An unusual inequality challenge

This challenge arose from idly thinking about what happens in Pythagoras' theorem if, instead of having precisely similar shapes on each side, only "approximately similar" shapes are used. Figure 1 suggests the general idea with circles on the sides of the triangle: we consider two other related possibilities at the end. The impetus from Pythagoras guided the write-up below which starts with the case of a right-angled triangle and the realisation that we could "push" this case to deal with all obtuse-angled triangles before looking separately at acute-angled triangles.

Figure 1 shows a triangle inside its circumcircle of radius R. The small circles are the biggest ones that fit inside each segment, with the circle opposite angle A having area α, etc. The challenge is to find a scalene triangle whose angles $A < B < C$ are each a multiple of 5° for which $\alpha + \beta > \gamma$ … (*). This restriction looks rather artificial but, perhaps surprisingly, ensures a unique solution.

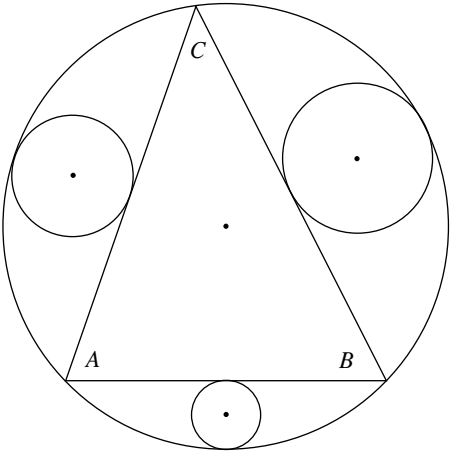


FIGURE 1

We note that the circle opposite A has radius $\frac{1}{2}R(1 - \cos A)$ so that $\alpha = \frac{1}{4}\pi R^2(1 - \cos A)^2$ or $\alpha = (1 - \cos A)^2$ with appropriate units of area. We will work mainly in degrees, switching to radians for calculations involving calculus.

First we deal with the case of obtuse-angled triangles where $C \geqslant 90°$. If $C = 90°$, then

$$\begin{aligned} \alpha + \beta &= (1 - \cos A)^2 + (1 - \cos B)^2 \\ &= (1 - \cos A)^2 + (1 - \sin A)^2 \\ &= 3 - 2\cos A - 2\sin A \\ &< 3 - 2 = (1 - \cos C)^2 = \gamma \end{aligned}$$

https://doi.org/10.1017/mag.2018.30 Published online by Cambridge University Press

since, by the triangle inequality in Figure 2, $2R < 2R\cos A + 2R\sin A$ for $A > 0°$.

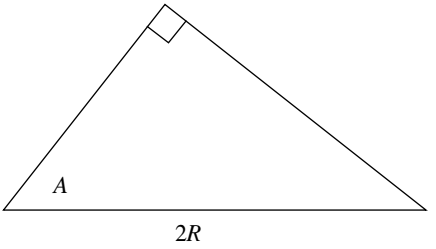


FIGURE 2

It follows that $\alpha + \beta < \gamma$ continues to hold when $C > 90°$. For, in Figure 3, inserting a diameter parallel to the longest side creates a right-angled triangle with angles $A', B', C' = 90°$ as shown. With notation as above, we then have $\alpha < \alpha', \beta < \beta', \gamma' < \gamma$ from which it follows that $\alpha + \beta < \alpha' + \beta' < \gamma' < \gamma$.

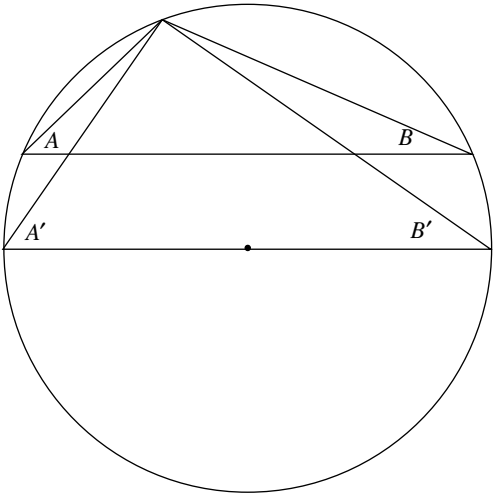


FIGURE 3

Our hunt for (*) thus narrows down to acute-angled triangles. Certainly an equilateral triangle satisfies (*) so, perturbing the angles slightly, shows that there certainly are scalene triangles satisfying (*). But they are quite rare. For, in the $B - A$ plane, they correspond to the region in Figure 4 defined by the inequalities $A < B$,

$$(1 - \cos A)^2 + (1 - \cos B)^2 > (1 + \cos(A + B))^2, A + 2B < 180°$$

(from $180° - (A + B) > B$).

https://doi.org/10.1017/mag.2018.30 Published online by Cambridge University Press

Notice that the line $A + 2B = 180°$ lies above the curve because, with $B = 90° - \frac{1}{2}A$,

$$(1 - \cos A)^2 + (1 - \cos B)^2 - (1 + \cos(A + B))^2$$

$$= (1 - \cos A)^2 + (1 - \cos(90° - \tfrac{1}{2}A))^2 - (1 + \cos(90° + \tfrac{1}{2}A))^2$$

$$= (1 - \cos A)^2 > 0$$

when $A > 0°$. (It is easy to check by implicit differentiation that the line is in fact the tangent to the curve at $A = 0°$.)

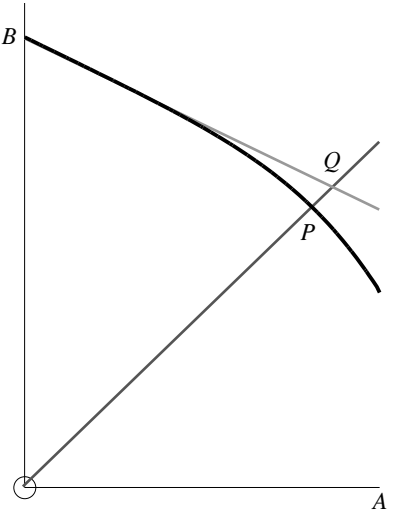


FIGURE 4: Graph showing the small curvilinear triangle of possibilities, *PQB*

The point $Q(60°,60°)$ in Figure 4 corresponds to an equilateral triangle; the point P is where $(1 - \cos A)^2 + (1 - \cos A)^2 = (1 + \cos 2A)^2 = 4\cos^4 A$ or $(2\cos^2 A - \sqrt{2}(1 - \cos A))(2\cos^2 A + \sqrt{2}(1 - \cos A)) = 0$ which solves to give $\cos A = \dfrac{2}{1 + \sqrt{1 + 4\sqrt{2}}}$ so that $A = B \approx 56.04°$, $C \approx 67.92°$. (The segment PQ therefore contains the triangle with $A = B = 1$ radian .) A quick spreadsheet search produces the possible scalene triangles with integer degree angles in Table 5; of these just one satisfies our original demand, namely $A = 55°$, $B = 60°$, $C = 65°$. In a teaching context, it is worth pointing out that restricting to discrete angles means that we sometimes lose in-between values: for example, the existence of solutions for $A = 33°$, $35°$ does not guarantee a solution for $A = 34°$.

https://doi.org/10.1017/mag.2018.30 Published online by Cambridge University Press

$A°$	$B°$	$C°$
33	73	74
35	72	73
36	71	73
37	71	72
38	70	72
39	70	71
40	69	71
41	69	70
42	68	70
43	67-68	70-69
44	67	69
45	66-67	69-68
46	65-66	69-68

$A°$	$B°$	$C°$
47	64-66	69-67
48	64-65	68-67
49	63-65	68-66
50	62-64	68-66
51	62-64	67-65
52	60-63	68-65
53	59-63	68-64
54	59-62	67-64
55	58-62	67-63
56	57-61	67-63
57	58-61	65-62
58	59-60	63-62
59	60	61

TABLE 5

One unusual feature of this problem is that if angle C is fixed, the sum of areas $\alpha + \beta$ is always maximised for degenerate triangles with $A = 0°$ (so it's always best to put your eggs into one basket!).

To see this, let $f(A) = \alpha + \beta = (1 - \cos A)^2 + (1 + \cos(A + C))^2$. Then

$$\begin{aligned} f'(A) &= 2\sin A(1 - \cos A) - 2\sin(A + C)\big(1 + \cos(A + C)\big) \\ &= -2\big[\sin(A + C) - \sin A\big] - \big[\sin(2A + 2C) + \sin 2A\big] \\ &= -4\cos\big(A + \tfrac{1}{2}C\big)\sin\tfrac{1}{2}C - 2\sin(2A + C)\cos C \\ &= -4\cos\big(A + \tfrac{1}{2}C\big)\sin\tfrac{1}{2}C - 4\cos\big(A + \tfrac{1}{2}C\big)\sin\big(A + \tfrac{1}{2}C\big)\big[\cos^2\tfrac{1}{2}C - \sin^2\tfrac{1}{2}C\big] \\ &= -4\cos\big(A + \tfrac{1}{2}C\big)\big[\sin\tfrac{1}{2}C\big(1 - \sin\tfrac{1}{2}C\sin\big(A + \tfrac{1}{2}C\big)\big) + \cos^2\tfrac{1}{2}C\sin\big(A + \tfrac{1}{2}C\big)\big] \end{aligned}$$

so that $f'(A) < 0$ when $0° < A < 90° - \frac{1}{2}C$. (This property gives an alternative proof that $\alpha + \beta < \gamma$ when C is obtuse.)

Related problems

1. If, instead of circles, we use the segments on each side, then the segment opposite angle A has area α given by $\alpha = \frac{1}{2}R^2(2A - \sin 2A)$ or $\alpha = 2A - \sin 2A$ with appropriate units of area. Here, it is obvious that if angle C is fixed, the sum of areas $\alpha + \beta$ is maximised for

https://doi.org/10.1017/mag.2018.30 Published online by Cambridge University Press

degenerate triangles with $A = 0°$. Figure 4 looks very similar when plotted with the segment-area formulae: the line $A + 2B = 180°$ is a tangent to the curve and the point corresponding to P satisfies $A = \frac{1}{4}\pi + \sin A \cos^3 A$ which may be solved by function iteration to give $A = B \approx 54.27°, C \approx 71.47°$.

2. If, instead of circles, we use the (isosceles) triangles of greatest area in each segment, then the triangle opposite angle A has area α given by $\alpha = R^2(1 - \cos A)\sin A$ or $\alpha = (1 - \cos A)\sin A$ with appropriate units of area. Once again, Figure 4 looks very similar when plotted with the triangle-area formulae: the line $A + 2B = 180°$ is a tangent to the curve and the point corresponding to P satisfies $2\cos^3 A + \cos A - 1 = 0$ which solves to give $A = B \approx 53.86°$, $C \approx 72.28°$. But there is a surprise in store when we look at the sum of areas $\alpha + \beta$ if angle C is fixed. Here, when $C > 60°$, $\alpha + \beta$ is maximised for degenerate triangles with $A = 0°$. However, when $2\sin^{-1}\frac{1}{4} \approx 28.96° < C < 60°$, there is a maximum value for $\alpha + \beta$ with $A < B$. And when $C < 2\sin^{-1}\frac{1}{4} \approx 28.96°$, the maximum value of $\alpha + \beta$ occurs when $A = B$.

Acknowledgement

It is a pleasure to record my thanks to the referee for several helpful suggestions for improvements.

10.1017/mag.2018.30

NICK LORD

Tonbridge School, Kent TN9 1JP

e-mail: *njl@tonbridge-school.org*

Three cameos involving *e*

The first two cameos below were inspired by The Mathematical Association's A-level poster called '*e*'. This occupies a prominent position on my classroom wall and features aspects of e such as its decimal expansion to 12 decimal places and the graphs of $y = e^x$ and $y = \ln x$ shown in Figure 1.

Cameo 1

I had just completed my usual introduction to e, ending up with a heuristic derivation of the formula $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ to give us a way of working out e. Normally, I just pick a large value of n such as a million for this calculation, but this time I decided to be more systematic and produced Table 1 below. The initial reaction of the class was that the first n initial digits of e were produced from using 10^n in the formula for e, but this stumbled at $n = 13$, so we quickly modified it to 'correct significant figures' meaning that, for example, when $n = 5$, both e and the approximation $\left(1 + 10^{-5}\right)^{10^5}$ agree as 2.7183 when both are rounded to 5 significant figures.

https://doi.org/10.1017/mag.2018.30 Published online by Cambridge University Press