Ergod. Th. & Dynam. Sys. (2013), **33**, 441–454
doi:10.1017/S0143385711001064
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Canonical compactifications for Markov shifts

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(*Received* 22 *August* 2011 *and accepted in revised form* 6 *November* 2011)

Abstract. We give a complete characterization of the compact metric dynamical systems that appear as boundaries of the canonical compactification of a locally compact countable state mixing Markov shift. Consider such a compact metric dynamical system. Then there is a pair of non-conjugate Markov shifts with conjugate canonical compactifications, one of which has the given compact system as canonical boundary.

1. *Definitions and background*

In this paper we always mean by Markov shift a mixing locally compact countable state Markov shift. These can be defined by considering a graph G with a countable set of vertices V and a countable set of directed edges E and such that at each vertex an incoming edge and an outgoing edge exist. Then the set

$$S = \{(x_i)_{i\in\mathbb{Z}} \in E^{\mathbb{Z}} \mid \text{terminal vertex of } x_i = \text{ initial vertex of } x_{i+1}\}$$

endowed with the product topology of the discrete topology on E together with the left shift map defines a Markov shift. We denote space and shift map with the same symbol; which one is meant will be always clear from the context. The Markov shift is mixing if and only if the graph is irreducible and aperiodic. Local compactness for mixing Markov shifts is equivalent to that the graph has at each vertex only finitely many edges coming in or out. Compactness is equivalent to a finite edge set. Starting from §2, we will mean by Markov shift always a non-compact Markov shift.

The books of Kitchens [**9**] and of Lind and Marcus [**11**] give a good overview of the basic results on the subclass of compact Markov shifts, called shifts of finite type (SFT), as well as an introduction to non-compact Markov shifts.

Markov shifts themselves are studied via their invariants like entropy or zeta function. A basic general problem is the characterization of invariants. For example, the numbers which arise as entropies of SFTs were characterized by Lind [**10**], and the functions which arise as zeta functions of SFTs were characterized by Kim *et al* [**5**]. In this paper, we solve the realization problem for the canonical boundary of a locally compact, non-compact, countable state Markov shift.

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Markov shifts have also been studied by way of their automorphism groups. This has been extensively done in the SFT setting, for example in [**2, 6, 8**], and finally the automorphism groups were a very important tool in the negative solution of William's conjecture [**7**]. Schraudner's work [**12, 13**] is the first in the general locally compact setting.

Automorphisms of Markov shifts are homeomorphisms from the shift space onto itself that commute with the shift map. The automorphisms of a Markov shift form a group that is a conjugacy invariant. Groups are understood by their actions, and we know very few computable actions defined in general for these groups. Each automorphism is a self conjugacy of the Markov shift, and thus induces a self conjugacy of the canonical boundary [**3**]. Schraudner used this fact to investigate the automorphism group via an automorphism group representation, which is a homomorphism from the automorphism group into the automorphism group of the canonical boundary system of the (non-compact) Markov shift [**12, 13**]. The image of this automorphism group representation is a conjugacy invariant subgroup of the automorphism group of the boundary system. This image group is considerably simpler and easier to study than the automorphism group of the Markov shift itself, and can be described explicitly in many important cases [**12**]. To progress in this investigation, the question thus arises of which compact metric systems do occur as the canonical boundary of a Markov shift, and how strongly it is related to the canonical compactification. We provide the complete answer in Theorem 2.4.

The canonical boundary of a Markov shift reflects the 'behaviour' of the shift 'near infinity'. We give an informal idea of the definition; for details and proofs we refer to [**3**]. Removing from the graph G a finite set of edges K_1 leaves a subgraph that splits into finitely many undirected components; at least one of them is an infinite component. Each infinite component has an undirected period that generalizes the notion of period for strongly connected graphs. This information about components and their periods is captured in a finite dynamical system $(Z_{K_1}, \vartheta_{K_1})$, with one ϑ_{K_1}-orbit of length k for each infinite component of undirected period k. Removing now a larger finite subset $K_2 \supset K_1$ of edges, we see that the K_2-components are contained in K_1-components, and the undirected period is a multiple of the previous one. Recording this information again in a finite dynamical system, the hierarchy of components thus induces a factor map $(Z_{K_2}, \vartheta_{K_2}) \to (Z_{K_1}, \vartheta_{K_1})$. Inductively, by choosing a sequence of finite subsets of edges $K_1 \subset K_2 \subset \cdots$ with union the whole edge set E, we obtain an inverse limit system $(Z(S), \vartheta_S) = \varprojlim (Z_{K_n}, \vartheta_{K_n})$. This dynamical compact metric system does not depend on the particular choice of the $(K_n)_n$ and is the canonical boundary of the Markov shift S. It will be attached in a canonical way to the Markov shift, to obtain the canonical compactification $(Z^*(S), \vartheta_S^*)$ in which $S \subset Z^*(S)$ as a dense open set, and $Z^*(S) - S = Z(S)$ [**3**].

2. *Results*

In this section, we characterize the canonical boundaries of a (non-compact) Markov shift, Corollary 2.5, and show in the main result that for each canonical boundary which occurs, there are two non-conjugate Markov shifts with conjugate canonical compactifications, one having the given boundary as its canonical boundary, and sharing other essential invariants of conjugacy, Theorem 2.4.

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By entropy of a Markov shift we mean Gurevic entropy. The Gurevic entropy $h(S)$ of a Markov shift S is the topological entropy of the 1-point compactification of the Markov shift, and agrees with the usual topological entropy in case the space of S is compact. It can be calculated by counting loops of length n at a fixed vertex of G [**4**].

A mixing SFT always has the property that removing any edge from a defining graph strictly lowers the entropy. Locally compact Markov shifts with this property are called strongly positive recurrent; see for example [**1**] for equivalent definitions.

We start this section by showing that there are strongly positive recurrent Markov shifts with conjugate canonical compactifications not being conjugate, Theorem 2.1. This theorem provides just one example with the stated properties; however, it serves as a prototype for the elaborate construction in the main result (Theorem 2.4). Nevertheless, Theorem 2.1 is not a consequence of the main theorem (Theorem 2.4), as the constructed shifts in the latter are not necessarily strongly positive recurrent.

The following implies, in particular, that the canonical compactification of a locally compact mixing Markov shift is not a complete invariant for topological conjugacy, since it provides two non-conjugate Markov shifts that have conjugate canonical compactifications. The invariant distinguishing the Markov shifts is here the zeta function; actually, the Markov shifts have different numbers of fixed points. Note also that the construction provides an example for a Markov shift T that is strongly positive recurrent, and yet has a proper open subshift, which again is a locally compact Markov shift, as a subsystem of the same entropy.

THEOREM 2.1. *There are two mixing locally compact Markov shifts S and T with the same finite Gurevic entropy, having for each $n > 1$ the same finite number of orbits of length n, being both strongly positive recurrent, with return time numbers of a distinguished vertex in a suitable graph presentation exactly the same, and having conjugate canonical compactifications, yet not being topologically conjugate. Furthermore, S can be chosen to be an open subshift of T.*

Proof. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers. We define the Markov shifts S and T, and verify the stated properties.

Let S be the Markov shift given by the following graph with vertex set

$$V_S = \mathbb{Z} \cup \{(n, i) \mid n \in \mathbb{N}, i \in \mathbb{Z}\} \cup \{(n, 0') \mid n \in \mathbb{N}\},$$

where there is:

- an edge from vertex 0 to vertex 0;
- an edge from vertex n to vertex $n + 1$, $n \in \mathbb{Z}$;
- and for each $n \in \mathbb{N}$:
  - an edge from $2n$ to $(n, 0)$ and an edge from $(n, 0)$ to $-2n$;
  - an edge from (n, i) to $(n, i + 1)$ for each $i \in \mathbb{Z}$, $i \geq 0$;
  - an edge from $(n, -i)$ to $(n, -i + 1)$ for each $i \in \mathbb{Z}$, $i \geq 2$;
  - an edge from $(n, -1)$ to $(n, 0')$;
  - an edge from (n, i) to $(n, -i + 1)$ and an edge from (n, i) to $(n, -i)$ for each $i \in \mathbb{Z}$, $i \geq 2$;
  - an edge from $(n, 1)$ to $(n, -1)$ and an edge from $(n, 1)$ to $(n, 0')$;

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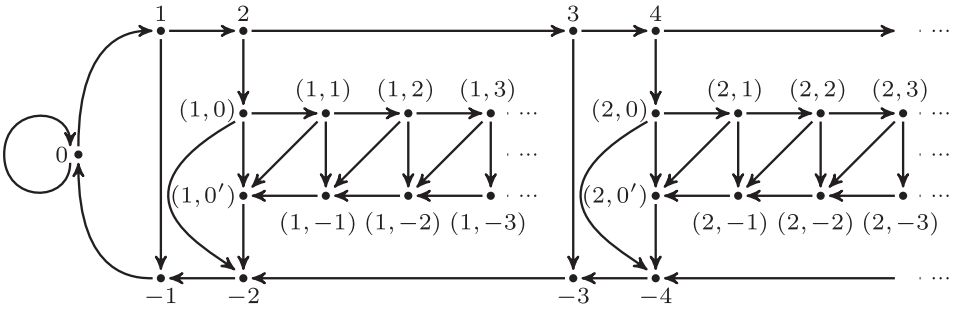


FIGURE 1. The graph for S.

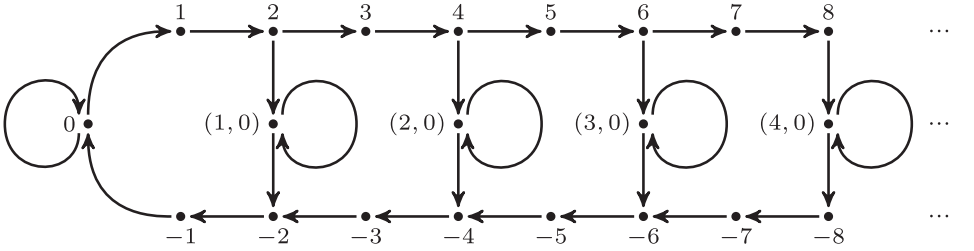


FIGURE 2. The graph for T.

- an edge from $(n, 0)$ to $(n, 0')$;
- an edge from $(n, 0')$ to $-2n$.

This defines an irreducible graph (Figure 1) with a fixed point loop, and thus S is a locally compact mixing Markov shift. Note that for each $n \in \mathbb{N}$ and each $k \in \mathbb{N}$, there is a unique path of length k from vertex $(n, 0)$ to vertex $(n, 0')$ that uses only vertices in $V_S - \mathbb{Z}$.

Let T be the Markov shift given by the following graph with vertex set

$$V_T = \mathbb{Z} \cup \{(n, 0) \mid n \in \mathbb{N}\},$$

where there is:

- an edge from vertex 0 to vertex 0;
- an edge from vertex n to vertex $n + 1$, $n \in \mathbb{Z}$;
- an edge from $2n$ to $(n, 0)$ and an edge from $(n, 0)$ to $-2n$ for each $n \in \mathbb{N}$;
- an edge from $(n, 0)$ to $(n, 0)$ for each $n \in \mathbb{N}$.

This defines a locally compact mixing Markov shift T (Figure 2).

Since S has only one fixed point, but T has infinitely many fixed points, S and T are not conjugate.

For each $n > 1$, an orbit of length n in either S or T is uniquely determined by a (possibly non-simple) loop of length n at vertex 0. Thus, the numbers of orbits of length n are determined by the numbers of first return loops at vertex 0. These numbers coincide in both graphs, and so S and T have the same finite number of orbits of length n for each $n > 1$.

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Since the degree of both graphs is bounded, both shifts have finite non-zero Gurevic entropy and, since the number of loops of length n at vertex 0 is the same in both graphs, the Gurevic entropies of S and T agree.

Denote by f_n the number of first return loops of length n at 0 (in the graph of either S or T). We determine these numbers in the graph of T. We see immediately $f_1 = 1$, $f_2 = f_3 = f_4 = f_5 = 0$. Now let $n \geq 6$. Write $n = 4 \cdot k + 2 + m$ for suitable $k \in \mathbb{N}$, $m \geq 0$. Then there is a unique first return loop at 0 of length n that goes from 0 to the vertex with the fixed point loop at vertex $(k, 0)$, uses that loop m times and goes back to 0. If $n = 4 \cdot p + 2 + r$ for some $r \in \{0, 1, 2, 3\}$, then there are exactly p choices for the parameter k, while m is fixed by this choice. So, we get $f_{4\cdot p+2} = f_{4\cdot p+3} = f_{4\cdot p+4} = f_{4\cdot p+5} = p$ for $p \in \mathbb{N}$.

So, the sequence f_n satisfies $(1/n) \log f_n \longrightarrow 0$ and the Markov shifts are thus strongly positive recurrent (see for example [**1**], where it is called SPR).

We now first determine the canonical boundaries $(Z(S), \vartheta_S)$ and $(Z(T), \vartheta_T)$ for S and T the canonical compactifications; for detailed definitions and a proof of the canonicity see [**3**].

For $m \geq 1$, let K_m be the subset of edges in S that has an initial vertex in $\{(n, i) \mid n \leq m$ and $(|i| \leq m$ or $i = 0')\} \cup \{n \mid |n| \leq 2m\}$. Then $K_m \subset K_{m+1}$ for each m and the union of these edge sets is the whole edge set of the graph of S. Removing the edges of K_m splits the remaining graph into $m + 1$ undirected infinite components, each of undirected period 1. Denote these components by $H_{m,n}$, $0 \leq n \leq m$, where for $n \geq 1$ the component $H_{m,n}$ is the one that contains the edges with initial vertices (n, i), $|i| > m$ and $H_{m,0}$ is the remaining component. Then we have $H_{m+1,n} \subset H_{m,n}$ for $1 \leq n \leq m$. This shows that $Z(S, (K_m)_{m\in\mathbb{N}}) = \{(n_k)_{k\in\mathbb{N}} \mid (\exists i : n_k = 0\ \forall k < i,\ n_k = i\ \forall k \geq i)$ or $(n_k = 0\ \forall k)\}$ and ϑ_S is the identity map on $Z(S, (K_m))$. Now consider the sequence $(N_m)_{m\in\mathbb{N}}$, where $N_m = (n_k^m)_{k\in\mathbb{N}} \in Z(S, (K_m))$ with $n_k^m = 0$ for $k < m$ and $n_k^m = m$ for $k \geq m$. Since $H_{m+1,n} \subset H_{m,0}$ for $0 \leq n \leq m$, $(N_m)_{m\in\mathbb{N}}$ converges to the sequence of all zeros. Thus, since the canonical boundary $Z(S)$ does not depend on the choice of the finite edge sets K_m [**3**, Lemma 3.4], we obtain that $(Z(S), \vartheta_S)$ is topologically conjugate to the dynamical system consisting of the space $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ with the identity map.

Removing any finite subset K of edges of the graph of T leaves a unique infinite undirected component of undirected period 1. Thus, the canonical boundary $(Z(T), \vartheta_T)$ is the 1-point system, and the canonical compactification T is the Alexandroff 1-point compactification.

We now establish a conjugacy between the canonical compactifications of S and T.

By [**3**, Theorem B and Lemma 1.3], the canonical compactification $(Z^*(S), \vartheta_S^*)$ of S can be written in the form $(S + Z(S), \vartheta_S^*)$, where ϑ_S^* is the shift map on S and the map ϑ_S on $Z(S)$, and an explicit formula for the metric on $S + Z(S)$ is given. In this metric on $S + Z(S)$, two points $x, y \in S$ are close if they coincide on a long interval around the zero coordinate or, for some large N, both zero coordinates can be connected by an undirected path of weight 0 in the complement of K_N. A point $x \in S$ and a point $y \in Z(S)$ are close in this metric if the zero coordinate of x is for some large N an edge in the K_N-component H_N that has a periodic label k_n and $y = (y_n)_{n\in\mathbb{N}}$ with $y_N = (H_N, k_N)$.

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We define a map $\Phi : S + Z(S) \to T + Z(T)$ as follows.

First, let $x \in S$; then $y = \Phi(x) \in T$ is defined as follows.

- If the initial vertex of x_i is $n \in \mathbb{Z}$, then y_i is the unique edge in the graph of T with initial vertex n and the same terminal vertex as x_i.
- If x_i has initial vertex (n, j) for some $n \in \mathbb{N}$, $j \in \mathbb{Z}$ and does not have terminal vertex $-2n$, then let y_i be the unique edge in T with initial and terminal vertex $(n, 0)$.
- If x_i has initial vertex $(n, 0)$ or $(n, 0')$ for some $n \in \mathbb{N}$ and terminal vertex $-2n$, then let y_i be the unique edge in the graph of T from $(n, 0)$ to $-2n$.

Now let $x \in Z(S)$; then $y = \Phi(x)$ is determined as follows.

- If $x = 1/n$, then let y be the unique fixed point in T that has all its edges with initial vertex $(n, 0)$, $n \in \mathbb{N}$.
- If $x = 0$, then let y be the unique point in $Z(T)$.

It remains to show that Φ is a homeomorphism from $S + Z(S)$ onto $T + Z(T)$ satisfying $\Phi \circ \vartheta_S^* = \vartheta_T^* \circ \Phi$.

The map $\Phi|_S : S \to T$ is continuous and shift commuting by definition; in fact, a 1-block map from S into T. Note that in the graph of S for each $n \in \mathbb{N}$ and each $k \geq 1$, there is a unique path of length k from vertex $(n, 0)$ to vertex $-2n$ having all its edges with initial vertex in $\{(n, i) \mid i \in \mathbb{Z}\} \cup \{(n, 0')\}$. This ensures that $\Phi|_S : S \to \Phi(S)$ is a homeomorphism and, thus, Φ embeds S into T with image being all of T except the countably many fixed points that have all their initial vertices at $(n, 0)$ for some $n \in \mathbb{N}$.

It remains to consider $\Phi|_{Z(S)} : Z(S) \to T + Z(T)$. Let x^m, $x \in Z(S)$ with $x^m \to x$. If $x \neq 0$, then $x^m = x$ for all large m and thus $\Phi(x^m) \to \Phi(x)$. If $x = 0$, then $\Phi(x) = y$ is the unique point in $Z(T)$ and $x^m = 0$ or $x^m = 1/n_m$ with the sequence n_m tending to ∞. Thus, $\Phi(x^m)$ is y or the unique fixed point in T that has all its edges with initial vertex $(n_m, 0)$. The topology on $Z^*(T)$ has the property that points in T with large zero coordinates are close to the boundary $Z(T)$, and thus these fixed points converge to the unique boundary point y as $m \to \infty$. This shows $\Phi(x^m) \to \Phi(x)$. Since $\Phi|_{Z(S)}$ is injective, this shows Φ is a homeomorphism between $S + Z(S)$ and $T + Z(T)$. And, since $\vartheta_T^*(y) = y$ for all $y \in \Phi(Z(S))$, we have that $\Phi \circ \vartheta_S^* = \vartheta_T^* \circ \Phi$. □

Remark 2.2. By [**1**, Theorem 6.4], S and T are almost isomorphic, meaning there are a transitive Markov shift R (which is not necessarily locally compact) and injective 1-block maps $R \longrightarrow S$, $R \longrightarrow T$ with magic words. The above proof actually shows that this common extension R can be chosen to be locally compact in this case, namely it can be chosen to be $R = S$, since the constructed embedding $S \longrightarrow T$ is an injective 1-block map with magic word being the edge from 0 to 1.

COROLLARY 2.3. *There are mixing locally compact Markov shifts S and T that are strongly positive recurrent with the same finite Gurevic entropy such that S embeds as an open subshift into T and T embeds as an open ϑ_S^*-invariant subset into $Z^*(S)$.*

Proof. The Markov shifts S and T from Theorem 2.1 satisfy the claim, since trivially T embeds as an open ϑ_T^*-invariant subset into its canonical compactification $Z^*(T)$, and the canonical compactifications of S and T are conjugate. □

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We now generalize the idea of the construction in the last theorem to obtain the main result of the paper.

THEOREM 2.4. *Let (Z, ϑ) be an inverse limit of finite dynamical systems. Then there are two mixing locally compact Markov shifts S and T such that:*

- *the canonical boundary $(Z(T), \vartheta_T)$ of T is topologically conjugate to (Z, ϑ);*
- *the canonical compactifications $(Z^*(S), \vartheta_S^*)$ of S and $(Z^*(T), \vartheta_T^*)$ of T are topologically conjugate;*
- *S can be embedded into T as an open subshift;*
- *the Gurevic entropies coincide: $h_G(S) = h_G(T)$;*
- *S and T are not conjugate.*

Proof. Since (Z, ϑ) is an inverse limit of finite dynamical systems, it can be written in the form

$$Z = \{(z_n)_{n\in\mathbb{N}} \in \Pi_{n\in\mathbb{N}} Z_n \mid z_n \in Z_n, \pi_{n+1}(z_{n+1}) = z_n, n \in \mathbb{N}\},$$

where Z_n are finite sets with the discrete topology and bijections $\vartheta_n : Z_n \to Z_n$, and $\pi_{n+1} : Z_{n+1} \to Z_n$ are maps with $\vartheta_n \circ \pi_{n+1} = \pi_{n+1} \circ \vartheta_{n+1}$ for all $n \in \mathbb{N}$ and the homeomorphism ϑ is given by $\vartheta((z_n)_{n\in\mathbb{N}}) = (\vartheta_n(z_n))_{n\in\mathbb{N}}$. Here Z carries the topology as a subset from the compact space $\Pi_{n\in\mathbb{N}} Z_n$.

We will construct first the Markov shift T, and obtain from it via 'replacing first return loops with a prototype graph' the desired Markov shift S.

We define the graph G for T inductively. We begin with a graph G_0 which consists of a unique vertex, called 0, and an edge from 0 to 0.

To construct G_1, consider (Z_1, ϑ_1). The set Z_1 splits into the orbits under ϑ_1, say $Z_1 = \dot{\bigcup}_{i=1}^{c_1} O_{1,i}$. Let $n_{1,i}$ be the length of the orbit $O_{1,i}$. Thus, $\sum_{i=1}^{c_1} n_{1,i} = \#Z_1$. Fix a multiple n_1 of $\Pi_{i=1}^{c_1} n_{1,i}$ greater than 4. This will ensure the component associated to $O_{1,i}$ will have a period $n_{1,i}$. The vertex set of G_1 is

$$V_1 = \{0\} \dot{\cup} \dot{\bigcup}_{i=1}^{c_1} (\{(1, i, k) \mid k \in \mathbb{Z} - \{0\} \text{ with } |k| \le n_1\} \dot{\cup} \{(1, i, *, k) \mid 0 \le k < n_{1,i}\}).$$

The edges are as follows:

(1) there is an edge from vertex 0 to vertex 0 and, for each $1 \le i \le c_1$, there is an edge from vertex 0 to vertex $(1, i, 1)$ and there is an edge from vertex $(1, i, -1)$ to vertex 0;

(2) for each $1 \le i \le c_1$ and for each $1 \le k < n_1$, there is an edge from $(1, i, k)$ to $(1, i, k + 1)$;

(3) for each $1 \le i \le c_1$ and for each $-n_1 \le k < -1$, there is an edge from $(1, i, k)$ to $(1, i, k + 1)$;

(4) for each $1 \le i \le c_1$, there is an edge from $(1, i, n_1)$ to $(1, i, *, 0)$ and an edge from $(1, i, *, 0)$ to $(1, i, -n_1)$;

(5) for each $1 \le i \le c_1$ such that $n_{1,i} > 1$, there is for each $0 \le k < n_{1,i} - 1$ an edge from $(1, i, *, k)$ to $(1, i, *, k + 1)$ and an edge from $(1, i, *, n_{1,i} - 1)$ to $(1, i, *, 0)$;

(6) for each $1 \le i \le c_1$ such that $n_{1,i} = 1$, there is an edge from $(1, i, *, 0)$ to $(1, i, *, 0)$.

The graph G_1 (Figure 3) is irreducible and aperiodic. Given now that we constructed G_{N-1}, we construct G_N as follows. The set Z_N is a disjoint union of the ϑ_N-orbits

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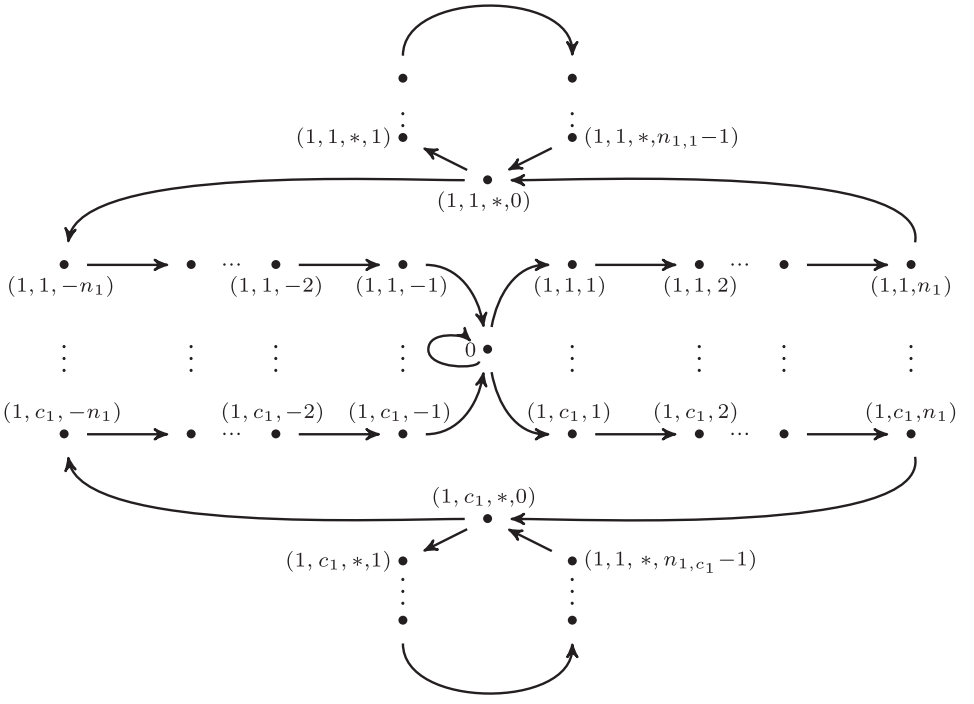


FIGURE 3. The graph G_1.

$O_{N,i}$, $i = 1, \ldots, c_N$ with orbit $O_{N,i}$ of length $n_{N,i}$. Fix a multiple n_N of $\Pi_{i=1}^{c_N} n_{N,i}$ greater than 4. Since $\vartheta_{N-1} \circ \pi_N = \pi_N \circ \vartheta_N$, the map π_N maps the orbit $O_{N,i}$ onto an orbit $O_{N-1,j(i)}$ for a suitable $1 \le j(i) \le c_{N-1}$. The vertex set of G_N is

$$V_N = V_{N-1} \dot\cup \dot\bigcup_{i=1}^{c_N} (\quad \{(N,i,k) \mid k \in \mathbb{Z} - \{0\} \text{ with } |k| \le n_N\} \\ \dot\cup \{(N,i,*,k) \mid 0 \le k < n_{N,i}\}).$$

For each $1 \le i \le c_N$, the following edges are added to the graph G_{N-1} using the enlarged vertex set V_N to obtain G_N:

(1) if $\pi_N(O_{N,i}) = O_{N-1,j}$, then there is an edge from vertex $(N-1, j, n_{N-1})$ to vertex $(N, i, 1)$ and an edge from (N, i, k) to $(N, i, k+1)$ for each $1 \le k < n_N$;
(2) if $\pi_N(O_{N,i}) = O_{N-1,j}$, then there is an edge from vertex $(N, i, -1)$ to vertex $(N-1, j, -n_{N-1})$ and an edge from (N, i, k) to $(N, i, k+1)$ for each $-n_N \le k < -1$;
(3) there is an edge from (N, i, n_N) to $(N, i, *, 0)$ and an edge from $(N, i, *, 0)$ to $(N, i, -n_N)$;
(4) if $n_{N,i} > 1$, there is, for each $0 \le k < n_{N,i} - 1$, an edge from $(N, i, *, k)$ to $(N, i, *, k+1)$ and an edge from $(N, i, *, n_{N,i} - 1)$ to $(N, i, *, 0)$;
(5) if $n_{N,i} = 1$, there is an edge from $(N, i, *, 0)$ to $(N, i, *, 0)$.

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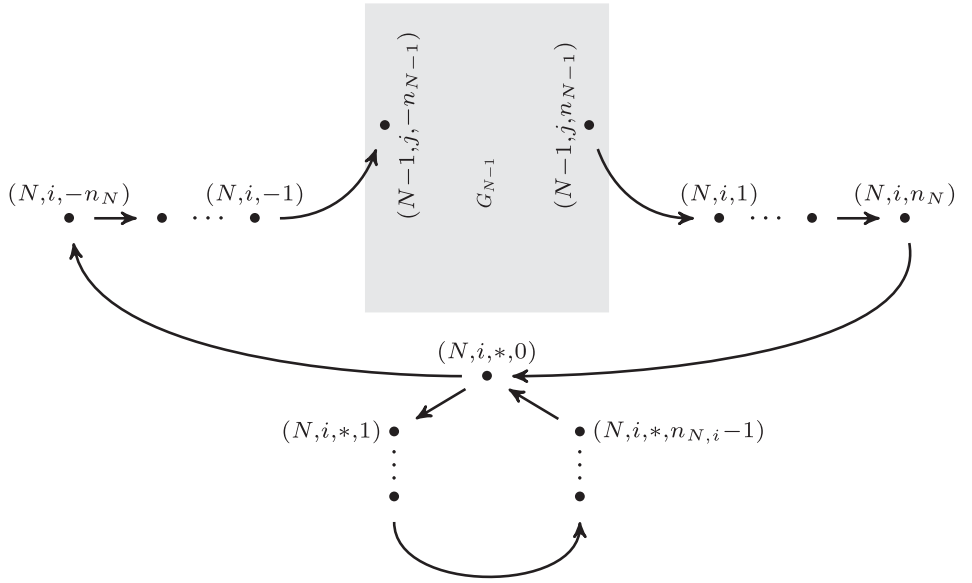


FIGURE 4. The edges, that are related to the orbit $O_{N,i}$, added to G_{N-1}, using the condition $\pi_N(O_{N,i}) = O_{N-1,j}$.

Since the graph G_{N-1} was irreducible, the graph G_N (Figure 4) is irreducible as well. Let T be the Markov shift defined by the graph G obtained from the inductive construction of the graphs G_N.

CLAIM. *$(Z(T), \vartheta_T)$ is topologically conjugate to (Z, ϑ).*

Let K_N be the finite set of edges of G_N. Then $K_N \subset K_{N+1}$ and the union of the K_N are all edges of G. Thus, $Z(T) = Z(T, (K_N))$ [**3**, Lemma 3.4, Theorem B]. We need to determine now the undirected infinite components and their undirected periods of the graph after removing the edges K_N. Consider an edge, say e, in $K_{M+1} - K_M$ for $M > N$. If the initial vertex of e is (M, j, n_M) for some $1 \le j \le c_M$, then there is a path from e to vertex $(M, j, -n_M)$ by (1), (2) and (3). If the initial vertex of e is $(M+1, i, k)$ for some $1 \le i \le c_{M+1}$, $1 \le k \le n_{M+1}$ and if $\pi_{M+1}(O_{M+1,i}) = O_{M,j}$ for some $1 \le j \le c_M$, then there is a path from e to vertex $(M, j, -n_M)$ by (1), (2) and (3). If the initial vertex of e is a vertex $(M+1, i, *, k)$ for some $1 \le i \le c_{M+1}$, $0 \le k \le n_{M+1,i}$, then, by (4), there is a path to vertex $(M+1, i, *, 0)$ and, thus, by (3) and (2), a path to vertex $(M, j, -n_M)$ when $\pi_{M+1}(O_{M+1,i}) = O_{M,j}$. So, we see for each edge $e \notin K_N$, there is a unique j such that there is a path from e to vertex $(N, j, -n_N)$. j is determined as follows. There is a unique M such that $e \in K_{M+1} - K_M$. e determines a unique orbit $O_{M+1,i}$, and we have $\pi_{N+1} \circ \cdots \circ \pi_{M+1}(O_{M+1,i}) = O_{N,j}$. So, after removing the edges from K_N, the graph splits into c_N infinite undirected components. To determine the periods of the undirected components, we fix a vertex, say $(N, i, *, 0)$, in the component associated to $O_{N,i}$ and determine the greatest common divisor of all weights of undirected loops at this vertex. As each n_M is a multiple of the product of the orbit lengths in Z_M, each such undirected loop has a weight being a multiple of n_N. By (4) and (5), there is a loop of length $n_{N,i}$

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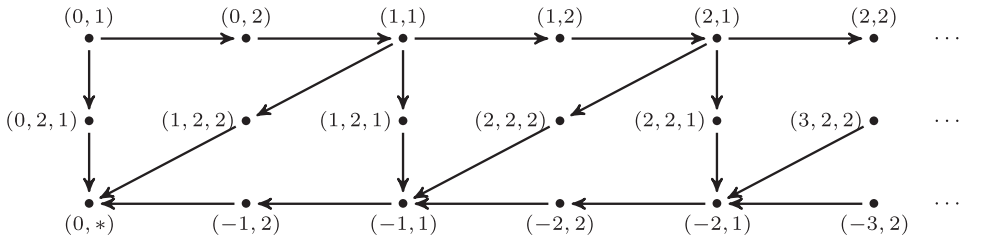


FIGURE 5. The graph $R(p)$ for $p = 2$.

at vertex $(N, i, *, 0)$. Thus, there is an undirected path (in fact a directed one) of weight $n_{N,i}$ and thus the undirected period of the component associated to the orbit $O_{N,i}$ is the greatest common divisor of $n_{N,i}$ and n_N, which is $n_{N,i}$, since n_N was chosen to be a multiple of $n_{N,i}$. The component of $G - K_N$ corresponding to the orbit $O_{N,i}$ sits inside the component $O_{N-1,j}$ of $G - K_{N-1}$ if and only if $\pi_N(O_{N,i}) = O_{N-1,j}$, by (1) and (2). This proves that the canonical boundary $(Z(T), \vartheta_T)$ is conjugate to (Z, ϑ).

We now define the Markov shift S. For that, we define for each $p \in \mathbb{N}$ a graph $R(p)$ as follows: the vertex set of $R(p)$ is

$$\begin{aligned} V_{R(p)} = &\{(n, i) \mid n \in \mathbb{Z}, i \in \{1, 2, \ldots, p\}\} \\ &\cup \{(n, i, j) \mid n \geq 1, i \in \{2, 3, \ldots, p\}, j \in \{1, 2\}\} \\ &\cup \{(0, i, 1) \mid i \in \{2, 3, \ldots, p\}\} \cup \{(0, *)\} \end{aligned}$$

and the edges are as follows:

(1) for each $n \in \mathbb{Z}$ and $1 \leq i < p$, there is an edge from vertex (n, i) to vertex $(n, i + 1)$ and there is an edge from (n, p) to $(n + 1, 1)$ for all $n \neq -1$, and there is an edge from $(-1, p)$ to $(0, *)$;

(2) for each $n \geq 0$, there is an edge from $(n, 1)$ to $(n, 2, 1)$ and, for each $2 \leq i < p$, an edge from $(n, i, 1)$ to $(n, i + 1, 1)$ and an edge from $(n, p, 1)$ to $(-n, 1)$ for all $n \geq 1$ and an edge from $(0, p, 1)$ to $(0, *)$;

(3) for each $n \geq 1$, there is an edge from $(n, 1)$ to $(n, 2, 2)$ and, for each $2 \leq i < p$, an edge from $(n, i, 2)$ to $(n, i + 1, 2)$ and an edge from $(n, p, 2)$ to $(-n + 1, 1)$ for all $n \geq 2$ and an edge from $(1, p, 2)$ to $(0, *)$.

Note that the graph of $R(p)$ (Figure 5) has the property that for each $n \geq 1$, there is a unique path in $R(p)$ of length $n \cdot p$ starting from vertex $(0, 1)$ and ending at vertex $(0, *)$. Furthermore, any path from $(0, 1)$ to $(0, *)$ uses an edge at most once, and has a length which is a multiple of p.

Now we can define S. We start with the graph G of T and, for each vertex $(N, i, *, 0)$, we remove the $n_{N,i}$ edges from the loop of length $n_{N,i}$ going through the vertices $(N, i, *, k)$, $0 \leq k < n_{N,i}$, that were defined in (4) and (5). Then we glue at vertex $(N, i, *, 0)$ a copy of the graph $R(p)$ with $p = n_{N,i}$ by identifying the vertex $(N, i, *, 0)$ of G with the vertex $(0, 1)$ of $R(p)$ and by adding an edge from the vertex $(0, *)$ of $R(p)$ to the vertex $(N, i, -n_N)$ of G. We refer to this newly added graph at the vertex $(N, i, *, 0)$ as the 'subgraph associated to the ϑ_N-orbit $O_{N,i} \subset Z_N$'. Doing this for each $N \geq 1$, $1 \leq i \leq c_N$, we obtain the graph H for S.

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Define now a shift commuting 1-block map $\Phi : S \to T$ by defining $\Phi(x)_0$. If the edge x_0 in H is an edge in G, too, then let $\Phi(x)_0 := x_0$. If x_0 is an edge in the copy $R(p)$ that is glued to the G vertex $(N, i, *, 0)$, then the initial vertex of x_0 is (n, k) or (n, k, j) for suitable $n \in \mathbb{Z}$, $1 \le k \le p$, $j \in \{1, 2\}$. Then let $\Phi(x)_0$ be the edge in G from vertex $(N, i, *, k-1)$ to vertex $(N, i, *, k \mod p)$. Finally, if x_0 is the edge starting in $(0, *)$, then let $\Phi(x)_0$ be the edge from $(N, i, *, 0)$ to $(N, i, -n_N)$ in G.

Since each $R(p)$ has the property that all paths from $(0, 1)$ to $(0, *)$ have a length that is a multiple of p, and each length occurs only once, we see that Φ sends S injectively into T, and the image $\Phi(S)$ is all of T except the periodic points that never visit the vertex 0 of G (those are the periodic points given by the loops at $(N, i, *, 0)$ defined in (4) and (5)). Since every point y in the image $\Phi(S)$ visits vertex 0 at least once, the zero coordinate of the preimage can be uniquely determined by either y_0 or by $y[-n, n]$, where n is determined by the distance to the vertex 0. Thus, Φ is an embedding of S into T and S sits inside T as an open subshift.

The entropies of S and T agree, since the numbers of first return loops at vertex 0 in either graph G or H are exactly the same, and thus the numbers of loops of length n at this vertex agree in both graphs.

It remains to prove that the canonical compactifications of S and T are conjugate. For that, we need to determine the canonical boundary $(Z(S), \vartheta_S)$ of S. For $N \in \mathbb{N}$, let K'_N be the finite subset of edges e of H such that either $e \in K_N$ (considered as an edge in G) or e is an edge in one of the subgraphs $R(n_{M,i})$ associated to the ϑ_M-orbit $O_{M,i} \subset Z_M$ and there is a path of length at most $n_{M,i} \cdot N$ from vertex $(M, i, *, 0)$ to the initial vertex of e with $M \le N$ and $1 \le i \le c_M$. This defines an increasing sequence of finite subsets of edges of H with the union being the whole set of edges. Thus, $(Z(S), \vartheta_S) = (Z(S, (K'_N)), \vartheta_S)$, [**3**, Lemma 3.4 and Theorem B]. Removing the edges of K'_N from the graph H, we see for each $M < N$, $1 \le i \le c_M$, an infinite component that is what is remaining after edge removal from the subgraph $R(n_{M,i})$ associated to the ϑ_M-orbit $O_{M,i} \subset Z_M$. This undirected infinite component has undirected period $n_{M,i} = \#O_{M,i}$. We refer to it as the $O_{M,i}$-component. Furthermore, we see one infinite undirected component containing the vertex $(N, i, 1)$ for each $1 \le i \le c_N$ and these components have undirected period $n_{N,i}$, as they contain the subgraph $R(n_{N,i})$ associated to the ϑ_N-orbit $O_{N,i} \subset Z_N$. Removing now the edges of K'_{N+1}, we first note that the $O_{M,i}$-component from the previous removal still exists, and does not split further, it just 'lost finitely many edges'. The components that contained the vertex $(N, i, 1)$ in the previous removal step now split into a new $O_{N+1,j}$-component for each j such that there is an ϑ_{N+1}-orbit $O_{N+1,j}$ with $\pi_{N+1}(O_{N+1,j}) = O_{N,i}$, where the new component contains the vertex $(N+1, j, 1)$. Defining now maps

$$\pi'_n : \dot{\bigcup}_{i=1}^{n+1} Z_i \to \dot{\bigcup}_{i=1}^{n} Z_i$$

by $\pi'_n(z) = z$ if $z \in \dot{\bigcup}_{i=1}^{n} Z_i$ and $\pi'_n(z) = \pi_n(z)$ if $z \in Z_{n+1}$, this description of the components and their periods means that

$$Z(S) = \left\{ (z_n)_{n\in\mathbb{N}} \;\middle|\; z_n \in \dot{\bigcup}_{i=1}^{n} Z_i,\ \pi'_n(z_{n+1}) = z_n,\ n \in \mathbb{N} \right\}$$

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and $\vartheta_S((z_n)_{n\in\mathbb{N}}) = (z'_n)_{n\in\mathbb{N}}$, where $z'_n = \vartheta_k(z_n)$, where k is the unique index such that $z_n \in Z_k$.

We now define a map $\varphi : S + Z(S) \to T + Z(T)$. For that, we pick for each N and each $1 \le i \le c_N$, a reference point $w_{N,i} \in O_{N,i} \subset Z_N$.

Let $\varphi(x) = \Phi(x)$ if $x \in S$. If $x = (z_n)_{n\in\mathbb{N}} \in Z(S)$ and there is some N such that $z_n = z_N$ for all $n \ge N$, then pick the smallest such N and pick $1 \le i \le c_N$ such that $z_N \in O_{N,i}$. Then there is a unique $0 \le k < n_{N,i}$ with $z_N = \vartheta_N^k(w_{N,i})$. Then let $\varphi(x) = y \in T$ be the unique point of period $n_{N,i}$ in T which visits the vertex $(N, i, *, 0)$ and never the vertex 0 and such that y_0 has initial vertex $(N, i, *, k)$. If $x = (z_n)_{n\in\mathbb{N}} \in Z(S)$ but there is no N such that $z_n = z_N$ for all $n \ge N$, then infinitely often $z_n \ne z_{n+1}$ and thus by the definition of the map π'_{n+1} we have that $z_n \in Z_n$, $z_{n+1} \in Z_{n+1}$ infinitely often. But, if $z_n \in Z_n$ and $z_{n+m} \in Z_{n+m}$, then, by the definition of the maps $\pi'_{n+m}, \ldots, \pi'_{n+1}$, we obtain $z_i \in Z_i$ for all $n \le i \le m$. Altogether we see that if $x = (z_n)_{n\in\mathbb{N}} \in Z(S)$ but there is no N such that $z_n = z_N$ for all $n \ge N$, then $z_n \in Z_n$ for all n. In this case, let $\varphi(x) = x \in Z(T)$.

We check now that φ is the desired conjugacy. Let $x, x' \in S + Z(S)$ with $\varphi(x) = \varphi(x') = y$. If $x \in S$, then $y \in T$ visits the vertex 0 at least once and thus, by definition of φ, we have $x' \in S$, too. Since Φ is injective, it follows that $x = x'$. In particular, $x \in S \iff x' \in S$. If $x \notin S$ but $y \in T$, then y is a periodic point that never visits vertex 0. Thus, there is a unique $N \in \mathbb{N}$ and $1 \le i \le c_N$ such that y visits the vertex $(N, i, *, 0)$. Then y_0 begins in vertex $(N, i, *, k)$ for some $0 \le k \le n_{N,i}$. Then $x = x' = (z_n)_{n\in\mathbb{N}}$, where $z_N = \pi_N(w_{N,i})$ and $z_n = z_N$ for all $n \ge N$. If $x \notin S$ and $y \notin T$, then $y = x = x'$. Thus, the map is injective.

Now let $y \in T + Z(T)$. If $y \in T$ and it visits the vertex 0, then $y \in \Phi(S)$ and thus has a preimage $x \in S$.

The image $\Phi(S)$ consists of all points in T that visit vertex 0 at least once. Thus, if $y \in T$ and it never visits the vertex 0, then $y \notin \Phi(S)$ and y is a periodic point, and there is a unique N and $1 \le i \le c_N$ and $0 \le k < n_{N,i}$ such that y_0 begins in vertex $(N, i, *, k)$. Then $x = (z_n)_{n\in\mathbb{N}}$, where $z_N = \vartheta_N^k(w_{N,i})$ and $z_n = z_N$ for all $n \ge N$ satisfies $\varphi(x) = y$. If $y \notin T$, then $y = (z_n)_{n\in\mathbb{N}}$ and since $z_n \in Z_n$ for all n, the point $x = (z_n)_{n\in\mathbb{N}}$ is an element of $Z(S)$ with $\varphi(x) = y$. Thus, the map is onto.

Let $x \in S + Z(S)$ and $y = \varphi(x)$.

If $x \in S$, then $y \in T$ and thus $\vartheta_T^* \circ \varphi(x) = \varphi \circ \vartheta_S^*(x)$ since $\vartheta_S^*|S$ and $\vartheta_T^*|T$ are the shift maps and $\varphi|_S = \Phi$ is shift commuting. If $x = (z_n)_{n\in\mathbb{N}} \in Z(S)$, where $z_N = \vartheta_N^k(w_{N,i})$ and $z_n = z_N$ for all $n \ge N$, then $\vartheta_T^* \circ \varphi(x) = \varphi \circ \vartheta_S^*(x)$ since $\vartheta_S^*(x) = (z'_n)_{n\in\mathbb{N}}$ with $z'_n = \pi_N(z_N) = \vartheta_N^{k+1}(w_{N,i})$ for all $n \ge N$, and $\vartheta_T^*(y)$ has as zero coordinate y_1 which begins in vertex $(N, i, *, k + 1 \mod n_{N,i})$. Finally, if $x = (z_n)_{n\in\mathbb{N}} \in Z(S)$ with $z_n \in Z_n$ for all n, then $y = x$. This holds for all such points x and, since $\vartheta_T^*(y) = \vartheta_S^*(x) \in Z(T)$, $\vartheta_T^* \circ \varphi(x) = \varphi \circ \vartheta_S^*(x)$.

So, the map φ commutes with the maps ϑ_S^* and ϑ_T^*.

Since the compactifications are compact metric spaces, it remains to show that φ is continuous. For that, let $x, x^m \in S + Z(S)$ with $x^m \to x$.

If $x \in S$, then, for all large m, the points $x^m \in S$ are, too, since S is open in $S + Z(S)$; thus $\varphi(x^m) = \Phi(x^m) \to \Phi(x) = \varphi(x)$, since Φ is continuous.

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If $x \notin S$ and $x = (z_n)_{n\in\mathbb{N}} \in Z(S)$, where $z_N = \vartheta_N^k(w_{N,i})$ and $z_n = z_N$ for all $n \geq N$, then $x^m \to x$ implies $x^m = x$ for all large enough m. Thus, again, $\varphi(x^m) \to \varphi(x)$.

If $x = (z_n)_{n\in\mathbb{N}} \in Z(S)$ with $z_n \in Z_n$ for all n, then $\varphi(x) = x$. And, $x^m \to x$ implies, that given N, for all large enough m, we have $x^m = (z'_n)_{n\in\mathbb{N}} \in Z(S)$ with $z'_n = z_n$ for all $n \leq N$ or $x^m \in S$ with x_0^m beginning in a vertex that has a periodic label $k \mod n_{N,i}$, where $z_N = \vartheta_N^k(w_{N,i})$; see the remark after Theorem B in [**3**]. Thus, $\varphi(x^m)$ is either a point in $Z(T)$ close to $\varphi(x)$ or it is a periodic point in T that has all its edges outside of K_N and the initial vertex of $\varphi(x^m)_0$ is a vertex of the form $(N, i, *, k)$. This point is close to $\varphi(x)$ as well, again by the remark in [**3**]. Thus, the map φ is continuous, and this finishes the proof that φ is a conjugacy between the canonical compactifications of S and T.

The Markov shift S has a rome, that is, there is a compact open $K \subset S$ such that for every $x \in S$ there is $n \in \mathbb{Z}$ such that $S^n x \in K$, which means that every point in S visits the rome K at least once. For example, one can choose for K the set of points $x \in S$ with x_0 having initial vertex 0.

A Markov shift R that is conjugate to S has a rome as well, since if $f : S \to R$ is a conjugacy, then $f(K) \subset R$ is compact open and, if $y \in R$, then there is $n \in \mathbb{Z}$ such that $S^n(f^{-1}(y)) \in K$ and thus $R^n y = R^n(f(f^{-1}(y))) = f(S^n(f^{-1}(y))) \in f(K)$.

The Markov shift T has no rome. Since having a rome is a conjugacy invariant, the Markov shifts S and T are not conjugate. □

As a corollary, we obtain the complete characterization of the dynamical systems that occur as the canonical boundary of some Markov shift.

COROLLARY 2.5. *Let (X, T) be a compact metric dynamical system. Then there is a mixing locally compact Markov shift S such that the canonical boundary $(Z(S), \vartheta_S)$ is topologically conjugate to (X, T) if and only if (X, T) is conjugate to an inverse limit of finite dynamical systems.*

Proof. '$\Rightarrow$': this follows from [**3**, Theorem B].

'$\Leftarrow$': this follows from Theorem 2.4. □

COROLLARY 2.6. *For each dynamical system (Z, ϑ) that is an inverse limit of finite dynamical systems, there are non-conjugate mixing locally compact Markov shifts S and T with the same finite Gurevic entropy and conjugate canonical compactifications, such that $(Z(T), \vartheta_T)$ is conjugate to (Z, ϑ) and S embeds as an open subshift into T and T embeds as an open ϑ_S^*-invariant subset into $Z^*(S)$.*

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