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MASS PROBLEMS AND INITIAL SEGMENT COMPLEXITY

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Abstract. By the complexity of a finite sequence of 0's and 1's we mean the Kolmogorov complexity, that is the length of the shortest input to a universal recursive function which returns the given sequence as output. By initial segment complexity of an infinite sequence of 0's and 1's we mean the asymptotic behavior of the complexity of its finite initial segments. In this paper, we construct infinite sequences of 0's and 1's with given recursive lower bounds on initial segment complexity which do not compute any infinite sequences of 0's and 1's with a significantly larger recursive lower bound on initial segment complexity. This improves several known results about randomness extraction and separates many natural degrees in the lattice of Muchnik degrees.

§1. Introduction.

1.1. Complexity and randomness. Complexity, roughly speaking, is the measure of how hard it is to compress or describe finite sequences of 0's and 1's. If a finite sequence of 0's and 1's is either simple or has many repeated patterns, we can use that knowledge to describe it in a short fashion; if however, a finite sequence of 0's and 1's is complex, then often the best way of describing it may be by listing the entire sequence. We can extend this notion to include infinite sequences of 0's and 1's in a natural way, by talking about the complexity of its finite initial segments. This is formalized by defining the complexity $\mathrm{K}(X \upharpoonright n)$ to be the minimum length of a computer input which outputs the first n bits of the infinite sequence X. The precise definitions will be given below, or the reader can see Downey and Hirschfeldt [5], Li and Vitányi [10], or Nies [14] for more details. Of particular interest are those infinite sequences of 0's and 1's with nontrivial recursive lower bounds on initial segment complexity.

Randomness, on the other hand, is the study of how typical an infinite sequence of 0's and 1's is. We know for example that for an infinite sequence of coin flips (assuming the coin is fair), the coin should almost certainly come up heads about half of the time. More precisely we say that with probability one, the limit frequency of heads will be $1/2$. A *random* infinite sequence of 0's and 1's (or *algorithmically random*, more precisely) is one which satisfies every "reasonable" property that occurs with probability 1. Of course, different interpretations of the word "reasonable" will

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lead to different notions of randomness. The most common form of algorithmic randomness is that of Martin-Löf randomness; see Downey and Hirschfeldt [5] for a precise definition of this notion.

There is a deep connection between randomness and complexity, which can be traced as far back as a family of theorems originally due to Levin and Schnorr. One of these theorems states that an infinite sequence of 0's and 1's X is Martin-Löf random exactly when its initial segment complexity is always nearly maximal, that is $\mathrm{K}(X \upharpoonright n) \geq n - O(1)$; Schnorr's original proof is given by Chaitin [4, Theorem R1]. This theorem is particularly interesting because Martin-Löf randomness is considered by many to be the most natural formalization of algorithmic randomness. More work has been done recently on relating randomness and complexity; particularly, characterizations of Martin-Löf randomness in terms of other (closely related) complexity notions have been discovered, as have characterizations of other natural randomness notions in terms of complexity.

1.2. Other notions of randomness. The preceding statements refer to randomness with respect to the fair coin probability measure. If we replace the fair coin measure with another recursive non-atomic probability measure, we find that a random infinite sequence of 0's and 1's will no longer have maximal complexity; in this case we will still be able to compute a lower bound on the growth rate of the initial segment complexity of the infinite sequence, as proved by Reimann [15, Proposition 4.11].

Besides randomness for probability measures, there are several notions of partial randomness. From an effectivization of Hausdorff dimension, we have several notions of s-randomness for rational $0 < s \leq 1$—that is randomness for the s-dimensional effective Hausdorff measure. For a treatment of this topic, see Downey and Hirschfeldt [5] or Reimann and Stephan [16]. One of these notions, weak s-randomness, is also related to initial segment complexity, as an infinite sequence of 0's and 1's X is weakly s-random exactly when its initial segment complexity is bounded below by s times its initial segment length, that is $\mathrm{K}(X \upharpoonright n) \geq sn - O(1)$. In the same vein there is a natural definition of randomness with respect to an unbounded recursive function $f : \mathbb{N} \to [0, \infty)$. Given a recursive unbounded function $f : \mathbb{N} \to [0, \infty)$ we introduce the notion of f-randomness and show that an infinite sequence of 0's and 1's X is f-random if and only if $\mathrm{K}(X \upharpoonright n) \geq f(n) - O(1)$.

1.3. Separating randomness and complexity notions. An important area of randomness and complexity theory is randomness extraction or information extraction. The main question of this field is: given a source of on average low complexity or randomness, is it possible to extract a higher amount of complexity or randomness on average? Partial answers to this question have allowed for separations of various randomness notions, although this work is far from complete.

One major result proves the existence of an infinite sequence of 0's and 1's with asymptotically maximal initial segment complexity which does not compute a Martin-Löf random.

THEOREM 1.1 (Greenberg and Miller [6]). *There exists an $X \in 2^{\mathbb{N}}$ such that*

$$\lim_{n \to \infty} \frac{K(X \upharpoonright n)}{n} = 1$$

but X computes no Martin-Löf random.

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Another result shows that for any rational s, there is an infinite sequence of 0's and 1's with a linear lower bound on initial segment complexity given by s, which does not compute any infinite sequence of 0's and 1's with a higher linear lower bound on initial segment complexity.

THEOREM 1.2 (Miller [12]). *For each computable* $0 < s < 1$ *there is an* $X \in 2^{\mathbb{N}}$ *such that* $K(X \upharpoonright n) \geq sn - O(1)$ *and if* $Y \leq_T X$ *and* $t > s$ *then* $K(Y \upharpoonright n) \not\geq tn - O(1)$.

These two results show us a limitation of randomness and complexity: a certain level of randomness or information will not reliably lead to a higher level of randomness or information.

In this paper we will give a generalization of Miller's result that additionally implies the result of Greenberg and Miller. This generalization will separate appropriate notions of f-randomness and g-randomness; since f-randomness corresponds to lower bounds on initial segment complexity, we will be able to prove that extracting even a small amount of information can be hard. Our main result, Theorem 5.1, is rather technical and difficult to state. The following is a special case of the theorem and shows its usefulness for separating complexity classes.

THEOREM 1.3. *For any recursive and unbounded* $f : \mathbb{N} \to [0, \infty)$ *such that* $f(n+1) \leq f(n) + 1$ *for all* n *and* $\lim_{n\to\infty} n - f(n) = \infty$ *there is an* $X \in 2^{\mathbb{N}}$ *such that* $K(X \upharpoonright n) \geq f(n) - O(1)$ *and if* $Y \leq_T X$ *then* $K(Y \upharpoonright n) \not\geq f(n) + 2\log_2 f(n) - O(1)$.

EXAMPLE 1.4. *There is an* $X \in 2^{\mathbb{N}}$ *such that* $K(X \upharpoonright n) \geq n - 2\log_2 n - O(1)$ *but* X *does not compute a Martin-Löf random*.

EXAMPLE 1.5. *Let* $\log_2^k$ *denote the* k*th iterated logarithm. For each* k, *there is an* $X \in 2^{\mathbb{N}}$ *such that* $K(X \upharpoonright n) \geq \log_2^k n - O(1)$ *but no* Y *computable from* X *satisfies* $K(Y \upharpoonright n) \geq \log_2^k n + 2\log_2^{k+1} n - O(1)$.

1.4. Mass problems. One ramification of our result is found in mass problems. Whereas the Turing degrees are equivalence classes of infinite sequences of 0's and 1's under reducibility, the Muchnik degrees or weak degrees are equivalence classes of sets of infinite sequences of 0's and 1's (subsets of $2^{\mathbb{N}}$, that is). See section 10 for a precise definition. Of specific interest are the Muchnik degrees of Π^0_1 (effectively closed) sets of infinite sequences of 0's and 1's. Whereas the semi-lattice of r.e. Turing degrees contains only two natural examples, there are many important Muchnik degrees arising from randomness, logic, and recursion theory; these have been studied extensively by Simpson [17, 18, 19]. It is well known that certain sets of complex infinite sequences of 0's and 1's are Π^0_1; our result shows that a great number of the Muchnik degrees arising from these sets are in fact distinct.

§2. Preliminaries and notation.

We assume basic familiarity with the foundations of recursion theory. We will use some non-standard notations, while most will be standard.

2.1. Infinite and finite strings. The following is standard. $\mathbb{N}$ is the set of natural numbers, which contains 0. We let $2^{\mathbb{N}}$ denote Cantor space, the set of all infinite sequences of 0's and 1's. An infinite sequence X can be written $X(0)X(1)\ldots$. We let $2^{<\mathbb{N}}$ denote the set of all finite sequences of 0's and 1's. For $\sigma \in 2^{<\mathbb{N}}$ we write $|\sigma|$ to denote the length of σ; a finite sequence σ can be written $\sigma(0)\sigma(1)\ldots\sigma(|\sigma|-1)$. For $\sigma, \tau \in 2^{<\mathbb{N}}$ we write $\sigma \subseteq \tau$ when $(\forall n < |\sigma|)(\sigma(n) = \tau(n))$ and we write

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$\sigma \subset \tau$ when $\sigma \subseteq \tau$ and $\sigma \neq \tau$. For $\sigma \in 2^{<\mathbb{N}}$ and $X \in 2^{\mathbb{N}}$ we write $\sigma \subset X$ when $(\forall n < |\sigma|)(\sigma(n) = X(n))$. For $X \in 2^{\mathbb{N}}$ and $k \in \mathbb{N}$ we define $X \upharpoonright k = \sigma$ for the unique $\sigma \subset X$ such that $|\sigma| = k$. For $\sigma \in 2^{<\mathbb{N}}$ we define $[\sigma] = \{X \in 2^{\mathbb{N}} \mid \sigma \subset X\}$. For a set $A \subseteq 2^{<\mathbb{N}}$ we define $[A] = \{X \in 2^{\mathbb{N}} \mid (\exists \sigma \in A)(\sigma \subset X)\}$. For $\sigma, \tau \in 2^{<\mathbb{N}}$ we write $\sigma \asymp \tau$ to denote that either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$; otherwise we write $\sigma \not\asymp \tau$. We write λ for the unique sequence of length 0.

We also use some nonstandard notation and definitions for strings. We define several special subsets of $2^{<\mathbb{N}}$. Specifically, we define $2^n = \{\sigma \in 2^{<\mathbb{N}} \mid |\sigma| = n\}$, $2^{\leq n} = \{\sigma \in 2^{<\mathbb{N}} \mid |\sigma| \leq n\}$, and $2^{>n} = \{\sigma \in 2^{<\mathbb{N}} \mid |\sigma| > n\}$. For $\sigma \in 2^{<\mathbb{N}}$ we define $[\sigma]^{<\mathbb{N}} = \{\tau \in 2^{<\mathbb{N}} \mid \sigma \subseteq \tau\}$. For a set $A \subseteq 2^{<\mathbb{N}}$ we define $[A]^{<\mathbb{N}} = \{\tau \mid (\exists \sigma \in A)(\sigma \subseteq \tau)\}$. For $A \subseteq 2^{<\mathbb{N}}$, we define the set $\widehat{A} = \{\sigma \in A \mid \neg(\exists \tau \subset \sigma)(\tau \in A)\}$.

2.2. Recursive and r.e. sets. A set $A \subseteq 2^{<\mathbb{N}}$ is *recursive* if the function

$$\chi_A(\sigma) = \begin{cases} 1 & \text{if } \sigma \in A, \\ 0 & \text{if } \sigma \notin A \end{cases}$$

is recursive. A set $A \subseteq 2^{<\mathbb{N}}$ is said to be *recursively enumerable* if it is the empty set or if there is a recursive function $\Phi\colon \mathbb{N} \to 2^{<\mathbb{N}}$ which enumerates A, that is $A = \{\sigma \mid (\exists n)(\Phi(n) = \sigma)\}$. Here we say that Φ *witnesses* the fact that A is r.e. If non-empty A is r.e., witnessed by Φ, then we define the set

$$A^k = \{\sigma \mid (\exists n < k)(\Phi(n) = \sigma)\}$$

for each $k \in \mathbb{N}$. The sets A^k are also called an *enumeration* of A. Note that the cardinality of each A^k is at most k.

The following is a non-standard definition. We say that an enumeration of a nonempty r.e. set $A \subseteq 2^{<\mathbb{N}}$ is *frugal* if it never enumerates new extensions of old elements. That is, A has a frugal enumeration if whenever σ belongs to A^t and $\tau \supset \sigma$ does not belong to A^t, then $\tau \notin A$.

LEMMA 2.1. *If $A \subseteq 2^{<\mathbb{N}}$ is nonempty and r.e., then there is an r.e. set $\widetilde{A} \subseteq 2^{<\mathbb{N}}$ with a frugal enumeration such that $\widetilde{A} \subseteq A$ and $[A] = [\widetilde{A}]$.*

PROOF. Let A be witnessed by Φ, and define

$$\widetilde{\Phi}(n) = \begin{cases} \Phi(n) & \text{if } \neg(\exists k < n)(\Phi(k) \subseteq \Phi(n)), \\ \widetilde{\Phi}(n-1) & \text{otherwise} \end{cases}$$

which is also a recursive function. Define $\widetilde{A} = \{\sigma \mid (\exists n)(\widetilde{\Phi}(n) = \sigma)\}$. Clearly $\widetilde{A} \subseteq A$ and $\widetilde{A}^k \subseteq A^k$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, $[\widetilde{A}^k] = [A^k]$, and so $[A] = [\widetilde{A}]$. The enumeration that defines $\widetilde{A}$ is clearly frugal. ⊣

So we will occasionally assume that r.e. sets have a frugal enumeration and that a given enumeration is frugal.

2.3. Prefix-free complexity. In general, partial recursive functions (those computed by Turing machines) may or may not halt for a given input. We write $\Psi\colon \subseteq D \to R$ to mean that Ψ is partial, and its domain of definition is some subset of D. For $x \in D$, we write $\Psi(x)\downarrow$ to mean that $\Psi(x)$ is defined (or that the computation halts); otherwise we write $\Psi(x)\uparrow$. A partial recursive machine $\Psi : \subseteq 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is *prefix-free* if whenever $\sigma \subset \tau$, at least one of $\Psi(\sigma)$ and $\Psi(\tau)$ is undefined. For any

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such Ψ, we define a function $\mathrm{K}_\Psi\colon 2^{<\mathbb{N}} \to \mathbb{N}\cup\{\infty\}$ by $\mathrm{K}_\Psi(\sigma) = \min\{|\tau| \mid \Psi(\tau) = \sigma\}$. It is a fact that there is a prefix-free $M\colon \subseteq 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, which has additively optimal complexity: for any prefix-free partial recursive $\Psi\colon \subseteq 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, there is a constant c_Ψ such that $(\forall\sigma \in 2^{<\mathbb{N}})(\mathrm{K}_M(\sigma) \leq \mathrm{K}_\Psi(\sigma) + c_\Psi)$. We fix some partial recursive $M\colon \subseteq 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ with this property and define a function $\mathrm{K}\colon 2^{<\mathbb{N}} \to \mathbb{N}$ by $\mathrm{K} = \mathrm{K}_M$. For $\sigma \in 2^{<\mathbb{N}}$, we define the prefix-free complexity of σ to be $\mathrm{K}(\sigma)$.

2.4. Measure. For a set $A \subseteq 2^{<\mathbb{N}}$ we define $\mu(A) = \sum_{\sigma\in\widehat{A}} 2^{-|\sigma|}$. Note that if $A, B \subseteq 2^{<\mathbb{N}}$ are such that $[A] = [B]$, then $\mu(A) = \mu(B)$. For a set $P \subseteq 2^{\mathbb{N}}$, we define an outer measure μ^* by $\mu^*(P) = \inf\{\mu(A) \mid A \subseteq 2^{<\mathbb{N}} \text{ and } P \subseteq [A]\}$. This gives rise to a measure μ on subsets of $2^{\mathbb{N}}$; all open and closed sets are measurable, for example. For a set $A \subseteq 2^{<\mathbb{N}}$, we have $\mu([A]) = \mu(A)$.

§3. Partial randmoness.

The following notions are generalizations of ones introduced primarily by Tadaki [20] and Calude, Staiger, and Terwijn [3], and further developed by Reimann [15]. The key new concepts we make use of are suitable functions, the increasing set, the notion of $\ll$ for functions, and a greatly modified version of optimal covers.

3.1. f-weights. Let $f\colon \mathbb{N} \to [0,\infty)$ be recursive and unbounded, and let $A \subseteq 2^{<\mathbb{N}}$. The *direct f-weight* of A is $\mathrm{dwt}_f(A) = \sum_{\sigma\in A} 2^{-f(|\sigma|)}$. The *vehement f-weight* of A is $\mathrm{vwt}_f(A) = \inf\{\mathrm{dwt}_f(B) \mid [B] \supseteq [A]\}$. The *prefix f-weight* of A is $\mathrm{pwt}_f(A) = \sup\{\mathrm{dwt}_f(P) \mid P \subseteq A \text{ is prefix-free}\}$.

For convenience we will write $\mathrm{dwt}_f(\sigma)$ to mean $\mathrm{dwt}_f(\{\sigma\})$ for any $\sigma \in 2^{<\mathbb{N}}$. Similarly, we may write $\mathrm{vwt}_f(\sigma)$ for $\mathrm{vwt}_f(\{\sigma\})$ and $\mathrm{pwt}_f(\sigma)$ for $\mathrm{pwt}_f(\{\sigma\})$.

3.2. f-randomness. Let $f\colon \mathbb{N} \to [0,\infty)$ be recursive and unbounded. A uniform r.e. sequence $\langle A_k \subseteq 2^{<\mathbb{N}} \mid k \in \mathbb{N}\rangle$ is called an *f-test* if $\mathrm{dwt}_f(A_k) \leq 2^{-k}$ for each $k \in \mathbb{N}$. We say that $X \in 2^{\mathbb{N}}$ is *f-random* if for every f-test $\langle A_k\rangle$ there is a k such that $X \notin [A_k]$; otherwise we say that X is *f-null*.

The notion of f-randomness mimics that of Martin-Löf randomness. Specifically if we take the function $f(n) = n$ the notions of f-randomness and Martin-Löf randomness coincide; likewise if we let $f(n) = sn$ for some fixed recursive $s > 0$, the notion of f-randomness coincides with that of weak s-randomness (mentioned above), introduced by Tadaki [20].

For two functions $f, g\colon \mathbb{N} \to [0,\infty)$ we say that f and g are *equivalent* if $|f(n) - g(n)| < O(1)$. It is straightforward to prove that if recursive and unbounded functions $f, g\colon \mathbb{N} \to [0,\infty)$ are equivalent, then a sequence is f-random if and only if it is g-random.

LEMMA 3.1. *For every recursive unbounded $f\colon \mathbb{N} \to [0,\infty)$, there is a recursive unbounded $g\colon \mathbb{N} \to \mathbb{N}$ such that f and g are equivalent.*

PROOF. There is a computable function $\Psi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ such that for all n and all t, if $\Psi(n,t) = \langle p,q\rangle$ then $|\frac{p}{q} - f(n)| \leq 2^{-t}$. Then for each $n \in \mathbb{N}$ let $\langle p_n, q_n\rangle = \Psi(n,1)$ and define $g(n) = \min\{m \in \mathbb{N} \mid \frac{p_n}{q_n} < m\}$. This provides a recursive definition of g. It is clear that f and g are equivalent as claimed. ⊣

It is worth pointing out that Martin-Löf randomness can be defined in terms of either martingales or Solovay tests. Likewise, there are other approaches to

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defining an appropriate notion of f-randomness. These other notions will not in general coincide with the notion of f-randomness defined above. See Downey and Hirschfeldt [5, Chapter 13], Reimann and Stephan [16], or Hudelson [7] for more on this topic.

§4. Initial segment complexity and partial randomness. In the previous section we introduced the notion of f-randomness. In this section we will see that, like Martin-Löf randomness, f-randomness is connected to prefix-free Kolmogorov complexity.

4.1. Initial segment complexity. If $f : \mathbb{N} \to [0, \infty)$ is recursive and unbounded and $X \in 2^{\mathbb{N}}$, we say that X is *f-complex* if $\mathrm{K}(X \upharpoonright n) \geq f(n) - O(1)$. More generally we say that X is *complex* if it is f-complex for some recursive and unbounded $f : \mathbb{N} \to [0, \infty)$. The topic of complex $X \in 2^{\mathbb{N}}$ is discussed in length by Kjos-Hanssen, Merkle, and Stephan [8].

Recall that $f, g : \mathbb{N} \to [0, \infty)$ are equivalent if $|f(n) - g(n)| < O(1)$. It is immediate that if recursive and unbounded functions $f, g : \mathbb{N} \to [0, \infty)$ are equivalent, then $X \in 2^{\mathbb{N}}$ is f-complex if and only it is g-complex.

The following is then a corollary of Lemma 3.1.

COROLLARY 4.1. *For every recursive and unbounded $f : \mathbb{N} \to [0, \infty)$, there is a recursive and unbounded $g : \mathbb{N} \to \mathbb{N}$ such that $X \in 2^{\mathbb{N}}$ is f-random if and only if it is g-random, and X is f-complex if and only if it is g-complex.*

Up until this point in the paper, we have allowed our functions to be real-valued; because of this result, we can replace each such function with an equivalent integer-valued function. So, for the remainder of the paper, we will work exclusively with integer-valued functions, that is recursive and unbounded functions $f : \mathbb{N} \to \mathbb{N}$. It should be noted that every result holds for recursive and unbounded $f : \mathbb{N} \to [0, \infty)$ as well, though.

The following lemma is standard. For a proof, see Downey and Hirschfeldt [5, Theorem 3.6.1].

LEMMA 4.2 (Kraft-Chaitin). *If $A \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$ is r.e. and $\sum_{\langle n,\sigma\rangle \in A} 2^{-n} < \infty$, then $K(\sigma) \leq n + O(1)$ for all $\langle n, \sigma\rangle \in A$.*

Another standard result is the following; the proof is straightforward.

LEMMA 4.3 (Kraft). $\sum_{\sigma \in 2^{<\mathbb{N}}} 2^{-K(\sigma)} < 1$.

The following theorem makes precise the sense in which f-randomness and f-complexity are related for recursive and unbounded $f : \mathbb{N} \to \mathbb{N}$. This result is a generalization of an unpublished theorem of Schnorr (see Chaitin [4]). A weak version of this was first proved by Tadaki [20], who only considered functions of the form $f(n) = sn$ for fixed recursive $0 < s < 1$; a more general version was given by Reimann [15] for non-decreasing functions. Reimann's proof uses discrete semimeasures, while we use arguments similar to those of Tadaki and Chaitin.

THEOREM 4.4 (Schnorr, Chaitin, Tadaki, Reimann). *If $f : \mathbb{N} \to \mathbb{N}$ is recursive and unbounded and $X \in 2^{\mathbb{N}}$, then X is f-random if and only if X is f-complex.*

PROOF. Let X be f-random. For each $k \in \mathbb{N}$ define

$$A_k = \{\sigma \in 2^{<\mathbb{N}} \mid \mathrm{K}(\sigma) \leq f(|\sigma|) - k\}.$$

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Then $\langle A_k \rangle$ is a uniformly r.e. sequence, by definition of K. For each $k \in \mathbb{N}$

$$\mathrm{dwt}_f(A_k) = \sum_{\sigma \in A_k} 2^{-f(|\sigma|)} \leq \sum_{\sigma \in A_k} 2^{-\mathrm{K}(\sigma)-k} \leq 2^{-k} \sum_{\sigma \in 2^{<\mathbb{N}}} 2^{-\mathrm{K}(\sigma)} < 2^{-k}.$$

Then $\langle A_k \rangle$ is an f-test. Since X is f-random, there is a $k \in \mathbb{N}$ such that $X \notin [A_k]$. Therefore $\mathrm{K}(X \upharpoonright n) > f(n) - k$ for all n, and so X is f-complex.

Now suppose that X is not f-random. Then there is an f-test $\langle A_k \rangle$ such that $X \in [A_k]$ for all $k \in \mathbb{N}$. Define a set $A \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$ by $A = \{\langle f(|\sigma|) - k, \sigma \rangle \mid \sigma \in A_{2k}\}$. A is r.e. and

$$\begin{aligned} \sum_{\langle n,\sigma \rangle \in A} 2^{-n} &= \sum_{k \in \mathbb{N}} \sum_{\sigma \in A_{2k}} 2^{-(f(|\sigma|)-k)} \\ &= \sum_{k \in \mathbb{N}} 2^k \mathrm{dwt}_f(A_{2k}) \leq \sum_{k \in \mathbb{N}} 2^{-2k+k} = \sum_{k \in \mathbb{N}} 2^{-k} < \infty. \end{aligned}$$

Therefore by Lemma 4.2, $\mathrm{K}(\sigma) \leq f(|\sigma|) - k + O(1)$ for all $\sigma \in A_{2k}$, where the constant $O(1)$ is independent of k. But for each $k \in \mathbb{N}$ there is a $\sigma \subset X$ such that $\sigma \in A_{2k}$, and so X is not f-complex. ⊣

This gives a characterization of complex $X \in 2^{\mathbb{N}}$ in terms of a randomness notion. Kjos-Hanssen, Merkle, and Stephan [8] characterized the complex $X \in 2^{\mathbb{N}}$ as exactly those X which truth-table compute diagonally non-recursive functions. It has also been shown by Binns [2, Theorem 2.13] that an effectively closed subset of $2^{\mathbb{N}}$ contains a complex X exactly when it contains a recursively perfect subset.

4.2. Order functions versus more general recursive functions. When discussing complex and f-complex points in $2^{\mathbb{N}}$, it is often customary [5, 8] to only consider *order functions*, those functions which are recursive, unbounded, and non-decreasing. Note the following.

LEMMA 4.5. *For any recursive and unbounded function $f : \mathbb{N} \to \mathbb{N}$ there is an order function $g : \mathbb{N} \to \mathbb{N}$ such that any X is f-random if and only if it is g-random.*

PROOF. Let $f : \mathbb{N} \to \mathbb{N}$ be recursive and unbounded. Define $g(0) = f(0)$ and

$$g(n) = \begin{cases} f(n) & \text{if } f(n) \geq g(n-1), \\ g(n-1) & \text{otherwise} \end{cases}$$

for every $n > 0$. Then g is clearly recursive, unbounded, and non-decreasing; therefore g is an order function.

Now suppose that X is f-random and let $\langle A_k \rangle$ be a g-test. For every n let m_n be the least number such that $f(m_n) = g(n)$; by induction on n, note that $m_n \leq n$ for all n. Then define $B_k = \{\sigma \upharpoonright m_{|\sigma|} \mid \sigma \in A_k\}$ for each k. The sequence $\langle B_k \rangle$ is uniformly r.e. since $\langle A_k \rangle$ is. Also, if $\sigma \in B_k$ then there is some $\tau \in A_k$ such that $\sigma = \tau \upharpoonright m_{|\tau|}$, in which case $f(|\sigma|) = g(|\tau|)$; from this it follows that

$$\mathrm{dwt}_f(B_k) = \sum_{\sigma \in B_k} 2^{-f(|\sigma|)} \leq \sum_{\tau \in A_k} 2^{-g(|\tau|)} = \mathrm{dwt}_g(A_k) \leq 2^{-k}$$

for each k. Then $\langle B_k \rangle$ is an f-test, and so $X \notin [B_k]$ for some k. But since $[B_k] \supseteq [A_k]$ it follows that $X \notin [A_k]$ either, and so X is g-random.

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Finally, let X be g-random and let $\langle A_k \rangle$ be an f-test. Since $f(n) \leq g(n)$ for each n it follows that $\langle A_k \rangle$ is also a g-test. Therefore $X \notin [A_k]$ for some k, and so X is f-random. ⊣

A similar theorem holds for recursive and unbounded functions $f : \mathbb{N} \to [0, \infty)$. Then we conclude that each theorem in this paper could be restated in terms of order functions, without any loss of generality in the results. However, this would not lead to any great simplification of the presentation of these results. For this reason, we will not require any restrictions to the class of order functions.

§5. **The Main Theorem.** Now we have enough machinery to state the main theorem, as well as several examples and corollaries of it. We will give the proof of the main theorem in the subsequent sections.

5.1. Suitable functions and the increasing set. Recall that for two functions $f, g : \mathbb{N} \to \mathbb{N}$ we say that f and g are equivalent if $|f(n) - g(n)| < O(1)$.

A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *suitable* if it is recursive, unbounded and $f(n+1) \leq f(n) + 1$ for all n. Note that if a recursive unbounded $f : \mathbb{N} \to \mathbb{N}$ has the property that $f(n+1) \leq f(n)+1$ for all sufficiently large n, then f is equivalent to a suitable function. We refer to the function $h : \mathbb{N} \to \mathbb{N}$ given by $h(n) = n$ for all n as the identity function. Up to a constant, the identity function is an upper bound on any suitable function.

If $f : \mathbb{N} \to \mathbb{N}$ is suitable and equivalent to the identity function, then any $X \in 2^{\mathbb{N}}$ is f-random if and only if it is Martin-Löf random. Martin-Löf randoms have been studied at great length in a number of papers and books; in this paper, we are more interested in notions of partial randomness, so we will focus on those suitable $f : \mathbb{N} \to \mathbb{N}$ which are not equivalent to the identity function.

A suitable function $f : \mathbb{N} \to \mathbb{N}$ which is not equivalent to the identity function has the additional property that $f(n+1) < f(n) + 1$ for infinitely many n; stated another way $(\forall c)(\exists n)(f(n) < n - c)$.

5.2. Functions which grow sufficiently faster than a given function. For recursive unbounded $f : \mathbb{N} \to \mathbb{N}$, the *increasing set* of f is the set $\mathcal{I}(f) = \{n \in \mathbb{N} \mid (\forall m < n)(f(m) < f(n))\}$. Note that for any recursive and unbounded $f : \mathbb{N} \to \mathbb{N}$, the restriction of f to $\mathcal{I}(f)$ is one-to-one. If f is suitable and $f(0) = 0$, then the restriction of f to $\mathcal{I}(f)$ is onto as well.

For recursive and unbounded $f, g : \mathbb{N} \to \mathbb{N}$ we say that $f \ll g$ if $\sum_{n \in \mathcal{I}(f)} 2^{f(n)-g(n)} < \infty$ is a recursive real. We will see in Lemma 8.3 that requiring this sum to be recursive allows us to carefully compare the f- and g-weights of certain r.e. sets.

Note that this sum is taken only over $\mathcal{I}(f)$, not over $\mathbb{N}$. It is also worth noting that this notion of $\ll$ is not necessarily transitive, and so should not be thought of as a true domination property. However, when restricted to order functions $\ll$ is transitive as well.

5.3. The main result and some examples.

THEOREM 5.1 (extracting complexity is hard). *If $f : \mathbb{N} \to \mathbb{N}$ is suitable and not equivalent to the identity function, there is an $X \in 2^{\mathbb{N}}$ such that X is f-complex and if $g : \mathbb{N} \to \mathbb{N}$ is recursive, $f \ll g$, and $Y \leq_T X$, then Y is not g-complex.*

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Theorem 1.3, restated as the following corollary, follows from this more general theorem.

COROLLARY 5.2. *If $f : \mathbb{N} \to \mathbb{N}$ is suitable and not equivalent to the identity function, and g is given by $g(n) = f(n) + 2\log_2 f(n)$ for all n, there is an $X \in 2^{\mathbb{N}}$ such that X is f-complex, and if $Y \leq_T X$ then Y is not g-complex.*

PROOF. We can assume without loss of generality that $f(0) = 0$. Then f restricted to $I(f)$ is one-to-one and onto. Then

$$\sum_{n\in\mathcal{I}(f)} 2^{f(n)-g(n)} = \sum_{n\in\mathcal{I}(f)} 2^{-2\log_2 f(n)} = \sum_{m\in\mathbb{N}} 2^{-2\log_2 m} = \sum_{m\in\mathbb{N}} \frac{1}{m^2}$$

which is a finite and recursive sum, and so $f \ll g$. Apply Theorem 5.1 to get X as claimed. ⊣

EXAMPLE 5.3. *Let $f : \mathbb{N} \to [0, \infty)$ be given by $f(n) = n - 2\log_2 n$ for all n. There is an $X \in 2^{\mathbb{N}}$ which is f-random but computes no Martin-Löf random.*

This provides an alternate proof of Theorem 1.1, since X has the property that $\lim_{n\to\infty} \frac{\mathrm{K}(X\upharpoonright n)}{n} = 1$. Moreover, this X has a natural and explicit recursive lower bound on initial segment complexity.

Recall that for each $k \in \mathbb{N}\setminus\{0\}$ we defined $\log_2^k n = \log_2 \log_2^{k-1} n$, where $\log_2^0 n = \log_2 n$.

EXAMPLE 5.4. *For fixed $k \in \mathbb{N}\setminus\{0\}$, let $f, g : \mathbb{N} \to [0, \infty)$ be given by $f(n) = \log_2^k(n)$ and $g(n) = \log_2^k(n) + 2\log_2^{k+1}(n)$. There is an $X \in 2^{\mathbb{N}}$ such that X is f-complex, and if $Y \leq_T X$ then Y is not g-complex.*

Theorem 5.1 also relates back to the effectivization of Hausdorff dimension mentioned before. By applying Theorem 5.1 (or Corollary 5.2) to the function $f(n) = sn$ for recursive $0 < s < 1$, we get an alternative proof of Theorem 1.2.

5.4. A note on Miller's construction. Many of the ideas in this paper were first introduced or partially developed by Miller [12]; indeed large parts of this argument are direct adaptations of arguments originally given by him.

The optimal covers we use are very similar to Miller's, while at the same time having several additional requirements placed on them; these additional requirements improve the usefulness of optimal covers when studying slow-growing functions. Similarly, a concept similar to the approximate cover was discussed by Miller; we have formalized it and used it to help construct optimal covers as well. Additionally, Lemmas 8.2 and 9.1 are straightforward generalizations of Miller's results. Lemmas 8.5 and 9.2 are also largely based on arguments given by Miller and follow similar structures to the ones set up by him. Finally, while the statement of Lemma 8.4 is a generalization of a lemma of Miller's, the proof itself is new.

§6. Notions of f-randomness for the unit interval. The following results will only be used in a single proof, specifically that of Lemma 8.4 and so the reader may skip this section. We wish to develop a notion of f-randomness for $x \in [0, 1]$. However, each $X \in 2^{\mathbb{N}}$ can be treated as a binary expansion of some $x \in [0, 1]$, and so we would like f-randomness to be preserved when moving between these two spaces.

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6.1. Cantor space and the unit interval. For $x \in [0,1)$ define $x^\star \in 2^{\mathbb{N}}$ to be such that $x^\star(n-1)$ is the n^{th} digit in the binary expansion of x for each $n \geq 1$. To ensure that this definition is well-formed, assume that the binary expansions of all dyadic rationals end in an infinite sequence of 0's, rather than 1's (e.g., $\frac{1}{2}$ has expansion $0.100\ldots$ rather than $0.011\ldots$).

For $X \in 2^{\mathbb{N}}$ we define $0.X \in [0,1]$ to be the real number $0 \leq 0.X \leq 1$ whose binary expansion has n^{th} digit $X(n-1)$ for all $n \geq 1$. Similarly, for $\sigma \in 2^{<\mathbb{N}}$ we define $0.\sigma \in [0,1]$ to be the real number $0 \leq 0.\sigma \leq 1$ whose binary expansion has n^{th} digit $\sigma(n-1)$ for $1 \leq n \leq |\sigma|$ and 0 otherwise. It is not hard to see that there are distinct $X, Y \in 2^{\mathbb{N}}$ such that $0.X = 0.Y$. In this case, the real number $0.X$ must be a dyadic rational, and so X, Y, and $0.X = 0.Y$ are all recursive.

For a closed interval $I \subseteq [0,1]$ we define $|I| = -\log_2(x-y)$ where $x = \max I$ and $y = \min I$. Define $\mathcal{J}$ to be the set of all closed intervals $I \subseteq [0,1]$ such that $\min I$, $\max I \in \mathbb{Q}$ and $|I| \in \mathbb{N}$. The set $\mathcal{J}$ can be coded as a subset of $\mathbb{N}^4$ where $(a,b,c,d) \in \mathbb{N}^4$
is a code for the interval $I = [\frac{a}{b}, \frac{c}{d}]$ when this makes sense and $|I| \in \mathbb{N}$. Note that for every $\sigma \in 2^{<\mathbb{N}}$, if we define $I = \{0.X \mid X \in 2^{\mathbb{N}} \text{ s.t. } \sigma \subset X\}$, then $I \in \mathcal{J}$ and $|I| = |\sigma|$. If $A \subseteq \mathcal{J}$, then we define $[A] = \bigcup_{I \in A} I$. We say that $A \subseteq \mathcal{J}$ is prefix-free if for all $I, J \in A$, the set $I \cap J$ is finite (I and J at most overlap at an endpoint, in other words).

LEMMA 6.1. *There is a partial recursive function* $\Phi\colon \subseteq \mathbb{N}^4 \to 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ *such that if* (a,b,c,d) *is a code for* $I \in \mathcal{J}$, *then* $\Phi(a,b,c,d) \downarrow$ *and* $\Phi(a,b,c,d) = (\sigma_1, \sigma_2)$ *where* σ_1, σ_2 *are such that* $I \subseteq \{0.X \mid X \in [\{\sigma_1, \sigma_2\}]\}$ *and* $|\sigma_1| = |\sigma_2| = |I|$.

PROOF. Fix $(a,b,c,d) \in \mathbb{N}^4$ to be a code for $I \in \mathcal{J}$, and let $n = |I|$. If there is a $\sigma \in 2^n$ such that $0.\sigma = a/b = \min I$, then define $\Phi(a,b,c,d) = (\sigma, \sigma)$. Note that in this case, we have $I = \{0.X \mid X \in [\sigma]\}$ as required.

If no such σ as above exists (that is, if $a/b = \min I$ is not a dyadic rational), let σ_1 be the unique string of length n such that $a/b < 0.\sigma_1 < c/d$. Let σ_2 be the predecessor of σ_1 in the lexicographical ordering of 2^n (e.g., the predecessor of 01110 in the lexicographical ordering of 2^5 would be 01101). Define $\Phi(a,b,c,d) = (\sigma_1, \sigma_2)$. If $x \in I$, then $x^\star \upharpoonright n$ must be either σ_1 or σ_2. Therefore $I \subseteq \{0.X \mid X \in [\{\sigma_1, \sigma_2\}]\}$ as required. Note that this construction is done recursively. ⊣

Let $(a,b,c,d) \in \mathbb{N}^4$ be a code for $I \in \mathcal{J}$, let $\Phi(a,b,c,d) = (\sigma_1, \sigma_2)$, and let $x \in [0,1]$ be irrational. In the previous lemma, note that we are ensuring that if $x \in I$ then $x^\star \in [\sigma_1] \cup [\sigma_2]$.

6.2. f-weights. Let $f : \mathbb{N} \to \mathbb{N}$ be recursive and unbounded, and let $A \subseteq \mathcal{J}$. The *direct f-weight* of A is $\mathrm{dwt}_f(A) = \sum_{I \in A} 2^{-f(|I|)}$. The *vehement f-weight* of A is $\mathrm{vwt}_f(A) = \inf\{\mathrm{dwt}_f(B) \mid [B] \supseteq [A]\}$. The *prefix f-weight* of A is $\mathrm{pwt}_f(A) = \sup\{\mathrm{dwt}_f(P) \mid P \subseteq A \text{ is prefix-free}\}$.

6.3. f-randomness. Let $f : \mathbb{N} \to \mathbb{N}$ be recursive and unbounded. A uniformly r.e. sequence $\langle A_k \subseteq \mathcal{J} \mid k \in \mathbb{N}\rangle$ is called an *f-test* if $\mathrm{dwt}_f(A_k) \leq 2^{-k}$ for each $k \in \mathbb{N}$. We say that $x \in [0,1]$ is *f-random* if for every f-test $\langle A_k \rangle$ there is a k such that $x \notin [A_k]$; otherwise we say that x is *f-null*.

LEMMA 6.2. *If* $f : \mathbb{N} \to \mathbb{N}$ *is recursive and unbounded and* $x \in [0,1]$, *then x is f-random if and only if $x^\star$ is f-random*.

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PROOF. If either x or $x^\star$ is recursive, then both will be; in this case, neither will be f-random. So we assume that neither x or $x^\star$ is recursive, and so in particular x is not a dyadic rational and has a unique binary expansion.

Suppose that x is f-null. Then there is an f-test $\langle A_k \subseteq \mathcal{J} \rangle$ such that $x \in [A_k]$ for all $k \in \mathbb{N}$. We define a sequence $\langle B_k \subseteq 2^{<\mathbb{N}} \mid k \in \mathbb{N} \rangle$ as follows. Let $\Phi\colon \mathbb{N}^4 \to 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ be the partial recursive function constructed in Lemma 6.1. Fix $k > 0$. For each $I \in A_k$, let (a, b, c, d) be a code for I and let $(\sigma_1, \sigma_2) = \Phi(a, b, c, d)$; put σ_1, σ_2 into B_{k-1}. This defines $\langle B_k \rangle$ as a uniformly r.e. sequence. For each $k > 1$,

$$\mathrm{dwt}_f(B_{k-1}) = \sum_{\sigma \in B_{k-1}} 2^{-f(|\sigma|)} \leq \sum_{I \in A_k} 2^{-f(|I|)+1} \leq 2^{-k+1}.$$

Therefore $\langle B_k \rangle$ is an f-test (on $2^{\mathbb{N}}$ this time). Now fix $k > 1$, and let $I \in \mathcal{A}_k$ be such that $x \in I$. Then for a code (a, b, c, d) for I, we have $(\sigma_1, \sigma_2) = \Phi(a, b, c, d)$, and both σ_1 and σ_2 are in B_{k+1} and $x^\star \in [\sigma_1] \cup [\sigma_2]$. Therefore $x^\star \in [B_k]$ for all $k \in \mathbb{N}$ and so $x^\star$ is f-null.

Now suppose that $x^\star$ is f-null. Then there is an f-test $\langle A_k \subseteq 2^{<\mathbb{N}} \mid k \in \mathbb{N} \rangle$ such that $x^\star \in [A_k]$ for all k. Define a sequence $\langle B_k \subseteq \mathcal{J} \mid k \in \mathbb{N} \rangle$ as follows. For each $k \in \mathbb{N}$ and each $\tau \in A_k$, put the interval $I = \{0.X \mid X \in [\tau]\}$ into B_k (this interval I is closed, has dyadic rational endpoints, and $|I| = |\tau|$). Then $\mathrm{dwt}_f(B_k) = \mathrm{dwt}_f(A_k)$ for all $k \in \mathbb{N}$ and so $\langle B_k \rangle$ is an f-test (on [0,1] in this case). If $X \upharpoonright n \in A_k$ then $0.X \in B_k$. Since $0.x^\star = x$, this implies that x is f-null. ⊣

The point of this lemma, and in fact this whole section, is that f-randomness in [0,1] is the same as f-randomness in $2^{\mathbb{N}}$. We will use this result later, particularly in Lemma 8.4.

§7. Optimal f-covers. We now introduce a generalization of (and an alternate approach to) the *optimal covers* used by Miller [12]. Many results about the original optimal covers remain valid in this more general setting.

7.1. Optimal and approximate f-covers. Miller's original idea of optimal covers was to pick a canonical set that is best (in a precise sense) at covering a given set. Here, we broaden the concept to allow for many optimal covers; indeed, many sets can be just as efficient at covering the given set. We also introduce a new concept of approximate covers, which is closely related.

If $A, B \subseteq 2^{<\mathbb{N}}$ are such that $[B] \supseteq [A]$, $\mathrm{dwt}_f(B) = \mathrm{vwt}_f(A)$, and $|\sigma| \in \mathcal{I}(f)$ for all $\sigma \in B$, then we say that B is an *optimal f-cover* of A. If $A, B \subseteq 2^{<\mathbb{N}}$ are r.e. sets such that for each k, $\widehat{B^k}$ is an optimal f-cover of A^k, then we say that B is an *approximate f-cover* of A.

Note that an optimal f-cover of A then has minimal direct f-weight among all covers of A; these two notions are closely related in the following way.

LEMMA 7.1. *Let $f : \mathbb{N} \to \mathbb{N}$ be recursive and let $A \subseteq 2^{<\mathbb{N}}$ be r.e. If $B \subseteq 2^{<\mathbb{N}}$ is an approximate f-cover of A, then $\widehat{B}$ is an optimal f-cover of A.*

PROOF. First, we make the following observations. If $\sigma \in \widehat{B}$ then $\sigma \in \widehat{B^k}$ for all sufficiently large k, and if $\sigma \notin \widehat{B}$ then $\sigma \notin \widehat{B^k}$ for all sufficiently large k. In other words, $\widehat{B}$ is the pointwise limit of the sets $\widehat{B^k}$. Also $\mathrm{dwt}_f(\widehat{B^{k+1}}) \geq \mathrm{dwt}_f(\widehat{B^k})$ for all k, since otherwise $\widehat{B^k}$ would fail to be an optimal f-cover of A^k.

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If it were the case that $\lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k}) < \mathrm{dwt}_f(\widehat{B})$, there would be some finite set $P \subseteq \widehat{B}$ such that $\lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k}) < \mathrm{dwt}_f(P)$ as well. Since we can pick an N sufficiently large such that $P \subseteq \widehat{B^k}$ for all $k > N$, this would imply

$$\lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k}) < \mathrm{dwt}_f(P) \leq \lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k})$$

which is a contradiction. Then $\lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k}) \geq \mathrm{dwt}_f(\widehat{B})$.

Now suppose that $\widehat{B}$ is not an optimal f-cover of A. Then there is a set $C \subseteq 2^{<\mathbb{N}}$ such that $[C] \supseteq [A]$ and $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\widehat{B})$. By the above, there is some k such that $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\widehat{B^k})$. Since $[C] \supseteq [A^k]$, $\widehat{B^k}$ is not an optimal f-cover of A^k. This is a contradiction, and so $\widehat{B}$ is an optimal f-cover of A. ⊣

7.2. Boundedly optimal and approximate f-covers. We should note that the existence of optimal f-covers and approximate f-covers is not guaranteed, even for finite sets. For example, if $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(n) = 2n$ for all n, then $\mathrm{vwt}_f(A) = 0$ for every $A \subseteq 2^{<\mathbb{N}}$, and so the only set with an optimal f-cover is $\emptyset$. For this reason we define the following concepts.

Let $f : \mathbb{N} \to \mathbb{N}$ be recursive. If $A, B \subseteq 2^{<\mathbb{N}}$ are such that

(1) $[B] \supseteq [A]$,
(2) $\mathrm{dwt}_f(B) = \inf\{\mathrm{dwt}_f(C) \mid [C] \supseteq [A] \text{ and } (\forall \sigma \in C)(\exists \tau \in A)(\sigma \subseteq \tau)\}$, and
(3) $|\sigma| \in \mathcal{I}(f)$ for all $\sigma \in B$,

then we say that B is a *boundedly optimal f-cover* of A. If $A, B \subseteq 2^{<\mathbb{N}}$ are r.e. sets such that for each k, $\widehat{B^k}$ is a boundedly optimal f-cover of A^k, then we say that B is a *boundedly approximate f-cover* of A.

Note that the infimum used to define boundedly optimal f-covers is closely related to the definition of vehement f-weight earlier in the paper; in this new definition we take the infimum over a greatly restricted selection of sets.

Unlike optimal f-covers and approximate f-covers, we can say for certain that every finite set has a boundedly optimal f-cover; moreover they can be found in a uniform enough way that every r.e. set with a frugal enumeration will have a boundedly approximate f-cover. This is achieved using the following lemmas.

LEMMA 7.2. *Let $f : \mathbb{N} \to \mathbb{N}$ be recursive. If $A \subseteq 2^{<\mathbb{N}}$ is finite, then there is a $B \subseteq 2^{<\mathbb{N}}$ such that B is a boundedly optimal f-cover of A.*

PROOF. Since A is finite, the infimum used to define the boundedly optimal f-cover becomes a minimum. So let $C \subseteq 2^{<\mathbb{N}}$ be such that

(1) $[C] \supseteq [A]$,
(2) $(\forall \sigma \in C)(\exists \tau \in A)(\sigma \subseteq \tau)$, and
(3) $\mathrm{dwt}_f(C) = \min\{\mathrm{dwt}_f(D) \mid [D] \supseteq [A] \text{ and } (\forall \sigma \in D)(\exists \tau \in A)(\sigma \subseteq \tau)\}$.

Define $B = \{\sigma \upharpoonright k \mid \sigma \in C,\ k \in \mathcal{I}(f),\ \text{and } f(k) = f(|\sigma|)\}$. Obviously

(1) $\mathrm{dwt}_f(B) \leq \mathrm{dwt}_f(C)$,
(2) $[B] \supseteq [A]$, and
(3) $(\forall \sigma \in B)(\exists \tau \in A)(\sigma \subseteq \tau)$.

So B is a boundedly optimal f-cover of A, as required. ⊣

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LEMMA 7.3. *Let $f : \mathbb{N} \to \mathbb{N}$ be recursive. If $B \subseteq 2^{<\mathbb{N}}$ is a boundedly optimal f-cover of $A \subseteq 2^{<\mathbb{N}}$ and $\sigma \in 2^{<\mathbb{N}}$ is such that $\neg(\exists \tau \in B \cup A)(\sigma \supset \tau)$, then $B \setminus [\sigma]^{<\mathbb{N}}$ is a boundedly optimal f-cover of $A \setminus [\sigma]^{<\mathbb{N}}$.*

PROOF. Note that $A = (A \setminus [\sigma]^{<\mathbb{N}}) \cup (A \cap [\sigma]^{<\mathbb{N}})$ and $B = (B \setminus [\sigma]^{<\mathbb{N}}) \cup (B \cap [\sigma]^{<\mathbb{N}})$. Suppose $B \setminus [\sigma]^{<\mathbb{N}}$ is not a boundedly optimal f-cover of $A \setminus [\sigma]^{<\mathbb{N}}$. Then there is a $C \subseteq 2^{<\mathbb{N}}$ such that $[C] \supseteq [A] \setminus [\sigma]$, $(\forall \tau \in C)(\exists \rho \in A \setminus [\sigma]^{<\mathbb{N}})(\tau \subseteq \rho)$, and $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(B \setminus [\sigma]^{<\mathbb{N}})$. But then define $D = C \cup (B \cap [\sigma]^{<\mathbb{N}})$ and note that $\mathrm{dwt}_f(D) < \mathrm{dwt}_f(B)$, $[D] \supseteq [A]$, and $(\forall \tau \in D)(\exists \rho \in A)(\tau \subseteq \rho)$ which is a contradiction of the assumption that B is a boundedly optimal f-cover of A. ⊣

LEMMA 7.4. *Let $f : \mathbb{N} \to \mathbb{N}$ be recursive and let $A \subseteq 2^{<\mathbb{N}}$ be an r.e. set with a frugal enumeration. Then there is an r.e. set $B \subseteq 2^{<\mathbb{N}}$ such that B is a boundedly approximate f-cover of A.*

PROOF. Recall what it means for the enumeration of A to be frugal: for any $\sigma \subset \tau$ and any t, if $\sigma \in A^t$ and $\tau \notin A^t$ then $\tau \notin A$. We define B by defining the set B^n recursively in n. If $A^1 = \{\sigma\}$, then we define $B^1 = \{\sigma \upharpoonright k\}$ where $k \in \mathcal{I}(f)$ is such that $f(k) = f(|\sigma|)$. If $[A^n] \subseteq [B^{n-1}]$, then define $B^n = B^{n-1}$. Otherwise, let $\sigma_n \in A^n \setminus A^{n-1}$ (such σ_n is unique, of course). Let k_n be the biggest $k \leq |\sigma_n|$ such that $\sigma_n \upharpoonright k$ belongs to some boundedly optimal f-cover of A^n. For later reference, we write C_n to denote some boundedly optimal f-cover of A^n such that $\sigma_n \upharpoonright k_n \in C_n$. Define $B^n = B^{n-1} \cup \{\sigma_n \upharpoonright k_n\}$.

Now we wish to show that B is a boundedly approximate f-cover of A. Obviously $\widehat{B^1}$ is a boundedly optimal f-cover of A^1. Now suppose that $\widehat{B^{n-1}}$ is a boundedly optimal f-cover of A^{n-1}. If $B^n = B^{n-1}$, then $\widehat{B^n}$ is trivially a boundedly optimal f-cover of A^n. If $B^n \neq B^{n-1}$, then by Lemma 7.3 and the fact that A has a frugal enumeration, $\widehat{B^{n-1}} \setminus [\sigma_n \upharpoonright k_n]^{<\mathbb{N}} = \widehat{B^n} \setminus [\sigma_n \upharpoonright k_n]^{<\mathbb{N}}$ is a boundedly optimal f-cover for $A^{n-1} \setminus [\sigma_n \upharpoonright k_n]^{<\mathbb{N}} = A^n \setminus [\sigma_n \upharpoonright k_n]^{<\mathbb{N}}$; by the same argument $C_n \setminus [\sigma_n \upharpoonright k_n]^{<\mathbb{N}}$ is a boundedly optimal f-cover for $A^n \setminus [\sigma_n \upharpoonright k_n]^{<\mathbb{N}}$. But then it follows that $\mathrm{dwt}_f(\widehat{B^n} \setminus \{\sigma_n \upharpoonright k_n\}) = \mathrm{dwt}_f(C_n \setminus \{\sigma_n \upharpoonright k_n\})$ and so $\mathrm{dwt}_f(\widehat{B^n}) = \mathrm{dwt}_f(C_n)$. This shows that $\widehat{B^n}$ is a boundedly optimal f-cover for A^n as required. ⊣

Note that the previous construction gives preferential treatment to longer strings: if both $\sigma \upharpoonright n$ and $\sigma \upharpoonright m$ are potential candidates to add to B at the same stage, $n < m$, and $n, m \in \mathcal{I}(f)$, then the construction will always pick $\sigma \upharpoonright m$ to be in B. This provides a uniformity to this construction, as we will see in Lemma 7.10.

7.3. Suitable functions guarantee the existence of optimal and approximate f-covers. Recall that $f : \mathbb{N} \to \mathbb{N}$ is suitable if it is recursive, unbounded, and $f(n+1) \leq f(n) + 1$ for all n. The following is the key lemma pertaining to suitable functions.

LEMMA 7.5. *If $f : \mathbb{N} \to \mathbb{N}$ is suitable, $\sigma \in 2^{<\mathbb{N}}$, and $B \subseteq [\sigma]^{<\mathbb{N}}$ is such that $[\sigma] \subseteq [B]$, then* $\mathrm{dwt}_f(\sigma) \leq \mathrm{dwt}_f(B)$.

PROOF. By compactness, we can assume that B is a finite set. We may also assume that B is prefix-free. Define $B_0 = B$. For each n, we define B_{n+1} in terms of B_n as follows. If $B_n = \{\sigma\}$ then define $B_{n+1} = B_n$; otherwise let $\tau^\frown 0, \tau^\frown 1 \in B_n$ be as long as possible. Then since f is suitable we have

$$\mathrm{dwt}_f(\tau) = 2^{-f(|\tau|)} \leq 2^{-f(|\tau|+1)+1} = 2 \cdot 2^{-f(|\tau|+1)} = \mathrm{dwt}_f(\{\tau^\frown 0, \tau^\frown 1\}).$$

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So define $B_{n+1} = (B_n \setminus \{\tau^\frown 0, \tau^\frown 1\}) \cup \{\tau\}$. We have $[B_{n+1}] = [B_n]$ and $\mathrm{dwt}_f(B_{n+1}) \leq \mathrm{dwt}_f(B_n)$. Each B_k is prefix-free as well. Since B is finite, there must be a stage k such that $B_k = \{\sigma\}$. Since $\mathrm{dwt}_f(B_k) \leq \mathrm{dwt}_f(B_0)$, this completes the proof. ⊣

The following is an immediate corollary.

COROLLARY 7.6. *If $f : \mathbb{N} \to \mathbb{N}$ is suitable, and $\sigma \in 2^{<\mathbb{N}}$ and $A, B \subseteq 2^{<\mathbb{N}}$ are such that $\mathrm{dwt}_f(\sigma) > \mathrm{vwt}_f(A)$ and B is an optimal f-cover of A, then $[\sigma] \nsubseteq [B]$.*

The following shows the importance of using suitable functions: a bounded strategy for covering a set is also an optimal one.

LEMMA 7.7. *If $f : \mathbb{N} \to \mathbb{N}$ is suitable, $A \subseteq 2^{<\mathbb{N}}$, and $B \subseteq 2^{<\mathbb{N}}$ is a boundedly optimal f-cover of A, then B is an optimal f-cover of A.*

PROOF. Let $C \subseteq 2^{<\mathbb{N}}$ be such that $[C] \supseteq [A]$. Define

$$D = \left\{\sigma \in 2^{<\mathbb{N}} \mid [\sigma] \subseteq [C] \text{ and } (\exists \tau \in A)(\sigma \subseteq \tau)\right\}$$

and then define $E = \widehat{D}$. It is clear that $[E] \supseteq [A]$ and $(\forall \sigma \in E)(\exists \tau \in A)(\sigma \subseteq \tau)$. For every $\sigma \in E$, let $C_\sigma = \{\tau \in C \mid \sigma \subseteq \tau\}$. By definition of E, it is true that $[C_\sigma] = [\sigma]$ for each $\sigma \in E$. So by Lemma 7.5

$$\mathrm{dwt}_f(E) = \sum_{\sigma \in E} \mathrm{dwt}_f(\sigma) \leq \sum_{\sigma \in E} \mathrm{dwt}_f(C_\sigma) \leq \mathrm{dwt}_f(C).$$

So $\mathrm{dwt}_f(E) \leq \mathrm{dwt}_f(C)$, and by construction $(\forall \sigma \in E)(\exists \tau \in A)(\sigma \subseteq \tau)$. Since this construction can be done for arbitrary $[C] \supseteq [A]$, this shows that

$$\mathrm{vwt}_f(A) = \inf\{\mathrm{dwt}_f(E) \mid [E] \supseteq [A] \text{ and } (\forall \sigma \in E)(\exists \tau \in A)(\sigma \subseteq \tau)\}.$$

Then every boundedly optimal f-cover is an optimal f-cover, as required. ⊣

COROLLARY 7.8. *If $f : \mathbb{N} \to \mathbb{N}$ is suitable and $A \subseteq 2^{<\mathbb{N}}$ is recursively enumerable, then A has an optimal f-cover.*

PROOF. Let C be an r.e. set with frugal enumeration such that $\widehat{C} = \widehat{A}$ (and so $[C] = [A]$ as well). Then by Lemma 7.4 the set C has a boundedly approximate f-cover B. Then for each k the set $\widehat{B^k}$ is a boundedly optimal f-cover of C^k, and so by Lemma 7.7 it is also an optimal f-cover of C^k. Therefore by Lemma 7.1 the set $\widehat{B}$ is an optimal f-cover of C. Since $[C] = [A]$, it follows that $\mathrm{vwt}_f(C) = \mathrm{vwt}_f(A)$ and so $\widehat{B}$ is an optimal f-cover of A. ⊣

Looking at the definitions introduced in this section, and the proof of Corollary 7.8, we have proven the following:

THEOREM 7.9. *Let $f : \mathbb{N} \to \mathbb{N}$ be suitable. For any r.e. set $A \subseteq 2^{<\mathbb{N}}$ with a frugal enumeration there is an r.e. set B with the following properties*:

(1) $[B] \supseteq [A]$,
(2) $[B^k] \supseteq [A^k]$ *for each* k,
(3) $\mathrm{dwt}_f(\widehat{B}) = \mathrm{vwt}_f(A)$,
(4) $\mathrm{dwt}_f(\widehat{B^k}) = \mathrm{vwt}_f(A^k)$ *for each* k, *and*
(5) $|\sigma| \in \mathcal{I}(f)$ *for all* $\sigma \in B$.

If A does not have a frugal enumeration, (1), (3), *and* (5) *still hold.*

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7.4. A canonical way to pick optimal and approximate f-covers. Let $f : \mathbb{N} \to \mathbb{N}$ be a suitable function. As mentioned before, a set might still have several optimal f-covers and approximate f-covers. For an r.e. set $A \subseteq 2^{<\mathbb{N}}$ with a fixed frugal enumeration, we refer to the set B constructed in Lemma 7.4 as *the* approximate f-cover of A, denoted $\mathcal{AC}_f(A)$. We refer to $\widehat{B}$ as *the* optimal f-cover of A, denoted $\mathcal{OC}_f(A)$. Note that the definition of $\mathcal{OC}_f(A)$ and $\mathcal{AC}_f(A)$ will depend on the enumeration of A we have fixed; the important thing is that we have a uniform way of generating these sets from a fixed frugal enumeration.

LEMMA 7.10. *If f is a suitable function, $A \subseteq 2^{<\mathbb{N}}$ is r.e. with a frugal enumeration, and $\sigma \in 2^{<\mathbb{N}}$ is such that $[\sigma] \not\subseteq [\mathcal{OC}_f(A)]$, then $\mathcal{OC}_f(A \cap [\sigma]^{<\mathbb{N}}) = \mathcal{OC}_f(A) \cap [\sigma]^{<\mathbb{N}}$, assuming $A \cap [\sigma]^{<\mathbb{N}}$ has the enumeration inherited from A.*

PROOF. First we claim that $\mathcal{OC}_f(A) \cap [\sigma]^{<\mathbb{N}}$ is an optimal f-cover of $A \cap [\sigma]^{<\mathbb{N}}$. If not, there is a set C such that $[C] \supseteq A \cap [\sigma]^{<\mathbb{N}}$ and $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\mathcal{OC}_f(A) \cap [\sigma]^{<\mathbb{N}})$. Then defining $B = (\mathcal{OC}_f(A) \setminus [\sigma]^{<\mathbb{N}}) \cup C$, we have $[B] \supseteq [A]$ and

$$\mathrm{dwt}_f(B) = \mathrm{dwt}_f(\mathcal{OC}_f(A)) - \mathrm{dwt}_f(\mathcal{OC}_f(A) \cap [\sigma]^{<\mathbb{N}}) + \mathrm{dwt}_f(C) < \mathrm{dwt}_f(\mathcal{OC}_f(A)).$$

But then $\mathcal{OC}_f(A)$ cannot be an optimal f-cover of A, which is a contradiction.

Then it remains to show that $\mathcal{OC}_f(A) \cap [\sigma]^{<\mathbb{N}}$ is *the* optimal f-cover of $A \cap [\sigma]^{<\mathbb{N}}$. For each k, consider $A^k \cap [\sigma]^{<\mathbb{N}}$ and A^k. If some set C is an optimal f-cover for A^k and contains no proper prefix of σ, then the set $C \cap [\sigma]^{<\mathbb{N}}$ will be an optimal f-cover for $A^k \cap [\sigma]^{<\mathbb{N}}$ by an argument similar to the above. Similarly, if D is an optimal f-cover of $A^k \cap [\sigma]^{<\mathbb{N}}$ and contains no proper prefix of σ then there is an optimal f-cover C of A^k such that $C \cap [\sigma]^{<\mathbb{N}} = D$. In summary, any $\tau \supseteq \sigma$ belongs to an optimal f-cover of $A^k \cap [\sigma]^{<\mathbb{N}}$ if and only if τ belongs to an optimal f-cover of A^k. Then in the constructions of $\mathcal{OC}_f(A) \cap [\sigma]^{<\mathbb{N}}$ and $\mathcal{OC}_f(A \cap [\sigma]^{<\mathbb{N}})$ (using Lemma 7.4), since the constructions always choose the longest available strings at each point, and each construction will have the same strings extending σ available, we will have $\mathcal{OC}_f(A \cap [\sigma]^{<\mathbb{N}}) = \mathcal{OC}_f(A) \cap [\sigma]^{<\mathbb{N}}$ as required. ⊣

7.5. Properties of approximate and optimal f-covers. When we defined optimal f-covers, it was evident that the property we were optimizing was direct f-weight. That is, an optimal f-cover of A has minimum direct f-weight among all covers of A. The following lemma shows that approximate f-covers are optimal in a weaker sense: an approximate f-cover of A has minimal prefix f-weight among all covers of A.

LEMMA 7.11. *Let f be suitable and let A and B be r.e. sets. If B is an approximate f-cover of A, then* $\mathrm{pwt}_f(B) = \mathrm{vwt}_f(A)$.

PROOF. Since $\widehat{B}$ is prefix-free and $[\widehat{B}] \supseteq [A]$, it is clear that $\mathrm{vwt}_f(A) \leq \mathrm{pwt}_f(B)$. So we need only show that $\mathrm{pwt}_f(B) \leq \mathrm{vwt}_f(A)$. Suppose that this property does not hold for some A, B as in the statement. Then there is a $k \in \mathbb{N}$ such that $\mathrm{pwt}_f(B^k) \leq \mathrm{vwt}_f(A^k)$, but $\mathrm{pwt}_f(B^{k+1}) > \mathrm{vwt}_f(A^{k+1})$. So let $P \subseteq B^{k+1}$ be a prefix-free set such that $\mathrm{dwt}_f(P) > \mathrm{vwt}_f(A^{k+1})$, and let $\sigma \in B^{k+1} \setminus B^k$ (such a σ is unique). It must be true that $\sigma \in P$, since otherwise we would have $P \subseteq B^k$ as well. Now define $P' = (P \setminus \{\sigma\}) \cup \widehat{C}$, where $C = \{\tau \in B^k \mid \sigma \subseteq \tau\}$. Clearly $P' \subseteq B^k$ and P' is prefix-free. Since the $\widehat{B^i}$ are optimal f-covers of the A^i we have

$$\mathrm{vwt}_f(A^{k+1}) - \mathrm{vwt}_f(A^k) = \mathrm{dwt}_f(\widehat{B^{k+1}}) - \mathrm{dwt}_f(\widehat{B^k}) = \mathrm{dwt}_f(\sigma) - \mathrm{dwt}_f(\widehat{C}).$$

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Recall that P was picked such that $\mathrm{dwt}_f(P) - \mathrm{vwt}_f(A^{k+1}) > 0$. Then we also have

$$\begin{aligned}\mathrm{pwt}_f(B^k) \geq \mathrm{dwt}_f(P') &= \mathrm{dwt}_f(P) - \mathrm{dwt}_f(\sigma) + \mathrm{dwt}_f(\widehat{C}) \\ &= \mathrm{dwt}_f(P) - \mathrm{vwt}_f(A^{k+1}) + \mathrm{vwt}_f(A^k) > \mathrm{vwt}_f(A^k)\end{aligned}$$

which is a contradiction since $\mathrm{pwt}_f(B^k) \leq \mathrm{vwt}_f(A^k)$. This completes the proof. ⊣

The following is a useful observation about the possible size of the vehement f-weight of a set $A \subseteq 2^{<\mathbb{N}}$.

LEMMA 7.12. *Let $f : \mathbb{N} \to \mathbb{N}$ be suitable. If $A \subseteq 2^{<\mathbb{N}}$ is an r.e. set with frugal enumeration then*

$$\mathrm{pwt}_f(\mathcal{AC}_f(A)) = \mathrm{vwt}_f(A) = \mathrm{dwt}_f(\mathcal{OC}_f(A)) \leq 1.$$

PROOF. This is immediate, since $[\lambda] \supseteq [A]$ and $\mathrm{dwt}_f(\lambda) = 2^{-f(0)} \leq 1$. ⊣

§8. Forcing with optimal f-covers. The original technique given by Miller [12] was to construct effectively closed subsets of $2^{\mathbb{N}}$ whose measures have small effective Hausdorff dimension. The idea with our forcing conditions will be to restrict the functions g for which the measures of these sets are g-random. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is suitable if it is recursive, unbounded, and $f(n+1) \leq f(n)+1$ for all n.

8.1. Defining f-conditions. Let $f : \mathbb{N} \to \mathbb{N}$ be suitable, $\sigma \in 2^{<\mathbb{N}}$, and let $S \subseteq 2^{<\mathbb{N}}$ be an r.e. set with a frugal enumeration. We say that the ordered pair $\langle \sigma, S \rangle$ is an *f-condition* if $[\sigma] \supsetneq [\mathcal{OC}_f(S)]$. In other words, $\langle \sigma, S \rangle$ is an f-condition if S is a subset of $[\sigma]^{<\mathbb{N}}$ whose optimal f-cover does not cover $[\sigma]$. If $\langle \sigma, S \rangle$ is an f-condition, define $P_{\langle \sigma, S \rangle} = [\sigma] \setminus [\mathcal{OC}_f(S)]$. Since $[\mathcal{OC}_f(S)] = [\mathcal{AC}_f(S)]$ is the union of an effective list of basic open neighborhoods, the set $P_{\langle \sigma, S \rangle}$ can be thought of as effectively closed. If $\langle \sigma, S \rangle$ and $\langle \tau, T \rangle$ are f-conditions, we say that $\langle \tau, T \rangle$ extends $\langle \sigma, S \rangle$, denoted $\langle \tau, T \rangle \preccurlyeq \langle \sigma, S \rangle$, when $P_{\langle \tau, T \rangle} \subseteq P_{\langle \sigma, S \rangle}$ and $\sigma \subseteq \tau$. This defines a partial order on the f-conditions.

8.2. Properties of f-conditions. The first thing we can note is the following basic result about f-conditions.

LEMMA 8.1 (extending an f-condition). *If $\langle \sigma, S \rangle$ is an f-condition, $\tau \supseteq \sigma$, and $[\tau] \nsubseteq [\mathcal{OC}_f(S)]$, then $\langle \tau, S \cap [\tau]^{<\mathbb{N}} \rangle$ is an f-condition and $\langle \tau, S \cap [\tau]^{<\mathbb{N}} \rangle \preccurlyeq \langle \sigma, S \rangle$.*

PROOF. This is a corollary of Lemma 7.10. ⊣

If $\langle \sigma, S \rangle$ is an f-condition, then there is some $X \in P_{\langle \sigma, S \rangle}$. In other words, the corresponding effectively closed set is non-empty. The following lemma proves a much stronger result, that in fact these effectively closed sets have positive measures.

LEMMA 8.2 (f-conditions have positive measures). *If $f : \mathbb{N} \to \mathbb{N}$ is a suitable function not equivalent to the identity function and $\langle \sigma, S \rangle$ is an f-condition, then $\mu(P_{\langle \sigma, S \rangle}) > 0$.*

PROOF. Since f is not equivalent to the identity function, for each c there exists an n such that $f(n) < n - c$, and so $f(n+1) < f(n) + 1$ for infinitely many n. If $\mathcal{OC}_f(S)$ is a finite set, the fact that $P_{\langle \sigma, S \rangle}$ is non-empty implies that it also has positive measure. So we assume that $\mathcal{OC}_f(S)$ is an infinite set.

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Let $n = |\sigma|$. Since $\mathcal{OC}_f(S) \neq \{\sigma\}$, $\mathrm{vwt}_f(S) \leq \mathrm{dwt}_f(\sigma) = 2^{-f(n)}$. Additionally, $|\tau| > |\sigma| = n$ for all $\tau \in \mathcal{OC}_f(S)$. Since $\mathcal{OC}_f(S)$ is infinite, it contains arbitrarily long strings; so there is some $\tau \in \mathcal{OC}_f(S)$ such that $f(|\tau|) < f(n) + |\tau| - n$. Then

$$\begin{aligned}\mu([\mathcal{OC}_f(S)]) &= \sum_{\tau\in\mathcal{OC}_f(S)} 2^{-|\tau|} < \sum_{\tau\in\mathcal{OC}_f(S)} 2^{-n+f(n)-f(|\tau|)} \\ &= 2^{-n+f(n)}\mathrm{dwt}_f(\mathcal{OC}_f(S)) \\ &= 2^{-n+f(n)}\mathrm{vwt}_f(S) \leq 2^{-n}.\end{aligned}$$

So $\mu([\mathcal{OC}_f(S)]) < \mu([\sigma])$ and therefore $\mu([\sigma] \setminus [\mathcal{OC}_f(S)]) > 0$ as required. ⊣

The following two lemmas give a taste of why we restrict to functions f and g such that $f \ll g$.

LEMMA 8.3. *Let $f : \mathbb{N} \to \mathbb{N}$ be suitable and let $g : \mathbb{N} \to [0, \infty)$ be recursive such that $f \ll g$. There is a recursive function $p : \mathbb{N} \to \mathbb{N}$ such that for any set A the following holds: if $|\sigma| \in \mathcal{I}(f)$ for all $\sigma \in A$ then $\mathrm{dwt}_g(A \cap 2^{>p(n)}) < 2^{-n}\mathrm{pwt}_f(A)$ for every n.*

PROOF. Fix n. Assume that $|\sigma| \in \mathcal{I}(f)$ for all $\sigma \in A$. Note that every set of the form $A_m = A \cap 2^m$ is prefix-free. In addition $A = \bigcup_m A_m$. Then for any k note

$$\begin{aligned}\mathrm{dwt}_g(A \cap 2^{>k}) &= \sum_{m>k} \mathrm{dwt}_g(A_m) \\ &= \sum_{m>k} \sum_{\sigma\in A_m} 2^{-g(m)} \\ &= \sum_{m>k} \sum_{\sigma\in A_m} 2^{f(m)-g(m)-f(m)} \\ &= \sum_{m>k} 2^{f(m)-g(m)}\mathrm{dwt}_f(A_m) \leq \sum_{\substack{m\in\mathcal{I}(f)\\ m>k}} 2^{f(m)-g(m)}\mathrm{pwt}_f(A).\end{aligned}$$

Since $f \ll g$ the sum $\sum_{m\in\mathcal{I}(f)} 2^{f(m)-g(m)} < \infty$ converges to a recursive real number. Then we can effectively in n find some number k such that

$$\sum_{\substack{m\in\mathcal{I}(f)\\ m>k}} 2^{f(m)-g(m)} < 2^{-n}.$$

Define $p(n) = k$ for such an effectively picked k. This defines p in a recursive fashion, and guarantees that $\mathrm{dwt}_g(A \cap 2^{>p(n)}) < 2^{-n}\mathrm{pwt}_f(A)$ for each n as required. ⊣

LEMMA 8.4 (the measure of a condition has restricted randomness). *If $f : \mathbb{N} \to \mathbb{N}$ is suitable, $\langle \sigma, S\rangle$ is an f-condition, and $g : \mathbb{N} \to [0, \infty)$ is recursive such that $f \ll g$, then $\mu(P_{\langle\sigma,S\rangle})$ is g-null.*

PROOF. For convenience, let $A = \mathcal{AC}_f(S)$; note that if A is finite this problem becomes trivial, and so we may assume that A is infinite. We may also assume that A is enumerated without repetition. Since $\mu(P_{\langle\sigma,S\rangle}) = \mu([\sigma]) - \mu([A])$ and $\mu([\sigma])$ is a diadic rational, there will be some N such that $\mu(P_{\langle\sigma,S\rangle})^\star(n) = 1 - \mu([A])^\star(n)$ for all $n > N$; then it is clear that $\mu(P_{\langle\sigma,s\rangle})$ is g-null if and only if $\mu([A])$ is. Then

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we will show that $\mu([A])$ is g-null. We will define a g-test $\langle B_k \rangle$ by building sets B_k^t uniformly in k and t and letting $B_k = \bigcup_t B_k^t$.

Let p be the recursive function guaranteed by Lemma 8.3. Note that since A is an approximate f-cover, we have $\mathrm{pwt}_f(A) \leq 1$ and so $\mathrm{dwt}_g(A \cap 2^{>p(k)}) \leq 2^{-k}$ for each k.

Defining B_k for fixed k. First define $a_0 = b_0 = 0$. For each $t > 0$ let τ_t be the unique new element enumerated into A^t, and define a_t, b_t as follows.

(1) If $|\tau_t| \leq p(k)$ then define $a_t = b_t = b_{t-1}$.
(2) If $|\tau_t| > p(k)$ and either $|\tau_{t-1}| \leq p(k)$ or $t = 1$ then define $a_t = \mu([A^t])$ and $b_t = a_t + 2^{-|\tau_t|} = \mu([A^t]) + 2^{-|\tau_t|}$.
(3) If $t \neq 1$, $|\tau_t| > p(k)$, and $|\tau_{t-1}| > p(k)$, then define $a_t = b_{t-1}$ and $b_t = a_t + 2^{-|\tau_t|} = b_{t-1} + 2^{-|\tau_t|}$.

Now define $B_k^0 = \emptyset$. For $t > 0$ define B_k^t as follows.

(1) If $a_t = b_t$, define $B_k^t = B_k^{t-1}$.
(2) If $a_t \neq b_t$ define $B_k^t = B_k^{t-1} \cup \{[a_t, b_t]\}$.

As stated before we define $B_k = \bigcup_t B_k^t$ for each k.

Proving that $\langle B_k \rangle$ is a g-test. It is clear that each B_k^t is defined uniformly in k and t, and so the sequence $\langle B_k \rangle$ is uniformly r.e. Moreover each interval $I \in B_k$ has rational endpoints and has $|I| = |\tau|$ for some $\tau \in A$. Finally

$$\mathrm{dwt}_g(B_k) = \sum_{I \in B_k} 2^{-g(|I|)} = \sum_{\substack{\tau \in A \\ |\tau| > p(k)}} 2^{-g(|\tau|)} = \mathrm{dwt}_g(A \cap 2^{>p(k)}) \leq 2^{-k}.$$

This completes the verification that $\langle B_k \rangle$ is a g-test.

Proving that $\langle B_k \rangle$ covers $\mu([A])$. Let k be fixed. Since $2^{\leq p(k)}$ is finite, let u be the largest stage such that $|\tau_{u-1}| \leq p(k)$, or let $u = 1$ if no such stage exists. In the construction we have $a_t = b_{t-1}$ for all $t > u$, and so $[a_u, b_t] \subseteq [B_k]$ for all $t \geq u$. By construction $a_u = \mu([A^u])$ and $b_u = \mu([A^u]) + 2^{-|\tau_u|}$. For all $t > u$ we have $|\tau_t| > p(k)$ and $|\tau_{t-1}| > p(k)$, and so $a_t = b_{t-1}$ and $b_t = a_t + 2^{-|\tau_t|}$. Note that $b_u - \mu([A_u]) = 2^{-|\tau_u|}$, and so for $t > u$

$$b_t - \mu([A_t]) \geq (b_{t-1} + 2^{-|\tau_t|}) - (\mu([A_{t-1}]) + 2^{-|\tau_t|}) = b_{t-1} - \mu(A_{t-1}) \geq 2^{-|\tau_u|}$$

by induction on t. Then in the limit

$$\mu([A]) + 2^{-|\tau_u|} = \lim_{t \to \infty} \mu([A_t]) + 2^{-|\tau_u|} \leq \lim_{t \to \infty} b_t$$

and so there must be some t such that $\mu([A]) < b_t$. Then for such t we have $\mu([A]) \in [a_u, b_t] \subseteq [B_k]$. Therefore $\mu([A])$ is not g-random, as required. ⊣

LEMMA 8.5 (common extension to compatible f-conditions). *Let $f : \mathbb{N} \to \mathbb{N}$ be a suitable function not equivalent to the identity function and let $\langle \sigma_1, S_1 \rangle, \ldots, \langle \sigma_n, S_n \rangle$ be f-conditions such that $\mu(P_{\langle \sigma_1, S_1 \rangle} \cap \cdots \cap P_{\langle \sigma_n, S_n \rangle}) > 0$. Then there is an f-condition $\langle \tau, T \rangle$ such that $\langle \tau, T \rangle \preccurlyeq \langle \sigma_i, S_i \rangle$ for each $1 \leq i \leq n$.*

PROOF. Clearly $\sigma_i \asymp \sigma_j$ for each i and j; so define $\sigma = \bigcup_{i \leq n} \sigma_i$ (the shortest element of $2^{<\mathbb{N}}$ such that $\sigma_i \subseteq \sigma$ for each i). Then let

$$P = P_{\langle \sigma_1, S_1 \rangle} \cap \cdots \cap P_{\langle \sigma_n, S_n \rangle} = [\sigma] \setminus [\mathcal{OC}_f(S_1) \cup \cdots \cup \mathcal{OC}_f(S_n)].$$

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Since $\mu(P)$ is positive, we can fix $b \in \mathbb{N}$ such that $\mu(P) \geq 2^{-b}$. Then for each $m \in \mathcal{I}(f)$ such that $m > b$ define the set

$$D_m = \{\tau \supseteq \sigma \mid (|\tau| = m) \wedge \neg(\exists \rho \subseteq \tau)(\rho \in \mathcal{OC}_f(S_1) \cup \cdots \cup \mathcal{OC}_f(S_n)\}.$$

If $|\tau| = m$ but $\tau \notin D_m$ then either $\tau \not\succ \sigma$ or $[\tau] \subseteq [\mathcal{OC}_f(S_i)]$ for some i. Either of these possibilities implies $P \cap [\tau] = \emptyset$. Since this is true for any τ of length m, we have $P \subseteq [D_m]$. Therefore $2^{-b} \leq \mu(P) \leq \#(D_m)2^{-m}$ and so $\#(D_m) \geq 2^{m-b}$.

Next, for each $\tau \in D_m$ define $T_\tau = \{\sigma \supseteq \tau \mid \sigma \in \mathcal{AC}_f(S_1) \cup \cdots \cup \mathcal{AC}_f(S_n)\}$. Since f is not equivalent to the identity function, the sequence $\langle k - f(k) \mid k \in \mathbb{N}\rangle$ tends to positive infinity. For fixed m observe that

$$\begin{aligned} n \geq \sum_{i=1}^{n} \mathrm{vwt}_f(S_i) &= \sum_{i=1}^{n} \mathrm{dwt}_f(\mathcal{OC}_f(S_i)) \\ &\geq \mathrm{dwt}_f(\mathcal{OC}_f(S_1) \cup \cdots \cup \mathcal{OC}_f(S_n)) \\ &\geq \sum_{\tau \in D_m} \mathrm{dwt}_f((\mathcal{OC}_f(S_1) \cup \cdots \cup \mathcal{OC}_f(S_n)) \cap [\tau]^{<\mathbb{N}}) \\ &\geq \sum_{\tau \in D_m} \mathrm{vwt}_f(T_\tau). \end{aligned}$$

Note that $\mathrm{vwt}_f(T_\tau) \leq 2^{-f(m)}$ for all $\tau \in D_m$. Assume that every $\tau \in D_m$ also satisfies $\mathrm{vwt}_f(T_\tau) = 2^{-f(m)}$. Then we have

$$n \geq \sum_{\tau \in D_m} \mathrm{vwt}_f(T_\tau) = \sum_{\tau \in D_m} 2^{-f(m)} \geq 2^{m-b}2^{-f(m)} = 2^{(m-f(m))-b}.$$

But $n < 2^{(m-f(m))-b}$ for some $m \in \mathcal{I}(f)$, and so this is a contradiction. So for this m and some $\tau \in D_m$ we must have $\mathrm{dwt}_f(\tau) > \mathrm{vwt}_f(T_\tau)$. Then $[\tau] \not\subseteq [\mathcal{OC}_f(T_\tau)]$ by Lemma 7.6. Since T_τ is an r.e. set, $\langle \tau, T_\tau\rangle$ is an f-condition. For each $1 \leq i \leq n$ we also have

$$P_{\langle \tau, T_\tau\rangle} \subseteq [\tau] \setminus [\mathcal{OC}_f(S_1) \cup \cdots \cup \mathcal{OC}_f(S_n)] \subseteq P \subseteq [\sigma_i] \setminus [\mathcal{OC}_f(S_i)] = P_{\langle \sigma_i, S_i\rangle}$$

and so $\langle \tau, T_\tau\rangle \preccurlyeq \langle \sigma_i, S_i\rangle$ for all $1 \leq i \leq n$ as required. ⊣

§9. Proving the Main Theorem.

9.1. Extracting additional complexity is difficult. Now we are ready to begin proving the main theorem.

LEMMA 9.1 (initial condition). *If $f : \mathbb{N} \to \mathbb{N}$ is suitable and*

$$S = \{\sigma \in 2^{<\mathbb{N}} \mid K(\sigma) \leq f(|\sigma|) - f(0)\},$$

then $\langle \lambda, S\rangle$ is an f-condition. Moreover, if $X \in P_{\langle \lambda, S\rangle}$, then X is f-complex.

PROOF. The set S is clearly r.e. Note that

$$\mathrm{vwt}_f(S) \leq \mathrm{dwt}_f(S) = \sum_{\sigma \in S} 2^{-f(|\sigma|)} \leq \sum_{\sigma \in S} 2^{-\mathrm{K}(\sigma)-f(0)} < 2^{-f(0)}.$$

This shows that $\mathrm{vwt}_f(S) < \mathrm{dwt}_f(\lambda)$ and so $\lambda \notin \mathcal{OC}_f(S)$. By Lemma 7.6, this shows that $\langle \lambda, S\rangle$ is an f-condition.

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If $X \in P_{\langle \lambda, S\rangle}$ then $X \notin [\mathcal{OC}_f(S)] \supseteq [S]$. Therefore $\mathrm{K}(X \upharpoonright n) > f(n) - f(0)$ for all n, and so X is f-complex. ⊣

LEMMA 9.2. *Let $f, g : \mathbb{N} \to \mathbb{N}$ be recursive and unbounded such that f is suitable and not equivalent to the identity function and $f \ll g$, and let $e, a \in \mathbb{N}$ be fixed. If $\langle \sigma, S\rangle$ is an f-condition, then there is an f-condition $\langle \sigma', S'\rangle \preccurlyeq \langle \sigma, S\rangle$ such that if $X \in P_{\langle \sigma', S'\rangle}$ and $\Phi_e^X \in 2^{\mathbb{N}}$, then $(\exists k)(K(\Phi_e^X \upharpoonright k) \leq g(k) - a)$.*

PROOF. Fix $b \in \mathbb{N}$ such that $2^{-b} < \mu(P_{\langle \sigma, S\rangle})$. Recall that $M :\subseteq 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is the prefix-free partial recursive function used to define K. Define a function $R :\subseteq 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ as follows. For $\rho \in 2^{<\mathbb{N}}$, simulate the computation of $M(\rho)$, letting $\pi = M(\rho)$ and $m = |\pi|$ if this computation halts. For each $\tau \in 2^{m-b}$ let $T_\tau = \{v \in [\sigma]^{<\mathbb{N}} \mid \tau \subseteq \Phi_e^v\}$, the set of extensions of σ computing τ via the e^{th} recursive functional. Each T_τ is an r.e. set. Effectively search for some $\tau \in 2^{m-b}$ such that $\mu(P_{\langle \sigma, S \cup T_\tau\rangle}) < 0.\pi$. This search can be performed effectively, since each measure is a right r.e. real number. When the first such τ is found, define $R(\rho) = \tau$. This completes the definition of R, which is partial recursive and prefix-free. Recall that there is a constant c such that $(\forall \tau \in 2^{<\mathbb{N}})(\mathrm{K}(\tau) \leq \mathrm{K}_R(\tau) + c)$. There is also a constant d such that $(\forall \pi \in 2^{\geq b})(\mathrm{K}(\pi) \leq \mathrm{K}(\pi \upharpoonright (|\pi| - b)) + d)$.

By Lemma 8.2, $\mu(P_{\langle \sigma, S\rangle})^\star$ (the binary expansion of the measure, turned into a sequence of 0's and 1's) is not g-complex, and so there exists an $\widetilde{m} > 0$ such that $\mathrm{K}(\mu(P_{\langle \sigma, S\rangle})^\star \upharpoonright \widetilde{m}) \leq g(\widetilde{m}) - a - c - d$. Define $m = \widetilde{m} + b$, and note that

$$\mathrm{K}(\mu(P_{\langle \sigma, S\rangle})^\star \upharpoonright m) \leq \mathrm{K}(\mu(P_{\langle \sigma, S\rangle})^\star \upharpoonright (m - b)) + d \leq g(m - b) - a - c.$$

So define $\pi = \mu(P_{\langle \sigma, S\rangle})^\star \upharpoonright m$ (in fact, we can find infinitely many values for m that work, so pick one that guarantees $\pi \neq 0^m$). Let $\rho \in 2^{<\mathbb{N}}$ be such that $M(\rho) = \pi$ and $|\rho| = \mathrm{K}(\pi) \leq g(m - b) - a - c$.

Now we show how to define σ' and S'.

Case 1: Suppose there is a $\tau \in 2^{<\mathbb{N}}$ such that $R(\rho) \downarrow = \tau$. By definition of R we have $\mu(P_{\langle \sigma, S \cup T_\tau\rangle}) < 0.\pi$, but $\mu(P_{\langle \sigma, S\rangle}) \geq 0.\pi$ and so

$$P_{\langle \sigma, S\rangle} \setminus P_{\langle \sigma, S \cup T_\tau\rangle} = [\mathcal{OC}_f(S \cup T_\tau)] \setminus [\mathcal{OC}_f(S)] \neq \emptyset.$$

Then define $\sigma' \in T_\tau$ to be such that $[\sigma'] \nsubseteq [\mathcal{OC}_f(S)]$. Such a σ' must exist, since otherwise $[\mathcal{OC}_f(S)] \supseteq [\mathcal{OC}_f(S \cup T_\tau)]$. By definition $\sigma' \supseteq \sigma$. Since T_τ is closed under extensions, we may also pick σ' so that it is a proper extension of σ. By Lemma 8.1, $\langle \sigma', S \cap [\sigma']^{<\mathbb{N}}\rangle$ is an f-condition and $\langle \sigma', S \cap [\sigma']^{<\mathbb{N}}\rangle \preccurlyeq \langle \sigma, S\rangle$, so we define $S' = S \cap [\sigma']^{<\mathbb{N}}$.

Now let $X \in P_{\langle \sigma', S'\rangle}$, and suppose that $\Phi_e^X \in 2^{\mathbb{N}}$. We have $\sigma' \subset X$ and $\sigma' \in T_\tau$ where $\tau = R(\rho)$. So by definition of T_τ, $\tau \subseteq \Phi_e^X$. Let $k = |\tau| = m - b > 0$, and note that

$$\begin{aligned}\mathrm{K}(\Phi_e^X \upharpoonright k) = \mathrm{K}(\tau) \leq\ & \mathrm{K}_R(\tau) + c \leq |\rho| + c = \mathrm{K}(\pi) + c \\ & \leq g(m - b) - a = g(k) - a\end{aligned}$$

as required.

Case 2: The computation of $R(\rho)$ does not halt. Then for each $\tau \in 2^{m-b}$ we have $\mu(P_{\langle \sigma, S \cup T_\tau\rangle}) \geq 0.\pi > 0$ and therefore $\langle \sigma, S \cup T_\tau\rangle$ is an f-condition extending $\langle \sigma, S\rangle$. Note that $\mu(P_{\langle \sigma, S\rangle}) \leq 0.\pi + 2^{-m}$, since otherwise π would be different in some bit, since $|\pi| = m$. So

$$\mu(P_{\langle \sigma, S\rangle} \setminus P_{\langle \sigma, S \cup T_\tau\rangle}) \leq 2^{-m}.$$

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Therefore

$$\mu\Big(\bigcap_{\tau\in 2^{m-b}} P_{\langle\sigma,S\cup T_\tau\rangle}\Big) = \mu\Big(P_{\langle\sigma,S\rangle} \setminus \bigcup_{\tau\in 2^{m-b}} \big(P_{\langle\sigma,S\rangle} \setminus P_{\langle\sigma,S\cup T_\tau\rangle}\big)\Big)$$
$$\geq \mu(P_{\langle\sigma,S\rangle}) - \sum_{\tau\in 2^{m-b}} \mu(P_{\langle\sigma,S\rangle} \setminus P_{\langle\sigma,S\cup T_\tau\rangle}) > 2^{-b} - 2^{m-b}2^{-m} = 0.$$

So by Lemma 8.5 there is an f-condition $\langle\sigma', S'\rangle$ extending $\langle\sigma, S\cup T_\tau\rangle$ for all $\tau\in 2^{m-b}$, which therefore extends $\langle\sigma,S\rangle$ as well. We can assume that this is a proper extension.

Now assume that $\Phi_e^X \in 2^{\mathbb{N}}$, and let $\tau = \Phi_e^X \upharpoonright (m-b)$. Since $\sigma\subset X$ there is some $\rho\subset X$ which is in T_τ. Therefore $X\in[T_\tau]\subset[\mathcal{OC}_f(S\cup T_\tau)]$ and therefore $X\notin P_{\langle\sigma,S\cup T_\tau\rangle}\supseteq P_{\langle\sigma',S'\rangle}$. So the requirement has been met. ⊣

Finally, we put together the pieces.

THEOREM 9.3 (extracting additional complexity is not possible). *If $f:\mathbb{N}\to\mathbb{N}$ is suitable and not equivalent to the identity function, there is an $X\in 2^{\mathbb{N}}$ such that X is f-complex and if $g:\mathbb{N}\to\mathbb{N}$ is recursive, $f\ll g$, and $Y\leq_T X$, then Y is not g-complex.*

PROOF. We build a sequence of f-conditions $\langle\langle\sigma_i,S_i\rangle \mid i\in\mathbb{N}\rangle$ such that $\langle\sigma_{i+1},S_{i+1}\rangle\preccurlyeq\langle\sigma_i,S_i\rangle$ for each $i\in\mathbb{N}$. Let $\langle g_i:\mathbb{N}\to\mathbb{N}\mid i\in\mathbb{N}\rangle$ be an enumeration of all unbounded recursive functions g such that $f\ll g$ (such an enumeration will not be effective). We let $R_{e,a,i}$ denote the requirement:

$$\Phi_e^X\in 2^{\mathbb{N}} \Rightarrow (\exists k) K(\Phi_e^X\upharpoonright k)\leq g_i(k)-a.$$

Stage 0. Let $\langle\sigma_0,S_0\rangle$ be the f-condition $\langle\lambda,S\rangle$ described in Lemma 9.1.

Stage $t+1=2^e3^a5^i$. Using Lemma 9.2, pick a condition $\langle\sigma_{t+1},S_{t+1}\rangle\preccurlyeq\langle\sigma_t,S_t\rangle$ satisfying the requirement $R_{e,a,i}$.

Stage $t+1\neq 2^e3^a5^i$. Define $\sigma_{t+1}=\sigma_t$ and $S_{t+1}=S_t$.

Now pick $X\in\bigcap_t[\sigma_t]$ (by compactness). X is f-complex by Lemma 9.1. Let $Y\leq_T X$ be arbitrary. Then there is some $e\in\mathbb{N}$ such that $Y=\Phi_e^X$ (and in particular, $\Phi_e^X\in 2^{\mathbb{N}}$). Then for each $a,i\in\mathbb{N}$ the condition $R_{e,a,i}$ was satisfied during the construction of the sequence $\langle\langle\sigma_i,S_i\rangle\mid i\in\mathbb{N}\rangle$, and $X\in P_{\langle\sigma_i,S_i\rangle}$ for every $i\in\mathbb{N}$. Then for each $a,i\in\mathbb{N}$ there exists a k such that $\mathrm{K}(Y\upharpoonright k)\leq g_i(k)-a$. So Y is not g_i-complex for any $i\in\mathbb{N}$, and therefore it is not g-complex for any recursive unbounded $g:\mathbb{N}\to\mathbb{N}$ such that $f\ll g$. ⊣

9.2. A similar theorem for other notions of Kolmogorov complexity. The main theorem in this paper is stated in terms of prefix-free complexity. The following alternate type of complexity is due to Zvonkin and Levin [22]. A function $\delta: 2^{<\mathbb{N}}\to[0,\infty)$ is a *continuous semimeasure* if $\delta(\lambda)\leq 1$ and $\delta(\sigma)\geq\delta(\sigma^\frown 0)+\delta(\sigma^\frown 1)$ for all σ; a continuous measure is *recursively enumerable* if the set $\{(\sigma,q)\in 2^{<\mathbb{N}}\times\mathbb{Q}\mid q<\delta(\sigma)\}$ is recursively enumerable. An r.e. continuous semimeasure δ is said to be *multiplicatively maximal* if for any other r.e. continuous semimeasure γ there is constant k such that $\gamma(\sigma)\leq k\cdot\delta(\sigma)$ for all σ. Such semimeasures must exist. Letting δ be a fixed multiplicatively maximal r.e. continuous semimeasure, for each σ the *a priori complexity of* σ is $\mathrm{KA}(\sigma)=-\log_2\delta(\sigma)$.

Let $f:\mathbb{N}\to\mathbb{N}$ be suitable. We define $X\in 2^{\mathbb{N}}$ to be *strongly f-complex* if $\mathrm{KA}(X\upharpoonright n)\geq f(n)-O(1)$. Then in Lemma 9.1, the set S can be replaced with the

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set $S' = \{\sigma \mid \mathrm{KA}(\sigma) \leq f(|\sigma|) - f(0)\}$ and the pair $\langle \sigma, S' \rangle$ is also an f-condition. Then in Theorem 5.1, K can be replaced by KA without changing the validity of the statement.

For a discussion of prefix-free, a priori, and other varieties of Kolmogorov complexity, see for example Uspensky and Shen [21] or Downey and Hirschfeldt [5].

§10. Application to mass problems. The following are notions which are originally due to Medvedev [11] and Muchnik [13], and can be thought of as a useful tool for studying degrees of unsolvability apart from the study of the Turing degrees. For an introduction to these topics as well as some motivation behind our interest in them, see Simpson [18]. Note also that the *weak degrees* discussed here are known elsewhere as *Muchnik degrees*.

Recall that Baire space, $\mathbb{N}^{\mathbb{N}}$, includes Cantor space, $2^{\mathbb{N}}$, as a subset; although our theory of complex infinite sequences of 0's and 1's is developed within Cantor space, we must briefly extend our focus to this larger space. A *mass problem* refers to a subset of $\mathbb{N}^{\mathbb{N}}$, specifically within the context of degrees of unsolvability and reducibility which we outline below.

DEFINITION 10.1 (weak reducibility). *Let $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$. We say that P is* weakly reducible *to Q, written $Q \leq_w P$, if for each $f \in P$ there is a partial recursive functional $\Psi\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\Psi(f)$ is defined and $\Psi(f) \in Q$. If $P \leq_w Q$ and $Q \leq_w P$, we say that P and Q are* weakly equivalent, *written $P \equiv_w Q$.*

DEFINITION 10.2 (weak degrees). *For a subset of Baire space P, we define the* weak degree *of P to be $\deg_w(P) = \{Q \subseteq \mathbb{N}^{\mathbb{N}} \mid P \equiv_w Q\}$. We write $\mathcal{D}_w$ for the set of all weak degrees.*

DEFINITION 10.3 (lattice of weak degrees). *$\mathcal{D}_w$ is a lattice under weak reducibility. The bottom degree, $\mathbf{0}$, is simply the degree of any set containing a recursive function. The top degree, $\boldsymbol{\infty}$, is the degree of the empty set. For two degrees $\boldsymbol{p} = \deg_w(P)$ and $\boldsymbol{q} = \deg_w(Q)$, the meet is $\inf(\boldsymbol{p}, \boldsymbol{q}) = \deg_w(P \cup Q)$, and the join is $\sup(\boldsymbol{p}, \boldsymbol{q}) = \deg_w(\{f \oplus g \mid f \in P, g \in Q\})$.*

In this last definition, the function $f \oplus g$ is defined by $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$ for all n.

DEFINITION 10.4 (degrees of nonempty Π^0_1 sets). *We define a special lattice $\mathcal{E}_w = \{\deg_w(P) \mid P \subseteq 2^{\mathbb{N}}$ is Π^0_1 and nonempty$\}$. This is a sublattice of $\mathcal{D}_w$. The bottom degree is again $\mathbf{0}$, the degree of a set containing a recursive function, and the top degree, $\boldsymbol{1}$, is the weak degree of the set of completions of Peano Arithmetic.*

Simpson [17, 18, 19] has studied these lattices in great depth. As we will see, many natural sets of infinite sequences of 0's and 1's whose degrees are in this lattice arise from randomness and recursion theory. Moreover, there are interesting connections and separations between these degrees. The following lemma, often called the Embedding Lemma, is useful for constructing natural examples of degrees in $\mathcal{E}_w$.

LEMMA 10.5 (Simpson [18, Lemma 3.3]). *Let $P \subseteq \mathbb{N}^{\mathbb{N}}$ be a Σ^0_3 set, and let $\boldsymbol{p} = \deg_w(P)$. Then for each $\boldsymbol{q} \in \mathcal{E}_w$ we have $\inf(\boldsymbol{p}, \boldsymbol{q}) \in \mathcal{E}_w$.*

Another important and useful lemma is the following.

LEMMA 10.6 (Simpson [17, Theorem 10.11]). *If $P \subseteq 2^{\mathbb{N}}$ is a nonempty and Σ^0_2 set then $\deg_w(P) \in \mathcal{E}_w$.*

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DEFINITION 10.7 (mass problems arising from initial segment complexity). *Let* $C_f = \{X \in 2^{\mathbb{N}} \mid X \text{ is } f\text{-complex}\}$, *where* $f : \mathbb{N} \to \mathbb{N}$ *is recursive and unbounded. We write* $\boldsymbol{c}_f$ *for the weak degree of* C_f. *We also define* $C = \{X \in 2^{\mathbb{N}} \mid X \text{ is complex}\}$, *and write* $\boldsymbol{c}$ *for the weak degree of* C.

LEMMA 10.8. *For any recursive and unbounded* $f : \mathbb{N} \to \mathbb{N}$, $\boldsymbol{c}_f \in \mathcal{E}_w$ *provided that* C_f *is nonempty. In addition,* $\boldsymbol{c} \in \mathcal{E}_w$.

PROOF. Observe that C_f is a Σ^0_2 set in Cantor space, since

$$X \in C_f \iff \exists k \forall n \forall s \forall \sigma \forall \tau \Big(\big(U(\sigma)[s] \downarrow = \tau \wedge |\sigma| < f(n) - k \big) \Rightarrow \tau \neq X \upharpoonright f(n) \Big).$$

So if C_f is nonempty then $\boldsymbol{c}_f \in \mathcal{E}_w$ by Lemma 10.6.

Observe next that C is a Σ^0_3 set in Baire space, since

$$X \in C \iff \exists e \exists k \forall n \forall s \forall \sigma \forall \tau \exists m (\varphi_{e,m}(n) \downarrow \in \{0,1\}) \wedge \Big(\big(U(\sigma)[s] \downarrow = \tau \wedge |\sigma| < \varphi_{e,m}(n) - k \big) \Rightarrow \tau \neq X \upharpoonright \varphi_{e,m}(n) \Big).$$

Now let f be any recursive function such that $\boldsymbol{c}_f \in \mathcal{E}_w$. Clearly each $X \in C_f$ is also an element of C, and so $\boldsymbol{c} < \boldsymbol{c}_f$. Then by Lemma 10.5, we have $\inf(\boldsymbol{c}, \boldsymbol{c}_f) = \boldsymbol{c} \in \mathcal{E}_w$. This completes the proof. ⊣

We now recall the following standard definitions from recursion theory.

DEFINITION 10.9 (diagonally nonrecursive functions). *We say that* $f : \mathbb{N} \to \mathbb{N}$ *is* diagonally nonrecursive, *or* DNR, *if for each* $e \in \mathbb{N}$, $\varphi_e(e) \neq f(e)$ (*if the computation halts, these values are not equal*). *We say that* f *is* recursively bounded DNR *if it is* DNR *and there is a recursive* $h : \mathbb{N} \to \mathbb{N}$ *such that* $f(n) < h(n)$ *for all* n.

LEMMA 10.10. *The degrees of each of the following sets in Baire space are in* $\mathcal{E}_w$.

(1) $D = \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is DNR}\}$.
(2) $D_h = \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is DNR and } f(n) < h(n) \text{ for all } n\}$ *for each recursive* $h : \mathbb{N} \to \mathbb{N} \setminus \{0,1\}$.
(3) $D_{REC} = \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is recursively bounded DNR}\}$.

We write $\boldsymbol{d}$, $\boldsymbol{d}_h$, *and* $\boldsymbol{d}_{REC}$ *for the weak degrees of* D, D_h, D_{REC} *respectively.*

PROOF. Let $h : \mathbb{N} \to \mathbb{N} \setminus \{0,1\}$ be recursive. Observe that D_h is a Π^0_1 set in Baire space, since

$$f \in D_h \iff \forall n \forall s (\varphi_{n,s}(n) \downarrow \Rightarrow \varphi_{n,s}(n) \neq f(n)) \wedge (f(n) < h(n)).$$

$\boldsymbol{1} \in \mathcal{E}_w$ can be characterized as the weak degree of the set of all $\{0,1\}$-valued diagonally non-recursive functions, which is denoted D_2 [17, Remark 7.10]. Obviously $D_2 \subseteq D_h$ and so $\boldsymbol{d}_h \leq \boldsymbol{1}$. By Lemma 10.5, we have $\boldsymbol{d}_h = \inf(\boldsymbol{d}_h, \boldsymbol{1}) \in \mathcal{E}_w$. The proofs that $\boldsymbol{d} \in \mathcal{E}_w$ and $\boldsymbol{d}_{\mathrm{REC}} \in \mathcal{E}_w$ can be done in a similar manner. ⊣

The reason these degrees are also of interest to us, is the following deep and surprising results.

THEOREM 10.11 (Kjos-Hanssen, Merkle, and Stephan [8]). *For each recursive unbounded* $f : \mathbb{N} \to \mathbb{N}$ *such that* C_f *is nonempty, there is a recursive* $g : \mathbb{N} \to \mathbb{N}$ *such that* $\boldsymbol{d}_g \leq \boldsymbol{c}_f$. *For each recursive unbounded* $g : \mathbb{N} \to \mathbb{N}$ *there is a recursive unbounded* $f : \mathbb{N} \to \mathbb{N}$ *such that* $\boldsymbol{c}_f \leq \boldsymbol{d}_g$. *Consequently,* $\boldsymbol{d}_{REC} = \boldsymbol{c}$.

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This is quite remarkable, as the connection between these two concepts is certainly less than obvious.

We also have the following result separating weak degrees.

THEOREM 10.12 (Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1, Theorem 1.9]). *Let $f : \mathbb{N} \to \mathbb{N} \setminus \{0, 1\}$ be recursive. Then there is a recursive $g : \mathbb{N} \to \mathbb{N} \setminus \{0, 1\}$ such that $\boldsymbol{d}_g \not\leq \boldsymbol{d}_f$ (there is some g-bounded DNR function which computes no f-bounded DNR function).*

So we see that there are natural separations of the degrees of certain classes of DNR functions, which in turn correspond roughly to our degrees of the form $\boldsymbol{c}_f$. One can indeed use the separations from Theorem 10.12 to get separations of the degrees $\boldsymbol{c}_f$, but they will not be as sharp as our result which follows. This is a direct corollary of our main result, Theorem 5.1.

COROLLARY 10.13. *If $f : \mathbb{N} \to \mathbb{N}$ is suitable, $g : \mathbb{N} \to \mathbb{N}$ is recursive, and $f \ll g$, then $\boldsymbol{c}_f \not\leq \boldsymbol{c}_g$.*

Not only does this provide a very good separation for these degrees arising from initial segment complexity, but together with Theorem 10.11 it also provides a new proof of 10.12. It is worth noting that the original proof of of that theorem is quite technical, and requires the technique of *bushy trees*, which were first used by Kumabe and Lewis [9] to prove the existence of a DNR function of minimal Turing degree. The alternate proof of Theorem 10.12 is as follows.

PROOF. Let $f : \mathbb{N} \to \mathbb{N} \setminus \{0, 1\}$ be recursive. By Theorem 10.11 there is a suitable function $p : \mathbb{N} \to \mathbb{N}$ such that $\boldsymbol{c}_p \leq \boldsymbol{d}_f$. Then let $q : \mathbb{N} \to \mathbb{N}$ be defined by $q(n) = \lfloor p(n)/2 \rfloor$ for all n. Since $q \ll p$ it follows from Corollary 10.13 that $\boldsymbol{c}_p \not\leq \boldsymbol{c}_q$. Finally, pick a recursive $g : \mathbb{N} \to \mathbb{N} \setminus \{0, 1\}$ such that $\boldsymbol{d}_g \leq \boldsymbol{c}_q$. If $\boldsymbol{d}_f \leq \boldsymbol{d}_g$ then $\boldsymbol{c}_p \leq \boldsymbol{c}_q$ as well, which is not the case. Then $\boldsymbol{d}_f \not\leq \boldsymbol{d}_g$ as required. ⊣

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