i=1}^d d_i\Psi(x(a))[x^i(b) - x^i(a)]$ for $(a,b) \in \Delta_T^2$. Then μ is a continuous function on Δ_T^2 such that

$$|\delta\mu(a,b,c)| \leq d \max_{i=1}^d \|d_i\Psi\|_\gamma \|x\|_{\alpha,0,T}^{\gamma+1}(b-a)^{\alpha(\gamma+1)}, \quad \forall (a,b,c) \in \Delta_T^3,$$

and using Taylor expansion of Ψ between $x(a)$ and $x(b)$

$$\begin{aligned} |\Psi(x(b)) - \Psi(x(a)) - \mu(a,b)| &\leq \sum_{i=1}^d \max_{\theta\in[0,1]} |d_i\Psi(x(a) + \theta[x(b) - x(a)]) - d_i\Psi(x(a))|\,|x(b) - x(a)|\,, \\ &\leq d \max_{i=1}^d \|d_i\Psi\|_\gamma \|x\|_{\alpha,0,T}^{\gamma+1}(b-a)^{\alpha(\gamma+1)}. \end{aligned}$$

The conclusion follows from uniqueness in Lemma 4.1. □

Let $x = (x_i)_{i=1}^d \in C^\alpha([0,T], \mathbb{R}^d)$ for $\alpha > 1/2$, then the iterated integrals are well defined and from Corollary 5.14 we obtain

Corollary 6.6. Let $x = (x_i)_{i=1}^d \in C^\alpha([0,T], \mathbb{R}^d)$ for $\alpha > 1/2$, then

$$\left| \int_{a\leq u_1 \leq \dots \leq u_n \leq b} \mathrm{d}x(u_1) \otimes \dots \otimes \mathrm{d}x(u_n) \right|_n \leq \frac{\|x\|_{\alpha,0,T}^n}{\alpha^{2(n-1)}(1 - 2^{-\frac{\alpha}{[\alpha]}})^{n-1}} \frac{|b-a|^{n\alpha}}{(\alpha n)!} \quad (a,b) \in \Delta_T^2.$$

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6.2. Differential equations driven by α Hölder continuous function with $\alpha > 1/2$.

We follow the proof of Lyons [21], in the simpler framework of α Hölder continuous functions, using simplifications given by Lemma 4.1. This approach is also followed by Lejay in [20].

First a definition of solution of differential equations is given. Second, a result of existence of solution is given. Uniqueness, convergence of Picard iteration and the regularity of the flow of solutions are stated under a stronger hypothesis on vector fields.

6.2.1. Definition

Suppose x is a α Hölder continuous path from $[0,T]$ in $\mathbb{R}^m$ with $\alpha > 1/2$ and let $(V^i)_{i=1}^m$ be functions from $\mathbb{R}^d$ into $\mathbb{R}^d$, γ Hölder continuous, with $\gamma > \frac{1}{\alpha} - 1$ and $a \in \mathbb{R}^d$.

Given a path z_0 in $\mathbb{R}^d$, α Hölder continuous, from Proposition 6.1, one may form a second path $z_1 = P_V^{T,a,x} z_0$, α Hölder continuous where

$$P_V^{T,a,x} z_0(t) := z_1(t) = a + \sum_{i=1}^m \int_0^t V^i(z_0(u)) \mathrm{d}x^i(u), \quad t \in [0,T].$$

We say we perform Picard iteration if we continue the process:

$$z_{n+1}(t) = a + \sum_{i=1}^m \int_0^t V^i(z_n(u)) \mathrm{d}x^i(u), \quad t \in [0,T].$$

Definition 6.1. We say that z_0 is a solution to the equation

$$\mathrm{d}z(t) = \sum_{i=1}^m V^i(z(u)) \mathrm{d}x^i(u), \quad z(0) = a, \quad t \in [0,T] \tag{6.36}$$

if and only if $z_1 = z_0$, or equivalently if z_0 is a fixed point of $P_V^{T,a,x}$.

6.2.2. Existence

In this section, we prove the results of [21] using almost the same proof.

Proposition 6.7. [21] Suppose $x \in C^\alpha([0,T],\mathbb{R}^m)$ with $\alpha > 1/2$ and let $(V^i)_{i=1}^m$ be γ Hölder functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\gamma > \frac{1}{\alpha} - 1$ and a be in $\mathbb{R}^d$.

Then, the differential equation (6.36) has a solution, z, in $C^\alpha([0,T],\mathbb{R}^m)$.

Remark 6.1. Davie, in [10], has shown examples of multiple solutions to equation (6.36) when $\gamma > \frac{1}{\alpha}$.

Remark 6.2 (Quoting [21])**.** "Altough it is quite standard to make an identification between solutions to differential equations and solutions to the associated integral equations, a little reflection should convince the reader that the approach makes an implicit assumption requiring verification. In equation (6.36), the path x belongs to a vector space with a prescribe coordinate chart, however each V^i is a vector field defined unambiguously without reference to a particular choice of coordinate chart; we therefore expect z to be a path on a manifold and to be independent of the choice of coordinates. However, the representation of z as the solution to an integral equation only make sense after having made a choice of coordinate chart. One should therefore look at the independence of the solution under changes in coordinates. Fortunately, it is easy to show that any solutions defined through the Youngs integrals are invariant under $(1+\alpha)$ Lipschitz changes of coordinates". This can be proved using Corollary 6.5.

Proof of Proposition 6.7. The proof is split in to three parts.

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- First, for $0 < \varepsilon < \frac{(\gamma+1)\alpha-1}{\gamma+1}$, there exists $T_1(a) > 0$ such that $P_V^{T_1(a),a,x}$ is a compact map from

$$\{z \in C([0,T_1(a)],\mathbb{R}^d), \quad z(0) = a, \quad \|z\|_{\alpha-\varepsilon,0,T_1(a)} \leq 1\}$$

into itself. Using a fix point argument, we prove that the integral equation (6.36) has a solution up to the time $T_1(a)$ in $C^{\alpha-\varepsilon}([0,T_1(a)],\mathbb{R}^d)$, which is α Hölder continuous.
- We prove an *a priori* estimation on the α Hölder norm of the solution.
- We construct a kind of "maximal solution" using concatenation of local solutions in the set α Hölder continuous functions.

1. Let $z_0 \in C^{\alpha-\varepsilon}([0,T],\mathbb{R}^d)$ be such that $z_0(0) = a$, then $V^i(z_0) \in C^{(\alpha-\varepsilon)\gamma}([0,T],\mathbb{R}^d)$ and for $i = 1,\dots,m$,

$$\|V^i(z_0)\|_{(\alpha-\varepsilon)\gamma,0,T} \leq \|V^i\|_\gamma \|z_0\|^\gamma_{\alpha-\varepsilon,0,T},$$
$$\|V^i(z_0)\|_{\infty,0,T} \leq \|V^i(a)\| + \|V\|_\gamma \|z_0\|^\gamma_{\alpha-\varepsilon,0,T} T^{\gamma(\alpha-\varepsilon)}.$$

According to Proposition 6.1, inequality (6.33), since $(\alpha-\varepsilon)\gamma + \alpha > 1$, z_1 is α, and also $\alpha-\varepsilon$ Hölder continuous with Hölder norm controlled by

$$\|z_1\|_{\alpha-\varepsilon,0,T} \leq T^\varepsilon \|x\|_{\alpha,0,T} \left[\{1+\theta((\alpha-\varepsilon)(\gamma+1)-1)\} T^{\gamma(\alpha-\varepsilon)} \|V\|_\gamma \|z_0\|^\gamma_{\alpha-\varepsilon,0,T}] + \sum_{i=1}^m \|V^i(a)\|\right].$$

Let $T_1(a)$ be defined by

$$T_1(a) = \left\{ \|x\|_{\alpha,0,T} \left[\{1+\theta((\alpha-\varepsilon)(\gamma+1)-1)\} \|V\|_\gamma T^{\gamma(\alpha-\varepsilon)} + \sum_{i=1}^m \|V^i(a)\|\right]\right\}^{-1/\varepsilon}, \tag{6.37}$$

then $P_V^{T_1(a),a,x}$ is a continuous operator from $\mathbb{C}$ into itself where

$$\mathbb{C} = \{z \in C^{\alpha-\varepsilon}([0,T_1(a)],\mathbb{R}), \quad z(0) = a, \quad \|z\|_{\alpha-\varepsilon,0,T_1(a)} \leq 1\}.$$

The set $\mathbb{C}$ is a compact, convex subset of $C^{\alpha-\varepsilon}([0,T_1(a)],\mathbb{R})$ endowed with the norm $N_{\alpha-\varepsilon,\infty,0,T}$. According to fix point theorem of Tychonoff's [30], $P_V^{T_1(a),a,x}$ has a fix point denoted by z^0 in $\mathbb{C}$. From Proposition 6.1, z^0 is also α Hölder continuous.

2. We prove now an estimation on the α Hölder norm of solution.

Proposition 6.8. Suppose $x \in C^\alpha([0,T],\mathbb{R}^m)$ with $\alpha > 1/2$ and let $(V^i)_{i=1}^m$ be γ Hölder functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\gamma > \frac{1}{\alpha} - 1$ and $a \in \mathbb{R}^d$. Let $S \in (0,T]$ such that there exits a fix point of $P_V^{S,a,x}$ denoted by z. Let

$$\Delta(S) = 1 \wedge \frac{1}{2[\|x\|_{\alpha,0,T}\{1+\theta(\alpha(\gamma+1)-1)\}\|V\|_\gamma]^{\frac{1}{\alpha\gamma}}}.$$

Then, for any $0 < \Delta < \Delta(S)$,

$$\|z\|_{\alpha,0,S} \leq \frac{S}{\Delta(S)} \times \exp\left(\frac{S}{\Delta(S)}\right) \times \left(1 + 2\|x\|_{\alpha,0,T}\|V(a)\| + \frac{e^2}{e-1}\right).$$

First, using change of variable Corollary 6.3, we prove the following lemma.

Lemma 6.9. Suppose $x \in C^\alpha([0,T],\mathbb{R}^m)$ with $\alpha > 1/2$ and let $(V^i)_{i=1}^m$ be γ Hölder functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\gamma > \frac{1}{\alpha} - 1$ and $a \in \mathbb{R}^d$. Let $S \in (0,T]$ such that there exits a fix point of $P_V^{S,a,x}$ denoted

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by z. Let $S' < S$ and $\tilde{x}$ and $\tilde{z}$ be the shift of x and z by S' :

$$\tilde{x}(t) := x(t+S'), \quad t \in [0, S-S']$$
$$\tilde{z}(t) := z(t+S').$$

Then, $\tilde{z}$ is a fix point of $P_V^{S-S', z(S'), \tilde{x}}$.

Proof of Lemma 6.9. For $t \in [0, S-S']$, the integral is additive and

$$\begin{aligned}\tilde{z}(t) &= a + \int_0^{t+S'} \sum_{i=1}^m V^i(z(u)) \mathrm{d}x^i(u), \\ &= z(S') + \int_{S'}^{t+S'} \sum_{i=1}^m V^i(z(u)) \mathrm{d}x^i(u).\end{aligned}$$

We perform the change of variable $u \mapsto u - S'$ and used the fact that the value of integral with respect to x does not depend of the initial value of x. Then,

$$\tilde{z}(t) = z(S') + \int_0^t \sum_{i=1}^m V^i(\tilde{z}(u)) \mathrm{d}\tilde{x}^i(u), \quad t \in [0, S-S'].$$

□

Proof of Proposition 6.8. Let $S > 0$, z be a fix point of $P_V^{S,a,x}$ and $n \in \mathbb{N}$ such that $n < \frac{S}{\Delta}$. According to Lemma 6.9, the restriction of z to $[n\Delta, (n+1)\Delta \wedge S]$ is a fix point of $P_V^{\Delta, z(n\Delta), \tilde{x}}$ (here $\tilde{x}(t) = x(t-n\Delta) - x(n\Delta)$, $t \in [0, \Delta]$) and from Proposition 6.1 and the definition of $\Delta(S)$ (using $u^\gamma \leq 1+u$ since $\gamma < 1$.) we obtain

$$\begin{aligned}\|z\|_{\alpha, n\Delta, (n+1)\Delta} &\leq \|x\|_{\alpha,0,T} \left[\{1 + \theta(\alpha(\gamma+1) - 1)\} \|V\|_\gamma \Delta^{\alpha\gamma} \|z\|^\gamma_{\alpha, n\Delta, (n+1)\Delta} + \|V(z(n\Delta))\| \right], \\ &\leq \frac{1}{2} \|z\|_{\alpha, n\Delta, (n+1)\Delta} + \|x\|_{\alpha,0,T} \|V(z(n\Delta))\|.\end{aligned}$$

We note that

$$\|V(z(n\Delta))\| \leq \|V(a)\| + \|V\|_\gamma \Delta^{\alpha\gamma} \sum_{i=0}^{n-1} \|z\|^\gamma_{\alpha, i\Delta, (i+1)\Delta}.$$

Using the definition of $\Delta(S)$ we obtain

$$\|z\|_{\alpha, n\Delta, (n+1)\Delta} \leq 2\|x\|_{\alpha,0,T} \|V(a)\| + \sum_{i=0}^{n-1} \|z\|_{\alpha, i\Delta, (i+1)\Delta}.$$

Using the discret Gronwall lemma, 6.10, we achieve the proof of Proposition 6.8. □

Lemma 6.10. Let $(u)_n$, $(f_n)_n$ and $(g_n)_n$ be non negative sequences such that

$$u_{n+1} \leq f_{n+1} + \sum_{k=0}^n g_k u_k, \quad \forall n \in \mathbb{N}$$

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then

$$u_{n+1} \leq \exp\left(\sum_{i=0}^{n} g_i\right) \times u_0 + \sum_{k=0}^{n} (f_{k+1} - f_k) \times \exp\left(\sum_{i=k+1}^{n} g_i\right), \quad \forall n \in \mathbb{N}.$$

Proof of Lemma 6.10. Let us define $S_0 = u_0$ and $S_n = f_n + \sum_{k=0}^{n-1} g_k u_k, \quad n \in \mathbb{N}^*$ then

$$S_{n+1} - S_n \leq f_{n+1} - f_n + g_n S_n.$$

By iteration on n, we obtain

$$S_{n+1} \leq \Pi_{i=0}^{n}(g_i + 1) \times S_0 + \sum_{k=0}^{n} \left(\Pi_{i=k+1}^{n}(g_i + 1)\right)(f_{k+1} - f_k), \quad n \in \mathbb{N}.$$

We conclude by using $(1 + u) \leq \exp u, \quad \forall u \geq 0$. □

3. Let us denote $T_0 = \min(\Delta(T), T_1(b), \quad b \in B(a, M))$, where $\Delta(T)$ and $M = \frac{T}{\Delta(T)} \times \exp\left(\frac{T}{\Delta(T)}\right) \times \left(1 + 2\|x\|_{\alpha,0,T}\|V(a)\| + \frac{e^2}{e-1}\right) T^\alpha$ are given in Proposition 6.8. Let us denote by $a^0 = a$, $T_0 = T_1(a)$, $\tilde{x}_0 = x$. We define by iteration $(\tilde{x}_n, z^n, a_n)_n$ where for all $n \in \{1, \ldots, \frac{T}{T_0} + 1\}$
   - $\tilde{x}_n := x(. + T \wedge nT_0) - x(T \wedge nT_0)$,
   - $a_n := z^{n-1}(T \wedge (n-1)T_0)$,
   - z^n is a fix point of $P_V^{T_0, a_n, \tilde{x}_n}$.

   Now z is the function defined by

$$z(t) := z^n(t - T \wedge nT_0), \quad t \in [T \wedge nT_0, T \wedge (n+1)T_0], \quad n \in \{0, \ldots, \frac{T}{T_0} + 1\}.$$

The function z is α Hölder continous and according to the change of variable Corollary 6.3, for $t \in [T \wedge nT_0, T \wedge (n+1)T_0]$,

$$\begin{aligned} z(t) &= z^n(t - T \wedge nT_0), \\ &= a_n + \sum_{i=1}^{m} \int_0^{t - T \wedge nT_0} V^i(z^n(u)) \mathrm{d}\tilde{x}^i(u), \\ &= a_n + \sum_{i=1}^{m} \int_{T \wedge nT_0}^{t} V^i(z(s)) \mathrm{d}x^i(s). \end{aligned}$$

By induction on n, we obain

$$z(t) = a_0 + \sum_{i=1}^{m} \int_0^t V^i(z(s)) \mathrm{d}x^i(s), \quad t \leq T.$$

That means that z is a fix point of $P_V^{T,a,x}$. □

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6.2.3. Uniqueness

As it is noted by Lyons in [21] and Davie in [10], and additional hypothesis is required on V^i, $i = 1, \dots, m$.

Definition 6.2. A function V from $\mathbb{R}^d$ into itself is said to be $(1+\gamma)$ Lipschitz, $0 < \gamma \leq 1$ if and only if:
V is Lipschitz continuous, and there exists a function g_V from $\mathbb{R}^d \otimes \mathbb{R}^d$ into $\mathbb{R}^d$, and a constant M such that

$$V(z) - V(\tilde{z}) = \sum_{i=1}^{d} (z^i - \tilde{z}^i) g_V^i(z, \tilde{z}) \quad z = (z^i)_{i=1}^d, \quad \tilde{z} = (\tilde{z}^i)_{i=1}^d \quad \in \mathbb{R}^d;$$
$$|g_V(z, \tilde{z})| \leq M |z - \tilde{z}|^\gamma.$$

This definition is taken in Stein [28] p. 176. An equivalent condition is the following:

Lemma 6.11. A function V from $\mathbb{R}^d$ into itself is $(1+\gamma)$ Lipschitz, $0 < \gamma \leq 1$ if and only if V is continuous, has continuous partial derivatives γ Hölder continuous.

The set of $(1+\gamma)$ Lipschitz functions is endowed with the semi-norm $\|.\|_{1+\gamma}$ defined as

$$\|V\|_{1+\gamma} = \|V\|_1 + \|g\|_\gamma.$$

Proposition 6.12. [21] Let $x \in C^\alpha([0,T], \mathbb{R}^m)$ with $\alpha > 1/2$ and let $(V^i)_{i=1}^m$ be $1+\gamma$ Lipschitz functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\gamma > \frac{1}{\alpha} - 1$ and $a \in \mathbb{R}^d$.
Then, the differential equation (6.36) has a unique solution, z, in the set of α Hölder continuous function of $[0,T]$ starting from a.

Proof of Proposition 6.12. Let z and $\tilde{z}$ be two solutions α Hölder continuous of (6.36) on $[0,T]$ and $y = z - \tilde{z}$. Note that for $i = 1, \dots, m$ and $t \in [0,T]$,

$$V^i(z(t)) - V^i(\tilde{z}(t)) = \sum_{j=1}^{d} y^j(t) g_{V^i}^j(z(t), \tilde{z}(t)).$$

Since $g_{V^i}^j(z, \tilde{z})$ is $\alpha\gamma$ Hölder continuous with $\alpha(\gamma+1) > 1$, then the following Young integral is well defined

$$y(t) = \sum_{i=1}^{m} \sum_{j=1}^{d} \int_0^t y^j(u) g_{V^i}^j(z(u), \tilde{z}(u)) \mathrm{d}x^i(u) \quad t \in [0,T].$$

According to Proposition 6.1 and Corollary 6.4,

$$\|y\|_{\alpha,0,T} \leq md \|x\|_{\alpha,0,T} \|y\|_{\alpha,0,T} \|g_V\|_\gamma [\|z\|_{\alpha,0,T} + \|\tilde{z}\|_{\alpha,0,T}] \{\theta(\alpha(\gamma+1) - 1) T^\alpha + 1\} T^{2\alpha - 1} [1 + \theta(2\alpha - 1)].$$

Then from Corollary 6.6, since $y(0) = 0$, for all $0 \leq s \leq t$,

$$|y(t) - y(s)| \leq C \left[\frac{t^2 - s^2}{2!} \right]^\alpha \|y\|_{\alpha,0,T},$$

where

$$C = \left[md \|x\|_{\alpha,0,T} \|g_V\|_\gamma [\|z\|_{\alpha,0,T} + \|\tilde{z}\|_{\alpha,0,T}] \{\theta((\gamma+1)\alpha - 1\} T^\alpha + 1\} (1 + 4^{2\alpha} \zeta(2\alpha)) \right].$$

Then, by induction, from Corollary 6.6, for all $0 \leq s \leq t$,

$$|y(t) - y(s)| \leq C^n \left[\frac{t^n - s^n}{n!} \right]^\alpha \|y\|_{\alpha,0,T}, \quad n \in \mathbb{N}^*.$$

We derive that $\|y\|_{\alpha,0,T} = 0$ and since $y(0) = 0$ then $y(t) = 0$ on $[0,T]$. □

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6.2.4. Convergence of Picard iteration

Proposition 6.13. [21] Let $x \in C^\alpha([0,T],\mathbb{R}^m)$ with $\alpha > 1/2$ and let $(V^i)_{i=1}^m$ be $1+\gamma$ Lipschitz functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\gamma > \frac{1}{\alpha} - 1$ and $a \in \mathbb{R}^d$.

Given a path z_0 in $\mathbb{R}^d$, α Hölder continuous and define by induction $z_{n+1} = P_V^{T,a,x}(z_n)$. Then, the sequence $(z_n)_n$ converges to z, the unique solution of differential equation (6.36), in the set of α Hölder continuous function of $[0,T]$ starting from a.

Proof of Proposition 6.13. First, we will prove that $\sup_n \|z_n\|_{\alpha,0,T} < +\infty$ (using twice iteration procedures) and conclude using Corollary 6.6

Note that from Proposition 6.1, for $n \in \mathbb{N}$,

$$\|z_{n+1}\|_{\alpha,0,t} \leq B(\|z_n\|_{\alpha,0,t} t^\alpha + 1), \quad t \in [0,T] \tag{6.38}$$

where

$$B = \|x\|_{\alpha,0,T} \max(\|V(a)\|; \|V\|_1(\theta(2\alpha-1)+1)).$$

Then, by iteration on n inequality (6.38) we prove that

$$\|z_n\|_{\alpha,0,t} \leq [t^\alpha B]^n \|z_0\|_{\alpha,0,t} + \left(\frac{[t^\alpha B]^{n+1} - 1}{t^\alpha B - 1}\right) B;$$

and for $t < B^{-\frac{1}{\alpha}}$

$$\sup_n \|z_n\|_{\alpha,0,t} < +\infty.$$

Assume that for $t < B^{-\frac{1}{\alpha}}$, $\sup_n \|z_n\|_{\alpha,0,Nt} < +\infty$. Note that from Proposition 6.1, for $N \in \mathbb{N}$,

$$\|z_{n+1}\|_{\alpha,Nt,(N+1)t} \leq B\left(\|z_n\|_{\alpha,Nt,(N+1)t} t^\alpha + 1 + \|z_n\|_{\alpha,0,Nt}\right). \tag{6.39}$$

Using iteration on n, we obtain from inequality (6.39) that $\sup_n \|z_n\|_{\alpha,Nt,(N+1)t} < +\infty$, Since, $\|.\|_{\alpha,s,t}$ is sub additive, then $\sup_n \|z_n\|_{\alpha,0,(N+1)t} < +\infty$. By iteration on $N \leq TB^{\frac{1}{\alpha}} + 1$ we conclude that $\sup_n \|z_n\|_{\alpha,0,T} < +\infty$.

Let us denote by $y_{n+1} = z_{n+1} - z_n$. Note that for $i = 1, \ldots, m$ and $t \in [0,T]$,

$$V^i(z_{n+1}(t)) - V^i(z_n(t)) = \sum_{j=1}^d y_{n+1}^j(t) g_{V^i}^j(z_{n+1}(t), z_n(t)).$$

Then,

$$y_{n+1}(t) = \sum_{i=1}^m \sum_{j=1}^d \int_0^t y_n^j(u) g_{V^i}^j(z_{n+1}(u), z_n(u)) \mathrm{d}x^i(u).$$

From Corollary 6.6, since $y_{n+1}(0) = 0$, for all $0 \leq s \leq t$,

$$|y_n(t) - y_n(s)| \leq C\left(\frac{t^1 - s^1}{1!}\right)^\alpha \|y_{n-1}\|_{\alpha,0,T},$$

where

$$C = 2d\|x\|_{\alpha,0,T}\|g_V\|_\gamma \sup_n \|z_n\|_{\alpha,0,T}\{\theta(2\alpha-1)+1\}^2(1+\zeta(2\alpha)).$$

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By induction, from Corollary 6.6, since $y(0)=0$, for all $0\leq s\leq t\leq T$,

$$\begin{aligned}|y_{n+1}(t)-y_{n+1}(s)| &\leq C^n\left(\frac{t^n-s^n}{n!}\right)^{\alpha}\|y_1\|_{\alpha,0,T},\\ &\leq C^n\left(\frac{T^{n-1}+1}{(n-1)!}\right)^{\alpha}\|y_1\|_{\alpha,0,T}|t-s|^{\alpha}.\end{aligned}$$

We derive that $(\sum y_n)_n$ (and $(z_n)_n$) converges to $\tilde{z}$ on the set of α Hölder continuous function starting from a. Since the map $z\mapsto P_V^{T,a,x}$ is continuous (see Proposition 6.1) then $\tilde{z}$ is the unique fix point of $P_V^{T,a,x}$. □

6.2.5. Continuity of the Itô map

Let $\alpha>1/2$ and let $(V^i)_{i=1}^m$ be $1+\gamma$ Lipschitz functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\gamma>\frac{1}{\alpha}-1$ and $a\in\mathbb{R}^d$.

Definition 6.3. For all $a\in\mathbb{R}^d$ and $x\in C^{\alpha}([0,T],\mathbb{R}^m)$, we note $\mathcal{I}_{V^1,\ldots,V^m}(x,a,T)$, and called it Itô map, the solution of

$$\mathrm{d}z(t)=\sum_{i=1}^m V^i(z(u))\mathrm{d}x^i(u),\quad z(0)=a,\quad t\in[0,T].$$

Proposition 6.14. [23] Let $\alpha>1/2$ and let $(V^i)_{i=1}^m$ be $1+\gamma$ Lipschitz functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\gamma>\frac{1}{\alpha}-1$ and $a\in\mathbb{R}^d$. The Itô map, $\mathcal{I}_{V^1,\ldots,V^m}(x,a,T)$, is locally Lipschitz from $C^{\alpha}([0,T],\mathbb{R}^m)\times\mathbb{R}^d$ into $C^{\alpha}([0,T],\mathbb{R}^d)$, where $\mathbb{R}^d$ endowed with the usual euclidian norm $|.|$ and $C^{\alpha}([0,T],\mathbb{R}^m)$ is endowed with $N_{\alpha,\infty,0,T}$.

Proof of Proposition 6.14. Let $R>0$ and

$$\mathbb{B}=\{(a,x),\quad a\in\mathbb{R}^d,\quad x\in C^{\alpha}([0,T],\mathbb{R}^m),\quad |a|\leq R,\quad N_{\alpha,0,T,\infty}(x)\leq R\}.$$

We will prove that uniformly in x, $\mathcal{I}_{V^1,\ldots,V^m}(x,a,T)$, is globally Lipschitz from $\mathbb{B}$ into $C^{\alpha}([0,T],\mathbb{R}^d)$. According to Proposition 6.8, $\mathcal{I}_{V^1,\ldots,V^m}(x,a,T)$ is bounded on $\mathbb{B}$:

$$\sup_{(a,x)\in\mathbb{B}}\|\mathcal{I}_{V^1,\ldots,V^m}(x,a,T)\|_{\alpha,0,T}<+\infty.$$

- Now, we will prove the continuity with respect to a.
  Since for $i=1,\ldots,m$; $j=1,\ldots,d$; g_i^j is γ Hölder continuous then for all (a,a') such that $|a|\leq R$, $|a'|\leq R$, and x such that $\|x\|_{\alpha,0,T}\leq R$, the function
  $g_i^j(\mathcal{I}_{V^1,\ldots,V^m}(x,a,T),\mathcal{I}_{V^1,\ldots,V^m}(x,a',T))$ is $\alpha\gamma$ Hölder continuous and

$$\max_{\{i=1,\ldots,m;\quad j=1,\ldots,d\}}\sup_{(a,x)}\sup_{(a',x)\in\mathbb{B}^2}\|g_i^j(\mathcal{I}_{V^1,\ldots,V^m}(x,a,T),\mathcal{I}_{V^1,\ldots,V^m}(x,a',T))\|_{\gamma\alpha}<+\infty.$$

  For $i=1,\ldots,m;\quad j=1,\ldots,d$ let h_i^j be given by

$$h_i^j(t)=\int_0^t g_i^j(\mathcal{I}_{V^1,\ldots,V^m}(x,a,T),\mathcal{I}_{V^1,\ldots,V^m}(x,a',T))(u)\mathrm{d}x^i(u),\quad t\in[0,T]$$

  from Proposition 6.1, h_i^j is α Hölder continuous and

$$H=dm[1+\{1+\theta(2\alpha-1)T^{\alpha}]\max_{\{i=1,\ldots,m;\quad j=1,\ldots,d\}}\sup_{(a,x)}\sup_{(a',x)\in\mathbb{B}}\|h_i^j\|_{\alpha,0,T}<+\infty.$$

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If $y = \mathcal{I}_{V^1,\dots,V^m}(x,a,T) - \mathcal{I}_{V^1,\dots,V^m}(x,a',T)$ then y is given by

$$y_l(t) = a_l - a'_l + \sum_{j=1}^{d}\sum_{i=1}^{m}\int_0^t y_j(u)\mathrm{d}h_i^j(u)^l, \quad t\in[0,T].$$

and using Proposition 6.1 and equation (6.33), we obtain

$$\|y\|_{\alpha,0,T} \le H[|a-a'| + \|y\|_{\alpha,0,T}].$$

Then using Corollary 6.6, we prove by iteration on n, using Corollary 5.2.6 for all $0\le s\le t\le 1$,

$$\begin{aligned}|y(t)-y(s)| \le \|a-a'\|\sum_{k=1}^{n} H^k(1+\zeta(2\alpha))^k\left(\frac{t^k-s^k}{(k!)}\right)^{\alpha}\\ + \|y\|_{\alpha,0,T}(1+\zeta(2\alpha))^{n+1}\left(\frac{t^{n+1}-s^{n+1}}{(n+1!)}\right)^{\alpha}.\end{aligned}$$

Then, when n goes to infinity,

$$\|\mathcal{I}_{V^1,\dots,V^m}(x,a,T) - \mathcal{I}_{V^1,\dots,V^m}(x,a',T)\|_{\alpha,0,T} \le |a-a'|H\sum_{k=1}^{\infty}\frac{H^k(1+\zeta(2\alpha))^k T^{\alpha k}}{((k-1)!)^{\alpha}}.$$

- We conclude with the continuity with respect to x.
Let x, $\tilde{x}$ in $C^{\alpha}([0,T],\mathbb{R}^m)$ and $a\in\mathbb{R}^d$ such that (a,x) and $(a,\tilde{x})$ belong to $\mathbb{B}$. Let us denote by $y = \mathcal{I}_{V^1,\dots,V^m}(x,a,T) - \mathcal{I}_{V^1,\dots,V^m}(\tilde{x},a,T)$ then y is given by for $t\in[0,T]$

$$\begin{aligned}y(t) = &\sum_{j=1}^{d}\sum_{i=1}^{m}\int_0^t y_j(u)g_j^i(\mathcal{I}_{V^1,\dots,V^m}(x,a,T),\mathcal{I}_{V^1,\dots,V^m}(\tilde{x},a,T))(u)\mathrm{d}x^i(u)\\ &+\int_0^t V^i(x(u)d[x-\tilde{x}](u).\end{aligned} \tag{6.40}$$

For $i=1,\dots,m$ and $j=1,\dots,d$ let h_i^j be given by

$$h_i^j(t) = \int_0^t g_j^i(\mathcal{I}_{V^1,\dots,V^m}(x,a,T),\mathcal{I}_{V^1,\dots,V^m}(\tilde{x},a,T))(u)\mathrm{d}x^i(u), \quad t\in[0,T]$$

and

$$D = dm(\|V\|_1 + \sup_{a,\|a\|\le R}\|V(a) + \max_{\{i=1,\dots,m;\ \ j=1,\dots,d\}}\sup_{(a,x)\ (a,\tilde{x})\in\mathbb{B}}\|h_i^j\|_{\alpha,0,T})(1+\theta(2\alpha-1))T^{\alpha}.$$

From expression (6.40) and Proposition 6.1, we obtain

$$\|y\|_{\alpha,0,T} \le D[\|y\|_{\alpha,0,T} + \|x-\tilde{x}\|_{\alpha,0,T}].$$

Then using Corollary 6.6, we prove by iteration on n, using expression (6.40) for all $0\le s\le t\le 1$,

$$\|\mathcal{I}_{V^1,\dots,V^m}(x,a,T) - \mathcal{I}_{V^1,\dots,V^m}(\tilde{x},a,T)\|_{\alpha,0,T} \le D\|x-\tilde{x}\|_{\alpha,0,T}\sum_{k=1}^{\infty}\frac{H^k(1+\zeta(2\alpha))^k T^{\alpha k}}{((k-1)!)^{\alpha}}.$$ □

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6.3. Conclusion

In order to define and solve and study the properties of flow of differential equations driven by a path α Hölder continuous we have

1. define "fundamental blocks" $\mu(a,b) = x(a)[y(b)-y(a)]$ (where x and y are in $C^\alpha([0,T],\mathbb{R})$ "almost additive" that means there exists C and $\varepsilon > 0$ such that
$$|\mu(a,b) - \mu(a,c) - \mu(c,b)| \leq C|b-a|^{1+\varepsilon} \quad \forall\ a \leq c \leq b.$$
2. Using sewing Lemma, we have build an additive functional $\mathcal{S}(\mu)$ and set $\int_a^b x(s)\mathrm{d}y(s) = \mathcal{S}(\mu)(a,b)$ such that
$$|\mu(a,b) - \int_s^b x(s)\mathrm{d}y(s)| \leq C\theta(\varepsilon)|b-a|^{1+\varepsilon} \quad \forall\ a \leq b.$$
Moreover, the integral fulfills for $(s,t) \in \Delta_T^2$
$$\left|\int_{s\leq u_1\leq\ldots\leq u_n\leq t} \mathrm{d}x(u_1)\ldots\mathrm{d}x(u_n)\right| \leq [\theta(\varepsilon)\|x\|_{\alpha,0,T}]^n \frac{(t-s)^{\alpha n}}{n!}.$$
3. Then, we have defined the solution of
$$\mathrm{d}z_t = \sum_{i=1}^m V(z_t)\mathrm{d}x_t^i, \quad z_0 \in V$$
as a fixed point of the integral operator $P_V^{z_0,x,T}$ where
$$P_V^{z_0,x,T}(t) = z_0 + \int_0^t \sum_{i=1}^m V(z_s)\mathrm{d}x_s^i, \quad t \in [0,T].$$
4. Using a fix point Theorem, we have prove local existence. The global existence came from *a priori* estimation on the solution.
5. Uniqueness and regularity of the solution is a consequence of a bound on the iterated integral.

We will do the same programm when x is a path α Hölder continuous, with $\alpha \in]1/3;1/2]$ in the next section and explain wath happens when $\alpha \leq 1/3$.

7. Integration: degree 2

Let $\alpha \in]1/3,1/2]$. Let W be $\mathbb{R}^m$, $m \geq 1$. The aim of this subsection is to define $\int \phi(X)\mathrm{d}X$, for $X \in \Omega H_\alpha^{(2)}(V)$, where $\phi : V \to \mathbf{L}(V,W)$ be a function which sends elements of V linearly to W-valued one-forms on V.

7.1. Introduction

Let x be an α Hölder continous path from $[0,T]$ info V and $\phi : V \to \mathbf{L}(V,W)$ γ Hölder continuous such that $\alpha(\gamma+1) > 1$. Let $\tilde{\mathcal{J}}_a(\phi,x)$ be
$$\tilde{\mathcal{J}}_a(\phi,x)_{a,b}^{(1)} = \phi(x(a)) \cdot X_{a,b}^{(1)}, \quad (a,b) \in \Delta_T^2,$$
$$X_{a,b}^{(1)} = x(b) - x(a).$$

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We compute

$$\delta(\tilde{\mathcal{J}}_a(\phi,x)^{(1)})_{a,c,b} = -(\phi(x(c)) - \phi(x(a)))(x(b) - x(c)), \quad \forall (a,c,b) \in \Delta_T^3.$$

The functional $\tilde{\mathcal{J}}_a(\phi,x)^{(1)}$ is almost additive if $\alpha > \frac{1}{2}$ and $\gamma > \frac{1}{\alpha} - 1$. In this case, (see section 6.1 on Young integral), $\int \phi(x)\mathrm{d}x = \mathcal{S}(\tilde{\mathcal{J}}_a(\phi,x))$.

When $\alpha \in]\frac{1}{3}, \frac{1}{2}]$, following the ideas sketched in the introduction, see section 3.2. we will assume that there exists X an multiplicative functional of order 2, α Hölder continuous over x. The functional $\tilde{\mathcal{J}}_a(\phi,x)$ will be replaced by $\mathcal{J}_a(\phi,X)$ where

$$\mathcal{J}_a(\phi,X)^{(1)}_{a,b} = \phi(x(a)) \cdot X^{(1)}_{a,b} + \mathrm{d}\Phi(x(a)) \cdot X^{(2)}_{a,b}, \quad (a,b) \in \Delta_T^2.$$

We will proved that $\mathcal{J}_a(\phi,X)$ is an almost multiplicative functional associated to $\varepsilon = (\gamma+1)\alpha - 1$. Then, according to Theorem 5.11, there exists a unique α Hölder continuous rough path $\mathcal{J}_a(\phi,X)$ denoted by $\int \phi(x)\mathrm{d}X$.

7.2. Construction of an integral then $\alpha \in]1/3, 1/2]$

Let $\phi : V \to \mathbf{L}(V,W)$ be a function which sends elements of V linearly to W-valued one-forms on V. Suppose that ϕ is continuously differentiable with derivative $\mathrm{d}\Phi$ and denote $\phi^i = d^{i-1}\phi, \quad i = 1,2$.

For $X \in \Omega H^{(2)}_{\alpha,T}(V)$, let us define $\mathcal{J}_a(\phi,X) \in C_\alpha(\Delta_T^2, T^{(2)}(W))$ where for $(s,t) \in \Delta_T^2$

$$\begin{aligned} \mathcal{J}_a(\phi,X)^{(1)}_{s,t} &:= \phi^1(X^{(1)}_{0,s}) \cdot X^{(1)}_{s,t} + \phi^2(X^{(1)}_{0,s}) \cdot X^{(2)}_{s,t}, \\ \mathcal{J}_a(\phi,X)^{(2)}_{s,t} &:= \phi^1(X^{(1)}_{0,s}) \otimes \phi^1(X^{(1)}_{0,s}) \cdot X^{(2)}_{s,t}. \end{aligned} \tag{7.41}$$

Assumption 7.1. Let $\phi : V \to \mathbf{L}(V,W)$. Suppose that ϕ is continuously differentiable with derivative $\mathrm{d}\Phi$ such that for all $(x,y) \in V$, $(v,w) \in V$,

$$\begin{aligned} \phi^1(x) \cdot v &= \phi^1(y) \cdot v + \phi^2(y) \cdot [(x-y) \otimes v] + R_1(y,x) \cdot v, \\ \phi^2(x) \cdot (v \otimes w) &= \phi^2(y) \cdot (v \otimes w) + R_2(y,x) \cdot (v \otimes w) \end{aligned}$$

and there exists γ such that $1/\alpha < \gamma \leq [1/\alpha] + 1$, $M > 0$ such that

$$\begin{aligned} &|\phi^i(x)|_{\mathbf{L}(V^{\otimes i},W)} \leq M(1 + |x|), \quad i = 1,2, \\ &|R_i(x,y)|_{\mathbf{L}(V^{\otimes i},W)} \leq M|x-y|^{\gamma - i}, \quad \forall (x,y) \in V, \quad \forall (v,w) \in V. \end{aligned}$$

Theorem 7.2. Theorem 5.2.1 and Remark 5.3.1 of [23]

Assume that $X \in \Omega H^{(2)}_{\alpha,T}(V)$. Let ϕ fulfilling Assumption 7.1, then $\mathcal{J}_a(\phi,X)$ given by (7.41) is an almost rough path α Hölder continuous in $T^{(2)}(W)$ and $\forall (s,t,u) \in \Delta_T^3$,

$$\left|\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(s,t,u)\right|_i \leq C(T,M,N_{\alpha,0,T}(X))N_{\alpha,0,T}(X)^{\gamma-1}|u-s|^{\gamma\alpha},$$

where

$$C(T,M,u) = \max\left(2M; M^2[4(1+T^\alpha u)^2 + u^{\gamma-2}T^{\alpha(\gamma-2)}]T^{(3-\gamma)\alpha}\right).$$

Proof of Theorem 7.2. Recall that

$$\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b) = \mathcal{J}_a(\phi,X)_{a,b} - \mathcal{J}_a(\phi,X)_{a,c} \otimes \mathcal{J}_a(\phi,X)_{c,b}, \quad (a,c,b) \in \Delta_T^3.$$

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(1) First, we study the level 1. Let $(a,c,b) \in \Delta_T^3$. Using the definition of the tensor product,

$$\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(1)} = \mathcal{J}_a(\phi,X)^{(1)}_{a,b} - \mathcal{J}_a(\phi,X)^{(1)}_{a,c} - \mathcal{J}_a(\phi,X)^{(1)}_{c,b}.$$

Using the definition of $\mathcal{J}_a(\phi,X)$ given in equation (7.41), we compute

$$\begin{aligned}\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(1)} = {} & \phi^1(X^{(1)}_{0,a})\cdot X^{(1)}_{a,b} + \phi^2(X^{(1)}_{0,a})\cdot X^{(2)}_{a,b} \\ & - \phi^1(X^{(1)}_{0,a})\cdot X^{(1)}_{a,c} - \phi^2(X^{(1)}_{0,a})\cdot X^{(2)}_{a,c} \\ & - \phi^1(X^{(1)}_{0,c})\cdot X^{(1)}_{c,b} - \phi^2(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b}.\end{aligned}$$

Using the Chen rule for X, see equation (5.10),

$$\begin{aligned}\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(1)} = {} & [\phi^1(X^{(1)}_{0,a}) - \phi^1(X^{(1)}_{0,c}) + \phi^2(X^{(1)}_{0,a})\cdot X^{(1)}_{a,c}]\cdot X^{(1)}_{c,b} \\ & + [\phi^2(X^{(1)}_{0,a}) - \phi^2(X^{(1)}_{0,c})]\cdot X^{(2)}_{c,b}.\end{aligned}$$

Introducing R_i $i=1,2$ we obtain

$$\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(1)} = -R_2(X^{(1)}_{0,a},X^{(1)}_{0,c})\cdot X^{(2)}_{c,b} - R_1(X^{(1)}_{0,a},X^{(1)}_{0,c})\cdot X^{(1)}_{c,b}. \tag{7.42}$$

Then, from hypothesis on R_1 and R_2, see Assumption 7.1,

$$\left|\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(1)}\right|_1 \le M\left[|X^{(1)}_{a,c}|_1^{\gamma-2}|X^{(2)}_{c,b}|_2 + |X^{(1)}_{a,c}|_1^{\gamma-1}|X^{(1)}_{c,b}|_1\right].$$

Under hypothesis on X,

$$\left|\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(1)}\right|_1 \le 2MN_{\alpha,0,T}(X)^\gamma|b-a|^{\alpha\gamma}. \tag{7.43}$$

(2) Second, we study the second level: from the definition of the tensor product,

$$\begin{aligned}\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(2)} = {} & [\mathcal{J}_a(\phi,X)_{a,b} - \mathcal{J}_a(\phi,X)_{a,c} - \mathcal{J}_a(\phi,X)_{c,b}]^{(2)} \\ & - \mathcal{J}_a(\phi,X)^{(1)}_{a,c}\otimes\mathcal{J}_a(\phi,X)^{(1)}_{c,b}.\end{aligned}$$

Then, using the definition of $\mathcal{J}_a(\phi,X)$, given in equation (7.41), we compute

$$\begin{aligned}\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(2)} = {} & \{\phi^1\otimes\phi^1\}(X^{(1)}_{0,a})\cdot[X^{(2)}_{a,b} - X^{(2)}_{a,c}] - \{\phi^1\otimes\phi^1\}(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b} \\ & - \left[\phi^1(X^{(1)}_{0,a})\cdot X^{(1)}_{a,c} + \phi^2(X^{(1)}_{0,a})\cdot X^{(2)}_{a,c}\right]\otimes\left[\phi^1(X^{(1)}_{0,c})\cdot X^{(1)}_{c,b} + \phi^2(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b}\right].\end{aligned}$$

Using the Chen rule (5.10) for X

$$\begin{aligned}\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(2)} = {} & [\{\phi^1\otimes\phi^1\}(X^{(1)}_{0,a}) - \{\phi^1\otimes\phi^1\}(X^{(1)}_{0,c})]\cdot X^{(2)}_{c,b} \\ & + \phi^1(X^{(1)}_{0,a})\otimes[\phi^1(X^{(1)}_{0,a}) - \phi^1(X^{(1)}_{0,c})]\cdot X^{(1)}_{a,c}\otimes X^{(1)}_{c,b} \\ & - \phi^2(X^{(1)}_{0,a})\cdot X^{(2)}_{a,c}\otimes\left[\phi^1(X^{(1)}_{0,c})\cdot X^{(1)}_{c,b} + \phi^2(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b}\right] \\ & - \phi^1(X^{(1)}_{0,a})\cdot X^{(1)}_{a,c}\otimes\phi^2(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b}.\end{aligned}$$

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We introduce R_1 and

$$\begin{aligned}\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(2)} =& -\phi^1(X^{(1)}_{0,a})\otimes[\phi^2(X^{(1)}_{0,a})\cdot X^{(1)}_{a,c}+R_1(X^{(1)}_{0,a},X^{(1)}_{0,c})]\cdot[X^{(2)}_{c,b}+X^{(1)}_{a,c}\otimes X^{(1)}_{c,b}]\\ &-[\phi^2(X^{(1)}_{0,a})\cdot X^{(1)}_{a,c})+R_1(X^{(1)}_{0,a},X^{(1)}_{0,c})]\otimes\phi^1(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b}\\ &-\phi^2(X^{(1)}_{0,a})\cdot X^{(2)}_{a,c}\otimes\left[\phi^1(X^{(1)}_{0,c})\cdot X^{(1)}_{c,b}+\phi^2(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b}\right]\\ &-\phi^1(X^{(1)}_{0,a})\cdot X^{(1)}_{a,c}\otimes\phi^2(X^{(1)}_{0,c})\cdot X^{(2)}_{c,b}. \end{aligned}\tag{7.44}$$

Using Assumption 7.1 ϕ^i and R^i for $i=1,2$,

$$|\delta_{T^2(V)}\mathcal{J}_a(\phi,X)(a,c,b)^{(2)}|_2\le\tilde{C}(M,T,N_{\alpha,0,T}(X))N_{\alpha,0,T}(X)^3|b-a|^{3\alpha}.\tag{7.45}$$

where

$$M^2\left(1+T^\alpha u\right)u^3\left[5(1+T^\alpha u+3u^{\gamma-2}\right]=M^2[4(1+T^\alpha u)^2+u^{\gamma-2}T^{\alpha(\gamma-2)}].$$

Since $3\alpha>\alpha\gamma>1$, from inequalities (7.43) and (7.45), we conclude that $\mathcal{J}_a(\phi,X)$ is almost multiplicative. □

From Theorems 5.11 and 7.2, we define the integral of a one form with respect to a functional in $\Omega H^{(2)}_{\alpha,T}$ for $\alpha>1/3$.

Definition 7.1. Let $X\in\Omega H^{(2)}_{\alpha,T}(V)$ and Φ a one-form fulfills Assumption 7.1, the integral of ϕ against the rough path X, denoted by $\int\phi(X)\mathrm{d}X$, or $\mathcal{J}(\phi,X)$ is the unique element of $\Omega H^{(2)}_{\alpha,T}(V)$ associated to the almost rough path $\mathcal{J}_a(\phi,X)\in C^\alpha(\Delta^2_T,T^{(2)}(W))$ where

$$\begin{aligned}\mathcal{J}_a(\phi,X)^{(1)}_{s,t}&=\phi^1(X^{(1)}_{0,s})\cdot X^{(1)}_{s,t}+\phi^2(X^{(1)}_{0,s})\cdot X^{(2)}_{s,t},\quad\forall(s,t)\in\Delta^2_T,\\ \mathcal{J}_a(\phi,X)^{(2)}_{s,t}&=\phi^1(X^{(1)}_{0,s})\otimes\phi^1(X^{(1)}_{0,s})\cdot X^{(2)}_{s,t}.\end{aligned}$$

Under Assumption 7.1, ϕ^1 and ϕ^2 are locally $\gamma-2>0$ Hölder continuous, then $\mathcal{J}_a(\Phi,.)$ is $\gamma-2>0$ Hölder continuous and from Proposition 5.10, we prove the continuity of $\mathcal{J}(\phi,.)$.

Proposition 7.3. Theorem 5.2.2 of [23] Let $\alpha\in]1/3,1/2]$, ϕ fulfills Assumption 7.1, then the map $\mathcal{J}(\phi,.)$ is continuous from $\Omega H^{(2)}_{\alpha,T}(V)$ into $\Omega H^{(2)}_{\alpha,T}(W)$.

Assumption 7.4. Let $\phi:V\to\mathbf{L}(V,W)$. Suppose that ϕ posseses all kth continuous derivatives $d^k\phi$ up to the degree 3 and for all $(x,y)\in V$, $(v,w)\in V$,

$$\begin{aligned}&|d^i\phi(x)|_{\mathbf{L}(V^{\otimes i},W)}\le M,\quad i=1,2,3,\\ &|d^i\phi(x)-d^i\phi(y)|_{\mathbf{L}(V^{\otimes i},W)}\le M|x-y|.\end{aligned}$$

Proposition 7.5. Let $\alpha\in]1/3,1/2]$, ϕ fulfills Assumption 7.4, then the map $\mathcal{J}(\phi,.)$ is locally Lipschitz continuous from $\Omega H^{(2)}_{\alpha,T}(V)$ into $\Omega H^{(2)}_{\alpha,T}(W)$.

Proof of Proposition 7.5. The map $X\mapsto\mathcal{J}_a(\Phi,X)$ is locally Lipschitz continuous from $\Omega H^{(2)}_{\alpha,T}(V)$ into $\Omega H^{(2)}_{\alpha,T}(W)$. Then, we have to prove that for all $K>0$, there exists a constant C_K depending only on K, α, T and M such that for all $(X,\tilde{X})\in B_{C^\alpha(\Delta^2_T,T^{(2)}(V))}(0,K)$, for $i=1,2$,

$$\left|\delta_{T^{(2)}(V)}\left[\mathcal{I}_a(\Phi,X)-\mathcal{I}_a(\Phi,\tilde{X})\right]^{(i)}(s,u,t)\right|\le C_KN_{\alpha,0,T}(X-\tilde{X})|t-s|^{3\alpha},\quad\forall(s,u,t)\in\Delta^3_T;\tag{7.46}$$

and conclude with Theorem 5.11.

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Under Assumption 7.4,

$$R_i(w,\tilde{w}) = \int_0^1 (1-\theta)^{2-i} d^2 f(w+\theta(\tilde{w}-w))\mathrm{d}\theta \cdot (w-\tilde{w})^{3-i}, \quad (w,\tilde{w}) \in W^2, \quad i=1,2.$$

Then, using the expression of $\delta_{T^{(2)}(V)}\mathcal{I}_a(\Phi,X)^{(1)}$ given in (7.42) and the expression of $\delta_{T^{(2)}(V)}\mathcal{I}_a(\Phi,X)^{(2)}$ given in (7.44), one can derive inequality (7.46). □

7.3. Change of variable formula

In this section, following [23] Section 5.4, we give a change of variable (or Itô formula) for rough paths.

Theorem 7.6. Theorem 5.4.1 of [23]
Let X be in $G\Omega H_{\alpha,T}(V)$, $1/3 < \alpha \leq 1/2$, and let f be a twice differentiable function from V into W such that $\mathrm{d}f$ fulfills Assumption 7.1. Then,

$$f(X_{0,t}^{(1)}) - f(X_{0,s}^{(1)}) = \left[\int \mathrm{d}f(X)\mathrm{d}X\right]_{s,t}^{(1)}, \quad \forall (s,t) \in \Delta_T^2. \tag{7.47}$$

Remark 7.1. It will be shown later that equation (7.47) is not true for non-geometric rough paths.

Proof of Theorem 7.6. Note that $\mathrm{d}f = \phi$ fulfills Assumption 7.1 and equation (7.47) is true for any smooth rough path X. Then, by continuity ot the two sides, see Proposition 7.3 for the right one, equation (7.47), for any geometric rough path. □

Corollary 7.7. Let $X = (1, X^{(1)}, X^{(2)})$ be in $\Omega H_{\alpha,T}^{(2)}(V)$ and let be Φ be additive continuous 2α Hölder in $V^{\otimes 2}$. Let us introduce $Y = (1, X^{(1)}, X^{(2)} + \Phi)$ and ϕ be a one form fulfilling Assumption 7.1, then for all $(s,t) \in \Delta_T^2$, we have

$$\left[\int \phi(Y)\mathrm{d}Y\right]_{s,t}^{(1)} = \left[\int \phi(X)\mathrm{d}X\right]_{s,t}^{(1)} + \int_s^t \phi^2(X_{0,r}^{(1)})\mathrm{d}\Phi_r,$$

and

$$\begin{aligned}\left[\int \phi(Y)\mathrm{d}Y\right]_{s,t}^{(2)} &= \left[\int \phi(X)\mathrm{d}X\right]_{s,t}^{(2)} + \int_s^t \phi^1(X_{0,r}^{(1)}) \otimes \phi^1(X_{0,r}^{(1)})\mathrm{d}\Phi_r, \\ &\quad + \int_s^t \left[\int_s^r \phi^2(X_{0,u}^{(1)})\mathrm{d}\Phi_r\right] \otimes \mathrm{d}Z_r + \int_s^t Z_{s,r} \otimes \phi^2(X_{0,r}^{(1)})\mathrm{d}\Phi_r,\end{aligned}$$

where the integrals involving Φ on the right-hand sides are Young's integrals and $Z = [\int \phi(X)\mathrm{d}X]^{(1)}$.

Proof of Corollary 7.7.

(1) First, we study the first level. From Remark 5.3,

$$\begin{aligned}\left[\int \phi(Y)\mathrm{d}Y\right]_{s,t}^{(1)} &= \lim_{|\mathcal{D}_m|\to 0} \sum_{i=1}^{k_m-1} \left\{\phi\left(X_{0,t_i^m}^{(1)}\right) \cdot X_{t_i^m,t_{i+1}^m}^{(1)} + \phi^2\left(X_{0,t_i^m}^{(1)}\right) \cdot X_{t_i^m,t_{i+1}^m}^{(2)}\right\} \\ &\quad + \lim_{|\mathcal{D}_m|\to 0} \sum_{i=1}^{k_m-1} \phi^2(X_{0,t_i^m}^1) \cdot \Phi_{t_i^m,t_{i+1}^m},\end{aligned} \tag{7.48}$$

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where $\mathcal{D}_m = \{t_1^m = s < \ldots < t_{k_m}^m = t\}$ and $|\mathcal{D}_m| = \sup_{i=1,\ldots,k_m-1} |t_i^m - t_{i+1}^m|$. Then according to Remark 5.3 and the definition of the Young integral the right member of equation (7.48) converges to $\left[\int \phi(X)\mathrm{d}X\right]_{s,t}^{(1)} + \int_s^t \phi^2(X_{0,r}^{(1)})\mathrm{d}\Phi_r$.

(2) The proof for the second level follows the same lines. □

7.4. Link with Young integral

Let x be in $C^\beta([0,T],V)$ with $\beta > \frac{1}{2}$, then for any $\alpha \in]\frac{1}{3}, \frac{1}{2}[$, $x \in C^\alpha([0,T],V)$.

Lemma 7.8. Let X be the functional over x such that

$$X_{a,b}^{(1)} := x(b) - x(a), \quad (a,b) \in \Delta_T^2,$$
$$X_{a,b}^{(2)} := \int_a^b (x(s) - x(a)) \otimes \mathrm{d}x(s)$$

is well defined, and belongs to $\Omega H_{\beta,T}^{(2)}(V)$.

Proof of Lemma 7.8. Since $\beta > \frac{1}{2}$, the integral of x with respect to itself used in the definition of $X^{(2)}$ exists as in Definition 7.1. The functional of the increments of x, $X^{(1)}$ is additive. Let $(a,c,b) \in \Delta_T^3$, from the Chalses and linear properties of Young integral,

$$\begin{aligned}
\delta_{T^{(2)}(V)}^{(2)} X_{a,c,b} &= \int_a^b (x(s) - x(a)) \otimes \mathrm{d}x(s) - \int_a^c (x(s) - x(a)) \otimes \mathrm{d}x(s) \\
&\quad - \int_c^b (x(s) - x(c)) \otimes \mathrm{d}x(s) - (x(c) - x(a)) \otimes (x(b) - x(c)), \\
&= \int_c^b [x(c) - x(a)] \otimes \mathrm{d}x(s) - (x(c) - x(a)) \otimes (x(b) - x(c)) \\
&= 0
\end{aligned}$$

and X is multiplicative.

According to inequality (6.33),

$$|X_{a,b}^{(2)}|_2 \leq [\|x\|_\infty + \theta(2\beta - 1)\|x\|_{\beta,0,T} T^\beta] \|x\|_{\beta,0,T} T^\beta |b-a|^{\beta 2}, \quad (a,b) \in \Delta_T^2,$$

and X belongs to $C^\beta(\Delta_T^2, T^{(2)}(V))$. □

The Young integral coincides with the "Rough path"integral introuced in Definition 7.1.

Proposition 7.9. Let ϕ be a one form fulfilling Assumption 7.1, and x be in $C^\beta([0,T],V)$ with $\beta > \frac{1}{2}$ such that $x(0) = 0$. Then,

$$\int_0^a \phi(x(s))\mathrm{d}x(s) = \mathcal{J}(\phi, X)_{0,a}^{(1)}, \quad a \in [0,T],$$
$$\int_0^a \phi(x(s)) \otimes [\int_0^s \phi(x(u))\mathrm{d}x(u)]\mathrm{d}x(s) = \mathcal{J}(\phi, X)_{0,a}^{(2)}.$$

Proof of Proposition 7.9. Note that for $(a,b) \in \Delta_T^2$

$$\left| \mathcal{J}_a(\phi, X)_{a,b}^{(1)} - \phi(x(a))(x(b) - x(a)) \right| \leq M(1 + \|x\|_\infty)[\|x\|_\infty + \theta(2\beta - 1)\|x\|_{\beta,0,T} T^\beta] |b-a|^{2\beta}.$$

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Since $\beta > \frac{1}{2}$, according to inequality (6.33) on the definition of the Young integral, and Definition 7.1 of $\mathcal{J}(\phi, X)$, $\mathcal{J}(\phi, X)^{(1)}_{a,b} = \int_a^b (\phi(x(s)) - \phi(x(a))\mathrm{d}x(s)$.

Note that from the definition of $X^{(2)}$, for $(a, b) \in \Delta_T^2$

$$\left|\mathcal{J}_a^{(2)}(\phi, X)_{a,b} - (\phi(x(a)) \otimes \phi(x(a)).x(a) \otimes (x(b) - x(a))\right| \leq M^2(1 + \|x\|_\infty)\theta(2\beta - 1)\|x\|^2_{\beta,0,T}|b - a|^{2\beta}.$$

Then, $\mathcal{J}^{(2)}(\phi, X)_{0,a} = \int_0^a \phi(x(s))^2 x(s)\mathrm{d}x(s)$, $a \in [0, T]$ and from Corollary 6.4, $\int_0^a \phi(x(s)) \otimes [\int_0^s \phi(x(u))\mathrm{d}x(u)]$ $\mathrm{d}x(s) = \mathcal{J}(\phi, X)^{(2)}_{0,a}$. □

It is worthy to note that the functional X defined in Lemma 7.8 is a geometric one. Let $m \in \mathbb{N}$, and x^m be the linear interpolation of x along the subdivision $(t_i^m := iT2^{-m}, \quad i = 0, \ldots, 2^m)$. Let X^m be the smooth functional of order 2 build on x^m.

Lemma 7.10. For ε such that $\beta - \varepsilon > \frac{1}{2}$, $(X^m)_m$ converges in $\Omega GH_{\beta-\varepsilon,T}(V)$ to X.

Proof of Lemma 7.10. Note that $\|x - x^m\|_{\beta-\varepsilon,0,T} \leq 3(T2^{-m})^\varepsilon \|x\|_{\beta,0,T}$ for $m \in \mathbb{N}$. Then, the conclusion follows from Proposition 5.10. □

8. Differential equations driven by α Hölder continuous 2 rough path with $1/3 < \alpha \leq 1/2$

In this section, we define and prove some results on differential equation of the form

$$dy_t = f(y_t)dx_t, y_0 \in \mathbb{R}^d,$$

where $x \in C^\alpha([0, T], \mathbb{R}^m)$ with $1/3 < \alpha \leq 1/2$.

The results contained in this section were originally proved by Lyons in [22]. Some proofs are also available in the books of Lyons and Qian [23] or Friz and Victoir [13].

We restrict ourself to the case $\alpha > 1/3$ as in [12]. Netherless, in the seminal paper of Lyons [22] and in the book [23], α is allowed to be in $]0, 1]$.

8.1. Notion of solution and reduction of the dimension of the problem

8.1.1. Notion of solution

Let $V = \mathbb{R}^m$ and $W = \mathbb{R}^d$. Let $f : V \to \mathbf{L}(V, W)$ be a function, which can be viewed as a linear map sending vector of V to a vector field on W. Let $X \in \Omega H^{(2)}_{\alpha,T}(V)$ be over x an α Hölder continuous path in V. Consider the following differential equation (initial value problem)

$$\begin{aligned} dy_t &= f(y_t)dx_t, \\ y(0) &= y_0. \end{aligned} \tag{8.49}$$

We have only defined integral of the form $\int f(Z)\mathrm{d}Z$ for rough path a Z, so equation (8.49) makes no sense in the rough path setting. To overcome this difficulty, the idea is to combine x and y together as a new path. We view equation (8.49) as

$$\begin{aligned} dX_t &= dX_t, \\ dY_t &= f(Y_t)dX_t, \quad Y_0 = y_0. \end{aligned} \tag{8.50}$$

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The initial condition of X is irrelevant, and therefore we simply take $X_0 = 0$. Define $\hat{f} : W \otimes V \oplus W \to \mathbf{L}(V \oplus W; V \oplus W)$ by

$$\hat{f}(y_0, x, y) \cdot (v, w) = (v, f(y + y_0).v), \quad \forall (y_0, x, y) \in W \oplus V \oplus W, \quad \forall (v, w) \in V \oplus W. \tag{8.51}$$

Lemma 8.1. Let f be a map from W into $\mathbf{L}(V, W)$ fulfilling Assumption 7.1, then for all $y_0 \in W$, $\hat{f}(y_0, .)$ is a map from $V \oplus W$ into $\mathbf{L}(V \oplus W, V \oplus W)$ fulfilling Assumption 7.1.

Proof of Lemma 8.1. Let $y_0 \in W$ fixed.

Since f is differentiable then $\hat{f}(y_0, .)$ is differentiable and for $(x, y), \quad (v_i, w_i) \in V \oplus W, \quad i = 1, 2$ we have

$$d\hat{f}(y_0, x, y) \cdot [(v_1, w_1); (v_2, w_2)] = (0_V, df(y + y_0) \cdot (v_1, w_2)) .$$

Let $\hat{R}_1(y_0, .)$ and $\hat{R}_2(y_0, .)$ be defined by for $(x, y), \quad (x', y')) \in V \oplus W$

$$\begin{aligned} \hat{R}_1(y_0, (x, y); (x', y')) \cdot (v_1, w_1) &= (0_V, R_1(y_0 + y, y_0 + y') \cdot v_1) \\ \hat{R}_2(y_0, (x, y); (x', y')) \cdot ((v_1, w_1); (v_2; w_2)) &= (0_V, R_2(y_0 + y, y_0 + y') \cdot (v_1; w_2)) \end{aligned} \tag{8.52}$$

then $(\hat{f}(y_0, .), d\hat{f}(y_0, .), \hat{R}_1, \hat{R}_2)$ fulfills the conditions of Assumption 7.1. □

Equation (8.50) can be written in the following more appreciating form

$$dZ_t = \hat{f}(y_0, Z_t) dZ_t. \tag{8.53}$$

Definition 8.1. Let $X \in \Omega H^{(2)}_{\alpha,T}(V)$, $Z \in \Omega H^{(2)}_{\alpha,T}(V \oplus W)$ is said to be a a solution to equation (8.50) with initial condition y_0 driven by X if

$$\begin{aligned} \Pi_{T^{(2)}(V)}(Z) &= X, \\ Z &= \int \hat{f}(y_0, Z) \mathrm{d}Z, \end{aligned}$$

where $\Pi_{T^{(2)}(V)}$ is the projector from $T^{(2)}(V \otimes W)$ on $T^{(2)}(V)$.

Note that the set

$$\{Z \in \Omega H^{(2)}_{\alpha}(V \oplus W), \quad \Pi_{T^{(2)}(V)}(Z) = X\}$$

is not empty, but not a convex set. Then, it seems to be difficult to prove existence of solution of equation (8.50) in the sens of Definition 8.1 using a fix point argument. Netherless, in the next subsection, we show that the projection on the convex set $\mathcal{C}_X$ (defined in Sect. 5.5) of a solution of equation (8.50) is a fix point of operator $P_f^{y_0,X,T} := \Pi_{\mathcal{C}_X} \circ \mathcal{J}(\hat{f}(y_0 + .), \Pi_{\mathcal{C}_X})$.

8.1.2. Reduction of the dimension of the problem

Recall that,

$$\hat{f}(y_0, x, y) \cdot (v, w) = (v, f(y + y_0).v), \quad \forall (x, y) \in V \oplus W, \quad \forall (v, w) \in V \oplus W.$$

then

$$\begin{aligned} \left[\hat{f}(y_0, x, y) \otimes \hat{f}(y_0, x, y)\right] \cdot [(v_1, w_1), (v_2, w_2)] &= [\hat{f}(y_0, x, y) \cdot (v_1, w_1)] \otimes [\hat{f}(y_0, x, y) \cdot (v_2, w_2)], \\ &= \begin{pmatrix} (v_1, v_2) & (f(y + y_0).v_1, v_2) \\ (v_1, f(y + y_0).v_2) & (f(y + y_0).v_1, f(y + y_0).v_2) \end{pmatrix}, \end{aligned}$$

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and

$$d\hat{f}(y_0,x,y)\cdot[(v_1,w_1);(v_2,w_2)]=(0_V,\mathrm{d}f(y+y_0)\cdot(v_1,w_2))\,.$$

Let $Z=(1,Z^{(1)},Z^{(2)})$ be a multiplicative functional in $T^{(2)}(V\otimes W)$. Then the almost rough path, $\mathcal{J}_a(\hat{f}(y_0,.),Z)$ used to define $\int\hat{f}(y_0,Z)\mathrm{d}Z$ depends only on $\Pi_{T^{(2)}(V)}(Z)$, $\Pi_W(Z)$ and $\Pi_{W\otimes V}(Z)$. Indeed, $\forall(s,t)\in\Delta_T^2$,

$$\begin{aligned}
&\Pi_V(\mathcal{J}_a(\hat{f}(y_0,.),Z))=\Pi_V(Z),\quad \Pi_{V^{\otimes 2}}(\mathcal{J}_a(\hat{f}(y_0,.),Z))=\Pi_{V^{\otimes 2}}(Z) \qquad (8.54)\\
&\Pi_W(\mathcal{J}_a(\hat{f}(y_0,.),Z))_{s,t}=f(y_0+\Pi_W(Z_{0,s}))\Pi_V(Z)_{s,t}+\mathrm{d}f(y_0+\Pi_W(Z_{0,s}))\Pi_{W\otimes V}(Z)_{s,t},\\
&\Pi_{W\otimes V}(\mathcal{J}_a(\hat{f}(y_0,.),Z))_{s,t}=(f(y_0+\Pi_W(Z_{0,s}))\otimes\mathbf{I}_V)\Pi_{V^{\otimes 2}}(Z)_{s,t},\\
&\Pi_{V\otimes W}(\mathcal{J}_a(\hat{f}(y_0,.),Z))_{s,t}=(\mathbf{I}_V\otimes f(y_0+\Pi_W(Z_{0,s})))\Pi_{V^{\otimes 2}}(Z)_{s,t},\\
&\Pi_{W\otimes W}(\mathcal{J}_a(\hat{f}(y_0,.),Z))_{s,t}=(f(y_0+\Pi_W(Z_{0,s}))\otimes f(y_0+\Pi_W(Z_{0,s})))\cdot\Pi_{V^{\otimes 2}}(Z)_{s,t}.
\end{aligned}$$

This observation leads us to used the results of Section 5.5. Let X be in $\Omega H_\alpha^{(2)}(V)$. Let $P_{a,f}^{y_0,X,T}$ be the operator from $\mathcal{C}_{X,T}$ into the set of applications from Δ_T^2 taking their values in $W\oplus(W\otimes V)$, where for $Y\in\mathcal{C}_{X,T}$,

$$\begin{aligned}
&\Pi_W(P_{a,f}^{y_0,X,T}(Y))_{s,t}:=f(y_0+Y_{0,s}^{(1)})X_{s,t}^{(1)}+df(y_0+Y_{0,s}^{(1)})\cdot Y_{s,t}^{(2)},\quad (s,t)\in\Delta_T^2, \qquad (8.55)\\
&\Pi_{W\otimes V}(P_{a,f}^{y_0,X,T}(Y))_{s,t}:=(f(y_0+Y_{0,s}^{(1)})\otimes\mathbf{I}_V)\cdot X_{s,t}^{(2)}.
\end{aligned}$$

The following lemma is a consequence of the proof of Theorem 7.2.

Lemma 8.2. Let $f:W\to\mathbf{L}(V,W)$ fulfills Assumption 7.1 and $X\in\Omega H_{\alpha,T}^{(2)}(V)$ and $\frac{1}{\gamma}<\alpha'\leq\alpha$ then $P_{a,f}^{y_0,X,T}$ is continuous from $\mathcal{C}_X^{\alpha'}$ into $\mathcal{AC}_X^{\alpha}$ and for $U=W$ or $W\oplus V$

$$\left|\Pi_U\delta_{\mathcal{C}_X}^{(i)}(P_{a,f}^{y_0,X,T}(Y))_{s,u,t}\right|\leq C_U^{\alpha,\alpha',T,X,y_0}(N_{\alpha',0,T}(Y))|t-s|^{\alpha'\gamma},\quad\forall(s,u,t)\in\Delta_T^3$$

where

$$\begin{aligned}
&C_W^{\alpha,\alpha',T,X,y_0}(u)=Mu^{\gamma-1}\left[u+N(X)_{\alpha,0,T}T^{\alpha-\alpha'}\right],\\
&C_{W\otimes V}^{\alpha,\alpha',T,X,,y_0}(u)=MT^{\alpha-\alpha'}uN_{\alpha,0,T}(X)2(1+\|y_0\|+uT^{\alpha'})(1+T^{\alpha-\alpha'}).
\end{aligned}$$

Let us denote $C^{\alpha,\alpha',T,X,y_0}:=\max(C_W^{\alpha,\alpha',T,X,y_0},C_W^{\alpha,\alpha',T,X,y_0})$.

Remark 8.1. If f and $\mathrm{d}f$ are bounded by M then $C_{W\otimes V}^{\alpha,\alpha',y_0,T,X}(u)$ is dominated by

$$C_{W\otimes V}^{\alpha,\alpha',T,X}(u)=2MuN_{\alpha,0,T}(X)T^{\alpha-\alpha'}(1+T^{\alpha-\alpha'}).$$

which doens't depend on y_0.

Proof of Lemma 8.2. From the definition of $\hat{R}_1(y_0,.)$ and $\hat{R}_2(y_0)$ given in (8.52) and identity (7.42) in the proof of Theorem 7.2, we obtain

$$\delta_{\mathcal{C}_X}^{(1)}P_{a,f}^{y_0,X,T}(Y)_{s,u,t}=-R_2(y_0+Y_{0,s}^{(1)},y_0+Y_{0,u}^{(1)})\cdot Y_{u,t}^{(2)}-R_1(y_0+Y_{0,s}^{(1)},y_0+Y_{0,u}^{(1)})\cdot X_{u,t}^{(1)}. \qquad (8.56)$$

From identity (7.44) in the proof of Theorem 7.2, we obtain

$$\begin{aligned}
\delta_{\mathcal{C}_X}^{(2)}P_{a,f}^{y_0,X,T}(Y)_{s,u,t}=&-\left[df(y_0+Y_{0,s}^{(1)})\otimes\mathbf{1}_V\right]\cdot\left[Y_{s,u}^{(1)}\otimes X_{u,t}^{(2)}+Y_{s,u}^{(2)}\otimes X_{u,t}^{(1)}\right]\\
&-(R_1(y_0+Y_{0,s}^{(1)},y_0+Y_{0,u}^{(1)})\otimes\mathbf{1}_V)\cdot X_{u,t}^{(2)}, \qquad (8.57)
\end{aligned}$$

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Then, the proof is a consequence of Assumption 7.1 on f, df, R_1 and R_2. □

Let us denote $P_f^{y_0,X,T} := \mathcal{E} \circ P_{a,f}^{y_0,X,T}$.

Corollary 8.3. Let $f : W \to \mathbf{L}(V,W)$ fulfills Assumption 7.1 and $X \in \Omega H^{(2)}_{\alpha,T}(V)$ and $\frac{1}{\gamma} < \alpha' \leq \alpha$ the map $P_f^{y_0,X,T}$ is continuous from $\mathcal{C}^{\alpha'}_{X,T}$ into $\mathcal{C}^{\alpha}_{X,T}$ and

$$\|P_f^{y_0,X,T}(Y)^{(i)}\|_{i\alpha,0,T} \leq C_f^{\alpha,\alpha',T,X,y_0,(i)}(N_{\alpha',0,T}(Y))$$

where

$$C_f^{\alpha,\alpha',T,X,y_0,(1)}(u) = C^{\alpha,\alpha',T,X,y_0}(u)\theta(\gamma\alpha'-1)T^{\alpha'\gamma-\alpha} + M(1+\|y_0\|+T^{\alpha'}u)[N_{\alpha,0,T}(X)+uT^{2\alpha'-\alpha}];$$
$$C_f^{\alpha,\alpha',T,X,y_0,(2)}(u) = T^{\alpha'\gamma-\alpha}Cste(C^{\alpha,\alpha',T,X,u}, v, \alpha'\gamma-1, T) + M(1+\|y_0\|+T^{\alpha'}u)N_{\alpha,0,T}(X)$$

$v = M(1+\|y_0\|+T^{\alpha'}u)(N_{\alpha,0,T}(X))(T^{\alpha-\alpha'}+uT^{\alpha'})$ and $Cste(C,\alpha,,T,X,u)$ is given in Theorem 5.11.

Proof of Corollary 8.3. Under Assumption 7.1, $P_{a,f}^{y_0,X,T}$ mapping continuously $\mathcal{AC}^{\alpha'}_{X,T}$ into $\mathcal{C}^{\alpha}_{X,T}$. Then from Corollary 5.17, the operator $P_f^{y_0,X,T}$ is continuous from $\mathcal{C}^{\alpha'}_{X,T}$ into $\mathcal{C}^{\alpha}_{X,T}$. □

Remark 8.2. If f and $\mathrm{d}f$ are bounded then,

$$C_f^{\alpha,\alpha',T,X,y_0,(1)}(u) \leq C^{\alpha,\alpha',T,X}(u)\theta(\gamma\alpha'-1)T^{\alpha'\gamma-\alpha} + M[N_{\alpha,0,T}(X)+uT^{2\alpha'-\alpha}];$$
$$C_f^{\alpha,\alpha',T,X,y_0,(2)}(u) \leq T^{\alpha'\gamma-\alpha}Cste(C^{\alpha,\alpha',T,X,u}, v, \alpha'\gamma-1, T) + MN_{\alpha,0,T}(X)$$

$v = M(N_{\alpha,0,T}(X))(T^{\alpha-\alpha'}+uT^{\alpha'})$.

Assume that $P_f^{y_0,X,T}$ has a fix point in $\mathcal{C}^{\alpha}_X$ denoted by Y. Let us introduce $\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)$ the functional in $T^{(2)}(V\oplus W)$ such that for $(s,t)\in\Delta_T^2$,

$$\begin{aligned}
\Pi_{T^{(2)}(V)}(\mathcal{J}_{a,p}(\hat{f}(Y_0,.),X,Y)) &= X,\\
\Pi_W(\mathcal{J}_{a,p}(\hat{f}(y_0,.),X,Y)_{s,t} &= f(y_0+Y_{0,s}^{(1)})\cdot X_{s,t}^{(1)} + df(y_0+Y_{0,s}^{(1)})\cdot Y_{s,t}^{(2)},\\
\Pi_{W\otimes V}(\mathcal{J}_{a,p}(\hat{f}(y_0,.),X,Y)_{s,t} &= (f(y_0+Y_{0,s}^{(1)})\otimes \mathbf{I}_V)\cdot X_{s,t}^{(2)},\\
\Pi_{V\otimes W}(\mathcal{J}_{a,p}(\hat{f}(y_0,.),X,Y)_{s,t} &= (\mathbf{I}_V\otimes f(y_0+Y_{0,s}^{(1)}))\cdot X_{s,t}^{(2)},\\
\Pi_{W\otimes W}(\mathcal{J}_{a,p}(\hat{f}(y_0,.),X,Y)_{s,t} &= (f(y_0+Y_{0,s}^{(1)})\otimes f(y_0+Y_{0,s}^{(1)}))\cdot X_{s,t}^{(2)}.
\end{aligned}$$

Lemma 8.4. Under Assumption 7.1, let X be in $\Omega H^{(2)}_{\alpha,T}(V)$.

1. If Z is a solution of equation (8.50) in the sens of Definition 8.1, then $\Pi_{W\oplus(W\otimes V)}(Z)$ is a fix point in $\mathcal{C}^{\alpha}_X$ of $P_f^{y_0,X,T}$.
2. If Y is a fix point in $\mathcal{C}^{\alpha}_{X,T}$ of $P_f^{y_0,X,T}$ then $\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)$ is an almost multiplicative functional. Moreover, $\mathcal{E}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)$ is a solution of equation (8.50) in the sens of Definition 8.1.

Proof of Lemma 8.4. We only have to prove the second point.

- First, we prove that $\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)$ is an almost multiplicative functional. Note that $\Pi_{W\oplus(W\otimes V)}((\mathcal{J}_{ap}(\hat{f}(y_0,.),\ (X,Y)) = P_{a,f}^{y_0,X,T}(Y)$. According to the proof of Lemma 8.2, we only have to study the projection on $V\otimes V$ and $W\otimes W$ of $\mathcal{J}_{ap}(\hat{f}(y_0,.),\ X,Y)$.

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From identity (7.44) in the proof of Theorem 7.2, we obtain

$$\Pi_{V\otimes W}\delta_{T^{(2)}(V\oplus W)}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y))_{s,u,t} = -[\mathbf{1}_V\otimes R_1(y_0+Y^{(1)}_{0,u},y_0+Y^{(1)}_{0,s})]\cdot[X^{(2)}_{u,t}+X^{(1)}_{s,u}\otimes X^{(1)}_{u,t}] - \left[\mathbf{1}_V\otimes \mathrm{d}f(y_0+Y^{(1)}_{0,u})\right].\left[(X^{(2)}_{s,t}-X^{(2)}_{s,u})\otimes Y^{(1)}_{s,u}+X^{(1)}_{u,t}\otimes Y^{(2)}_{s,u}\right].$$

From Assumption 7.1 on R_1 and $\mathrm{d}f$, there exists a constant C such that

$$\left|\Pi_{V\otimes W}\delta_{T^{(2)}(V\oplus W)}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y))_{s,u,t}\right| \le C|t-s|^{3\alpha},\quad (s,u,t)\in\Delta^3_T.$$

From identity (7.44) in the proof of Theorem 7.2, we obtain

$$\begin{aligned}\Pi_{W\otimes W}\delta_{T^{(2)}(V\oplus W)}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)))_{s,u,t} &= -f(y_0+Y^{(1)}_{0,s})\cdot X^{(1)}_{s,u}\otimes df(y_0+Y^{(1)}_{0,u})\cdot Y^{(2)}_{u,t}\\ -[df(y_0+Y^{(1)}_{0,s})\cdot Y^{(1)}_{s,u}&+R_1(y_0+Y^{(1)}_{0,s},y_0+Y^{(1)}_{0,u})]\otimes f(y_0+Y^{(1)}_{0,u})\cdot X^{(2)}_{u,t}\\ -f(y_0+Y^{(1)}_{0,s})&\otimes[\mathrm{d}f(y_0+Y^{(1)}_{0,s})\cdot Y^{(1)}_{s,u}+R_1(y_0+Y^{(1)}_{0,s},y_0+Y^{(1)}_{0,u})]\cdot[X^{(2)}_{s,t}-X^{(2)}_{u,s}]\\ -df(y_0+Y^{(1)}_{0,s})&\cdot Y^{(2)}_{s,u}\otimes[f(y_0+Y^{(1)}_{0,u})\cdot X^{(1)}_{u,t}+df(y_0+Y^{(1)}_{0,u})\cdot Y^{(2)}_{u,t}].\end{aligned}$$

From Assumption 7.1 on R_1 and $\mathrm{d}f$, there exists a constant C such that

$$\left|\Pi_{W\otimes W}\delta_{T^{(2)}(V\oplus W)}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y))_{s,u,t}\right| \le C|t-s|^{3\alpha},\quad (s,u,t)\in\Delta^3_T.$$

Then, $\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)$ is an almost multiplicative functional.

- Second, using uniqueness in Theorems 5.11 and 5.16, we derive

$$\Pi_{W\oplus(W\otimes V)}(\mathcal{E}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y))) = \mathcal{E}(P^{y_0,X,T}_{a,f}(Y)) = P^{y_0,X,T}_{f}(Y) = Y.$$

Moreover, the two following almost multiplicative functionals are equal

$$\mathcal{J}_a(\hat{f}(y_0,.),\mathcal{E}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y))) = \mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)$$

as their associated multiplicative functionals

$$\mathcal{J}(\hat{f}(y_0,.),\mathcal{E}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y))) = \mathcal{E}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y)).$$

Since the projection on $T^{(2)}(V)$ of $\mathcal{E}(\mathcal{I}_{ap}(\hat{f}(y_0,.),X,Y))$ is X, then $\mathcal{E}(\mathcal{J}_{ap}(\hat{f}(y_0,.),X,Y))$ is a solution of equation (8.50) in the sens of Definition 8.1. □

8.2. Existence

We are now in position to obtain the existence of solution of equation (8.50) in the sens of Definition 8.1. We prove local existence, and if f and $\mathrm{d}f$ are bounded, we derive global existence.

Proposition 8.5. Let $f : W \to \mathbf{L}(V,W)$ fulfilling Assumption 7.1, and $X\in\Omega H^{(2)}_\alpha(V)$. There exists a time $T_1^{X,f,y_0}>0$ depending on X, f y_0, γ and α such that equation (8.50) in the sens of Definition 8.1 has a solution in $\Omega H^{(2)}_\alpha(V\oplus W)$ up to the time $T_1^{X,f,y_0}>0$. Moreover, if f and $\mathrm{d}f$ are bounded by M, T_1^{X,f,y_0} does not depend on y_0.

Proof of Proposition 8.5. We will construct T_1^{X,f,y_0} such that $P_f^{y_0,X,T_1^{X,f,y_0}}$ has a fix point and we conclude with Lemma 8.4.

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Then, for $T \leq 1$, $C^{\alpha,\alpha',T,X,y_0}$ is bounded on $\bar{B}_{\mathcal{C}^{\alpha'}_{X,T}}(0,1)$ by

$$C_1 = \max_{u\in[0,1],i=1,2} \left(C_W^{\alpha,\alpha',T,X,y_0,(i)}(u) \right).$$

Let T_1^{X,f,y_0} be given by

$$T_1^{X,f,y_0} = \min\left(1, T, \frac{1}{C_1^{\alpha-\alpha'}}\right).$$

From Lemma 8.2 and Corollary 8.3, $P_f^{X,y_0,T_1^{X,f,y_0}}$ is a continuous map from $\bar{B}_{\mathcal{C}^{\alpha'}_{X,T_1^{X,f,y_0}}}(0,1)$ into itself. Since $\mathcal{C}_{X,T_1^{X,f,y_0}}$ is a convex set, then $\bar{B}_{\mathcal{C}^{\alpha'}_{X,T_1^{X,f,y_0}}}(0,1)$ is a compact convex set. According to the fix point theorem of Tychonov, $P_f^{X,y_0,T_1^{X,f,y_0}}$ has a fix point denoted by Y. Moreover, from Corollary 8.3, we derive that Y belongs to $C^{\alpha}_{X,T_1^{X,f,y_0}}$.

If f and $\mathrm{d}f$ are bounded, then from remarks 8.1 and 8.2, C_1 and T_1^{X,f,y_0} are independent of y_0. □

Corollary 8.6. Let $f : W \to \mathbf{L}(V,W)$ fulfilling Assumption 7.1 such that f and $\mathrm{d}f$ are bounded by M,, and $X \in \Omega H_\alpha^{(2)}(V)$. Then equation (8.50) has a solution in $\Omega H_\alpha^{(2)}(V \oplus W)$ up to time T in the sens of Definition 8.1.

Remark 8.3. Up to my knowledge, the existence of a global solution of equation (8.50) driven by X when f is not bounded is a open problem (See [20]).

Proof of Corollary 8.6. Let $T > 0$. Let $T_1 := T_1^{X,f,y_0} > 0$ be given by Proposition 8.5 when f is bounded. We will used the notations and the results of Lemma 5.15 on concatenation of functionals.

Let $N = \left[\frac{T}{T_1^{X,f,y_0}}\right]$, $S_i = T_1^{X,f,y_0}$ for $i = 0,\dots,N$ and $S_{N+1} = T - NT_1^{X,f,y_0}$.

Let ${}^0Z$ be a solution of equation (8.50) with initial condition y_0 and driven signal ${}^0X$ on the time interval $[0,T_1]$.

For $i = 1,\dots,N$ ${}^iZ$ is a solution of equation (8.50) with initial condition $y_0 + \sum_{j=0}^{i-1} \Pi_V({}^jZ)_{0,S_{j+1}}$ and driven signal ${}^iX$ on the time interval $[0,S_{i+1}]$ and $Z = {}^0Z \otimes \ldots \otimes {}^NZ$.

Then, by construction

$$Z_{0,s}^{(1)} = \sum_{j=0}^{i_s-1} {}^jZ_{0,S_j}^{(1)} + {}^{i_s-}Z_{0,s-T_{i_s-}}^{(1)}, \quad \text{for} \quad s \in [0,T]$$

and

$${}^i\mathcal{J}_a(\hat{f}(y_0,.),Z) = \mathcal{J}_a(\hat{f}(y_0 + \sum_{j=0}^{i-1} \Pi_V({}^jZ)_{0,S_{j+1}};.),\ {}^iZ), \quad \text{for} \quad i = 0,\dots,N.$$

Since ${}^iZ$ is a solution, for $i = 0,\dots,N$,

$$\mathcal{E}({}^i\mathcal{J}_a(\hat{f}(y_0,.),Z)) = {}^i\mathcal{E}(\mathcal{J}_a(\hat{f}(y_0,.),Z)) = {}^iZ,$$

and using point 3 of Lemma 5.15, $Z = \mathcal{E}(\mathcal{J}_a(\hat{f}(y_0,.),Z))$. The last equality means that Z is a solution of equation (8.50) in $\Omega H_{\alpha,T}^{(2)}(V \oplus W)$ up to time T in the sens of Definition 8.1. □

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With the same lines, we prove the following flow property.

Let Z be a multiplicative (resp. an almost multiplicative) functional in $T^{(2)}(V \oplus W)$ defined on Δ_2^T. Then, for $0 < S < T$ if ${}^S Z = (X_{a+S;b+S}, \quad (a,b) \in \Delta^2_{T-S})$ is again a multiplicative (resp. an almost multiplicative) functional.

Lemma 8.7. Let $\phi : V \to \mathbf{L}(V,W)$ fulfills Assumption 7.4 and $X \in \Omega H^{(2)}_\alpha(V)$ if Z is a solution of equation (8.50) in $\Omega H^{(2)}_\alpha(V \oplus W)$ up to time T in the sens of Definition 8.1, then for $0 < S < T$, ${}^S Z$ s a solution of equation (8.50) in $\Omega H^{(2)}_\alpha(V \oplus W)$ driven by ${}^S X$ with initial condition $y_0 + \Pi_V Z_{0,S}$ up to time $T - S$ in the sens of Definition 8.1.

8.3. Uniqueness

As it is noted in Davie [10], equation has not in general an unique solution under Assumption 7.1.

Theorem 8.8. Let $f : W \to \mathbf{L}(V,W)$ fulfills Assumption 7.4. Let $X \in \Omega H^{(2)}_{\alpha,T}(V)$ then equation (8.50) has a unique in $\Omega H^{(2)}_{\alpha,T}(V \oplus W)$ in the sens of Definition 8.1.

The proof of Theorem 8.8 relies on the following flow property (see Lemma 8.7) and local uniqueness.

Proof of Theorem 8.8. We only prove the local uniqueness.

- Let $K \leq 1$. Assume we have proved that the existence of a constant C depending on T, α, γ, X, K such that $P_f^{y_0,X,T}$ is CT^α Lipschitz continuous on $\overline{B}_{\mathcal{C}^\alpha_X}(0,K)$. Then, for $T_1 = [2C]^{\frac{-1}{\alpha}} > 0$, $P_f^{y_0,X,T_1}$ is $\frac{1}{2}$ Lipschitz continuous on $\overline{B}_{\mathcal{C}^\alpha_X,T_1}(0,K)$ and has a unique fix point Y.
- Let Z and $\tilde{Z}$ be two solution of equation (8.50) in $\Omega H^{(2)}_{\alpha,T_1}(V \oplus W)$ in the sens of Definition 8.1. Let $K \leq (1 + \max(N_{\alpha,0,T}(Z)), N_{\alpha,0,T}(\tilde{Z})))$. From point (1) of Lemma 8.4, $\Pi_{W\oplus(W\otimes V)}(Z)$ and $\Pi_{W\oplus(W\otimes V)}(\tilde{Z})$ are two fix points of $P_f^{y_0,X,T_1}$ on $\overline{B}_{\mathcal{C}^\alpha_X,T_1}(0,K)$. Then, $\Pi_{W\oplus(W\otimes V)}(Z)$ and $\Pi_{W\oplus(W\otimes V)}(\tilde{Z})$ are equal on $\Delta^2_{T_1}$. From point (2) of Lemma 8.4, $Z = \tilde{Z}$ on $\Delta^2_{T_1}$. □

Now, we study the Lipshitz property of $P_f^{y_0,X,T}$ on $\overline{B}_{\mathcal{C}^\alpha_X}(0,K)$ under Assumption 7.4.

Proposition 8.9. If f fulfills Assumption 7.4 and $K > 0$. Let $X \in \Omega H^{(2)}_{\alpha,T}(V)$ then $P_f^{y_0,X,T}$ is Lipschitz continuous on $\overline{B}_{\mathcal{C}^\alpha_{X,T}}(0,K)$ with Lipschitz constant $C_K T^\alpha$.

Proof of Proposition 8.9.

- In one hand, according to the definition of $P_{a,f}^{y_0,X,T}$ given in (8.55), and the fact that f and $d^i f$, $i = 1,2$ are bounded by M, $\Pi_W(P_{a,f}^{y_0,X,T})$ is Lipschitz continuous on $\overline{B}_{\mathcal{C}^\alpha_X}(0,K)$ with Lipschitz constant
$$T^\alpha M \left[\|X^{(1)}\|_{\alpha,0,T} + KT^\alpha + T^\alpha \right];$$
and $\Pi_{W\otimes V}(P_{a,f}^{y_0,X,T})$ is Lipschitz continuous on $\overline{B}_{\mathcal{C}^\alpha_X}(0,K)$ with Lipschitz constant $M\|X^{(2)}\|_{2\alpha,0,T}T^\alpha$.
- In the other hand, we study $\delta_{\mathcal{C}_X}(P_{a,f}^{y_0,X,t})$. Recall equations (8.56) and (8.57) in the proof of Lemma 8.2: for $(s,u,t) \in \Delta^3_T$,
$$\delta^{(1)}_{\mathcal{C}_X}(P_{a,f}^{y_0,X,T},Y)_{s,u,t} = -R_2(y_0 + Y^{(1)}_{0,s}, y_0 + Y^{(1)}_{0,u}) \cdot Y^{(2)}_{u,t} - R_1(y_0 + Y^{(1)}_{0,s}, y_0 + Y^{(1)}_{0,u}) \cdot X^{(1)}_{u,t},$$
and
$$\delta^{(2)}_{\mathcal{C}_X}(P_{a,f}^{y_0,X,T},Y)_{s,u,t} = -\left[\mathrm{d}f(y_0 + Y^{(1)}_{0,s}) \otimes \mathbf{1}_V \right] \cdot \left[Y^{(1)}_{s,u} \otimes X^{(2)}_{u,t} + Y^{(2)}_{s,u} \otimes X^{(1)}_{u,t} \right] - R_1(y_0 + Y^{(1)}_{0,s}, y_0 + Y^{(1)}_{0,u}) \otimes \mathbf{1}_V \cdot X^{(2)}_{u,t}.$$

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Under Assumption 7.4,

$$R_i(w,\tilde{w}) = \int_0^1 (1-\theta)^{2-i} d^2 f(w+\theta(\tilde{w}-w))\mathrm{d}\theta \cdot (w-\tilde{w})^{3-i}, \quad (w,\tilde{w}) \in W^2, \quad i=1,2.$$

Then, for $(Y,\tilde{Y}) \in \overline{B}_{\mathcal{C}_X^\alpha}(0,K)$, for all $(s,u,t) \in \Delta_T^3$,

$$|\delta^{(1)}_{\mathcal{C}_X}[P^{y_0,X,T}_{a,f}(Y) - P^{y_0,X,T}_{a,f}(\tilde{Y}))]_{s,u,t}|_1 \leq \tilde{C}_{L,K} N_{\alpha,0,T}(Y-\tilde{Y})|t-s|^{3\alpha},$$
$$|\delta^{(2)}_{\mathcal{C}_X}[P^{y_0,X,T}_{a,f}(Y) - P^{y_0,X,T}_{a,f}(\tilde{Y})]_{s,u,t}|_2 \leq \tilde{C}_{L,K} N_{\alpha,0,T}(Y-\tilde{Y})|t-s|^{3\alpha},$$

where

$$\tilde{C}_{L,K} = 6M(1+K^2)(1+N_{\alpha,0,T}(X)^2).$$

- Then, from Theorem 5.16,

$$N_{\alpha,0,T}(Y-\tilde{Y} - P^{y_0,X,T}_{a,f}(Y) + P^{y_0,X,T}_{a,f}(\tilde{Y})) \leq C_{L,K} T N_{\alpha,0,T}(Y-\tilde{Y}),$$

where

$$C_{L,K} = \tilde{C}_{L,K}[1 + 2KT^\alpha\theta(3\alpha-1)(N_{\alpha,0,T}(X) + KT^\alpha + T^\alpha) + 2K\theta(3\alpha-1)^2\tilde{C}_{l,K}T^{3\alpha}] \times \theta(3\alpha-1)(T^\alpha + T^{2\alpha}).$$

Moreover, $P_f^{y_0,X,t}$ is Lipschitz continuous on $\overline{B}_{\mathcal{C}_X^\alpha}(0,K)$ with Liptschitz constant

$$T^\alpha M\left[N_{\alpha,0,T}(X) + KT^\alpha + T^\alpha\right] + C_{L,K}.$$

□

8.4. Continuity of the Itô map

Let $\alpha > 1/3$ and let f be $W \to L(V,W)$ bounded fulfilling Assumption 7.4.
For all $y_0 \in \mathbb{R}^d$ and $X \in \Omega H_{\alpha,T}(V)$, we note $\mathcal{I}_f(X,y_0,T)$, and called it Itô map, the solution of

$$Z = y_0 + \int \hat{f}(Z)\mathrm{d}Z.$$

Using Corollary 4.4 or more precisely Corollary 5.18, one can prove

Proposition 8.10. [23] Let $\alpha > 1/3$ and let f be C_b^3. The Itô map, $\mathcal{I}_f(x,a,T)$, is locally Lipschitz from $\mathbb{R}^d \times \Omega H_{\alpha,T}(V)$ into $\Omega H_{\alpha,T}(V)$.

8.5. Conclusion

The integral $\int \Phi(X)\mathrm{d}X$ can be also developped when X is α Hölder continuous geometric rough path using this approach with $\alpha \leq 1/3$. But unfortunately, up to our knowledge, the fixed point argument does not work. An interested reader should read the books of [23] or [13].

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References

[1] F. Baudoin, *An introduction to the geometry of stochastic lows*. Imperial Press College, London (2004).

[2] J. Bertoin, Sur une intégrale pour les processus à α-variation bornée. *Ann. Probab.* **17** (1999) 1521–1535.

[3] K.-T. Chen, Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula. *Ann. Math.* **65** (1957) 163–178.

[4] K.-T. Chen, Integration of paths; a faithful representation of paths by non-commutative formal power series. *Trans. Amer. Math Soc.* 89 (1958) 395-407.

[5] K.-T. Chen, Integration of paths, *Bull. Amer. Math. Soc.* **83** (1977) 831–879.

[6] Z. Ciesielski, G. Kerkyacharian and B. Roynette, Quelques espaces fonctionnels associés à des processus gaussiens. *Studia Math.* **107** (1993) 171–204.

[7] L. Coutin and Z. Qian, Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields* **122** (2002) 108–140.

[8] L. Coutin and N. Victoir, Enhanced Gaussian processes and applications. *ESAIM Probab. Stat.* **13** (2009) 247–260.

[9] A.M. Davie, Differential equations driven by rough paths: an approach via discrete approximation. *Appl. Math. Res. Express. AMRX* **2** (2007) abm009, 40.

[10] A.M. Davie, Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN* **24** (2007) rnm124, 26.

[11] H. Doss, Liens entre équations différentielles stochastiques et ordinaires. *Ann. Inst. Henri Poincaré Sect. B (N.S.)* **13** (1977) 99–125.

[12] D. Feyel and A. de La Pradelle, Curvilinear integrals along enriched paths. *Electron. J. Probab.* **11** (2006) 860–892 (electronic).

[13] P. Friz and N. Victoir, *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications*. Cambridge University Press (2008).

[14] P. Friz and N. Victoir, Differential equations driven by Gaussian signals. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010) 369–413.

[15] M. Gubinelli, Controlling rough paths. *J. Funct. Anal.* **216** (2004) 86–140.

[16] Y. Hu and D. Nualart, Rough path analysis *via* fractional calculus. *Trans. Amer. Math. Soc.* **361** (2009) 2689–2718.

[17] Y. Inahama and H. Kawabi, Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths. *J. Funct. Anal.* **243** (2007) 270–322.

[18] A. Lejay, An introduction to rough paths. *Séminaire de probabilités, XXXVII* **1832** (2003) 1–59.

[19] A. Lejay, Yet another introduction to rough paths. *Séminaire de Probabilités, Lect. Notes in Maths* **XLII** (2009) 1–101.

[20] A. Lejay, On rough differential equations. *Electron. J. Probab.* **14** (2009) 341–364.

[21] T. Lyons, Differential equations driven by rough signals. I. An extension of an inequality of L.C. Young. *Math. Res. Lett.* **1** (1994) 451–464.

[22] T.J. Lyons, Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* **14** (1998) 215–310.

[23] T. Lyons and Zhongmin Qian, *System control and rough paths*. Oxford Mathematical Monographs. Oxford University Press, Oxford, Oxford Science Publications (2002).

[24] T. Lyons, M. Caruana and T. Lévy, *Differential equations driven by rough paths* Ecole d'été de probabilités de Saint-Flour **XXXIV** (2004), *Lectures Notes in Math* **1908**. J. Picard Ed., Springer, Berlin (2007).

[25] I. Nourdin, A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one, in *Séminaire de probabilités* **XLI**, *Lecture Notes in Math.* **1934**. Springer, Berlin (2008) 181–197.

[26] D. Nualart and A. Răşcanu, Differential equations driven by fractional Brownian motion. *Collect. Math.* **53** (2002) 55–81.

[27] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon (1993). Theory and applications, Edited and with a foreword by S.M. Nikolski Ed., Translated from the 1987 Russian original, Revised by the authors.

[28] E.M. Stein, *Singular integrals and differentiability properties of functions*, *Princeton Mathematical Series* **30**. Princeton University Press, Princeton, N.J. (1970).

[29] H. Sussmann, On the gap between deterministic and stochastic ordinary differential equations. *Ann. Probab.* **6** (1978) 19–41.

[30] A. Tychonoff, Ein Fixpunktsatz. *Math. Ann.* **111** (1935) 767–776.

[31] L.C. Young, An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** (1936) 251–282.

[32] M. Zähle, On the link between fractional and stochastic calculus, in *Stochastic dynamics*, Bremen (1997), Springer, New York (1999) 305–325.

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