Finite groups generated by reflections, and their subgroups generated by reflections. By H. S. M. COXETER, Ph.D., Trinity College.

[*Received* 11 August, *read* 29 October 1934.]

In connection with his work on singularities of surfaces, Du Val asked me to enumerate certain subgroups in the symmetry groups of the "pure Archimedean" polytopes n_{21} $(n < 5)$, namely those subgroups which are generated by reflections. For the sake of completeness, I have enumerated such subgroups of all the discrete groups generated by reflections (including the symmetry groups of the *regular* polytopes). The work involved being somewhat intricate, several slips would have been overlooked but for the information that Du Val was able to supply from the (apparently remote) theory of surfaces.

The references to my previous papers are as follows:

1. —— "The polytopes with regular-prismatic vertex figures", *Phil. Trans. Royal Soc.* (A), 229 (1930), 329—425.
2. —— "Discrete groups generated by reflections", *Annals of Math.* 35 (1934), 588—621.
3. —— "Wythoff's construction for uniform polytopes", *Proc. London Math. Soc.* (in the press).

1. A discrete group generated by reflections may be either *reducible* or *irreducible**. In the former case it is the direct product of several *irreducible components*, which are themselves generated by reflections. Every irreducible group generated by reflections occurs in a definite list of twenty special types†, ten finite and ten infinite.

Two alternative notations are used: one, a system of dots and links that may be called a *graph*‡, and one (easier to print, but less systematic) consisting of numbers enclosed within square brackets (or, in one of the infinite cases, within a complete square). Such numbers are the periods of the rotations that arise as products of pairs of reflections. It is sometimes convenient to classify the irreducible groups according to the greatest period-number that occurs. Apart from the "k-gonal dihedral group", $[k]$, and the infinite groups $[\infty]$, $[3, 6]$, every one of these groups is either *trigonal, tetragonal* or *pentagonal*, the trivial group $[\,]$ being reckoned as trigonal (*qua* $[3^0]$).

2. In the present paper, being chiefly concerned with the *finite* groups, we simplify the "bracket" notation by writing $[3^{n,p,q}]$ for $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}$. After rearrangement in accordance with the

* **2**, 595. † **2**, 601. ‡ **2**, 619.

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

above classification, the list of finite irreducible groups becomes

$$[3^n], \quad [3^{n,1,1}], \quad [3^{2,2,1}], \quad [3^{3,2,1}], \quad [3^{4,2,1}],$$
$$[3^n, 4], \quad [3, 4, 3],$$
$$[3, 5], \quad [3, 3, 5],$$
$$[k].$$

There is some overlapping between the ten types, since

$$[3^1] = [3], \quad [3^2] = [3, 3] = [3^{0,1,1}]^*$$

and

$$[3^0, 4] = [4].$$

When $n = 0$, $[3^n]$ is [ ], the group of order 2, generated by a single reflection. We shall find it useful to make the further conventions

$$[3^{-1,1,1}] = [\ ] \times [\ ] = [\ ]^2, \quad [3^{-1}, 4] = [\ ],$$

which are justified by the graphical notation. We generally suppose that $k \geqslant 3$, but the conventions

$$[2] = [\ ]^2, \quad [1] = [\ ]$$

are useful in summarizing the subgroups of $[k]$.

In each of the trigonal and pentagonal groups, and in $[k]$ (k odd), all reflections are conjugate†. But the reflections in each of the remaining groups fall into two non-conjugate sets, belonging to other groups in accordance with the following scheme:

$$[3^n, 4] \quad — [3^{n-1,1,1}] + [\ ]^{n+2},$$
$$[3, 4, 3] \quad — 2\,[3^{1,1,1}],$$
$$[k]\ (k \text{ even}) — 2\,[\tfrac{1}{2}k].$$

3. In any discrete group generated by reflections, we seek those subgroups which are themselves generated by reflections. Since every subgroup of a reducible group is a direct product of subgroups of irreducible components, there will be no real loss of generality in taking the whole group to be irreducible. Every subgroup of a trigonal group is trigonal; but a tetragonal group has both trigonal and tetragonal subgroups, and a pentagonal both trigonal and pentagonal.

By *obvious* subgroups, we mean such as are generated by subsets of the set of generating reflections of the whole group. The graph for an obvious proper subgroup is derived from the graph for the whole group by removing one or more of the dots (along with any links that emanate from the dot or dots).

4. Let G denote any irreducible *finite* group generated by reflections. Each set of (say) r conjugate reflections (i.e. the set of *all* reflections, if G is trigonal or pentagonal) involves a symmetrical

* More generally, $[3^{p+q}] = [3^{0,p,q}]$. † **2**, 610.

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

set of r primes through one point O of the Euclidean space in which the group G operates. By taking points at unit distance in both directions along the normals at O to these primes, we obtain $2r$ points, which may be regarded as the vertices of a polytope Π (centre O). Every operation of G is a symmetry of this polytope, and Table I shows that Π is in each case uniform*, and derivable from the group by Wythoff's construction. Moreover, this table necessarily includes all those uniform polytopes which have the special property that their vertices occur in opposite pairs whose joins are perpendicular to primes of symmetry.

The polytope Π has unit circum-radius, by definition. Its edge-length, when different from unity, is inserted (after the symbol) in the second column of the table.

On examining the third column, we see that only one or two dots are ringed in each graph. Now† the dots in the graph represent particular primes $a, b, c, \ldots$ through O, namely the primes of the generating reflections. These meet in lines $OA, OB, OC, \ldots$; OA lies in all the primes save a, and so on.

5. Consider first the case in which Π is obtained by ringing just one dot, say that which represents a. This means that the point A (at unit distance from O) is a typical vertex of Π. But the vertices of Π lie on the normals through O to all the reflecting primes. In particular, one of the primes is perpendicular to OA, and so parallel to the perpendicular bisector (of OA), which we shall call o.

The primes $o, a, b, c, \ldots$ bound a (Euclidean) simplex, and the reflections in them generate an infinite group, say G^+. Every proper subset of these primes is congruent to a set of primes (through O) belonging to G. Hence, the reflections in the primes of such a subset generate a group which is identical with some subgroup of G, except that its centre is not at O. In other words, the obvious proper subgroups of G^+, abstractly considered, are also subgroups of G.

The graph for G^+ is derived from the graph for G by inserting an extra dot to represent o. The o-dot is linked to the a-dot, but to no others (on account of the perpendicularity). By omitting the a-dot (and its links) from the graph for G^+, we obtain that subgroup of G which leaves one prime invariant (viz. the prime through O parallel to o).

By ringing the o-dot of the graph for G^+, we obtain the symbol for a degenerate polytope Π^+, having OA for an edge. The vertices of Π^+ form a *lattice*, since the product of reflections in o and in the parallel prime through O is a translation that takes O to A.

* 3, § 1. † 3, § 5.

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

It is easily seen that Π is the vertex figure* of Π^+ (taken at O). If A' is the image of A by reflection in a, AA' is an edge of Π, and $\ldots AOA' \ldots$ is a plane face of Π^+. If this plane face is $\{j\}$†, the length AA' is $2\cos(\pi/j)$, and the angle oa is π/j. Thus the oa link of the graph for G^+ is marked j (or is unmarked, if $AA' = 1$). This shows how the mark against the extra link depends on the edge-length of Π.

When j turns out to be fractional, G^+ is not discrete, and Π^+ is infinitely dense; but the formal derivation of subgroups remains valid.

6. If, on the other hand, Π is obtained by ringing both the a-dot and the b-dot in the graph for G, then its typical vertex, say P, lies on the bisector of the angle AOB. We now define o to be the perpendicular bisector of OP, and proceed as before, observing, however, that o is no longer perpendicular to b. Thus, in the graph for G^+, the o-dot is linked to both the a-dot and the b-dot. (The two links are in every case marked alike, owing to the symmetrical nature of the graph for G.) The subgroup of G that leaves one prime invariant is now obtained by omitting both the a- and b-dots from the graph for G^+.

By ringing the o-dot, we obtain the symbol for a degenerate polytope Π^+ (of unit or infinite density), having OP for an edge. As before, Π^+ is a lattice-polytope, and Π is its vertex figure. If P' is the image of P by reflection in a (or b), PP' is an edge of Π, and $\ldots POP' \ldots$ is a plane face of Π^+. If this plane face is $\{j\}$, the length PP' is $2\cos(\pi/j)$, and the angle oa (or ob) is π/j. Thus both the oa link and the ob link are marked j (or are unmarked, if $PP' = 1$).

7. On looking through the fourth column of Table I, we observe that it includes all the irreducible infinite discrete groups generated by reflections. Since every finite subgroup of G^+ is also a subgroup of G, it follows that the enumeration of subgroups of the finite groups generated by reflections will automatically provide the enumeration of finite subgroups of the infinite groups as well.

8. Table II gives those subgroups of G which are derived from the graph for G^+ by omitting just one dot in every possible way. (These are obvious proper subgroups of G^+, but not of G.) Each line of the table begins with G itself (derived by omitting the o-dot). $[3^n]$ stands alone, since in this case the dots in the graph for G^+ are all alike.

By omitting more than one dot, we only obtain obvious subgroups of these subgroups. But the table enables us to discover further subgroups in great profusion. For example, since $[3^{1,1,1}]^2$

* 1, 336. † E.g., a pentagram ($j = \frac{2}{5}$).

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

is a subgroup of $[3^{5,1,1}]$, and $[3^{5,1,1}]$ of $[3^{4,2,1}]$, therefore $[3^{1,1,1}]^2$ is a subgroup of $[3^{4,2,1}]$. (It will appear later that, by carrying this process through exhaustively, we could obtain *all* the subgroups generated by reflections, except those of [3, 3, 5] and $[k]$.)

9. We now pass to a more elaborate but really exhaustive method for discovering subgroups.

When G is tetragonal, we let the symbol Π include *both* the corresponding polytopes (in their proper relative position). Table III gives the most convenient coordinates for the vertices of Π.

If g is any irreducible subgroup generated by reflections, the actual reflections of g must occur among those of G, and therefore the corresponding polytope (or pair of polytopes) ϖ is inscribed in Π (Table IV). If ϖ is of lower dimensions, we notice those vertices of Π which lie in the absolutely perpendicular space (through the centre), in order to find what other subgroups can be combined with g to form reducible subgroups of G. In particular, we immediately obtain a set of *principal* subgroups (Table V), such that all other (proper) subgroups of G are proper subgroups of these. By arranging the actual groups G in the same order as in Table I, we can refer back for the subgroups of each principal subgroup, and so ultimately find *all* the subgroups generated by reflections (Table VI). The chief difficulty lies in distinguishing between subgroups that occur in two or more non-conjugate types; for this purpose, we have to refer again to Table IV and examine the polytopes in detail.

10. When G is trigonal, the distance between two vertices of Π can only take the values $1, \sqrt{2}, \sqrt{3}, 2$. (Certain extra values occur when G is tetragonal or pentagonal.) This verifies our statement that a trigonal group has only trigonal subgroups.

When the last column of Table IV contains several entries separated by commas, the polytopes represented lie in absolutely perpendicular spaces. For example, points forming a square are entered as "$\{\}2, \{\}2$" (rather than $\{4\}\sqrt{2}$), because they correspond to a subgroup $[\,] \times [\,]$. (On the other hand, points forming an octagon are entered as "$2\{4\}\sqrt{2}$", because they correspond to a subgroup [4].)

Most of the inscribed polytopes lie in subspaces that are obtained by equating several coordinates to each other or to zero. In order to show that ea_4 can be inscribed in $\{3, 3, 5\}\ \tau^{-1}$, we take the vertices of $\{3, 3, 5\}$ in the alternative form* and observe that the points

$$(\tau, \tau, \tau, -\tau^{-2}), \quad (2, 0, 0, -2), \quad (\tau^{-2}, -\tau, -\tau, -\tau)$$

are the vertices of $ea_4 2\sqrt{2}$.

* 1, 346 (bottom of page).

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

11. It is interesting to compare Tables II and V. The *principal* subgroups of $[3^n]$ do not appear in Table II, because they are *obvious* subgroups of $[3^n]$; similarly in other cases. On the other hand, the subgroup $[3^2]^2 \times [\ ]$ of $[3^{3,2,1}]$ does not appear in Table V, because $[3^2]^2$ is a subgroup of $[3^{3,1,1}]$ $(p = 3)$; similarly in other cases.

The only non-obvious principal subgroups that do not appear in Table II are:

$$[3, 3, 5] — [3]^2, \quad [3^{1,1,1}], \quad [3^3];$$
$$[k] — [k/p].$$

12. In Table VI (iii), the subgroups of $[3^{n,1,1}]$ are given in a remarkably compact form. We are even able to distinguish between intrinsically identical subgroups of different type. For example, $[3^{2,1,1}]$ has non-conjugate subgroups

$$[3^{-1,1,1}] \ (p = 2) \quad \text{and} \quad [\ ]^2 \ (q_1 = q_2 = 2),$$
$$[3^{0,1,1}] \quad (p = 3) \quad \text{and} \quad [3^2] \ (q = 4).$$

But the notation fails to indicate that two subgroups $[3^{n+1}]$ may be non-conjugate. (These, however, cannot be said to be of different *type*, since they are conjugate in the larger group $[3^{n+1}, 4]$, of which $[3^{n,1,1}]$ is a self-conjugate subgroup.)

In Table VI (vii), the two types of reflection that occur in $[3^n, 4]$ are distinguished as

$$[3^{-1}, 4] \quad \text{and} \quad [3^0].$$

The infinite groups

$$[4, 3^n, 4] \quad \text{and} \quad \begin{bmatrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{bmatrix}$$

have precisely the same subgroups, since each is a subgroup of the other (the latter of index 2 in the former, and the former of index 2^{n+1} in the latter. These are, in fact, the symmetry groups of δ_{n+3} and $h\delta_{n+3}$*. Taking the vertices of δ_{n+3} to be the totality of points having $n + 2$ integral coordinates, we can distribute them between two $h\delta_{n+3}\sqrt{2}$'s by fixing the parity of the sum of the coordinates. On the other hand, we can distribute them between 2^{n+2} $\delta_{n+3}2$'s by fixing the parity of each coordinate. Hence the vertices of $h\delta_{n+3}\sqrt{2}$ can be distributed among 2^{n+1} $\delta_{n+3}2$'s).

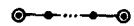
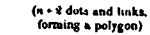
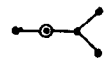
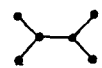
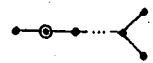
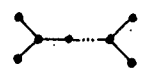
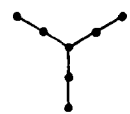
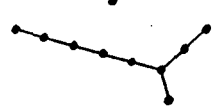
13. Having found the finite subgroups of each infinite group G^+ (as subgroups of G), there is evidently a simple rule for deducing *infinite* subgroups of G^+. If g is a finite subgroup

* 1, 365.

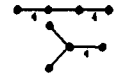
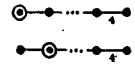
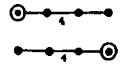
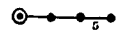
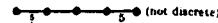
https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

TABLE I.

G	II (in two notations)		G^+
$[\ ]$	$\{\ \}\ 2$ (line, length 2)		
$[3]$	$ea_2=\{6\}$ (hexagon)		
$[3,\ 3]$	$ea_3=t_1\beta_3=\begin{Bmatrix}3\\4\end{Bmatrix}$ (cuboctahedron)		
$[3^n]$	ea_{n+1}*		 (n + 2 dots and links, forming a polygon)
$[3^{1,1,1}]$	$t_1\beta_4=\{3,\ 4,\ 3\}$ (24-cell)		
$[3^{2,1,1}]$	$t_1\beta_5$		
$[3^{n,1,1}]$	$t_1\beta_{n+3}$		
$[3^{2,2,1}]$	$1_{22}=(IA)_6$		
$[3^{3,2,1}]$	$2_{31}=(SA)_7$		
$[3^{4,2,1}]$	$4_{21}=(PA)_8$		

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

[4]	$2\{4\}\sqrt{2}$ (two reciprocal squares)		
[3, 4]	$\beta_3\sqrt{2}+t_1\beta_3$ (octahedron and cuboctahedron)		
$[3^n, 4]$	$\beta_{n+2}\sqrt{2}+t_1\beta_{n+2}$		
[3, 4, 3]	2 {3, 4, 3} (two reciprocal 24-cells)		
[5]	$\{10\}\tau^{-1}$† (decagon)		 (not discrete)
[3, 5]	$\begin{Bmatrix}3\\5\end{Bmatrix}\tau^{-1}$ (icosidodecahedron)		 (not discrete)
[3, 3, 5]	$\{3, 3, 5\}\tau^{-1}$ (600-cell)		 (not discrete)
$[k]$ (k odd, $=2l+1$)	$\{2k\}\,2\sin(\pi/2k)$		 (not discrete if $k>3$)
$[k]$ (k even, $=2l$)	$2\{k\}\,2\sin(\pi/k)$		 (not discrete if $k>6$)

* **1**, 366. This polytope is called $t_{0,n}\alpha_{n+1}$ in **3**; Mrs Stott calls it $e_nS(n+2)$. The vertices $(1, -1; 0^n)$ of the $e\alpha_{n+1}\sqrt{2}$ $(1, 0^n, -1)$ are transposed by reflection in the prime $x_1=x_2$; such reflections generate $[3^n]$ (see **2**, 606).

† $\tau^{-1}=\frac{1}{2}(\sqrt{5}-1)$.

(generated by reflections), then g^+ is an infinite subgroup; if $f \times g$ is a finite subgroup, then

$$f \times g^+, \quad f^+ \times g, \quad f^+ \times g^+$$

are infinite subgroups; and so on. (E.g., [3, 6] has the subgroups

$$[\,], \quad [\infty\,],$$
$$[\,]^2, \quad [\infty\,] \times [\,], \quad [\infty\,]^2,$$
$$[3], \quad \boxed{3^3},$$
$$[6], \quad [3, 6].$$

The fundamental regions for these subgroups are easily recognized when we draw a plane net of equilateral triangles, with all their medians.)

It is a plausible conjecture that we have here a *complete* enumeration of the subgroups generated by reflections in the infinite groups. Du Val has proved the following theorem, which justifies this conjecture.

14. *If g^+ is a subgroup of G^+, then g is a subgroup of G.*

The polytope Π^+ constitutes a lattice whose translation-group G^T is generated by unit translations parallel to its edges (since any two vertices can be joined by a chain of edges). All these translations are operations of G^+, since every edge is perpendicularly bisected by a prime of symmetry and there is also a prime of each family through each vertex. In other words, perpendicular to each system of parallel edges there is a system of parallel primes, the distance between consecutive primes being half the edge-length, so that successive reflections in consecutive primes generate the translations. Thus G^T just consists of those operations of G^+ which are translations. Now, if g^+ is any subgroup of G^+, the common subgroup of g^+ and G^T is a group of translations g^T, which generates the lattice polytope ϖ^+ corresponding to g^+. Since g^T is a subgroup of G^T, the polytope ϖ^+ is inscribed in Π^+.

Consider all the primes of g^+ that pass through one vertex of ϖ^+ (which is also a vertex of Π^+). Reflections in these generate g, whereas reflections in all the primes of G^+ that pass through this vertex generate G. But all the reflections of g^+ occur among those of G^+. Hence g is a subgroup of G.

15. When G is trigonal, its graph can be taken to represent the tree of neighbourhoods of a singular point on a surface*. The *first* neighbourhood is then represented by the dot or dots that are ringed in the symbol for Π. (See the third column of Table I.)

* Du Val, *Proc. Camb. Phil. Soc.* 30 (1934), 460—465.

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

TABLE II.

$[3^n]$.
$[3^{n,1,1}]$, $[3^{n-p,1,1}]\times[3^{p-3,1,1}]$ $(p\geqslant 2^*)$.
$[3^{2,2,1}]$, $[3^4]\times[\ ]$, $[3]^3$.
$[3^{3,2,1}]$, $[3^{3,1,1}]\times[\ ]$, $[3^4]\times[3]$, $[3^2]^2\times[\ ]$, $[3^6]$.
$[3^{4,2,1}]$, $[3^{3,2,1}]\times[\ ]$, $[3^{2,2,1}]\times[3]$, $[3^{2,1,1}]\times[3^2]$, $[3^3]^2$, $[3^4]\times[3]\times[\ ]$, $[3^6]\times[\ ]$, $[3^{5,1,1}]$, $[3^7]$.
$[3^n,4]$, $[3^{n-p},4]\times[3^{p-2},4]$ $(p\geqslant 1\dagger)$, $[3^{n-p},4]\times[3^{p-3,1,1}]$ $(p\geqslant 2)$, $[3^{n-1,1,1}]$.
$[3,4,3]$, $[3,4]\times[\ ]$, $[3]^2$, $[3^2]\times[\ ]$, $[3^2,4]$.
$[3,5]$, $[\ ]^3$.
$[3,3,5]$, $[3,5]\times[\ ]$, $[5]^2\ddagger$.
$[k]$ (k odd)§.
$[k]$ (k even), $[\ ]^2$.

* $[3^{-1,1,1}]=[\ ]^2$, $[3^{0,1,1}]=[3^2]$.
† $[3^{-1},4]=[\ ]$, $[3^0,4]=[4]$.
‡ Arising as $[\frac{5}{2}]\times[5]$.
§ $[k/l]$ is the same as $[k]$, l being prime to k. (See **2**, 596.) Similarly $[5,\frac{5}{2}]$ and $[3,\frac{5}{2}]$ are the same as $[3,5]$.

TABLE III.

Π	Vertices
ea_m	$(1, 0^{m-1}, -1)\sqrt{\frac{1}{2}}$
$t_1\beta_m$	$\pm(1, 1, 0^{m-2})\sqrt{\frac{1}{2}}$
1_{22}	$(0,0,0,0,0,0;\ 1,-1)\sqrt{\frac{1}{2}}$, $(1,0,0,0,0,-1;\ 0,0)\sqrt{\frac{1}{2}}$,* $(1,1,1,-1,-1,-1;\ 1,-1)\frac{1}{2}\sqrt{\frac{1}{2}}$
2_{31}	$(1,0,0,0,0,0,0,-1)\sqrt{\frac{1}{2}}$, $(1,1,1,1,-1,-1,-1,-1)\frac{1}{2}\sqrt{\frac{1}{2}}$
4_{21}	$\pm(1,1,0,0,0,0,0,0)\sqrt{\frac{1}{2}}$, $\frac{1}{2}(\pm)(1,1,1,1,1,1,1,1)\frac{1}{2}\sqrt{\frac{1}{2}}$*
$\beta_m\sqrt{2}+t_1\beta_m$	$\pm(1, 0^{m-1})$, $\pm(1, 1, 0^{m-2})\sqrt{\frac{1}{2}}$
$2\{3,4,3\}$	$\pm(1,0,0,0)$, $\pm(1,1,1,1)\frac{1}{2}$, $\pm(1,1,0,0)\sqrt{\frac{1}{2}}$†
$\left\{\begin{matrix}3\\5\end{matrix}\right\}\tau^{-1}$	$\pm(1,0,0)$, $\pm(\tau,1,\tau^{-1})'\frac{1}{2}$‡
$\{3,3,5\}\tau^{-1}$	$\pm(1,0,0,0)$, $\pm(1,1,1,1)\frac{1}{2}$, $\pm(\tau,1,\tau^{-1},0)'\frac{1}{2}$

* The symbol $\frac{1}{2}(\pm)$ indicates that we must take *either* every combination of an even number of negative signs *or* every combination of an odd number.
† Cf. D. E. Littlewood, "The groups of the regular solids in n dimensions", *Proc. London Math. Soc.* (2), 32 (1931), 13; G. de B. Robinson, "On the orthogonal groups in four dimensions", *Proc. Camb. Phil. Soc.* 27 (1931), 43.
‡ $\tau=\frac{1}{2}(\sqrt{5}+1)$. The dash indicates *even* permutation.

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

TABLE IV.

Parent polytope, Π	Inscribed polytopes, ϖ	In the absolutely perpendicular space
{6}	{ } 2	Nothing
$\begin{Bmatrix}3\\4\end{Bmatrix}$	{ } 2 {6}	{ } 2 Nothing
ea_{n+1}	{ } 2 ea_r ea_{n-1} ea_n	ea_{n-1} ea_{n-r} { } 2 Nothing
{3, 4, 3}	{ } 2 {6} $\begin{Bmatrix}3\\4\end{Bmatrix}$	{ } 2, { } 2, { } 2 Nothing Nothing
$t_1\beta_5$	{ } 2 {6} $\begin{Bmatrix}3\\4\end{Bmatrix}$ (*qua* $t_1\beta_3$) „ (*qua* ea_3) {3, 4, 3} ea_4	$\begin{Bmatrix}3\\4\end{Bmatrix}$, { } 2 { } 2, { } 2 { } 2, { } 2 Nothing Nothing Nothing
$t_1\beta_{n+3}$	{ } 2 ea_{r+2} $(0 \leqslant r \leqslant n-3)$ $_1\beta_{r+3}$ „ ea_n $t_1\beta_{n+1}$ ea_{n+1} $t_1\beta_{n+2}$ ea_{n+2}	$t_1\beta_{n+1}$, { } 2 $t_1\beta_{n-r}$ $t_1\beta_{n-r}$ { } 2, { } 2 { } 2, { } 2 Nothing Nothing Nothing
1_{22}	{ } 2 {6} $\begin{Bmatrix}3\\4\end{Bmatrix}$ {3, 4, 3} ea_4 $t_1\beta_5$ ea_5	ea_5 {6}, {6}, { } 2, { } 2 Nothing { } 2 Nothing { } 2

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

TABLE IV (*continued*).

Parent polytope, II	Inscribed polytopes, ϖ	In the absolutely perpendicular space
2_{31}	$\{\}2$	$t_1\beta_6$
	$\{6\}$	ea_5
	$\begin{Bmatrix}3\\4\end{Bmatrix}$	$\begin{Bmatrix}3\\4\end{Bmatrix}$, $\{\}2$
	$\{3, 4, 3\}$	$\{\}2$, $\{\}2$, $\{\}2$
	ea_4	$\{6\}$
	$t_1\beta_5$	$\{\}2$
	ea_5 $(x_1=x_2=x_3)$	$\{6\}$
	„ $(x_1=x_2=0)$	$\{\}2$
	$t_1\beta_6$	$\{\}2$
	ea_6	Nothing
	1_{22}	Nothing
	ea_7	—
4_{21}	$\{\}2$	2_{31}
	$\{6\}$	1_{22}
	$\begin{Bmatrix}3\\4\end{Bmatrix}$	$t_1\beta_5$
	$\{3, 4, 3\}$	$\{3, 4, 3\}$
	ea_4	ea_4
	$t_1\beta_5$	$\begin{Bmatrix}3\\4\end{Bmatrix}$
	ea_5	$\{6\}$, $\{\}2$
	$t_1\beta_6$	$\{\}2$, $\{\}2$
	ea_6	$\{\}2$
	1_{22}	$\{6\}$
	$t_1\beta_7$	Nothing
	ea_7	$\{\}2$ or Nothing*
	2_{31}	$\{\}2$
	$t_1\beta_8$	—
	ea_8	—
$2\{4\}\sqrt{2}$	$\{\}2$	$\{\}2$
$\beta_3\sqrt{2}+t_1\beta_3$	$\{\}2$ $(x_1=x_2=0)$	$2\{4\}\sqrt{2}$
	„ $(x_1=x_2, x_3=0)$	$\{\}2$, $\{\}2$
	$\{6\}$	Nothing
	$2\{4\}\sqrt{2}$	$\{\}2$
	$\begin{Bmatrix}3\\4\end{Bmatrix}$	—

* We may take ea_7 in the prime $\Sigma x=0$, using either the "even" or the "odd" coordinates for 4_{21}.

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

TABLE IV (*continued*).

Parent polytope, Π	Inscribed polytopes, ϖ	In the absolutely perpendicular space
$\beta_{n+2}\sqrt{2}+t_1\beta_{n+2}$	$\{\ \}2$	$\beta_{n+1}\sqrt{2}+t_1\beta_{n+1}$ or $\beta_n\sqrt{2}+t_1\beta_n$, $\{\ \}2$
	$2\{4\}\sqrt{2}$	$\beta_n\sqrt{2}+t_1\beta_n$ *
	ea_{r+1} $(1\leqslant r\leqslant n-2)$	$\beta_{n-r}\sqrt{2}+t_1\beta_{n-r}$
	$\beta_{r+2}\sqrt{2}+t_1\beta_{r+2}$,,	$\beta_{n-r}\sqrt{2}+t_1\beta_{n-r}$
	ea_n	$\{\ \}2$
	$\beta_{n+1}\sqrt{2}+t_1\beta_{n+1}$	$\{\ \}2$
	ea_{n+1}	Nothing
	$t_1\beta_{n+2}$	—
$2\{3, 4, 3\}$	$\{\ \}2$	$\beta_3\sqrt{2}+t_1\beta_3$
	$\{6\}$	$\{6\}$
	$2\{4\}\sqrt{2}$	$2\{4\}\sqrt{2}$
	$\beta_3\sqrt{2}+t_1\beta_3$	$\{\ \}2$
	$\beta_4\sqrt{2}+t_1\beta_4$	—
$\{10\}\tau^{-1}$	$\{\ \}2$	Nothing
$\begin{Bmatrix}3\\5\end{Bmatrix}\tau^{-1}$	$\{\ \}2$	$\{\ \}2$, $\{\ \}2$
	$\{6\}$	Nothing
	$\{10\}\tau^{-1}$	Nothing
$\{3, 3, 5\}\tau^{-1}$	$\{\ \}2$	$\begin{Bmatrix}3\\5\end{Bmatrix}\tau^{-1}$
	$\{6\}$	$\{6\}$
	$\{10\}\tau^{-1}$	$\{10\}\tau^{-1}$
	$\begin{Bmatrix}3\\4\end{Bmatrix}$	Nothing
	$\begin{Bmatrix}3\\5\end{Bmatrix}\tau^{-3}$	$\{\ \}2$
	$\{3, 4, 3\}$	—
	ea_4	—
$\{2k\}\,2\sin(\pi/2k)$ or $2\{k\}2\sin(\pi/k)$	$\{\ \}2$	$\{\ \}2$ if k is even, otherwise Nothing
	$\{2l\}\,2\sin(\pi/2l)$ (l any factor of k, greater than 2)	—

* When $n=2$, $2\{4\}\sqrt{2}$.

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

TABLE V.

G	Principal subgroups
$[3]$	$[\ ]$
$[3^2]$	$[\ ]^2$, $[3]$
$[3^n]$	$[3^{n-p}]\times[3^{p-2}]$ $(p\geqslant 2)$, $[3^{n-1}]$
$[3^{1,1,1}]$	$[\ ]^4$, $[3^2]$
$[3^{2,1,1}]$	$[3^2]\times[\ ]^2$, $[3^{1,1,1}]$, $[3^3]$
$[3^{n,1,1}]$	$[3^{n-p,1,1}]\times[3^{p-3,1,1}]$ $(p\geqslant 2)$, $[3^{n-1,1,1}]$, $[3^{n+1}]$
$[3^{2,2,1}]$	$[3^4]\times[\ ]$, $[3]^3$, $[3^{2,1,1}]$
$[3^{3,2,1}]$	$[3^{3,1,1}]\times[\ ]$, $[3^4]\times[3]$, $[3^{2,2,1}]$, $[3^6]$
$[3^{4,2,1}]$	$[3^{3,2,1}]\times[\ ]$, $[3^{3,2,1}]\times[3]$, $[3^3]^2$, $[3^{5,1,1}]$, $[3^7]$
$[4]$	$[\ ]^2$
$[3, 4]$	$[4]\times[\ ]$, $[3^2]$
$[3^n, 4]$	$[3^{n-p}, 4]\times[3^{p-2}, 4]$ $(p\geqslant 1)$, $[3^{n-1,1,1}]$
$[3, 4, 3]$	$[3]^2$, $[3^2, 4]$
$[5]$	$[\ ]$
$[3, 5]$	$[\ ]^3$, $[3]$, $[5]$
$[3, 3, 5]$	$[3, 5]\times[\ ]$, $[3]^2$, $[5]^2$, $[3^{1,1,1}]$, $[3^3]$
$[k]$	$[k/p]$ (p any prime factor of k)*

* $[1]=[\ ]$, $[2]=[\ ]^2$.

TABLE VI.

(i) Subgroups of $[3^n]$,
or finite subgroups of $[3^n]^+ = \boxed{3^{n+2}}$:

$$\Pi[3^{p-2}] \quad (\Sigma p \leqslant n+2,\ p\geqslant 2).$$

(ii) Subgroups of $[3^{1,1,1}]=\begin{bmatrix}3\\3\\3\end{bmatrix}$,

or finite subgroups of $[3^{1,1,1}]^+ = [3^{1,1,1,1}] = \begin{bmatrix}3\\3\\3\\3\end{bmatrix}$:

$[\ ]$ $[\ ]^2$ $[\ ]^3$ $[\ ]^4$	$[3]$ $[3^2]$ $[3^{1,1,1}]$

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

(iii) Subgroups of $[3^{n,1,1}]=\begin{bmatrix}3^n\\3\\3\end{bmatrix}$ $(n>1)$,

or finite subgroups of $[3^{n,1,1}]^+=\begin{bmatrix}3^{n-1},3\\3\\3\end{bmatrix}$

$$\Pi[3^{p-3,1,1}]\times\Pi[3^{q-2}] \quad (\Sigma p+\Sigma q\leqslant n+3,\ p\geqslant 2,\ q\geqslant 2).$$

(iv) Subgroups of $[3^{2,2,1}]=\begin{bmatrix}3,3\\3,3\\3\end{bmatrix}$,

or finite subgroups of $[3^{2,2,1}]^+=[3^{2,2,2}]=\begin{bmatrix}3,3\\3,3\\3,3\end{bmatrix}$

$[\ ]$	$[3]\times[\ ]$	$[3^2]$	$[3^4]$
$[\ ]^2$	$[3]\times[\ ]^2$	$[3^2]\times[\ ]$	$[3^4]\times[\ ]$
$[\ ]^3$	$[3]^2$	$[3^2]\times[\ ]^2$	$[3^{1,1,1}]$
$[\ ]^4$	$[3]^2\times[\ ]$	$[3^3]$	$[3^{2,1,1}]$
$[3]$	$[3]^3$	$[3^3]\times[\ ]$	$[3^{2,2,1}]$

(v) Subgroups of $[3^{3,2,1}]=\begin{bmatrix}3,3,3\\3,3\\3\end{bmatrix}$,

or finite subgroups of $[3^{3,2,1}]^+=[3^{3,3,1}]=\begin{bmatrix}3,3,3\\3,3,3\\3\end{bmatrix}$:

$[\ ]$	$[3^2]$	
$[\ ]^2$	$[3^2]\times[\ ]$ (two types)	$[3^5]$
$[\ ]^3$ (two types)	$[3^2]\times[\ ]^2$ (two types)	$[3^6]$
$[\ ]^4$ (two types)	$[3^2]\times[\ ]^3$	$[3^{1,1,1}]$
$[\ ]^5$	$[3^2]\times[3]$	$[3^{1,1,1}]\times[\ ]$
$[\ ]^6$	$[3^2]\times[3]\times[\ ]$	$[3^{1,1,1}]\times[\ ]^2$
$[\ ]^7$	$[3^2]^2$	$[3^{1,1,1}]\times[\ ]^3$
$[3]$	$[3^2]^2\times[\ ]$	$[3^{2,1,1}]$
$[3]\times[\ ]$	$[3^3]$	$[3^{2,1,1}]\times[\ ]$
$[3]\times[\ ]^2$	$[3^3]\times[\ ]$	$[3^{3,1,1}]$
$[3]\times[\ ]^3$	$[3^3]\times[3]$	$[3^{3,1,1}]\times[\ ]$
$[3]^2$	$[3^4]$ (two types)	$[3^{2,2,1}]$
$[3]^2\times[\ ]$	$[3^4]\times[\ ]$ (two types)	$[3^{3,2,1}]$
$[3]^3$	$[3^4]\times[3]$	

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

(vi) Subgroups of $[3^{4,2,1}] = \begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$,

or finite subgroups of $[3^{4,2,1}]^+ = [3^{5,2,1}] = \begin{bmatrix} 3, 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$:

$[\ ]$	$[3^2]\times[3]$	$[3^{1,1,1}]$
$[\ ]^2$	$[3^2]\times[3]\times[\ ]$	$[3^{1,1,1}]\times[\ ]$
$[\ ]^3$	$[3^2]\times[3]\times[\ ]^2$	$[3^{1,1,1}]\times[\ ]^2$
$[\ ]^4$ (two types)	$[3^2]^2$ (two types)	$[3^{1,1,1}]\times[\ ]^3$
$[\ ]^5$	$[3^2]^2\times[\ ]$	$[3^{1,1,1}]\times[\ ]^4$
$[\ ]^6$	$[3^2]^2\times[\ ]^2$	$[3^{1,1,1}]\times[3]$
$[\ ]^7$	$[3^3]$	$[3^{1,1,1}]\times[3^2]$
$[\ ]^8$	$[3^3]\times[\ ]$	$[3^{1,1,1}]^2$
$[3]$	$[3^3]\times[\ ]^2$	$[3^{2,1,1}]$
$[3]\times[\ ]$	$[3^3]\times[3]$	$[3^{2,1,1}]\times[\ ]$
$[3]\times[\ ]^2$	$[3^3]\times[3]\times[\ ]$	$[3^{2,1,1}]\times[\ ]^2$
$[3]\times[\ ]^3$	$[3^3]\times[3^2]$	$[3^{2,1,1}]\times[3]$
$[3]\times[\ ]^4$	$[3^3]^2$	$[3^{2,1,1}]\times[3^2]$
$[3]^2$	$[3^4]$	$[3^{3,1,1}]$
$[3]^2\times[\ ]$	$[3^4]\times[\ ]$ (two types)	$[3^{3,1,1}]\times[\ ]$
$[3]^2\times[\ ]^2$	$[3^4]\times[\ ]^2$	$[3^{3,1,1}]\times[\ ]^2$
$[3]^3$	$[3^4]\times[3]$	$[3^{4,1,1}]$
$[3]^3\times[\ ]$	$[3^4]\times[3]\times[\ ]$	$[3^{5,1,1}]$
$[3]^4$	$[3^5]$	$[3^{2,2,1}]$
$[3^2]$	$[3^5]\times[\ ]$	$[3^{2,2,1}]\times[\ ]$
$[3^2]\times[\ ]$	$[3^6]$ (two types)	$[3^{2,2,1}]\times[3]$
$[3^2]\times[\ ]^2$ (two types)	$[3^6]\times[\ ]$	$[3^{3,2,1}]$
$[3^2]\times[\ ]^3$	$[3^7]$	$[3^{3,2,1}]\times[\ ]$
$[3^2]\times[\ ]^4$		$[3^{4,2,1}]$

(vii) Subgroups of $[3^n, 4]$,

or finite subgroups of $[4, 3^n, 4]$ or of $\begin{bmatrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{bmatrix}$:

$$\Pi[3^{p-2}, 4]\times\Pi[3^{q-3,1,1}]\times\Pi[3^{r-2}] \quad (\Sigma p+\Sigma q+\Sigma r\leqslant n+2,\ p\geqslant 1,\ q\geqslant 2,\ r\geqslant 2)$$

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press

(viii) Subgroups of [3, 4, 3],
or finite subgroups of [3, 3, 4, 3]:

$[\ ]$	$[3]$	$[3^{1,1,1}]$	$[3, 4]$
$[\ ]^2$ (two types)	$[3]\times[\ ]$	$[4]$	$[3, 4]\times[\ ]$
$[\ ]^3$ (two types)	$[3]^2$	$[4]\times[\ ]$	$[3, 3, 4]$
$[\ ]^4$ (two types)	$[3^2]$	$[4]\times[\ ]^2$	$[3, 4, 3]$
	$[3^2]\times[\ ]$	$[4]^2$	

(ix) Subgroups of [3, 5]:

$[\ ]$	$[3]$
$[\ ]^2$	$[5]$
$[\ ]^3$	$[3, 5]$

(x) Subgroups of [3, 3, 5]:

$[\ ]$	$[3]$	$[3^3]$	$[5]^2$
$[\ ]^2$	$[3]\times[\ ]$	$[3^{1,1,1}]$	$[3, 5]$
$[\ ]^3$	$[3]^2$	$[5]$	$[3, 5]\times[\ ]$
$[\ ]^4$	$[3^2]$	$[5]\times[\ ]$	$[3, 3, 5]$

(xi) Subgroups of [6],
or finite subgroups of [3, 6]:

$[\ ]$	$[3]$
$[\ ]^2$	$[6]$

(xii) Subgroups of $[k]$:

$[l]$ (l any factor, including 1 and k).

https://doi.org/10.1017/S0305004100012718 Published online by Cambridge University Press