ON 3N POINTS IN A PLANE

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In this note I prove the following theorem:

THEOREM 1. *Given* $3N$ *points in a plane, we can divide them into* N *triads such that, when we form a triangle with the points of each triad, the* N *triangles will all have a common point.*

The proof depends on three lemmas; the first two are well known, and are quoted without proof; the third I believe to be new, and leads directly to the proof of the theorem.

E^n denotes the solid unit ball in n-dimensional space R^n; Σ^{n-1} is the surface of E^n, and S^{n-1} too denotes the surface of a sphere.

LEMMA 1. (Fixed point theorem.) *If a continuous vector field be defined for all points of* E^n, *with the vectors pointing inwards at all points of* Σ^{n-1}, *then there is a point inside* E^n *at which the field vanishes.*

LEMMA 2. (Caratheodory.) *Given a set* A *in Euclidean* m*-space and any point* P *in the convex cover of* A, *we can find* $(m+1)$ *points* $a_0, \ldots, a_m$ *belonging to* A *which span a (closed) simplex containing* P ($a_0, \ldots, a_m$ *need not be distinct*).

LEMMA 3. *Let a mass-distribution in* E^n *be defined by an integrable density function* $\rho(x)$. (Thus $\rho(x) = 0$ for $x \notin E^n$.) *Then we can find a point* r *inside* E^n *so that every closed half-space with* r *on its boundary will contain at least* $1/(n+1)$ *of the total mass.*

Proof. Suppose the lemma untrue, so that for every point q of E^n there is at least one half-space which contains less than $1/(n+1)$ of the mass. We will obtain a contradiction. We first note that, given a point q, there is a one-one correspondence between the unit vectors from q and the half-spaces with q on their boundary; in fact, to the vector j will correspond the half-space given by $j \cdot (x-q) \geqslant 0$. Fixing our point q, we give a vector j the weight

$$w(j,q) = \left(\lambda - \frac{1}{n+1}\right)$$

if the corresponding half-space contains a proportion λ of the mass, where

$$\lambda < \frac{1}{n+1};$$

otherwise, we give it weight zero. Then $w(j,q)$ is continuous as a function of j and q, being an integral. We can regard the unit vectors j as points of a unit sphere S^{n-1}; take the integral

$$\int w(j,q)\, j\, dS^{n-1}$$

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over the whole of S^{n-1}. This integral gives a vector $k(q)$; and $k(q)$ is a continuous function of q.

We have thus defined a continuous vector field $k(q)$, whose vectors evidently point inwards at all points of Σ^{n-1}. By Lemma 1, there is a point, r say, where the field vanishes. Consider the vectors from r as points of a unit sphere S^{n-1}. Denote by A the set of points j of S^{n-1} which were given a non-zero weight $w(j,r)$. By the hypothesis we desire to contradict, A is non-empty, and by the definition of the weighting function, A is open in S^{n-1}.

We assert that the centre of S^{n-1} lies in $H(A)$, the convex cover of A. For if not, there is a closed half-space (and hence an open half-space, since A is open) which contains A and has the centre of S^{n-1} on its boundary; that is, the set A is contained in a single hemisphere of S. But in this case, the vector integral

$$\int_{j \in A} w(j,r) j \, dS = k(r)$$

cannot possibly vanish, and we have a contradiction.

Applying Lemma 2, it follows that we can choose $(n+1)$ points $j_0, \ldots, j_n$ of A which span a simplex containing the centre of S^{n-1}. Every vector accordingly makes an angle less than or equal to $\frac{1}{2}\pi$ with at least one of $j_0, j_1, \ldots, j_n$; it follows that the $(n+1)$ closed half-spaces

$$j_i \cdot (x - r) \geqslant 0 \quad (i = 0, \ldots, n)$$

together account for the whole of space. But by our construction, they each contain less than $1/(n+1)$ of the total mass. We have thus obtained the contradiction we need.

COROLLARY 3·1. *The lemma remains true even if the mass distribution is no longer specified by a density function, e.g. we may allow concentrated point masses.*

In fact, such a distribution D will be the limit of a sequence of distributions corresponding to density functions; call such a sequence of density functions $\{\rho_n\}$. Then for each ρ_n there is a point r_n with the property described in the lemma; $\{r_n\}$ is a bounded infinite sequence of points, and so has at least one limit point, r say. We now see easily that any closed half-space with its boundary passing through r contains at least $1/(n+1)$ of D.

COROLLARY 3·2. *Let Y be a finite set consisting of M points in n-space, and suppose that $M > (n+1)R$. Then there is a point common to all the closed half-spaces which contain at least $(M-R)$ points of Y.*

Indeed, considering the set Y as a distribution of M equal point-masses, we see that all closed half-spaces containing $(M-R)$ points of Y must contain the point r constructed in Lemma 3 and its first corollary; for, if one did not, there would be another half-space opposite to it which had r on its boundary but contained no more than R points.

Proof of Theorem 1. We are given $3N$ points in a plane. By the corollary above, there is a point P common to all half-planes which contain $(2N+1)$ of the given points. We can set up a coordinate system with origin at P, so that the $3N$ given points have polar coordinates $(r_1, \theta_1), (r_2, \theta_2), \ldots, (r_{3N}, \theta_{3N})$. Number off the $3N$ points in order of increasing θ, and call them $b_1, \ldots, b_{3N}$. Now we assign the points to N triads, putting b_i

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and b_j into the same triad if and only if $i \equiv j \pmod{N}$. In order to prove the theorem, it will be enough to show that the triangle $b_i b_{i+N} b_{i+2N}$ contains P for all $i = 1, \ldots, N$. This is easy to prove; in fact, if it were false, we would have $\theta_{j+N} - \theta_j > \pi$ for some j (suffices taken modulo $3N$); but there are at most $(N-1)$ points with θ between θ_{j+N} and θ_j, and so there is a half-plane containing $(2N+1)$ of the given points, but not P. This is a contradiction.

We conclude with a few remarks and references.

(1) By a slight modification in the above argument, we can prove the apparently stronger result:

THEOREM 1*. *Given a plane set Y consisting of $(3N-2)$ points, we can split it into N subsets $Y_1, \ldots, Y_N$ so that there is a point common to all the convex covers $H(Y_i)$.*

(2) Theorem 1* is very similar to the results of Rado (7). Given a dimension n and an integer N we may define $f(n, N)$ as the smallest integer with the following property: *Let Y be any finite n-dimensional set consisting of M points, where $M \geqslant f(n, N)$. Then we can split Y into N subsets $Y_1, \ldots, Y_N$ so that there is a point common to all the convex covers $H(Y_i)$.* It is easy to see that

$$f(n, N) \geqslant (n+1)N - n,$$

and in fact
$$f(1, N) = 2N - 1. \tag{1}$$

Rado's principal result is
$$f(n, N) \leqslant 2f(n-1, N) - n, \tag{2}$$

and he combines this with (1) to obtain his Theorem 1, which asserts

$$f(n, N) \leqslant 2^n(N-2) + n + 2; \tag{3}$$

in particular, when $N = 2$ this is $f(n, 2) = n + 2$, which is substantially Helly's theorem. The two-dimensional case of Rado's theorem is $f(2, N) \leqslant 4N - 4$; our Theorem 1* improves this to the best possible result

$$f(2, N) = 3N - 2,$$

and of course we can do rather better than (3) by combining this with (2). This improvement is hardly a valuable one—but it would be nice to know whether

$$f(n, N) = (n+1)N - n$$

is true in general.

Lemma 3 leads by a very simple argument to an estimate for $f(n, N)$ which is substantially stronger than Rado's when n and N are large. Let Y be an n-dimensional set consisting of $[n(n+1)(N-1)+1]$ points; then by Corollary 3·1 there is a point p such that every closed half-space containing p contains at least $[(N-1)n+1]$ points of Y. p is in the convex cover of Y, so by Lemma 2 there is a subset Y_1 of Y consisting of at most $(n+1)$ points such that $H(Y_1)$ is a simplex containing p, and every half-space containing p contains at least $[(N-1)n+1-n]$ points of $Y - Y_1$. Repeating the argument $(N-1)$ times, there are $(N-1)$ disjoint subsets $Y_1, \ldots, Y_{N-1}$ of Y such that p is common to $H(Y_1), \ldots, H(Y_{N-1})$, and every half-space containing p contains a point of $Y - \bigcup_{r=1}^{N-1} Y_r = Y_N$, say; thus p is in $H(Y_N)$. It follows that

$$f(n, N) \leqslant Nn(n+1) - n^2 - n + 1.$$

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(3) Lemma 3 may be found more interesting than Theorem 1; so far as I know, it is new. The plane case of the lemma was proved in 1945 by Neumann (5); his proof, though elementary, is long, and does not extend to higher dimensions. Lemma 3 is clearly best possible, as we see when we consider the distribution consisting of $(n+1)$ point-masses.

(4) The method of Lemma 3 may be applied to several other problems which involve estimating

$$\min_x \max_j [g(x,j)],$$

where x runs through the points of E^n, j runs through the unit vectors corresponding to the points of an S^{n-1}, and $g(x,j)$ is a function of some class $\mathscr{G}$. For instance, Neumann (6) considered the plane case of another problem of this type. Let K be a closed convex body with interior points. Let p be a point inside K; then to any vector j there corresponds a directed chord of K through p. Suppose that p divides this chord so that the ratio of the segment in the positive direction from p to the whole chord is $\lambda:1$. Then λ is a continuous function of p and j for p in the interior of K. Let $r(p)$ be the maximum of λ for all vectors j; obviously $r(p)$ is a continuous function, $\frac{1}{2} \leqslant r(p) \leqslant 1$, and by compactness $r(p)$ attains its minimum r^*. By a straightforward application of the ideas of Lemma 3, we can prove

LEMMA 4. *Suppose that $r(q) = r^*$. Then there are $(n+1)$ vectors $j_0, \ldots, j_n$, possibly not all distinct, for all of which*

$$\lambda(q,j) = r(q) = r^*,$$

and which (viewed as points of S^{n-1}) span a simplex containing the centre of S.

From this lemma, we deduce

COROLLARY 4·1.

$$r^* \leqslant \frac{n}{n+1}, \tag{4}$$

with equality if and only if K is a simplex.

In fact, suppose that the chord through q parallel to j_i meets the boundary of K in s_i and t_i, so that qs_i is in the direction of j_i. Then q is in the simplex $s_0 s_1 \ldots s_n$. Drop duplicates, so that q lies in a simplex $s_0 s_1 \ldots s_m$, where $s_0, \ldots, s_m$ are all distinct, and $m \leqslant n$. Let the barycentric coordinates of q with respect to this simplex be $(\alpha_0, \alpha_1, \ldots, \alpha_m)$, and let the line $s_i q$ meet the opposite face of the simplex in the point u_i. Then

$$s_i q / s_i u_i = 1 - \alpha_i.$$

Also, since K is convex,

$$s_i u_i \leqslant s_i t_i \quad \text{for} \quad i = 0, \ldots, m; \tag{5}$$

and so

$$r^* = \frac{s_i q}{s_i t_i} \leqslant \frac{s_i q}{s_i u_i} = 1 - \alpha_i. \tag{6}$$

Summing (6) over $i = 0, \ldots, m$, we see that

$$(m+1)\, r^* \leqslant m + 1 - \Sigma \alpha_i = m,$$

so that

$$r^* \leqslant \frac{m}{m+1} \leqslant \frac{n}{n+1}.$$

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This proves (4). Further, equality can be necessary in (4) only if $m = n$ and there is equality in (5) and (6) for every i. This implies that $\alpha_i = 1/(n+1)$ and $t_i = u_i$ for each i. Thus K is just the simplex $s_0 s_1 \dots s_n$, and q is its barycentre.

Neumann (6) proved this corollary in the plane case. The n-dimensional inequality (4) was proved very simply by Süss (8), and independently by Hammer (3); both of them showed that, if g is the centroid of K,

$$r(g) \leqslant \frac{n}{n+1},$$

and from this (4) follows *a fortiori*; neither author gave the conditions for equality in (4). Another simple proof of (4) is suggested by Eggleston (2) as an exercise.

We may also apply Lemma 4 to sets of constant width; in this way we obtain yet another proof of Jung's theorem, that a set of diameter D is contained in a sphere of radius R, with $R \leqslant [n/\{2n+2\}]^{\frac{1}{2}} D$; see Jung (4). One very simple proof of this theorem has been given by Blumenthal and Wahlin (1), and an even more elegant one is contained in (2).

In conclusion, I would like to thank Professor Eggleston for his entertaining lectures, which led me to perpetrate this note.

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