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A note on the Julia set of a rational function

By S. D. LETHERMAN and R. M. W. WOOD
Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL

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1. *Introduction*

The purpose of this note is to present a few facts about the Julia set of a rational function that are well known to the experts in the subject of complex dynamics but whose documented exposition in the literature seems to need a little clarification. For example, under conditions set out in Theorem 1, the Julia set of a rational function can be expressed as the limit of a sequence of finite sets. In particular, for certain choices of a point α, the Julia set is the limit as n increases of the inverse image sets $R^{-n}(\alpha)$. This formulation is widely exploited in the backwards iteration algorithm to produce computer illustrations of Julia sets (see for example Section 5·4 of [**14**]).

A reference for use of this fact about Julia sets, in the context of iterated function systems, is Section 7·2 of [**2**]. The proof itself is essentially in [**3**] but Beardon does not mention the Hausdorff metric in his treatise. A variation of the statement and a proof of Theorem 1 requiring Sullivan's Classification Theorem are given in Section 5·1 of [**12**] where they appear as a step in an alternative characterization of hyperbolic maps.

We show in Section 3 how to use two theorems in [**3**] to obtain Theorem 1 and a number of corollaries.

In Section 4 we shall apply the formula of Theorem 1 to study the continuity of the *Julia function*

$$J : \mathbb{C} \to \mathscr{H}$$

which associates to each point c in the parameter plane $\mathbb{C}$ the Julia set J_c of the quadratic function $P_c(z) = z^2 + c$. The continuity of this function is relevant in computer graphics especially at points on the boundary ∂M of the Mandelbrot set M and has been studied by a number of authors [**1**, **6**, **8**, **9**]. Here the expression of Julia set as the limit of inverse image sets of a point gives a method of proof which avoids assumptions about non-hyperbolic maps in the interior of the Mandelbrot set.

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2. *Background material*

Our main reference is [**3**]. We shall just quote here some basic facts in order to make sense of statements made in this paper. Let n be reserved for non-negative

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integers. Given a rational map $R: \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the Riemann sphere $\mathbb{C}_\infty$ we let R^n denote the n-fold composite of R with itself and let $R^{-n}(G)$ be the inverse image of a set $G \subset \mathbb{C}_\infty$ under R^n. Conventionally R^0 is the identity map.

The main body of iteration theory of a rational function R on the Riemann sphere applies to functions of degree at least two and we shall assume this condition throughout. The goal of the theory is to study the dynamics of R on $\mathbb{C}_\infty$. This involves, in particular, tracing the forward and backward orbits

$$\bigcup_{n>0} R^n(z), \quad \bigcup_{n\geqslant 0} R^{-n}(z)$$

of a point $z \in \mathbb{C}_\infty$ and investigating typical problems like the existence of $\lim_{n\to\infty} R^n(z)$ and the meaning of $\lim_{n\to\infty} R^{-n}(z)$. If $\alpha = \lim_{n\to\infty} R^n(z)$ exists then α is a *fixed point* of R and the set of z such that $\alpha = \lim_{n\to\infty} R^n(z)$ is called the *basin of attraction* of α.

According to the classical dichotomy, the Riemann sphere splits into a disjoint union of two sets

$$\mathbb{C}_\infty = J \cup F$$

both of which are invariant under forward and backward iteration by R (and are thus called *completely invariant*). The *Fatou set* F of R is the maximal open set on which the family of iterates R^n for $n \geqslant 0$ is equicontinuous. The *Julia set* J is the complement of F. It is known that J is an uncountably infinite compact set on which the dynamics of R is chaotic. Arbitrarily close points in J can have forward orbits that behave in a very different manner; whereas all points in the same connected component of the Fatou set have the same future.

Of special interest in the dynamics of R are points with finite forward orbits. The points of a cycle of period n are fixed points of R^n and the derivative of R^n has constant value on such a cycle. Cycles are classified into *attractive*, *repelling* or *indifferent* according as the value of the modulus of the derivative of R^n is less than, greater than or equal to 1 at points of the cycle. If the derivative is zero then the cycle is called *super-attractive* (in [3] super-attractive excludes attractive). The indifferent cycles are further subclassified into the *parabolic case* where the derivative of R^n is a root of unity (also called the rational case) and the *irrational case*. Attractive cycles lie in the Fatou set, repelling cycles and parabolic cycles lie in the Julia set. There are examples of irrational cycles where the dynamics of R^n near a fixed point on the cycle is conformally equivalent to a Euclidean rotation through an irrational angle. These are called *Siegel cycles* or *linearizable irrationally indifferent cycles* and they belong to the Fatou set. According to the Classification Theorem of Sullivan [13] all points of the Fatou set are eventually drawn by iteration towards an attractive or parabolic cycle or into a cycle of Siegel discs or Herman rings. Up to conformal equivalence a Herman ring is a Euclidean annulus in which R^n acts by rotation.

Points where the derivative of R is zero are called *critical points*. They have a strong influence on the global dynamics of R. For example the basin of attraction of an attractive or parabolic cycle must contain a critical point.

In the case of a polynomial P, the point at infinity is always a super-attractive fixed point and therefore a member of the Fatou set. The point at infinity is *exceptional*, meaning that its forward and backward orbits are finite. Up to conjugacy by Möbius transformations exceptional points can only occur for polynomials and the functions z^{-d}, where the origin and the point at infinity are exceptional. Herman rings cannot occur for polynomials but Siegel discs can.

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The connected component of the Fatou set containing ∞ is also the basin of attraction of ∞ in the case of a polynomial P. The complementary set is called the *filled-in Julia set* of P. It contains all bounded orbits and is completely invariant. The Julia set of P is the boundary of the filled-in Julia set.

The most familiar family of polynomials considered in this subject is the quadratic family $P_c(z) = z^2 + c$ for complex values of c. There are a number of difficult problems concerned with the topology of the Julia sets J_c associated with the functions P_c in this family. The first main classification into topological types is with reference to the *Mandelbrot set* M which is defined as the subset of those c in control space for which J_c is connected. For pictures of the Mandelbrot set with associated Julia sets see for example [**11**].

The only critical point of $P_c(z) = z^2 + c$ in the finite part of dynamic space is the origin. The Mandelbrot set can alternatively be described as the set of points c in parameter space for which the forward orbit of the origin is bounded. If P_c has an attractive or parabolic orbit in the finite plane then it can have only one such orbit and the origin is attracted to that orbit. The parameters c for which P_c has an attractive cycle constitute an open set in the interior of M. Its components are called *hyperbolic components*. It is conjectured that these form the entire interior of M. The parameter c when viewed in dynamic space is on the forward orbit of the origin because $P_c(0) = c$. If the origin is periodic then so is c and satisfies a polynomial equation $P_c^n(c) = c$ for some n. Such points c lie in hyperbolic components of the control space and are called *centres of hyperbolic components*.

If P_c possesses an indifferent cycle then c lies in the Mandelbrot boundary. Such points are dense in the ∂M. A point c is called a *Misiurewicz point* if the origin is preperiodic, in other words some iterate of the origin is periodic, but the origin itself is not periodic. Misiurewicz points lie on Mandelbrot boundary and are dense in ∂M. They can be approximated arbitrarily closely by hyperbolic centres [**5**].

In order to handle the topology of sets we shall use the *Hausdorff metric* on the collection $\mathscr{H}$ of all non-empty compact subsets of the sphere. Intuitively a set A is close to a set B in the Hausdorff metric if each point of A is close to some point of B and each point of B is close to some point of A. The technical definition goes as follows: let $d(x, y)$ be the standard distance between two points x, y in the sphere; let $d(x, Y)$ denote the distance of x from the set Y. Then the Hausdorff distance is defined by

$$h(A,B) = \max\{\max_{a\in A} d(a,B), \max_{b\in B} d(b,A)\}$$

(see for example [**2**]).

3. *The Julia set as a limit*

We aim to prove Theorem 1, using results in [**3**] which stem from Montel's Theorem in complex analysis rather than Sullivan's quasiconformal surgery techniques.

THEOREM 1. *Let $R(z)$ be a rational function of degree at least 2 in the complex variable z and let G be a non-empty compact subset of the Riemann sphere $\mathbb{C}_\infty$ which does not contain an exceptional point or any accumulation point of the iterates $R^n(z)$ of any z in the Fatou set of R. Then the Julia set J of R is given by the formula*

$$J = \lim_{n\to\infty} R^{-n}(G),$$

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where the limit is with respect to the Hausdorff metric on non-empty compact subsets of $\mathbb{C}_\infty$ *and* $R^{-n}(G)$ *denotes the inverse image of* G *under* R^n.

To make the nomenclature easier, we shall call a point which is an accumulation point of $\{R^n(z): n \in \mathbb{N}\}$ for some z in the Fatou set of R an *orbital cluster point*. An exceptional point of a rational function R is always an orbital cluster point unless R is Möbius conjugate to $z \mapsto 1/z$.

As a special case of Theorem 1 we have

COROLLARY 2. *Let* α *be a point of* $\mathbb{C}_\infty$ *which is not an orbital cluster point. Then*

$$J = \lim_{n\to\infty} R^{-n}(\alpha).$$

The condition about orbital cluster points in the statement of Theorem 1 ensures that, for instance, G contains no points in Siegel discs. Such a point would always have backward iterates on circles inside such discs, and so would lie a finite distance from the Julia set. The backwards orbit could not therefore converge to the Julia set. However, there can be no such point in the basin of an attractive cycle, with the possible exception of the cycle itself.

The proof of Theorem 1 comes from the following two results. The first, theorem 4·2·8 of [3], and the remark that follows it, show that the Julia set is indeed the limit of backward iteration of a set G satisfying the given conditions but the limit here is meant in a weaker sense of semi-continuity.

THEOREM 3. *Given an open set* U *with* $J \subset U$, *and a compact set* G *containing no orbital cluster points of* R, *there exists a positive integer* N *such that* $R^{-n}(G) \subset U$ *for all* $n > N$.

This result tells us that points of $R^{-n}(G)$ can be made arbitrarily close to J for n large enough but this is only one half of the requirement for the Hausdorff limit. To furnish us with the other half we need to quote the following theorem 6·9·4 of [3].

THEOREM 4. *Let* W *be a domain meeting* J *and suppose compact set* G *contains no exceptional points of* R. *Then for all* n *sufficiently large* $G \subset R^n(W)$.

We are now in a position to prove Theorem 1 for G a compact set satisfying the hypotheses. Let $\epsilon > 0$. First of all we cover the Julia set J with a finite number of discs

$$D_\epsilon(z_1), \ldots, D_\epsilon(z_r)$$

of radius ϵ and centres $z_1, \ldots, z_r$ on J. This is possible by compactness of J. Let $U = \bigcup D_\epsilon(z_i)$ be the union of the discs. According to Theorem 3 there exists an integer N_0 such that $R^{-n}(G) \subset U$ for all $n > N_0$. According to Theorem 4 there exist integers $N_1, \ldots, N_r$ such that for each i, the intersection of $D_\epsilon(z_i)$ with $R^{-n}(G)$ is non-empty for all $n > N_i$. Choosing $N > N_i$ for $i = 0, \ldots, r$ we see that, for all $n > N$, every point of $R^{-n}(G)$ is within ϵ of some point of J, namely the centre of one of the discs $D_\epsilon(z_i)$ and at the same time each point of J is within 2ϵ of some point of $R^{-n}(G)$ because the intersection of this set with each disc is non-empty. This completes the proof of Theorem 1. ∎

Now that we have obtained the expression $J = \lim_{n\to\infty} R^{-n}(\alpha)$ under some

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sufficient conditions on α, it is natural to investigate how to work with these conditions and to what extent they are necessary. For instance, we have observed that an α which converges to an attractive cycle under iteration by R will generally satisfy the hypotheses of Theorem 1. More specifically, we can look at the case in which R is a member of the quadratic family $P_c(z) = z^2 + c$ with Julia set denoted J_c. Here any choice of α in the Fatou component at infinity apart from infinity itself will do the job, but it would be useful to find a single value of α which works for most of the P_c.

THEOREM 5. *The following conditions are equivalent*: [A] *The Julia set of* P_c *is given by the formula* $\lim_{n\to\infty} P_c^{-n}(0)$. [B] 0 *is not periodic under* P_c.

Proof. Case (i). First, take 0 to be in the Julia set. Here, both conditions [A] and [B] always hold. The Julia set is completely invariant, so $P_c^{-n}(0) \subset J_c$ for all n and taking the origin to be the compact set G we obtain the conclusion of Theorem 3. Exceptional points of polynomials must be members of the Fatou set, so Theorem 4 still applies. Consequently Corollary 2 is true with $\alpha = 0$ which gives us condition [A]. We obtain condition [B] by observing that if the origin is periodic then it must be a member of super-attracting cycle and so must be in the Fatou set.

Case (ii). 0 is in the Fatou set. Suppose condition [B] fails; that is, 0 is periodic with some period p. Since 0 must be an element of $P_c^{-kp}(0)$ for all k it follows that 0 must be an element of $\lim_{n\to\infty} P_c^{-n}(0)$ if this limit exists, which then cannot be the Julia set of P_c. Hence [A] fails.

Now suppose that condition [B] holds. From Sullivan's Classification Theorem we know that points in the Fatou set are eventually drawn under iteration into attracting, parabolic or Siegel cycles. A result of Herman[7] states that if a critical point of a polynomial of the form P_c is drawn to a Siegel cycle, then this critical point must be in the Julia set. Thus 0 may only fall into an attracting or parabolic cycle. If 0 is in the basin of an attracting cycle, the only way it can be an orbital cluster point is by being a member of this cycle, which is ruled out by condition [B]. If 0 falls into a parabolic cycle, contained in the Julia set, then again it cannot be an orbital cluster point. Hence Corollary 2 gives us condition [A]. ∎

Remark. In other words the Julia set of P_c is given by the limit of backward iteration of the finite critical point except when the point c is a hyperbolic centre.

To illustrate how results about Julia sets can be derived from Theorem 1, here is an elementary corollary which is well known.

PROPOSITION 6. *If* P *is a polynomial whose critical points are all contained in its filled-in Julia set, then the Julia set of* P *is connected.*

This follows because we can choose G to be a circle centred at 0 and of large enough radius that it is contained in the Fatou component at infinity. Then G satisfies the conditions of Theorem 1 so $J = \lim_{n\to\infty} P^{-n}(G)$. Also $P^{-1}(G)$ is another simple closed loop since G encircles all the critical values of P (thinking in the complex plane), and is once again contained in the Fatou component at infinity by complete invariance. Thus each of the sets $P^{-n}(G)$ is connected. Because the limit of connected sets in the Hausdorff metric is connected, we have that the limit of the above sequence, the Julia set of P, is also connected.

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4. *Continuity of the Julia set*

We now begin our investigation of continuity with reference to the work in the previous sections and prove the following result

THEOREM 7. (*i*) *The function* $J: \mathbb{C} \to \mathscr{H}$ *is continuous in the complement and in the interior of* M.

(*ii*) *The Julia function is not continuous at any point* c *on* ∂M *for which* $0 \notin J_c$. *It is not continuous at points* c *in* ∂M *where the Fatou set of* P_c *has a Siegel disc cycle.*

The continuity question for the Julia function has been considered by a number of authors [**1**, **8**] who show that the Julia function is continuous where the Julia set is equal to the filled-in Julia set. This is a consequence of the fact that the filled-in Julia set is an upper semi-continuous function whereas the Julia function itself is lower semi-continuous. For example it is known that the Julia function is continuous at Misiurewicz points on ∂M.

In the quadratic case there is no shortage of points α that can be chosen to satisfy the hypotheses of Corollary 2. Now for a fixed n the set $P_c^{-n}(\alpha)$ is the set of solutions of the monic polynomial equation $P_c^n(z) = \alpha$ of degree 2^n and varies continuously with c in the Hausdorff metric. To make further progress towards the proof of Theorem 7(*i*) we need to understand how the limiting process in Corollary 2 depends on c in the case of $R = P_c$. Following Brolin [**5**] and other authors we introduce the family $Q_{n,j}(c)$ of roots of the polynomial equations $P_c^n(z) = \alpha$ extracted as analytic functions of the parameter c in a suitable disc as n ranges over the positive integers and for each stage n the integer j ranges from 1 to 2^n. The details now follow.

THEOREM 8. *Let* D *be a disc in parameter space and* α *a point such that the equation*

$$P_c^n(c) = \alpha$$

has no solutions for any integer $n \geqslant 0$ *and* $c \in D$. *Then for* $n > 0$ *the roots of the polynomial equation* $P_c^n(w) = \alpha$ *can be extracted continuously giving rise to* 2^n *analytic branches* $w = Q_{n,j}(c)$ *satisfying the identity* $P_c^n(Q_{n,j}(c)) = \alpha$ *for all* $c \in D$ *and* $1 \leqslant j \leqslant 2^n$.

The proof is by induction on n, using the general fact that if $A(c)$ is an analytic function of c defined in a disc on which $A(c)$ never assumes the value zero then the two square roots $\pm\sqrt{A(c)}$ are well defined analytic functions on that disc. The induction starts with the observation that the equation

$$c = \alpha$$

has no solution for c in D by our assumption in case $n = 0$. Hence the square roots

$$Q_{1,1}(c) = \sqrt{(\alpha - c)}, \quad Q_{1,2}(c) = -\sqrt{(\alpha - c)}$$

are well defined and analytic functions on D. Assume at stage n that the functions $Q_{n,j}(c)$ have been defined to satisfy the required conditions. If the equation $Q_{n,j}(c) = c$ has a solution for $c \in D$ then $P_c^n(Q_{n,j}(c)) = \alpha = c$ which is ruled out as above. Hence the functions

$$Q_{n+1,j}(c) = \sqrt{(Q_{n,j}(c) - c)}, \quad Q_{n+1,j+2^n}(c) = -\sqrt{(Q_{n,j}(c) - c)}$$

are well defined and analytic for $1 \leqslant j \leqslant 2^n$. These 2^{n+1} functions satisfy the identity

$$\begin{aligned} P_c^{n+1}(Q_{n+1,j}(c)) &= P_c^n(P_c(Q_{n+1,j}(c)) = P_c^n((Q_{n+1,j}(c))^2 + c) \\ &= P_c^n((Q_{n,j}(c)) = \alpha, \end{aligned}$$

for all $c \in D$ and $1 \leqslant j \leqslant 2^{n+1}$. This completes the inductive proof.

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The functions $Q_{n,j}(c)$ pick out the roots of the polynomial equation $P_c^n(w) = \alpha$ continuously for $c \in D$. With the notation and hypotheses of Theorem 8 we have the improved result:

THEOREM 9. *The family $Q_{n,j}$ is equicontinuous on D.*

To prove this result we show in fact that the functions $Q_{n,j}$ form a bounded family. The function P_c is super-attractive at infinity. In fact it is easy to check directly that there exists a positive number B such that for $|z| > B$ we have $|P_c^n(z)| > |z|$ for all $n > 0$. In particular if B is chosen larger than $|\alpha|$ then the equation $P_c^n(\beta) = \alpha$ cannot have a solution for any β with $|\beta| > B$. Furthermore B can be chosen independently of c in the disc D. This is equivalent to saying that β is not in the image of any function $Q_{n,j}$. For suppose otherwise: then $\beta = Q_{n,j}(c)$ for some n, j and $c \in D$. Applying P_c^n and the definition of $Q_{n,j}$ gives $P_c^n(\beta) = \alpha$ which contradicts the above statement. In particular the family $Q_{n,j}$ misses three points and therefore by Montel's Theorem the family is equicontinuous.

Now we are in a position to complete the proof of Theorem 7 (i). Note first of all that by definition of the functions $Q_{n,j}$ in Theorem 8 we immediately have the identity

$$P_c^{-n}(\alpha) = \bigcup_{j=1}^{2^n} Q_{n,j}(c).$$

Given $\epsilon > 0$, we use Theorem 9 with D sufficiently small to ensure that in the standard metric d for parameter space the following condition holds for the individual roots:

$$d(Q_{n,j}(c_0), Q_{n,j}(c_1)) < \epsilon$$

for all c_0, c_1 in D and all n, j. Taking the union of the roots over j, this implies that in terms of the Hausdorff metric

$$h(P_{c_0}^{-n}(\alpha), P_{c_1}^{-n}(\alpha)) < \epsilon$$

for all c_0, c_1 in D and all n. Now for c_0, c_1 fixed in D we use Corollary 2 to find N for which

$$h(P_{c_0}^{-N}(\alpha), J_{c_0}) < \epsilon, \quad h(P_{c_1}^{-N}(\alpha), J_{c_1}) < \epsilon.$$

Then by the usual argument with the triangle inequality

$$h(J_{c_0}, J_{c_1}) \leqslant h(J_{c_0}, P_{c_0}^{-N}(\alpha)) + h(P_{c_0}^{-N}(\alpha), P_{c_1}^{-N}(\alpha)) + h(P_{c_1}^{-N}(\alpha), J_{c_1}) < 3\epsilon$$

which completes the proof of Theorem 7 (i) providing that we can find values of α which satisfy the hypotheses of Corollary 2 and Theorem 8. The following result provides such values.

THEOREM 10. (i) *Let D be a disc in the complement of the Mandelbrot set. Then $\alpha = 0$ satisfies the conditions of Corollary 2 and Theorem 8.*

(ii) *Let D be a disc in the interior of the Mandelbrot set. Then any α with $|\alpha| > 2$ satisfies the conditions of Corollary 2 and Theorem 8.*

To prove part (i) we recall that, for c in the complement of the Mandelbrot set, the origin converges to ∞ under iteration by P_c. In particular the origin is not an accumulation point of the iterates of any other point. This verifies the hypotheses of

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Corollary 2. With reference to Theorem 8 suppose $P_c^n(c) = 0$ for some $c \in D$ and some $n \geqslant 0$. Since $c \neq 0$ and $c = P_c(0)$ this implies that the origin is periodic, contradicting the previous observation that the origin converges to ∞. This verifies the conditions of Theorem 8.

To prove part (ii) of the theorem, we use the fact that for $|c| \leqslant 2$, in particular for $c \in M$, the Julia set J_c lies in the disc of radius 2 centre the origin in dynamic space. Hence α belongs to the component of the Fatou set at ∞ and the hypotheses of Corollary 2 follow. On the other hand for c in the Mandelbrot set the forward orbit of the origin is bounded by the disc $|z| \leqslant 2$. Since $c = P_c(0)$ it follows that the equation $P_c^n(c) = \alpha$ has no solutions under the given conditions. This verifies the hypotheses of Theorem 8 and completes the proof of Theorem 7(*i*).

The methods used to prove Theorem 7(*i*) break down for points on the boundary of the Mandelbrot set. In fact the family of functions $P_c^n(c)$ is not a normal family on D and the disc D will therefore be expanded as $n \to \infty$ to fill up the whole plane. The hypotheses of Theorem 8 cannot therefore be realized in this case. This does not of course rule out the possibility that the Julia function is continuous at some points on the Mandelbrot boundary by a different argument. However it is easy to see that the Julia function cannot be continuous everywhere on ∂M. Consider a point $c \in \partial M$ for which $0 \notin J_c$. Then the origin is a positive distance from the Julia set. On the other hand we can find a sequence of Misiurewicz points c_n converging on c such that $0 \in J_{c_n}$ (see section 9 of [**4**]). Hence the sequence of Julia sets J_{c_n} does not converge to J_c in the Hausdorff metric. This proves Theorem 7(*ii*). ∎

The same argument can be applied in a slightly different way to Julia sets with Siegel discs. Consider a point c_0 on the boundary of the main cardioid of the Mandelbrot set (chapter 2 of [**11**]) such that one of the fixed points u_0 of P_{c_0} is irrationally indifferent and lies in the Fatou set. Such points are known to exist and the component of the Fatou set containing u_0 is then a Siegel disc whose boundary is part of the Julia set J_{c_0}. In particular u_0 is a positive distance from J_{c_0}. On the other hand there are points c on the boundary of the main cardioid arbitrarily close to c_0 for which the corresponding fixed point u is close to u_0 but is of the parabolic type (rationally indifferent). In this case u is in the Julia set J_c. Hence as c approaches c_0 through parabolic points the Julia set J_c remains bounded away from J_{c_0} in the Hausdorff metric. Hence the Julia function is not continuous at these points c which give rise to Siegel discs.

On the other hand, it has been shown that the Julia function is continuous at Misiurewicz points. Their Julia sets are *dendrites* – connected and locally connected sets with empty interior and simply-connected complement, and the continuity can be shown by results on set-valued functions. In [**1**] they work with the *upper semi-continuous regularization* J^* of the Julia function defined by

$$J_c^* = \bigcap_{r>0} \left[\overline{\bigcup_{|c-c'|<r} J_{c'}} \right],$$

which contains the Julia set of P_c and is contained by the filled-in Julia set of P_c. At Misiurewicz points it is enough that the Julia set is the same as the filled-in Julia set to establish continuity of the Julia function. Similar work has also been done on continuity of the Julia set by Kriete[**8**].

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