Econometric Theory, **25**, 2009, 739–747. Printed in the United States of America.
doi:10.1017/S0266466608090270

CONVERGENCE TO STOCHASTIC POWER INTEGRALS FOR DEPENDENT HETEROGENEOUS PROCESSES

RICKARD SANDBERG
Stockholm School of Economics
and
Sveriges Riksbank

Building on work of Hansen (1992, *Econometric Theory* 8, 489–501), we show weak convergence for power transformations of integrated processes, with possibly serially correlated and heterogeneously distributed increments, to stochastic power integrals. The theory is applicable when testing the unit root or cointegration hypothesis in nonlinear systems by regression-based test statistics.

1. INTRODUCTION

The asymptotic theory for normalized integrated processes, (V_n), and associated stochastic integrals, $\int V_n dV_n$, has been studied extensively in the unit root and cointegration literature and plays a key role when deriving the limiting distribution for regression-based statistics in integrated time series. It is well known that V_n converges weakly in the Skorohod topology to a Brownian motion V under mild conditions, but the convergence of $\int V_n dV_n$ to $\int V dV$ is typically violated when V_n is not a martingale; see, for instance, Phillips (1987) and Hansen (1992). This might, in particular, be an issue in econometric applications where one would like to accommodate for serial correlation and conditional heteroskedasticity. Hansen (1992) relaxes the martingale assumption, considers the convergence of $\int V_n dV_n$ when the sequence of increments of V_n is strong mixing, and shows that the limiting result is $\int V dV + \mathcal{B}$, where $\mathcal{B}$ is a bias term.

The aim of this paper is to extend some of the results in Hansen (1992) by deriving asymptotics for a class of stochastic integrals that we refer to as stochastic

This research has been supported by Jan Wallander's Post Doctoral Scholarship W2005-0103:1. The paper was written during my time as a guest researcher at the Swedish Central bank, and I am thankful for a stimulating environment at their research department. I thank two anonymous referees for constructive and helpful suggestions that improved this paper. I also thank Tomas Björk and Annastiina Silvennoinen, who provided useful remarks. The views expressed in this paper are solely the responsibility of the author and should not be interpreted as reflecting the views of the Executive Board of Sveriges Riksbank. Address correspondence to Rickard Sandberg, Department of Economic Statistics, Stockholm School of Economics, SE-113 83, Stockholm, Sweden; e-mail: rickard.sandberg@hhs.se.

© 2009 Cambridge University Press 0266-4666/09 $15.00

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

power integrals.[1] This class is characterized by the fact that the integrand is a general power transformation of a scaled normalized integrated process.[2] The main contributions of the paper are therefore to characterize the resulting bias term for these power integrals by using the martingale difference approximation for strong mixing innovations by Hansen (1992) and also to establish an additional moment condition for weak convergence to hold.

The motivation for the nonlinear transformations proposed in this paper stems from the growing literature on testing the unit root and cointegration hypothesis in nonlinear autoregressive models based on Taylor approximations; see Kapetanios, Shin, and Snell (2003, 2006), Kiliç (2004), and He and Sandberg (2006), among others. However, the only results yet published in the literature are based on first-order approximations,[3] which in practice might offer an overly simple solution and will frequently generate tests with deficient power properties. Results for higher order approximations are therefore desirable, and asymptotics for the scaled power transformations of V_n must be established. The class of nonlinear models to which the theory in this paper applies are general and include, for instance, smooth transition autoregression (STAR) and time-varying STAR type models with a (possibly) nonlinear trend.

A few words on the notation: $\mathcal{C}^{k+1}(R)$ abbreviates the space of continuously differentiable functions of order $k+1$ on $R \subseteq \mathbb{R}$, $\mathcal{D}[0,1]$ denotes the space of càdlàg functions on $[0,1]$, $\Rightarrow$ signifies weak convergence with respect to the Skorohod metric (as defined in Billingsley, 1968), $\|X\|_p = (\mathrm{E}|X|^p)^{1/p}$ denotes the L_p-norm, and $B(r) = \sigma W(r)$ abbreviates a Brownian motion on $[0,1]$ with variance σ^2 and where $W(r)$ is a standard Wiener process.

The rest of the paper is organized as follows. Section 2 defines the stochastic power integrals. Theoretical results are found in Section 3. Mathematical proofs are given in the Appendix.

2. STOCHASTIC POWER INTEGRALS

Consider the stochastic array $(V_{nt}, : 1 \leq t \leq n; n \geq 1)$ where V_{nt} defines the normalized stochastic partial sum process

$$V_{nt} \equiv \frac{1}{\sqrt{n}} V_t, \qquad V_t \equiv \sum_{i=1}^{t} v_i.$$

The preceding array is transformed into a stochastic element on $[0,1]$ by defining $V_n(s) \equiv V_{n[ns]}$ where $0 \leq s \leq 1$ and $V_n(s) \in \mathcal{D}[0,1]$. For v_i to be possibly serially correlated and heterogeneously distributed the following assumption is imposed.

Assumption 1. For some $b > \beta > 2$, (v_i) is a zero-mean, strong mixing sequence with mixing coefficients α_m of size $-b\beta/(b-\beta)$ and $\sup_{i \geq 1} \|v_i\|_b = C < \infty$. In addition, $\lim_{n\to\infty} n^{-1}\mathrm{E}V_n^2 = \sigma^2 < \infty$ and $\mathrm{E}V_n^{2q-4} = O(n^{q-2})$ for some integer $q > 3$.

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

All conditions in Assumption 1 are standard except for the boundness condition $\mathrm{E}V_n^{2q-4} = O(n^{q-2})$, which is needed in the coming discussion because integrals involving qth-order power transformations of V_n are considered. Furthermore, a strong law of large numbers for strong mixing asserts that (McLeish, 1975) $n^{-1}\sum_{i=1}^n v_i^2 \overset{a.s.}{\to} \sigma_v^2$, where $\sigma_v^2 = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n \mathrm{E}v_i^2 < \infty$. In addition, the long-run variance σ^2 can be decomposed as $\sigma^2 = \sigma_v^2 + 2\Lambda$, where Λ defines the covariance $\Lambda \equiv \lim_{n\to\infty} n^{-1}\sum_{i=1}^n \sum_{j=i+1}^\infty \mathrm{E}v_i v_j$.

We are now ready to state the integrals of interest. Let V_{nt} and $V_n(s)$ be defined as before; then a ds-power integral and a stochastic $dV_n(s)$-power integral on $[0, s]$ are given by

$$\int_0^s \varphi_n^p(r) V_n^q(r)\, dr \equiv n^{-(p+q/2+1)} \sum_{i=1}^{[ns]} i^p V_i^q \tag{1}$$

and

$$\int_0^s \varphi_n^p(r) V_n^q(r)\, dV_n(r) \equiv n^{-(p+q/2+1/2)} \sum_{i=1}^{[ns]} i^p V_i^q v_{i+1}, \tag{2}$$

respectively, where $p, q \in \mathbb{Z}_+$ and $\varphi_n(r) : [0, 1] \mapsto [0, 1]$ is a step function defined by $\varphi_n(r) \equiv ([nr]+1)/n$ where $[nr]+1 = i$ holds on each subinterval $(i-1)/n \le r < i/n$.[4] It should be mentioned that the integral in (1) is an ordinary Riemann integral, and applying the continuous mapping theorem yields that

$$\int_0^s \varphi_n^p(r) V_n^q(r)\, dr \Rightarrow \int_0^s r^p B(r)^q dr, \tag{3}$$

as $n \to \infty$; see, for example, Sims, Stock, and Watson (1990). The reason for reintroducing the integral on the form (1) is just a matter of convenience because it appears in the (nonlinear) applications we have in mind and its properties are as such desirable.[5] In fact, the integrals (1) and (2) arise naturally when approximations of nonlinear dynamic time series models are considered. To take an example, define a stochastic process $(y_t : t \in \mathbb{Z}_+)$ that accommodates nonlinear deterministics in addition to nonlinear dynamics by

$$y_t \equiv x_t'\pi_0 + x_t'\pi_1 G_1(s_{1t}) + x_t'\pi_2 G_2(s_{2t}) + x_t'\pi_3 G_1(s_{1t}) G_2(s_{2t}) + v_t \tag{4}$$

with observed values at $t = 1, ..., n$, $x_t = (1, t, y_{t-1})'$, and $\pi_i = (\pi_{i1}, \pi_{i2}, \pi_{i3})' \in \mathbb{R}^3$ for $i = 0, ..., 3$ such that it is assumed that the process in (4) is stationary or stable. Furthermore, G_1 and G_2 are nonlinear functions assumed to satisfy $G_1(s_{1t}) \in \mathcal{C}^{k+1}(R_1)$ and $G_2(s_{2t}) \in \mathcal{C}^{l+1}(R_2)$ where $s_{it} \in R_i$ and $k, l \ge 1$ and $d^m G_i(0)/ds_{it}^m \ne 0$ for some $m \ge 1$. Leading examples in econometric applications assume that G_i is a cumulative distribution function (c.d.f.) where $s_{1t} = t$ and $s_{2t} = y_{t-1}$ are used as transition variables (see, e.g., Granger and Teräsvirta, 1993; Jansen and Teräsvirta, 1996). Substituting for kth and lth-order Taylor approximations of $G_1(t)$ and $G_2(y_{t-1})$, respectively, into (4) yields an auxiliary regression model with covariates (ignoring the remainder) $\tilde{X}_t = (\tilde{x}_{1t}', \tilde{x}_{2t}', \tilde{x}_{3t}')'$

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

where $\tilde{x}_{1t} = (1, t, ..., t^{k+1})'$, $\tilde{x}_{2t} = (y_{t-1}, y_{t-1}^2, ..., y_{t-1}^{l+1})'$, and $\tilde{x}_{3t}' = (t, ..., t^{k+1})' \otimes (y_{t-1}, y_{t-1}^2, ..., y_{t-1}^{l+1})'$.[6] For regression-based test statistics, the elements $\sum \tilde{X}_t \tilde{X}_t'$ and $\sum \tilde{X}_t v_t$ have an essential role, and the reason for $p, q \in \mathbb{Z}_+$ in the preceding integrals becomes evident.

3. THEORETICAL RESULTS

The following theorem entails large-sample properties for the stochastic integral in (2).

THEOREM 1. *If Assumption 1 holds with* $\beta = \max(3, 2q)$,[7] *then*

$$\int_0^s \varphi_n^p(r) V_n^q(r)\, dV_n(r) \Rightarrow \int_0^s r^p B(r)^q dB(r) + q\Lambda \int_0^s r^p B(r)^{q-1} dr \tag{5}$$

as $n \to \infty$, *where* $p, q \in \mathbb{Z}_+$.

Remarks on Theorem 1. We notice that Hansen (1992), in his applications 4.1 (Processes) and 4.2 (Products of $I(1)$ Processes), proves the asymptotics in (5) for $p = 0$ and $q = 1, 2$. Furthermore, because of the mixing assumption the covariance Λ appears as a factor in the bias term in (5), but in the special cases of martingale difference or independent and identically distributed (i.i.d.) innovations this bias is trivially eliminated because $\sigma^2 = \sigma_v^2$ holds. By (5) it also becomes evident that our bias term is of a somewhat more complicated form compared to the related results in Hansen (1992). Finally, in practice it is likely that Λ appears as a nuisance parameter in the regression-based test statistics. To operationalize these tests we can use the relationship $\sigma^2 = \sigma_v^2 + 2\Lambda$ to obtain a consistent estimator of Λ by combining the Newey–West (for details, see Newey and West, 1987) estimator of σ^2 with the result $n^{-1} \sum_{i=1}^n \hat{v}_i^2 \xrightarrow{a.s.} \sigma_v^2$ ($\hat{v}_i$ is the sample residual).

An equivalent representation for the limiting result in (5) can be obtained by using Itô calculus and is given in the corollary that follows.

COROLLARY 1. *Assume that the conditions in Theorem 1 hold; then*

$$\begin{aligned}\int_0^s \varphi_n^p(r) V_n^q(r) dV_n(r) &\\ \Rightarrow \frac{\sigma^{q+1}}{q+1} &\left(s^p W(s)^{q+1} - p \int_0^s r^{p-1} W(r)^{q+1} dr \right.\\ &\left. - \frac{q(q+1)}{2} \frac{\sigma_v^2}{\sigma^2} \int_0^s r^p W(r)^{q-1} dr \right)\end{aligned} \tag{6}$$

as $n \to \infty$, *where* $p, q \in \mathbb{Z}_+$.

We will next illuminate some applications of Taylor approximations in popular nonlinear models with integrated covariates and indicate how the resulting

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

asymptotics can be obtained by applying Theorem 1 with $s = 1$. First, Kapetanios et al. (2003, 2006) test the unit root and a nonlinear cointegration related hypothesis in an exponential smooth transition autoregression model using its first-order approximation. Their results in Theorem 2.1 and Theorem 3.1, respectively, are deduced letting $p = 0$ with $q = 1, ..., 6$.[8] Second, Kiliç (2004) tests for nonlinearities utilizing first-order approximations of a smooth transition autoregression and an exponential STAR model when data are highly persistent. The convergence results in his Theorem 1 follow by setting $p = 0$ with $q = 1, ..., 6$.[9] Third, He and Sandberg (2006) employ third-order approximations to test the unit root hypothesis in a logistic STAR model with a nonlinear trend. Studying $p \leq 6$ and $q \leq 2$ yields the large-sample results in their Theorem 1.[10] Finally, and as a topic for further research, the order of Taylor series approximations for the tests discussed in previous examples can easily be increased by further applying Theorem 1. Thereby it is simple to implement new regression-based test statistics, which in many cases will have better power properties than existing tests, and establish new limiting distributions.

NOTES

1. It should be pointed out that the results of Hansen (1992) are for normalized vector processes, whereas our results are for normalized scalar processes. It is, however, straightforward to extend our results to the vector case.

2. Nonlinear transformations of integrated time series have been considered elsewhere in the literature; see, for instance, Granger and Hallman (1991), Park and Phillips (1999, 2001), Pötscher (2004), and de Jong and Wang (2005). Notice though that they derive results only for Lebesgue integrals, not for stochastic integrals.

3. One exception is the work of He and Sandberg (2006), who base their results on a third-order Taylor approximation.

4. It is assumed that the zero element is contained in $\mathbb{Z}_+$.

5. We will also need the limiting results for the integral in (1) to prove the convergence results for (2).

6. Our approach is similar to the one that Ramsey (1969) adopted for his RESET (Regression Equation Specification Error Test) specification test of a linear conditional mean. The main difference is that we are interested in the case of $I(1)$ regressors and resulting inference therefore becomes nonstandard.

7. The condition $\beta = \max(3, 2q)$ is a sufficient but not a necessary condition. For instance, let $p = q = 0$ in (5); then $\int_0^s dV_n(r) \Rightarrow B(s)$ as long as $\beta > 2$ (Herrndorf, 1985).

8. The results of Kapetanios et al. (2003, 2006) are less general because they are derived under the assumption that (v_i) is an i.i.d. sequence.

9. The results by Kiliç (2004) are more restrictive because they rely upon the assumption that v_i is i.i.d.

10. The results of He and Sandberg (2006) allow for serially correlated errors only.

REFERENCES

Billingsley, P. (1968) *Convergence of Probability Measures*. Wiley.

de Jong, R.M. & C.-H. Wang (2005) Further results on the asymptotics for nonlinear transformations of integrated time series. *Econometric Theory* 21, 413–430.

Granger, C.W.J. & J. Hallman (1991) Nonlinear transformations of integrated time series. *Journal of Time Series Analysis* 12, 207–224.

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

Granger, C.W.J. & T. Teräsvirta (1993) *Modelling Nonlinear Economic Relationships*. Oxford University Press.
Hansen, B.E. (1992) Convergence to stochastic integrals for dependent heterogeneous processes. *Econometric Theory* 8, 489–501.
He, C. & R. Sandberg (2006) Dickey-Fuller type of tests against non-linear dynamic models. *Oxford Bulletin of Economics and Statistics* 68, 835–861.
Herrndorf, N. (1985) A functional central limit theorem for strongly mixing sequences of random variables. *Probability Theory and Related Fields* 69, 541–550.
Jansen, E.S. & T. Teräsvirta (1996) Testing parameter constancy and super exogeneity in econometric equations. *Oxford Bulletin of Economics and Statistics* 58, 735–763.
Kapetanios, G., Y. Shin, & A. Snell (2003) Testing for a unit root in the nonlinear STAR framework. *Journal of Econometrics* 112, 359–379.
Kapetanios, G., Y. Shin, & A. Snell (2006) Testing for cointegration in nonlinear smooth transition error correction models. *Econometric Theory* 22, 279–303.
Kiliç, R. (2004) Linearity tests and stationarity. *Econometrics Journal* 7, 55–62.
Kurtz, T.G. & P. Protter (1991) Weak limit theorems for stochastic integrals and stochastic differential equations. *Annals of Probability* 19, 1035–1070.
McLeish, D.L. (1975) A maximal inequality and dependent strong laws. *Annals of Probability* 3, 829–839.
Newey, W.K. & K.D. West (1987) A simple, positive definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica* 55, 277–301.
Park, J.Y. & P.C.B. Phillips (1999) Asymptotics for nonlinear transformations of integrated time series. *Econometric Theory* 15, 269–298.
Park, J.Y. & P.C.B. Phillips (2001) Nonlinear regressions with integrated time series. *Econometrica* 69, 117–161.
Phillips, P.C.B. (1987) Time series regression with a unit root. *Econometrica* 55, 277–301.
Pötscher, B.M. (2004) Nonlinear functions and convergence to Brownian motion: Beyond the continuous mapping theorem. *Econometric Theory* 20, 1–22.
Ramsey, J.B. (1969) Tests for specification errors in classical linear least squares regression analysis. *Journal of the Royal Statistical Society, series B* 31, 350–371.
Sims, C.A., J.H. Stock, & M.W. Watson (1990) Inference in linear time series with some unit roots. *Econometrica* 58, 113–144.

APPENDIX

Before proving Theorem 1 some prerequisites are needed. Set $\mathcal{F}_t = \sigma(v_i : i \leq t)$ to be the smallest σ-algebra containing the past history of (V_{nt}, v_t) for all n. Denote $\mathrm{E}(X|\mathcal{F}_i)$ by $\mathrm{E}_i X$, and define $\epsilon_i \equiv \sum_{k=0}^{\infty}(\mathrm{E}_i v_{i+k} - \mathrm{E}_{i-1} v_{i+k})$ and $z_i \equiv \sum_{k=1}^{\infty} \mathrm{E}_i v_{i+k}$. Then the decomposition

$$v_i = \epsilon_i + z_{i-1} - z_i \tag{A.1}$$

holds true with ϵ_i an $\mathcal{F}_i$-measurable sequence with $\mathrm{E}_{i-1}\epsilon_i = 0$, such that $(\epsilon_i, \mathcal{F}_i)$ is a martingale difference sequence. Moreover, define a new stochastic array, associated with ϵ_i, by $(Y_{nt}, : 1 \leq t \leq n; n \geq 1)$ where $Y_{nt} \equiv Y_t/\sqrt{n}$, $Y_t \equiv \sum_{i=1}^{t} \epsilon_i$, and $Y_n(s) = Y_{n[ns]} \in \mathcal{D}[0,1]$ with $0 \leq s \leq 1$.

Proof of Theorem 1. Using the martingale approximation (A.1) we obtain

$$\int_0^s \varphi_n^p(r) V_n^q(r)\, dV_n(r) = \int_0^s \varphi_n^p(r) V_n^q(r)\, dY_n(r) + B_n^*(s), \tag{A.2}$$

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

where

$$B_n^*(s) = \frac{1}{n^{p+q/2+1/2}} \sum_{i=1}^{[ns]} (i^p V_i^q - (i-1)^p V_{i-1}^q) z_i - \frac{[ns]^p}{n^{p+q/2+1/2}} V_{[ns]}^q z_{[ns]+1}.$$

If Assumption 1 holds and noticing that $(\epsilon_i, \mathcal{F}_i)$ is a martingale difference sequence, Theorem. 3.1 of Hansen (1992) (or Theorem 2.1 of that paper, which is a special case of several results by Kurtz and Protter, 1991) is applicable to the stochastic integral on the right-hand side of (A.2) and yields

$$\int_0^s \varphi_n^p(r) V_n^q(r)\, dY_n(r) \Rightarrow \int_0^s r^p B(r)^q\, dB(r). \tag{A.3}$$

To analyze the bias term $B_n^*(s)$ we start with its sum and apply the binomial theorem to $V_i^q = (V_{i-1} + v_i)^q$ and $(i-1)^p$ to obtain

$$\begin{aligned} &\frac{1}{n^{p+q/2+1/2}} \sum_{i=1}^{[ns]} (i^p V_i^q - (i-1)^p V_{i-1}^q) z_i \\ &\quad = \frac{1}{n^{p+q/2+1/2}} \left(\sum_{i=1}^{[ns]} \sum_{j=1}^{q} i^p c_j^q V_{i-1}^{q-j} v_i^j z_i - \sum_{i=1}^{[ns]} \sum_{k=1}^{p} (-1)^k i^{p-k} c_k^p V_{i-1}^q z_i \right), \end{aligned} \tag{A.4}$$

where c_j^q and c_k^p are binomial coefficients. Moreover, define $e_i \equiv v_i z_i - \Lambda$; then the first sum on the right-hand side in (A.4) with $j = 1\ (< q)$ can be written as

$$\begin{aligned} &\frac{1}{n^{p+q/2+1/2}} \sum_{i=1}^{[ns]} i^p c_1^q V_{i-1}^{q-1} v_i z_i \\ &\quad = \frac{1}{n^{p+q/2+1/2}} \sum_{i=1}^{[ns]} i^p c_1^q V_{i-1}^{q-1} e_i + \frac{1}{n^{p+q/2+1/2}} \sum_{i=1}^{[ns]} i^p c_1^q V_{i-1}^{q-1} \Lambda. \end{aligned} \tag{A.5}$$

The first term on the right-hand side of (A.5) can be analyzed by Theorem 3.3 of Hansen (1992) because $V_n^{q-1}(s) \Rightarrow B^{q-1}(s)$ holds by the continuous mapping theorem and e_i by Theorem 3.2 of Hansen (1992) satisfies the conditions in Theorem 3.3; hence

$$\sup_{s\in[0,1]} \left| \frac{1}{n^{p+q/2+1/2}} \sum_{i=1}^{[ns]} i^p c_1^q V_{i-1}^{q-1} e_i \right| \le c_1^q \sup_{s\in[0,1]} \left| \frac{1}{n} \sum_{i=1}^{[ns]} \left(\frac{V_{i-1}}{\sqrt{n}} \right)^{q-1} e_i \right| \xrightarrow{p} 0$$

as $n \to \infty$. Furthermore, the result in (3) is applicable to the second expression on the right-hand side of (A.5) and implies

$$\frac{1}{n^{p+q/2+1/2}} \sum_{i=1}^{[ns]} i^p c_1^q V_{i-1}^{q-1} \Lambda \Rightarrow q\Lambda \int_0^s r^p B(r)^{q-1} dr.$$

Consider next the first sum on the right-hand side of (A.4) with $j = 2\ (< q)$. By the triangle inequality and Hölder's inequality we find

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

$$\begin{aligned}
\mathrm{E}\sup_{s\in[0,1]}\left|\frac{1}{n^{p+q/2+1/2}}\sum_{i=1}^{[ns]} i^p c_2^q V_{i-1}^{q-2} v_i^2 z_i\right| &\le \mathrm{E}\sup_{s\in[0,1]}\frac{1}{n^{p+q/2+1/2}}\sum_{i=1}^{[ns]}\left|i^p c_2^q V_{i-1}^{q-2} v_i^2 z_i\right| \\
&\le \frac{1}{n^{q/2+1/2}}\sum_{i=1}^{n}\mathrm{E}\left|c_2^q V_{i-1}^{q-2} v_i^2 z_i\right| \\
&\le \frac{c_2^q}{n^{q/2+1/2}}\sum_{i=1}^{n}\left\|V_{i-1}^{q-2}\right\|_2\left\|v_i^2 z_i\right\|_2 \to 0
\end{aligned} \tag{A.6}$$

as $n\to\infty$. To see this, note that $\|v_i^2 z_i\|_2 \le \|v_i\|_6^2\|z_i\|_6 < \infty$ holds true because $\|v_i\|_6 < \infty$ by Assumption 1 and $\|z_i\|_6 \le 6C\sum_{k=1}^{\infty}\alpha_k^{1/6-1/b} < \infty$ holds by Minkowski's inequality and McLeish's strong mixing inequality (McLeish, 1975). Moreover,

$$\frac{1}{n^{q/2+1/2}}\left\|V_{i-1}^{q-2}\right\|_2 \le \frac{1}{n^{3/2}}\sum_{i=1}^{n}\left(n^{-(q-2)}\mathrm{E}V_n^{2q-4}\right)^{0.5} \to 0$$

as $n\to\infty$, because $n^{-(q-2)}\mathrm{E}V_n^{2q-4} = O(1)$ by Assumption 1 and is thus bounded.

Furthermore, it is easily shown by arguments similar to those in (A.6) that terms corresponding to $j = 3, ..., q-1$ in the first sum on the right-hand side of (A.4) also converge to zero. It is, however, illuminating to give the proof for $j = q-1$ because it determines the moment condition in Theorem 1 when $q > 1$. Hence, for $j = q-1$ we have

$$\mathrm{E}\sup_{s\in[0,1]}\left|\frac{1}{n^{p+q/2+1/2}}\sum_{i=1}^{[ns]} i^p c_{q-1}^q V_{i-1} v_i^{q-1} z_i\right| \le \frac{c_{q-1}^q}{n^{q/2+1/2}}\sum_{i=1}^{n}\left\|V_{i-1}\right\|_2\left\|v_i^{q-1} z_i\right\|_2 \to 0$$

as $n\to\infty$. This is a result that follows because $\sum_{i=1}^n\|V_{i-1}\|_2 \le n^{-3/2}\sum_{i=1}^n(n^{-1}\mathrm{E}V_n^2)^{0.5}$ where $n^{-1}\mathrm{E}V_n^2 = O(1)$ by Assumption 1, and $\|v_i^{q-1}z_i\|_2 \le \|v_i\|_{2q}^{q-1}\|z_i\|_{2q} < \infty$ because $\|v_i\|_{2q}^{q-1}$ is bounded by Assumption 1 with $\beta = 2q$ and $\|z_i\|_{2q} \le 6C\sum_{k=1}^{\infty}\alpha_k^{1/2q-1/b} < \infty$.

The last term ($j = q$) in the sum on the right-hand side of (A.4) yields

$$\begin{aligned}
\mathrm{E}\left|\sup_{s\in[0,1]}\frac{1}{n^{p+q/2+1/2}}\sum_{i=1}^{[ns]} i^p v_i^q z_i\right| &\le \mathrm{E}\sup_{s\in[0,1]}\frac{1}{n^{p+q/2+1/2}}\sum_{i=1}^{[ns]}\left|i^p v_i^q z_i\right| \\
&\le \frac{1}{n^{q/2+1/2}}\sum_{i=1}^{n}\left|v_i^q z_i\right| \\
&\le \frac{1}{n^{q/2+1/2}}\sum_{i=1}^{n}\|v_i\|_{(q+1)/q}^q\|z_i\|_{q+1} \to 0
\end{aligned}$$

as $n\to\infty$, because $\|v_i\|_{q+1} < \infty$ by Assumption 1 and $\|z_i\|_{q+1} \le 6C\sum_{k=1}^{\infty}\alpha_k^{1/(q+1)-1/b} < \infty$.

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press

Moving on to the second sum on the right-hand side of (A.4) with $k = 1$, we have

$$\begin{aligned}\frac{1}{n^{p+q/2+1/2}}\sum_{i=1}^{[ns]} c_1^p i^{p-1} V_{i-1}^q z_i &\le \frac{c_1^p}{n^{q/2+3/2}}\sum_{i=1}^{[ns]}\left|V_{i-1}^q z_i\right| \\ &\le \frac{c_1^p}{n^{q/2+3/2}}\sum_{i=1}^{n}\left|V_{i-1}^q z_i\right| \\ &\le \frac{c_1^p}{n^{q/2+3/2}} n \sup_{i\le n}\left|V_{i-1}^q z_i\right| \\ &\le c_1^p \sup_{i\le n}\left|\left(\frac{V_{i-1}}{\sqrt{n}}\right)^q\right| \sup_{i\le n}\left|\frac{1}{\sqrt{n}} z_i\right| \xrightarrow{p} 0 \end{aligned} \tag{A.7}$$

as $n \to \infty$, because $\sup_{i\le n}|(V_{i-1}/\sqrt{n})^q| = O_p(1)$ and $\sup_{i\le n}|z_i/\sqrt{n}| = o_p(1)$ using result (A.3) in the proof of Theorem 3.1 of Hansen (1992). Moreover, the terms corresponding to $k = 2, ..., p$ in the second sum on the right-hand side of (A.4) also converge to zero by arguments similar to those in (A.7).

Finally, observe that the last term in the expression for $B_n^*(s)$ obeys

$$\sup_{s\in[0,1]}\left|\frac{[ns]^p}{n^{p+q/2+1/2}} V_{[ns]}^q z_{[ns]+1}\right| \le \sup_{t\le n}\left|\left(\frac{V_t}{\sqrt{n}}\right)^q\right| \left|\frac{1}{\sqrt{n}} z_{t+1}\right| \xrightarrow{p} 0 \tag{A.8}$$

as $n \to \infty$, because $\sup_{t\le n}|(V_t/\sqrt{n})^q| = O_p(1)$ and $\sup_{t\le n}|z_t/\sqrt{n}| = o_p(1)$.

Combine the results in (A.2)–(A.8) to obtain the result in (5). ■

Proof of Corollary 1. Notice first that $dB(r) = \sigma\, dW(r)$ holds, implying $\int_0^s r^p B(r)^q dB(r) = \sigma^{q+1}\int_0^s r^p W(r)^q dW(r)$. Define $g_2 : [0,1]\times\mathbb{R} \mapsto \mathbb{R}$ by $g_2(r, W) = r^p W^{q+1}$; then Itô's lemma yields

$$\begin{aligned}&\sigma^{q+1} s^p W(s)^{q+1} \\ &\quad = \sigma^{q+1}\left(p\int_0^s r^{p-1} W(r)^{q+1} dr + (q+1)\int_0^s r^p W(r)^q dW(r)\right. \\ &\qquad \left. + \frac{q(q+1)}{2}\int_0^s r^p W(r)^{q-1} dr\right)\end{aligned}$$

Rearrange terms to obtain

$$\begin{aligned}&\sigma^{q+1}\int_0^s r^p W(r)^q dW(r) \\ &\quad = \sigma^{q+1}\left(\frac{s^p W(s)^{q+1}}{q+1} - \frac{p}{q+1}\int_0^s r^{p-1} W(r)^{q+1} dr - \frac{q}{2}\int_0^s r^p W(r)^{q-1} dr\right).\end{aligned} \tag{A.9}$$

Moreover, the bias term in (5) can be written as

$$q\Lambda\int_0^s r^p B(r)^{q-1} dr = q\frac{(\sigma^2 - \sigma_v^2)}{2}\sigma^{q-1}\int_0^s r^p W(r)^{q-1} dr. \tag{A.10}$$

Combining the results in (A.9) and (A.10) gives the desired result. ■

https://doi.org/10.1017/S0266466608090270 Published online by Cambridge University Press