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Lyapunov graph continuation

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Abstract. In this paper the Poincaré–Hopf inequalities are shown to be necessary and sufficient conditions for an abstract Lyapunov graph L to be continued to an abstract Lyapunov graph of Morse type. The Lyapunov graphs considered may represent smooth flows on closed orientable n-manifolds, $n \geq 2$. The continuation which is presented by means of a constructive algorithm, is shown to be unique in dimensions two and three. In all other dimensions, the exact number of possible continuations of L are presented.

1. *Introduction*

In this paper, closed orientable n-manifolds M, $n \geq 2$ are considered. Given a smooth flow $\phi_t : M \to M$, there exists a smooth function $f : M \to \mathbb{R}$ associated to this flow with the property that it decreases along orbits outside the chain recurrent set $\mathcal{R}$; that is, if $x \notin \mathcal{R}$ then $f(\phi_t(x)) < f(\phi_s(x))$ given that $t > s$ and is constant on connected components of $\mathcal{R}$. This function is defined as a *Lyapunov function*. We refer to ϕ_t as a *gradient-like* flow with respect to f because of the properties above.

A set $S \subset M$ is invariant if $\phi_t(S) = S$ for all $t \in \mathbb{R}$. A compact set $N \subset M$ is an *isolating neighborhood* if $\mathrm{inv}(N, \phi) = \{x \in N : \phi_t(x) \subset N,\ \forall t \in \mathbb{R}\} \subset \mathrm{int}\, N$. A compact set N is an *isolating block* if $N^- = \{x \in N : \phi_{[0,t)}(x) \not\subset N, \forall t > 0\}$ is closed and $\mathrm{inv}(N, \phi) \subset \mathrm{int}\, N$. An invariant set S is called an *isolated invariant set* if it is a maximal invariant set in some isolating neighborhood N; that is, $S = \mathrm{inv}(N, \phi)$.

A component R of $\mathcal{R}$ of the flow ϕ_t is an example of an invariant set. We will work under the hypothesis that $\mathcal{R}$ is the finite union of isolated invariant sets R_i. If f is a Lyapunov function associated to a flow and $c = f(R)$, then for $\varepsilon > 0$ the component of $f^{-1}[c - \varepsilon, c + \varepsilon]$ that contains R is an isolating neighborhood for R. Take $(N, N^-) = (f^{-1}[c - \varepsilon, c + \varepsilon],\ f^{-1}(c - \varepsilon))$ as an index pair for R. The Conley index is defined as the homotopy type of N/N^-. The Conley homology index is denoted by $CH_*(S)$ and its rank denoted by $h_* = \mathrm{rank}\, CH_*(S)$. For more details see [**2**].

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Lyapunov graphs were first introduced by Franks [**6**] and have proven to be an excellent bookkeeping device of dynamical and topological information of the flow and the phase space. A *Lyapunov graph* is a finite, connected, oriented graph with no oriented cycles and with labeled vertices and edges.

A Lyapunov function $f : M \to \mathbb{R}$ associated to a flow, determines a Lyapunov graph by the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. Therefore, $M/\sim_f$ is a Lyapunov graph. It is possible to choose f so that each critical level contains a component R of $\mathcal{R}$. A point on $M/\sim_f$ is a *vertex point* if under the equivalence relation it corresponds to a level set containing a component R of $\mathcal{R}$. All other points are *edge points*. Each edge represents a codimension one submanifold Q of M times an open bounded interval I, $Q \times I$. In order to retain some topological information of $Q \times I$, the edges will be labeled with the Betti numbers of Q. All homology groups will be computed with $\mathbb{Z}_2$ coefficients.

An *abstract Lyapunov graph* is an oriented graph with no oriented cycles such that each vertex v is labeled with a list of non-negative integers $\{h_0(v) = k_0, \ldots, h_n(v) = k_n\}$. Whenever $h_j(v) = 0$, it will be omitted from the list. Also, the labels on each edge $\{\beta_0 = 1, \beta_1, \ldots, \beta_{n-2}, \beta_{n-1} = 1\}$ must be a collection of non-negative integers satisfying Poincaré duality and if $n - 1$ is even then $\beta_{(n-1)/2}$ is even.

An *abstract Lyapunov graph of Morse type* is defined later, but roughly speaking, it is an abstract Lyapunov graph with all vertices labeled with non-degenerate singularities, i.e. $\{h_j(v) = 1\}$ for some j.

Next, the notion of *vertex explosion* is established in order to define continuation of abstract Lyapunov graphs. Let v be a vertex on an abstract Lyapunov graph labeled with $\{h_0(v), h_1(v), \ldots, h_n(v)\}$. A vertex v can be exploded if v can be removed and replaced by an abstract Lyapunov graph I of Morse type with k vertices. The graph I must respect the orientations and labels of the incoming and outgoing edges of v. In other words, the new graph obtained must be oriented and with no oriented cycles. The incoming (outgoing) edges of v, must be incoming (outgoing) edges on vertices of I and all labels on the edges must respect the restrictions of the Morse type vertices. Moreover,

$$h_\lambda(v) = \sum_{j=1}^{k} h_\lambda(v_j), \text{ for all } \lambda.$$

An abstract Lyapunov graph admits a *continuation* to an abstract Lyapunov graph of Morse type if each vertex can be exploded.

The underlying idea in this paper is to describe under which conditions an abstract Lyapunov graph can be continued to an abstract Lyapunov graph of Morse type. These conditions are the Poincaré–Hopf inequalities which are shown to be necessary and sufficient for the continuation to occur. This is done by presenting an algorithm which not only constructs the continuation but also provides the number of possible continuations.

THEOREM 1.1. *Every abstract Lyapunov graph that satisfies the Poincaré–Hopf inequalities at each vertex can be continued to an abstract Lyapunov graph of Morse type.*

This paper is organized as follows. Section 2 will introduce the techniques developed in [**3**]. The main idea is to develop a Morse theoretic approach of analyzing the changes to

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the manifold as handles are attached. For this purpose, a connection is drawn between the attachment of handles and the passage through a vertex on a Lyapunov graph L. The different types of attachments of handles will correspond to different labels on vertices of L.

In §3 a graph filtration is defined which corresponds to a filtration on the manifold and permits the proof of the generalized Morse inequalities using the ranks of the Conley homology index. Moreover, this filtration permits the translation of information of the flow to the graph and *vice versa*. The Poincaré–Hopf inequalities are deduced from an analysis of long exact sequences of index pairs.

In §4 an algorithm for the explosion of a vertex v is presented. The algorithm describes the number of different types of vertices that will replace v. This algorithm will differ slightly if the vertex v is on a graph that represents a flow on a manifold of odd dimension, even dimension equal to 0 mod 4 or even dimension equal to 2 mod 4. Each step of the algorithm imposes a series of restrictions described by linear equations, thus forming three families of linear systems whose solution is the number of different types of vertices which will substitute v in the explosion.

In §5 the linear systems will be treated rigorously using integer linear programming techniques. In other words, it will be proved that the three different families of linear systems always have non-negative integer solutions provided certain inequalities are satisfied. Moreover, the exact number of non-negative integral solutions is obtained.

In §6 it is shown that the inequalities in §5 required for the existence of non-negative integer solutions are precisely the Poincaré–Hopf inequalities.

2. *Handle theory*

A description of how handles can be attached will be given in this section. Denote by D^j the closed unit j-disk, i.e. the j-ball. Also, the boundary of D^j, ∂D^j is the $(j-1)$-sphere S^{j-1}.

Let $\overline{N} = N^- \times [0, 1]$ be the n-manifold obtained as a collar of N^- and $H = D^\ell \times D^{n-\ell}$. Let $\theta : \partial D^\ell \times D^{n-\ell} \to \partial \overline{N}$ be an embedding which defines the new manifold $N' = \overline{N} \cup_\theta H$, which is the result of attaching an ℓ-handle to $\overline{N}$. The following loose notation is used: $N' = \overline{N} \cup H^{(\ell)}$ or $N' = \overline{N} \cup H$, where $\partial N' = N^+ \cup N^-$.

In terms of a Lyapunov graph L, the attaching of a handle corresponds to the passage through a vertex in the opposite direction of the orientation of the graph. Hence a vertex of L represents the singularity. The vertex together with its labeled incident edges represents N'. The incoming edges represent $N^+ \times J$ and the outgoing edges represent $N^- \times J$ where J is an open interval.

The following definition distinguishes the effect on the Betti numbers of N^+ and N^- once the handle has been attached.

A handle containing a singularity of index ℓ or the corresponding vertex on L is called ℓ-disconnecting, in short ℓ-d, if this handle has the algebraic effect of increasing the ℓth Betti number of N^+ or the corresponding β_ℓ label on the incoming edge, respectively. A handle containing a singularity of index ℓ or the corresponding vertex on L is called $(\ell-1)$-*connecting*, in short $(\ell-1)$-c, if this handle has the algebraic effect of decreasing the $(\ell-1)$th Betti number of N^+ or the corresponding $\beta_{\ell-1}$ label on the incoming edge,

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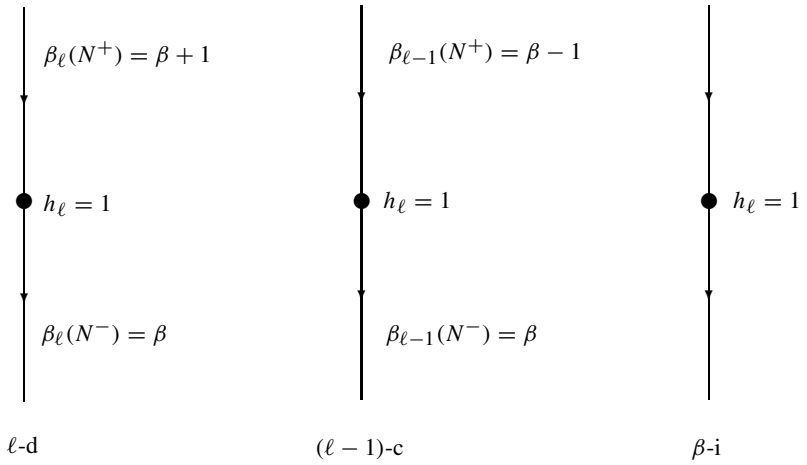


FIGURE 1. The three possible algebraic effects.

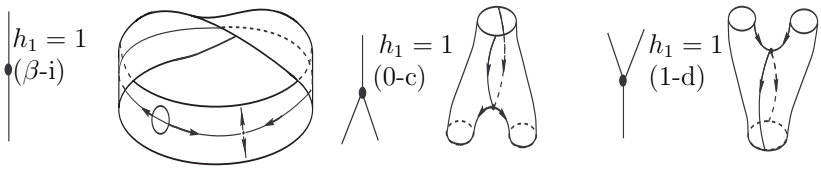


FIGURE 2. Isolating blocks for saddles in dimension 2.

respectively. A handle containing a singularity of index ℓ or the corresponding vertex on L is called β-*invariant*, in short β-i, if all Betti numbers are kept constant; see Figure 1.

The isolating blocks N' of a singularity of index $h_1 = 1$ in a manifold of dimension 2 of types β-i, 0-c and 1-d are shown in Figure 2 (see [**5**]). Note the presence of an extra edge, in the 0-c and 1-d case precisely, because the Betti numbers that decreased and increased, respectively, control the number of connected components. It is shown in Theorem 2.1 that on orientable even-dimensional manifolds of dimension n, β-i singularities can only occur if $n = 0 \bmod 4$.

The isolating block N' of a singularity of index $h_1 = 1$ in an orientable manifold of dimension three corresponding to the 0-c case and the 1-d case are as in Figure 3 (see [**4**]).

In this section part of the main result in [**3**] will be used.

THEOREM 2.1. *Let $\phi_t : M \to M$ be a Morse flow with Lyapunov function $f : M \to \mathbb{R}$. Let v be a vertex of the associated Lyapunov graph L. Let N be a isolating block containing only one singularity that corresponds to v. Let N^-, N^+ be the exiting and entering components of N. If v is labeled as an index ℓ singularity then:*

(1) *v is ℓ-d, $(\ell - 1)$-c or β-i;*
(2) *the sum of the labels on the incoming edges incident to v (i.e. the total Betti number of N^+) changes with respect to the sum of the labels on the outgoing edges incident to v (i.e. the total Betti number of N^-) by ± 2 or 0;*
(3) *v cannot be β-i if $n \neq 0 \bmod 4$.*

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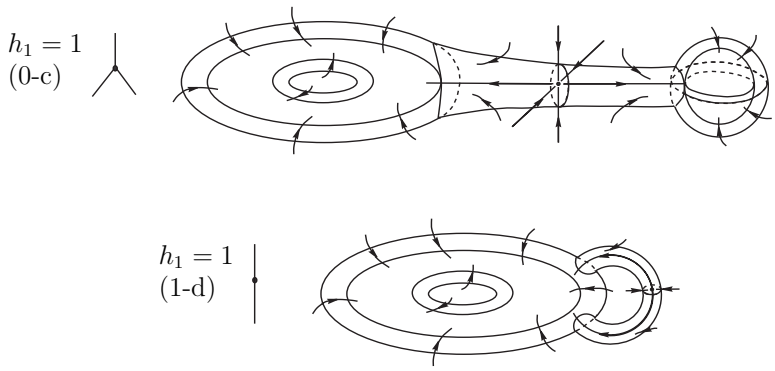


FIGURE 3. Isolating blocks for saddles in dimension 3.

An *abstract Lyapunov graph of Morse type* is an abstract Lyapunov graph that satisfies the following.

(1) Every vertex is labeled with $h_j = 1$ for some $j = 0, \ldots, n$.

(2) The number of incoming edges, e^+, and the number of outgoing edges, e^-, of a vertex must satisfy:
   (a) if $h_j = 1$ for $j \neq 0, 1, n-1, n$ then $e^+ = 1$ and $e^- = 1$;
   (b) if $h_1 = 1$ then $e^+ = 1$ and $0 < e^- \leq 2$; if $h_{n-1} = 1$ then $e^- = 1$ and $0 < e^+ \leq 2$;
   (c) if $h_0 = 1$ then $e^- = 0$ and $e^+ = 1$; if $h_n = 1$ then $e^+ = 0$ and $e^- = 1$.

(3) Every vertex labeled with $h_\ell = 1$ must be of type ℓ-d or $(\ell-1)$-c. Furthermore if $n = 2i = 0 \bmod 4$ and $h_i = 1$ then v may be labeled with β-i.

The specific effects of the ℓ-d, $(\ell-1)$-c or β-i vertices on the β_ℓ^+ labels on the incoming edge is specified in a series of Propositions in [**3**]. Let v be the vertex labeled with $h_\ell = 1$ on L, where L represents a flow whose phase space is an n-dimensional manifold M. The dimension of M, n, is referred to as the *ambient* dimension. Whenever M is an even-dimensional manifold and ℓ coincides with half its dimension or whenever M is an odd-dimensional manifold and ℓ coincides with either of the two middle dimensions then the analysis is slightly more elaborate. These cases will be referred to as the middle-dimensional cases and the results are summarized in the following proposition.

PROPOSITION 2.1. (1) *Let $n = 2i$ and $\ell \neq i$ or $n = 2i+1$ and $\ell \neq i, i+1$. Then either:*

(a) *v is ℓ-d, then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \text{for all } k \neq \ell, n-\ell-1, \\ \beta_k(N^+) - 1 & \text{for } k = \ell, n-\ell-1; \end{cases}$$

or else

(b) *v is $(\ell-1)$-c, then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \text{for all } k \neq \ell-1, n-\ell, \\ \beta_k(N^+) + 1 & \text{for } k = \ell-1, n-\ell. \end{cases}$$

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(2) *If* $n = 2i$ *and* $\ell = i$ *then:*

(a) *if* v *is* β*-*i *then*

$$\beta_k(N^-) = \beta_k(N^+) \quad \textit{for all } k;$$

(b) *if* v *is* ℓ*-d then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \textit{for all } k \neq i, i-1, \\ \beta_k(N^+) - 1 & \textit{for } k = i, i-1; \end{cases}$$

or else

(c) *if* v *is* $(\ell - 1)$*-c then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \textit{for all } k \neq i, i-1, \\ \beta_k(N^+) + 1 & \textit{for } k = i, i-1. \end{cases}$$

(3) *If* $n = 2i + 1$ *and* $\ell = i$ *then:*

(a) *if* v *is* ℓ*-d then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \textit{for all } k \neq i, \\ \beta_k(N^+) - 2 & \textit{for } k = i; \end{cases}$$

or else

(b) *if* v *is* $(\ell - 1)$*-c then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \textit{for all } k \neq i+1, i-1, \\ \beta_k(N^+) + 1 & \textit{for } k = i+1, i-1. \end{cases}$$

(4) *If* $n = 2i + 1$ *and* $\ell = i + 1$ *then:*

(a) *if* v *is* $(\ell - 1)$*-c then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \textit{for all } k \neq i, \\ \beta_k(N^+) + 2 & \textit{for } k = i; \end{cases}$$

or else

(b) *if* v *is* ℓ*-d then*

$$\beta_k(N^-) = \begin{cases} \beta_k(N^+) & \textit{for all } k \neq i+1, i-1, \\ \beta_k(N^+) - 1 & \textit{for } k = i+1, i-1. \end{cases}$$

To illustrate the use of ℓ-d and $(\ell - 1)$-c vertices consider the following explosion of a vertex labeled with $\{h_1 = 3, h_2 = 3, h_3 = 2, h_4 = 2\}$ and edges labeled with non-negative integers as in Figure 4. The ambient dimension is $n = 5$.

It will be shown that this vertex admits two different explosions as in Figure 5.

3. *Poincaré–Hopf Inequalities*

In §3.1 the classical Morse Inequalities are recalled. In §3.2 a graph filtration is defined, which corresponds to a flow-induced filtration on M. This makes it possible to transfer information from the flow on M to the Lyapunov graph and *vice versa*. Finally, this

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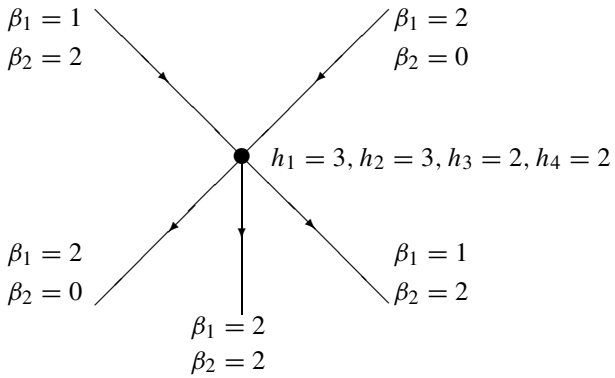


FIGURE 4. Vertex to be exploded.

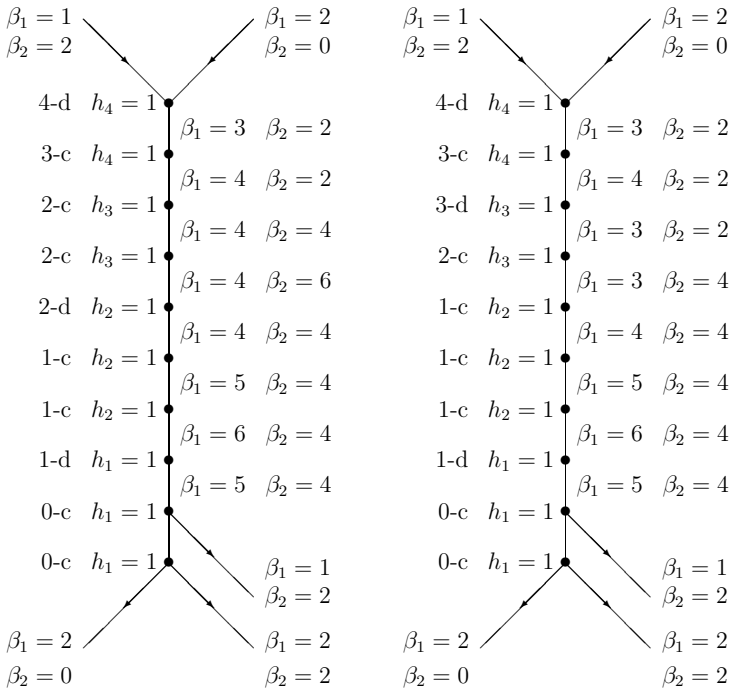


FIGURE 5. Vertex explosions.

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filtration is used to prove the generalized Morse inequalities. In §3.3 the Poincaré–Hopf inequalities will be proven. These inequalities, in contrast to the Morse inequalities, also relate the homology indices of an isolated invariant set to the Betti numbers of the entering and exiting sets of an isolating block. Due to the graph filtration in §3.2 these results can be transferred to the graph.

3.1. *Morse Inequalities.* The relationship between the topology of M and the critical points of a real valued function on M were described by the classical Morse inequalities. In this section, basic results will be stated and further details can be found in [**8**].

Definition 3.1. Let S be a function from certain pairs of spaces to the integers. S is *subadditive* if whenever $X \supset Y \supset Z$ we have $S(X, Z) \leq S(X, Y) + S(Y, Z)$. S is *additive* if equality holds.

Considering $\beta_\lambda(X, Y)$ as the λth Betti number of (X, Y), that is $\beta_\lambda(X, Y) = \text{rank}(H_\lambda(X, Y))$ and $\chi(X, Y) = \sum(-1)^\lambda \beta_\lambda(X, Y)$ as the Euler characteristic, it can be shown by analyzing the long exact sequence of (X, Y, Z) that β_λ is subadditive and χ is additive.

LEMMA 3.1. *If S is subadditive and $X_0 \subset \cdots \subset X_n$, then $S(X_n, X_0) \leq \sum_{i=1}^n S(X_i, X_{i-1})$. If S is additive then equality holds. If $X_0 = \emptyset$, then $S(X_n) \leq \sum_{i=1}^n S(X_i, X_{i-1})$ and equality holds if S is additive.*

Proof. The proof is done by induction on n. □

Given a compact manifold M and a differentiable function on M with non-degenerate isolated critical points, define a filtration $\emptyset = M^{a_0} \subset M^{a_1} \subset \cdots \subset M^{a_k} = M$, where $a_1 < \cdots < a_k$. In this case, the sum of the ranks of $H_*(M_{a_i}, M_{a_{i-1}})$ is $\sum_{i=1}^k \beta_\lambda(M_{a_i}, M_{a_{i-1}}) = C_\lambda$.

Since β_λ is subadditive and χ is additive for the filtration $\emptyset = M^{a_0} \subset M^{a_1} \subset \cdots \subset M^{a_k} = M$, the following results are obtained:

(1) weak Morse inequalities

$$\beta_\lambda(M) \leq C_\lambda;$$

(2) Poincaré–Hopf equality

$$\sum(-1)^\lambda \beta_\lambda(M) = \sum(-1)^\lambda C_\lambda;$$

where C_λ denotes the number of critical points of index λ.

Define the subadditive function $S_\lambda(X, Y) = \beta_\lambda(X, Y) - \beta_{\lambda-1}(X, Y) + - \cdots \pm \beta_0(X, Y)$. The Morse inequalities are obtained by using this function on the filtration $\emptyset = M^{a_0} \subset M^{a_1} \subset \cdots \subset M^{a_k} = M$:

$$S_\lambda(M) \leq \sum_{i=1}^n S_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda - C_{\lambda-1} + - \cdots \pm C_0$$

or in other words,

$$\beta_\lambda(M) - \beta_{\lambda-1}(M) + - \cdots \pm \beta_0(M) \leq C_\lambda - C_{\lambda-1} + - \cdots \pm C_0.$$

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3.2. *Graph filtration.* In this section we describe an ordering procedure similar to topological ordering of a finite connected oriented graph with no oriented cycles.

Given a smooth flow on M with Lyapunov graph L, a graph filtration on L will be defined. This filtration induces a filtration on M.

Definition 3.2. Let $v_1^1, v_1^2, \ldots, v_1^{k_1}$ be vertices of L of outdegree 0. Consider the incoming edges incident to these vertices. Denote by $N_1^1, N_1^2, \ldots, N_1^{k_1}$ the isolating blocks associated to these vertices and edges. Define $M_1 = \bigcup_{i=1}^{k_1} N_1^i$. Connect all vertices v_1^j, $j = 1, \ldots, k_1$ with the vertices v_2^j, $j = 1, \ldots, k_2$ such that the outgoing edges of these vertices are the incoming edges of the vertices v_1^j, $j = 1, \ldots, k_1$. Consider the outgoing and incoming edges of the vertices v_2^j, $j = 1, \ldots, k_2$. Denote by $N_2^1, N_2^2, \ldots, N_2^{k_2}$ the isolating blocks associated to these vertices and their incident edges. Define $M_2 = M_1 \cup \bigcup_{i=1}^{k_2} N_2^i$. Repeat this process until all vertices of the graph are connected. This defines a filtration on M, $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ which will be referred to as *graph filtration*,

$$M_\ell = M_{\ell-1} \cup \bigcup_{i=1}^{k_\ell} N_\ell^i.$$

The *homotopical Conley index* of the maximal invariant set represented by v_i^j will be denoted by $\mathcal{H}(v_i^j)$, and its homology $H_*(\mathcal{H}(v_i^j))$ is the Conley homology index and is denoted by $CH_*(v_i^j)$. Each vertex v_i^j of the graph is labeled with the ranks of the Conley homology indices, $h_*(v_i^j)$. If $CH_*(v_i^j)$ is trivial, then for simplicity $h_*(v_i^j) = 0$ will be omitted.

Given a smooth flow on M with Lyapunov graph L, a flow line on M corresponds to an oriented path on L. Given that there is no oriented path between two vertices v and w in L then there is no connecting orbit between Λ_v and Λ_w.

It is important to note that the choice of the graph filtration $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ implies that there is no oriented path connecting vertices that are at the same level $\{v_i\}_{j=1}^{k_i}$ and, therefore, the Λ_i^j associated to these vertices have no connecting orbits. Hence, this filtration defines an index pair (M_i, M_{i-1}) for the isolated invariant sets associated to $\bigcup_{j=1}^{k_i} v_i^j$, $\bigcup_{j=1}^{k_i} \Lambda_i^j$. This can easily be seen since (M_i, M_{i-1}) is a pair of compact spaces such that:

(1) $cl(M_i, M_{i-1})$ is an isolating neighborhood for the disjoint union of the isolated invariant sets $\bigcup_{j=1}^{k_i} \Lambda_i^j$;
(2) M_{i-1} is positively invariant in M_i, that is if $x \in M_{i-1}$ and $\phi_{[0,T]}(x) \subset M_i$ then $\phi_{[0,T]}(x) \subset M_{i-1}$;
(3) M_{i-1} is the exiting set for the flow, that is, if $x \in M_i$ and $\phi_{[0,\infty]}(x) \not\subset M_i$ then there is a $T > 0$ such that $\phi_{[0,T]}(x) \subset M_i$ and $\phi_T(x) \subset M_{i-1}$ (this last condition is vacuously satisfied).

Using the graph filtration and the subadditivity of β_λ and the additivity of χ as in the previous section, the following results are obtained:

(1) generalized weak Morse inequalities

$$\beta_\lambda(M) \leq h_\lambda(M);$$

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(2) generalized Poincaré–Hopf equality

$$\chi(M) = \sum_{\lambda=0}^{n} (-1)^{\lambda} h_{\lambda}(M);$$

where $\operatorname{rank}(CH_{\lambda}(M_i, M_{i-1})) = h_{\lambda}(M_i, M_{i-1})$ and $\sum_i h_{\lambda}(M_i, M_{i-1}) = h_{\lambda}(M)$.

Define the subadditive function S_{λ} as in the previous section and using the graph filtration the generalized Morse inequalities are obtained:

$$S_{\lambda}(M) \leq \sum_{i=1}^{\ell} S_{\lambda}(M_i, M_{i-1}) = h_{\lambda}(M) - h_{\lambda-1}(M) + - \cdots \pm h_0(M);$$

that is,

$$\beta_{\lambda}(M) - \beta_{\lambda-1}(M) + - \cdots \pm \beta_0(M) \leq h_{\lambda}(M) - h_{\lambda-1}(M) + - \cdots \pm h_0(M).$$

3.3. *Poincaré–Hopf inequalities.* Given a flow φ_t and an isolated invariant set $\Lambda \subset N^n$, it will be assumed that the inverse flow φ_{-t} has an isolated invariant set Λ' with the property that

$$h_i(\Lambda) = \dim CH_i(\Lambda) = \dim CH_{n-i}(\Lambda') = h_{n-i}(\Lambda').$$

There are many examples of isolated invariant sets that satisfy the above requirement. For instance, if a flow φ_t has a singularity P of Conley index S^k ($h_k = 1$), then the singularity P for the reverse flow φ_{-t} has Conley index S^{n-k} ($h_{n-k} = 1$). The same holds for a periodic orbit γ of a flow φ_t with Conley index $S^k \vee S^{k+1}$ ($h_k = h_{k+1} = 1$). The reverse flow φ_{-t} has γ as a periodic orbit of Conley index $S^{n-k-1} \vee S^{n-k}$ ($h_{n-k-1} = h_{n-k} = 1$). Hence the dual of $h_k = 1$ is $h_{n-k} = 1$ and the dual of $h_{k+1} = 1$ is $h_{n-(k+1)} = 1$.

This can also be seen to hold for three-dimensional flows with a suspension of the subshift of finite type σ with $n \times n$ matrix A. The suspension will be denoted by $\tilde{\sigma}$. The homology Conley index of $\tilde{\sigma}$ is

$$CH_1(\tilde{\sigma}) = F^n/(I - A)F^n, \quad CH_2(\tilde{\sigma}) = \ker(I - A) \quad \text{on } F^n.$$

Hence,

$$h_1 = \dim CH_1(\tilde{\sigma}) = \dim F^n/(I - A)F^n = \dim \ker(I - A) = \dim CH_2(\tilde{\sigma}) = h_2.$$

By considering the reverse flow, $\tilde{\sigma}^{-1}$ is now the suspension of the subshift of finite type σ^{-1} with the transpose of A, A^{t} as its matrix. Also, the suspension of the subshift of finite type for the inverse flow has the following homology indices:

$$CH_1(\tilde{\sigma}^{-1}) = F^n/(I - A^{\mathrm{t}})F^n, \quad CH_2(\tilde{\sigma}^{-1}) = \ker(I - A^{\mathrm{t}}) \quad \text{on } F^n.$$

Hence, given that $\overline{h}_i$ represents the dimension of homology Conley index for the inverse flow,

$$\overline{h}_1 = \dim F^n/(I - A^{\mathrm{t}})F^n = \dim \ker(I - A^{\mathrm{t}}) = \overline{h}_2.$$

Since $\ker(I - A^{\mathrm{t}}) \cong \ker(I - A)$, this implies $h_1 = h_2 = \overline{h}_1 = \overline{h}_2$. Therefore, h_1 is dual to $\overline{h}_2$.

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Let (N, N^-) be an index pair for Λ and (N, N^+) an index pair for Λ'.

Consider the following long exact sequences for the pairs (N, N^-) and (N, N^+):

$$\begin{aligned}
0 \to\ & H_n(N^-) \xrightarrow{i_n} H_n(N) \xrightarrow{p_n} H_n(N, N^-) \xrightarrow{\partial_n} H_{n-1}(N^-) \xrightarrow{i_{n-1}} H_{n-1}(N) \xrightarrow{p_{n-1}} \\
& \to H_{n-1}(N, N^-) \xrightarrow{\partial_{n-1}} H_{n-2}(N^-) \xrightarrow{i_{n-2}} H_{n-2}(N) \xrightarrow{p_{n-2}} H_{n-2}(N, N^-) \xrightarrow{\partial_{n-2}} \cdots \\
& \xrightarrow{\partial_4} H_3(N^-) \xrightarrow{i_3} H_3(N) \xrightarrow{p_3} H_3(N, N^-) \xrightarrow{\partial_3} H_2(N^-) \xrightarrow{i_2} H_2(N) \xrightarrow{p_2} H_2(N, N^-) \xrightarrow{\partial_2} \\
& \to H_1(N^-) \xrightarrow{i_1} H_1(N) \xrightarrow{p_1} H_1(N, N^-) \xrightarrow{\partial_1} H_0(N^-) \xrightarrow{i_0} H_0(N) \xrightarrow{p_0} H_0(N, N^-) \to 0
\end{aligned} \tag{1}$$

$$\begin{aligned}
0 \to\ & H_n(N^+) \xrightarrow{i'_n} H_n(N) \xrightarrow{p'_n} H_n(N, N^+) \xrightarrow{\partial'_n} H_{n-1}(N^+) \xrightarrow{i'_{n-1}} H_{n-1}(N) \xrightarrow{p'_{n-1}} \\
& \to H_{n-1}(N, N^+) \xrightarrow{\partial'_{n-1}} H_{n-2}(N^+) \xrightarrow{i'_{n-2}} H_{n-2}(N) \xrightarrow{p'_{n-2}} H_{n-2}(N, N^+) \xrightarrow{\partial'_{n-2}} \cdots \\
& \xrightarrow{\partial'_4} H_3(N^+) \xrightarrow{i'_3} H_3(N) \xrightarrow{p'_3} H_3(N, N^+) \xrightarrow{\partial'_3} H_2(N^+) \xrightarrow{i'_2} H_2(N) \xrightarrow{p'_2} H_2(N, N^+) \xrightarrow{\partial'_2} \\
& \to H_1(N^+) \xrightarrow{i'_1} H_1(N) \xrightarrow{p'_1} H_1(N, N^+) \xrightarrow{\partial'_1} H_0(N^+) \xrightarrow{i'_0} H_0(N) \xrightarrow{p'_0} H_0(N, N^+) \to 0.
\end{aligned} \tag{2}$$

It is an elementary result that given a long exact sequence of vector spaces,

$$\xrightarrow{h} A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \cdots \to D \to 0,$$

$\operatorname{rank} \operatorname{Im} h + \operatorname{rank} \operatorname{Im} i = \operatorname{rank} A$. This follows from the fact that $\operatorname{rank} A = \operatorname{rank} \operatorname{Im} i + \operatorname{rank} \ker i$ and from the exactness of the sequence $\ker i = \operatorname{Im} h$. Hence,

$$\operatorname{rank} h = \operatorname{rank} A - \operatorname{rank} B + \operatorname{rank} C - + \cdots \pm \operatorname{rank} D \geq 0.$$

It will also be assumed that $h_i = 0$ for $i = 0, n$. This last assumption will be removed at the end of the section. Applying these arguments to the long exact sequences of the pairs (N, N^-) and (N, N^+), and considering that $H_n(N) = H_n(N^\pm) = 0$, $\operatorname{rank} H_i(N, N^-) = h_i$, $\operatorname{rank} H_i(N, N^+) = h_{n-i}$, $\operatorname{rank} H_0(N^-) = e^-$, $\operatorname{rank} H_0(N^+) = e^+$, $\operatorname{rank} H_0(N) = 1$ and $\operatorname{rank}(H_i(N^\pm)) = B_i^\pm$. Hence, the following equation is obtained:

$$\begin{aligned}
\operatorname{rank} \partial_n = {} & B^-_{n-1} - \operatorname{rank}(H_{n-1}(N)) + h_{n-1} - B^-_{n-2} + \operatorname{rank}(H_{n-2}(N)) - h_{n-2} \cdots \\
& \pm B_3^- \pm \operatorname{rank}(H_3(N)) \pm h_3 \pm B_2^- \pm \operatorname{rank}(H_2(N)) \pm h_2 \\
& \pm B_1^- \pm \operatorname{rank}(H_1(N)) \pm h_1 \pm e^- \pm 1.
\end{aligned} \tag{3}$$

In the following section, the analysis of the long exact sequences of the pairs will be considered in two separate cases, namely when $\dim N$ is odd or even. In the former case, the classical Poincaré–Hopf equality is obtained.

3.3.1. *Odd-dimensional case, $n = 2i + 1$.* Poincaré duality on the boundary of N implies that $B^-_{(2i-j)} = B^-_j$. Also, since $\operatorname{rank} \partial_{(2i+1)} = 0$, it follows from (3) that

$$\begin{aligned}
2e^- - 2B_1^- + 2B_2^- - + \cdots \pm 2B^-_{i-1} \pm B_i^- + h_{2i} - h_{2i-1} + - \cdots - h_3 + h_2 - h_1 \\
= \sum_{j=0}^{2i} (-1)^j \operatorname{rank}(H_j(N)).
\end{aligned} \tag{4}$$

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Similarly, using the long exact sequence of the pair (N, N^+) and using the duality of the indices, $h_j(\Lambda) = h_{[(2i+1)-j]}(\Lambda')$ the following equation holds:

$$2e^+ - 2B_1^+ + 2B_2^+ - + \cdots \pm 2B_{i-1}^+ \pm B_i^+ + h_1 - h_2 + - \cdots - h_{2i-2} + h_{2i-1} - h_{2i} = \sum_{j=0}^{2i} (-1)^j \operatorname{rank}(H_j(N)). \quad (5)$$

Subtracting (4) from (5) and dividing by two, the following equation holds:

$$(e^+ - e^-) - (B_1^+ - B_1^-) + - \cdots \pm (B_{i-1}^+ - B_{i-1}^-) \pm \frac{(B_i^+ - B_i^-)}{2} - h_{2i} + h_{2i-1} - + \cdots \pm h_i \pm \cdots - h_2 + h_1 = 0 \quad (6)$$

which can be represented in short by

$$\mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \sum_{j=1}^{2i} (-1)^j h_j \quad (7)$$

where

$$\mathcal{B}^+ = \frac{(-1)^i}{2} B_i^+ \pm B_{i-1}^+ \pm \cdots - B_1^+$$
$$\mathcal{B}^- = \frac{(-1)^i}{2} B_i^- \pm B_{i-1}^- \pm \cdots - B_1^-.$$

This is an alternative form of the classical Poincaré–Hopf equality, see [**3**] for more details.

For our purposes, it is sufficient to analyze the long exact sequence of the pairs starting at mid-dimension.

Analyzing the ranks of p_i in (1), the following holds:

$$\begin{aligned} \operatorname{rank} p_i &= h_i - B_{i-1}^- + \operatorname{rank}(H_{i-1}) - h_{i-1} + B_{i-2}^- - \operatorname{rank}(H_{i-2}) + h_{i-2} - + \cdots \\ &\quad \pm B_2^- \pm \operatorname{rank}(H_2) \pm h_2 \pm B_1^- \pm \operatorname{rank}(H_1) \pm h_1 \pm e^- \pm 1 \geq 0 \\ &\Rightarrow h_i \geq B_{i-1}^- - B_{i-2}^- + - \cdots \pm B_2^- \pm B_1^- + h_{i-1} - h_{i-2} + - \cdots \pm h_2 \pm h_1 \\ &\quad \pm e^- \pm 1 - \operatorname{rank}(H_{i-1}) + \operatorname{rank}(H_{i-2}) - + \cdots \pm \operatorname{rank}(H_2) \pm \operatorname{rank}(H_1). \quad (8) \end{aligned}$$

Similarly, by considering $\operatorname{rank} \partial_i'$ in (2) and using the duality of the indices, the following inequality holds:

$$\begin{aligned} &- \operatorname{rank}(H_{i-1}) + \operatorname{rank}(H_{i-2}) - + \cdots \pm \operatorname{rank}(H_2) \pm \operatorname{rank}(H_1) \\ &\quad \geq -B_{i-1}^+ + B_{i-2}^+ - + \cdots \pm B_2^+ \pm B_1^+ - h_{i+2} - + \cdots \pm h_{2i-1} \\ &\quad \pm h_{2i} \pm e^+ \pm 1. \quad (9) \end{aligned}$$

Substituting (9) in (8), the following inequality holds:

$$\begin{aligned} h_i \geq &-(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\ &- (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \cdots \pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \\ &\pm (e^+ - e^-). \quad (10) \end{aligned}$$

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Analogously, analyzing p_i' and ∂_i as above and using the duality of the indices, the following inequality is obtained:

$$\begin{aligned} h_{i+1} \geq -[&-(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\ &- (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \cdots \\ &\pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \pm (e^+ - e^-)]. \end{aligned} \tag{11}$$

Note that in equation (10), h_i is greater than or equal to an integer number and in equation (11), h_{i+1} is greater than or equal to the opposite of that number. Obviously, since h_i and h_{i+1} are non-negative integers, one of the inequalities is redundant. This will continue to occur in the following inequalities.

The remaining inequalities will be obtained by analyzing two pairs of maps at a time,

$$\{[(p_{i-1}, \partial_{i-1}'), (p_{i-1}', \partial_{i-1})], \ldots, [(p_2, \partial_2'), (p_2', \partial_2)]\}.$$

Note that the analysis of these pairs provide the inequalities in both the odd-dimensional case, $n = 2i + 1$, as well as the even-dimensional case, $n = 2i$, and will be written with respect to n.

Hence, the last pair to be analyzed is (p_2, ∂_2') and (p_2', ∂_2). The analysis of the former pair implies

$$h_2 \geq -(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (e^+ - e^-), \tag{12}$$

while the analysis of the latter implies

$$h_{n-2} \geq -[-(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (e^+ - e^-)]. \tag{13}$$

The last inequalities are obtained by analyzing p_1 and p_1':

$$e^- \leq h_1 + 1, \quad e^+ \leq h_{n-1} + 1. \tag{14}$$

The analysis of the long exact sequences of the isolating blocks has provided a collection of inequalities, equality (7) and inequalities (10), (11) and successively up to inequalities (12), (13) and (14) which will be called the Poincaré–Hopf inequalities for the odd-dimensional case.

3.3.2. *Even-dimensional case, $n = 2i$.* In this section, the inequalities are obtained in the same fashion, that is by analyzing the long exact sequences of the pairs $(N, N^{\pm})$ where $\dim N = 2i$.

The inequalities will be obtained by analyzing two pairs of maps at a time,

$$\{[(p_i, \partial_i'), (p_i', \partial_i)], \ldots, [(p_2, \partial_2'), (p_2', \partial_2)]\}.$$

The last inequalities are obtained by analyzing p_1 and p_1'.

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Therefore, the Poincaré–Hopf inequalities for the even-dimensional case are identical to the odd-dimensional case except for the middle dimension:

$$\begin{aligned} h_i \geq & -(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \cdots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\ & - (h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \cdots \\ & \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (e^+ - e^-) \\ h_i \geq & -[-(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \cdots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\ & - (h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \cdots \\ & \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (e^+ - e^-)]. \end{aligned} \tag{15}$$

Moreover, in the case $n = 2 \bmod 4$ it is necessary to require that

$$h_i - \sum_{j=1}^{i-1} (-1)^{j+1} (B^+_j - B^-_j) - \sum_{j=1}^{i-1} (-1)^j (h_{2i-j} - h_j) + (e^- - e^+) \tag{16}$$

be even.

It is interesting to note that a Poincaré–Hopf equality does not exist in the even-dimensional case.

By removing the restriction of $h_n = h_0 = 0$, we obtain the Poincaré–Hopf inequalities in all its generality. Essentially we add or subtract a term $(h_n - h_0)$ to or from the inequalities (10)–(15) and to or from equalities (7) and (16). Hence, the general Poincaré–Hopf inequalities are represented by (17), (18) and (20) below:

$$\mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \sum_{j=0}^{2i+1} (-1)^j h_j \tag{17}$$

where

$$\begin{aligned} \mathcal{B}^+ &= \frac{(-1)^i}{2} B^+_i \pm B^+_{i-1} \pm \cdots - B^+_1 \\ \mathcal{B}^- &= \frac{(-1)^i}{2} B^-_i \pm B^-_{i-1} \pm \cdots - B^-_1 \end{aligned}$$

and

$$h_i - \sum_{j=1}^{i-1} (-1)^{j+1} (B^+_j - B^-_j) - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) + (e^- - e^+) \tag{18}$$

must be even for $2i = 2 \bmod 4$.

The Poincaré–Hopf inequality in the case $n = 2$ is

$$h_1 - h_2 - h_0 + 2 - (e^+ + e^-) \geq 0 \tag{19}$$

and even. If M is non-orientable, the Poincaré–Hopf inequality is the same; however, the expression on the right-hand side of the inequality (19) need not be even.

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This inequality (19) appears in [**5**]:

$$
\begin{cases}
n=2i+1 \begin{cases}
h_i \geq -(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\qquad -(h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \cdots \pm (h_{2i-1} - h_2) \\
\qquad \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^+ - e^-)] \\
h_{i+1} \geq -[-(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\qquad -(h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \cdots \pm (h_{2i-1} - h_2) \\
\qquad \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^+ - e^-)]]
\end{cases} \\
n=2i \begin{cases}
h_i \geq -(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \cdots \\
\qquad \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^+ - e^-)] \\
h_i \geq -[-(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \cdots \\
\qquad \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^+ - e^-)]]
\end{cases} \\
\qquad \vdots \\
\begin{cases}
h_j \geq -(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \cdots \\
\qquad \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \\
h_{n-j} \geq -[-(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + - \cdots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \cdots \\
\qquad \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)]]
\end{cases} \\
\qquad \vdots \\
\begin{cases}
h_2 \geq -(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \\
h_{n-2} \geq -[-(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-)]
\end{cases} \\
\begin{cases}
h_1 \geq h_0 - 1 + e^- \\
h_{n-1} \geq h_n - 1 + e^+.
\end{cases}
\end{cases}
\tag{20}
$$

4. *Explosion algorithms*

In this section a vertex explosion algorithm is presented for a *saddle type vertex* v, $h_0 = h_n = 0$, $n \geq 2$ and h_i, $0 < i < n$ a non-negative integer. Recall that the vertex represents an isolated invariant set of saddle type of a flow on a manifold of dimension n. This algorithm will be used in §4.2 to extend the result to generalized saddle-type vertices, attractor and repeller vertices. The initial two steps of the algorithm do not depend on the parity of n. However, the third step will distinguish three cases: n odd, $n = 0 \bmod 4$, $n = 2 \bmod 4$, producing three different algorithms. Each algorithm will generate a linear system whose solutions are precisely the number of different (ℓ-d, ℓ-c, β-i) vertices of Morse type used in the explosion of the vertex v. At the end of this section we treat the special case $n = 2$.

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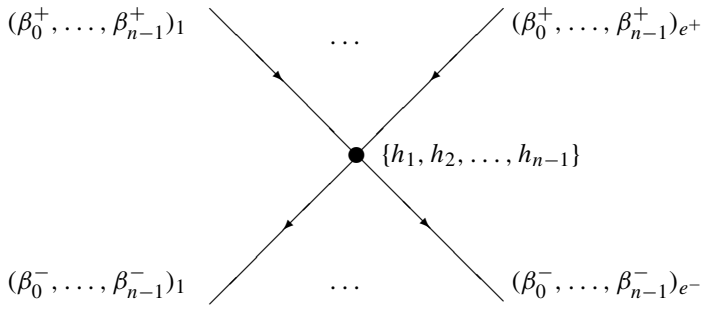


FIGURE 6. Vertex to be exploded.

4.1. *Vertex explosion algorithm.* Let v be a saddle-type vertex labeled with $\{h_1, h_2, \ldots, h_{n-1}\}$ and incoming edges labeled with $\{(\beta_0^+, \ldots, \beta_{n-1}^+)_i\}_{i=1}^{e^+}$ and outgoing edges labeled with $\{(\beta_0^-, \ldots, \beta_{n-1}^-)_i\}_{i=1}^{e^-}$, where i denotes the edge. Let $B_j^+ = \sum_{i=1}^{e^+}(\beta_j^+)_i$ and $B_j^- = \sum_{i=1}^{e^-}(\beta_j^-)_i$ (see Figure 6). Observe that $B_0^- = e^-$ e $B_0^+ = e^+$.

The algorithm will have three basic parts:

(1) adjusting the incident edges;
(2) the linear explosion without middle dimensions; and
(3) middle dimension explosion (n odd, $n = 0 \bmod 4$, $n = 2 \bmod 4$).

The first part of the algorithm consists in adjusting the incident edges of the graph. For this purpose we will define two graphs G^+ and G^-. The graph G^+ has the property that it has e^+ incoming edges and one outgoing edge. It will be formed using only singularities of index $n-1$ of type $(n-1)$-d. The graph G^- has the property that it has e^- outgoing edges and one incoming edge. It will be formed using only singularities of index 1 of type 0-c.

The second part of the algorithm consists of joining the outgoing edge of G^+ to a linear graph L^+ formed by Morse-type vertices labeled with $h_{n-j} = 1$ and $j <$ middle dimensions. Similarly, the incoming edge of G^- will be joined to a linear graph L^- formed by Morse type vertices labeled with $h_j = 1$ and $j <$ middle dimensions. As the singularities are inserted, recall that the variation of the labels on the edges is always considered in the opposite direction of the orientation of the graph.

The third part of the algorithm consists in inserting the middle-dimensional singularities, if there are any, and joining L^+ to L^- if they have not already been joined in the previous part. Of course, all of this must be done so that the weights on the last two edges to be joined coincide.

We will proceed to describe the algorithm.

Algorithm

Step 0—Adjusting the incident edges: G^- and G^+

Let $h_1^c = e^- - 1$. By choosing this number of vertices labeled with 1-singularities,

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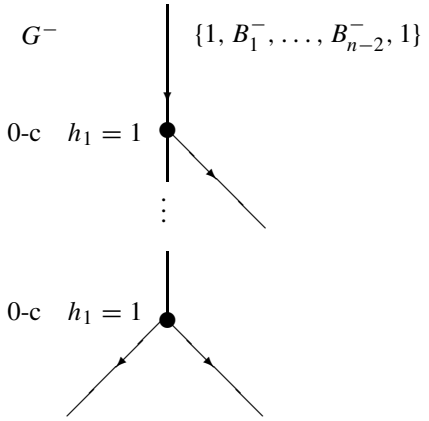


FIGURE 7. Outgoing edges exploded.

G^- is formed with e^- outgoing edges and one incoming edge. Singularities of type 0-c do not alter the β_i with $0 < i < n-1$. This type of singularity decreases β_0 and, by duality, β_{n-1}. Hence, the incoming edge of G^- has $B_0^- = B_{n-1}^- = 1$ and $B_j^- = \sum_{i=1}^{e^-}(\beta_j^-)_i$ with $j = \{1, \ldots, n-2\}$ (see Figure 7).

Similarly, the graph G^+ is formed with $h_{n-1}^d = e^+ - 1$ vertices labeled with $(n-1)$-singularities. This graph will have e^+ incoming edges and one outgoing edge. These singularities do not alter β_i, $0 < i < n-1$. The outgoing edge of G^+ will be labeled with $B_0^+ = B_{n-1}^+ = 1$ and $B_j^+ = \sum_{i=1}^{e^+}(\beta_j^+)_i$ with $j = \{1, \ldots, n-2\}$.

The linear explosion without middle dimensions (Step 1, . . . ,Step ℓ)
Step 1—The adjusting of B_1 and by duality B_{n-2}
The label of the incoming edge of G^- is labeled with

$$\{1, B_1^-, B_2^-, \ldots, B_{\text{mid}}^-, \ldots, B_{n-2}^-, 1\}.$$

In order to adjust B_1^- and by duality B_{n-2}^- add a linear graph L_1^- to G^-. This is done by inserting h_1^d vertices $h_1 = 1$ of type 1-d to G^-. Hence, the label on the incoming edge of the last vertex of this type inserted will be $B_1^- + h_1^d$. Observe that the insertion of vertices $h_2 = 1$ of type 1-c will decrease B_1^-. Thus, after the insertion of h_2^c vertices $h_2 = 1$ of type 1-c, $G^- \cup L_1^-$ is formed. The label on the incoming edge of $G^- \cup L_1^-$ will be $B_1^- + h_1^d - h_2^c$. By duality the label B_{n-2}^- has been modified to $B_{n-2}^- + h_1^d - h_2^c$.

The label of the outgoing edge of G^+ is labeled with

$$\{1, B_1^+, B_2^+, \ldots, B_{\text{mid}}^+, \ldots, B_{n-2}^+, 1\}.$$

In order to adjust B_1^+ and by duality B_{n-2}^+ add a linear graph L_1^+ to G^+. This is done by inserting h_{n-1}^c vertices $h_{n-1} = 1$ of type $(n-2)$-c. After the insertion of these vertices the label on the outgoing edge of the last vertex of this type will be $B_{n-2}^+ + h_{n-1}^c$. Subsequently, add h_{n-2}^d vertices $h_{n-2} = 1$ of type $(n-2)$-d forming $G^+ \cup L_1^+$. After the insertion of

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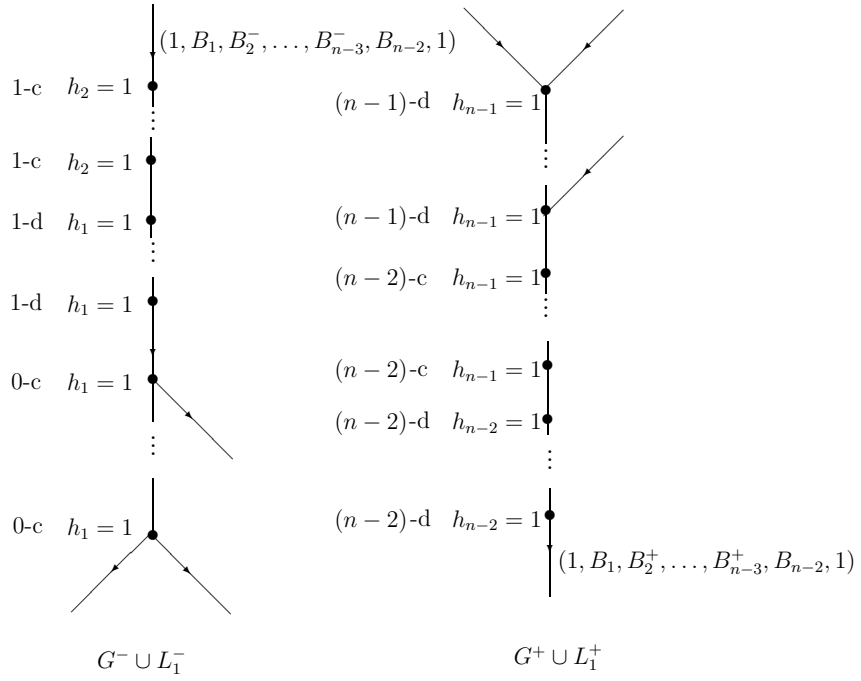


FIGURE 8. Linear explosion: L^- and L^+.

these vertices the label on the outgoing edge of $G^+ \cup L_1^+$ will be $B_{n-2}^+ - h_{n-2}^d + h_{n-1}^c$. By duality the label B_1^+ has been modified to $B_1^+ - h_{n-2}^d + h_{n-1}^c$.

Since the insertion of any other type of vertex will not alter the first and the $(n-2)$th Betti number, it is necessary that

$$B_1 = B_1^- + h_1^d - h_2^c = B_1^+ - h_{n-2}^d + h_{n-1}^c. \tag{21}$$

If the above equality is true then by duality the following equality holds:

$$B_{n-2} = B_{n-2}^- + h_1^d - h_2^c = B_{n-2}^+ + h_{n-2}^d - h_{n-1}^c.$$

Thus the labels of B_1 and B_{n-2} have been adjusted. The incoming edge of $G^- \cup L_1^-$ has the following labels:

$$\{1, B_1, B_2^-, \ldots, B_\ell^-, \ldots, B_{\text{mid}}^-, \ldots, B_{n-\ell-1}^-, \ldots, B_{n-3}^-, B_{n-2}, 1\}$$

and the outgoing edge of $G^+ \cup L_1^+$ has the following labels:

$$\{1, B_1, B_2^+, \ldots, B_\ell^+, \ldots, B_{\text{mid}}^+, \ldots, B_{n-\ell-1}^+, \ldots, B_{n-3}^+, B_{n-2}, 1\},$$

see Figure 8.

This process of adjusting B_ℓ^+ and $B_{n-\ell-1}^+$ will be repeated in increasing order except in the middle dimensions. The general procedure will be presented in the next step.

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Step ℓ—The adjusting of B_ℓ and by duality $B_{n-\ell-1}$

At this point, it is assumed by induction that the adjustments of B_j for $j < \ell$ and by duality B_{n-j-1} for $j < \ell$ have been made in increasing order for j. Hence, several linear graphs have been added to G^+ forming a graph $G^+ \cup \bigcup_{i=1}^{\ell-1} L_i^+$ at this point whose outgoing edge is labeled with

$$\{1, B_1, \ldots, B_{\ell-1}, B_\ell^+, \ldots, B_{\text{mid}}^+, \ldots, B_{n-\ell-1}^+, B_{n-\ell}, \ldots, B_{n-2}, 1\}.$$

Similarly, several linear graphs have been added to G^- forming a linear graph $G^- \cup \bigcup_{i=1}^{\ell-1} L_i^-$ at this point whose incoming edge is labeled with

$$\{1, B_1, \ldots, B_{\ell-1}, B_\ell^-, \ldots, B_{\text{mid}}^-, \ldots, B_{n-\ell-1}^-, B_{n-\ell}, \ldots, B_{n-2}, 1\}.$$

In order to adjust B_ℓ and by duality $B_{n-\ell-1}$, insert h_ℓ^d vertices $h_\ell = 1$ of type ℓ-d to $G^- \cup \bigcup_{i=1}^{\ell-1} L_i^-$. Hence, the label on the incoming edge of the last vertex of this type inserted will be $B_\ell^- + h_\ell^d$. Observe that the insertion of vertices of type ℓ-c will decrease B_ℓ^-. Thus, after the insertion of $h_{\ell+1}^c$ vertices $h_{\ell+1} = 1$ of type ℓ-c the label on the incoming edge of $G^- \cup \bigcup_{i=1}^{\ell} L_i^-$ will be $B_\ell^- + h_\ell^d - h_{\ell+1}^c$. By duality the label $B_{n-\ell-1}^-$ has been modified to $B_{n-\ell-1}^- + h_\ell^d - h_{\ell+1}^c$.

Similarly, the insertion of $h_{n-\ell}^c$ vertices $h_{n-\ell} = 1$ of type $(n-\ell-1)$-c will produce an outgoing edge labeled with $B_{n-\ell-1}^+ + h_{n-\ell}^c$. Subsequently, the insertion of $h_{n-\ell-1}^d$ vertices $h_{n-\ell-1} = 1$ of type $(n-\ell-1)$-d will form $G^+ \cup \bigcup_{i=1}^{\ell} L_i^+$ and the labels on its outgoing edge is $B_{n-\ell-1}^+ - h_{n-\ell-1}^d + h_{n-\ell}^c$. By duality the label B_ℓ^+ has been modified to $B_\ell^+ - h_{n-\ell-1}^d + h_{n-\ell}^c$.

Since the insertion of any other type of vertex will not alter the ℓth and $(n-\ell-1)$th Betti numbers, it is necessary that

$$B_\ell = B_\ell^- + h_\ell^d - h_{\ell+1}^c = B_\ell^+ - h_{n-\ell-1}^d + h_{n-\ell}^c. \tag{22}$$

If the above equality is true, then by duality the following equality holds:

$$B_{n-\ell-1} = B_{n-\ell-1}^- + h_\ell^d - h_{\ell+1}^c = B_{n-\ell-1}^+ - h_{n-\ell-1}^d + h_{n-\ell}^c.$$

The labels of B_ℓ and $B_{n-\ell-1}$ for $0 < \ell <$ mid have all been adjusted. It remains to adjust the middle-dimensional labels. This is done in the next step.

Middle-dimensional explosion

At this point the adjustments of the labels at the middle dimensions must be made. It is necessary to consider the case when $n-1$ is even and hence there is only one middle-dimensional label $B_{(n-1)/2}$ and the case when $n-1$ is odd and there are two middle-dimensional labels.

Case $n-1$ even. Let $n = 2i+1$. At this point there are two graphs $G^- \cup \bigcup_{j=1}^{i-1} L_j^-$ with incoming edge labeled with $\{1, B_1, B_2, \ldots, B_{i-1}, B_i^-, B_{i+1}, \ldots, B_{2i-1}, 1\}$ and $G^+ \cup \bigcup_{j=1}^{i-1} L_j^+$ with outgoing edge labeled with $\{1, B_1, B_2, \ldots, B_{i-1}, B_i^+, B_{i+1}, \ldots, B_{2i-1}, 1\}$.

To adjust B_i^- insert h_i^d vertices $h_i = 1$ of type i-d to the incoming edge of $G^- \cup \bigcup_{j=1}^{i-1} L_j^-$. Hence, the label on the incoming edge of the last vertex inserted of this type is $B_i^- + 2h_i^d$.

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Thus, after the insertion of h^c_{i+1} vertices $h_{i+1} = 1$ of type i-c to the outgoing edge of $G^+ \cup \bigcup_{j=1}^{i-1} L^+_j$, the label on the outgoing edge of last vertex inserted of this type is $B^+_i + 2h^c_{i+1}$.

Moreover, it is necessary that

$$B_i = B^-_i + 2h^d_i = B^+_i + 2h^c_{i+1}. \tag{23}$$

Since, the labels on the outgoing edge of one of the graphs now coincide entirely with the labels on the incoming edge of the other graph, they can be joined to form a connected graph.

Conclusion. Hence, at the end of this adjustment, (21), (22), (23), the following linear system must be solved for $\{h^c_1, h^d_1, \ldots, h^c_{2i}, h^d_{2i}\}$, in order for the algorithm to work:

$$\begin{cases} e^- - 1 - h^c_1 = 0 \\ \{h_j = h^c_j + h^d_j, \quad j = 1, \ldots, 2i \\ e^+ - 1 - h^d_{2i} = 0 \\ \begin{cases} -(B^+_1 - B^-_1) + h^d_1 - h^c_2 - h^c_{2i} + h^d_{2i-1} = 0 \\ -(B^+_2 - B^-_2) + h^d_2 - h^c_3 - h^c_{2i-1} + h^d_{2i-2} = 0 \\ \vdots \\ \dfrac{-(B^+_i - B^-_i)}{2} + h^d_i - h^c_{i+1} = 0. \end{cases} \end{cases} \tag{24}$$

Case $n-1$ odd. Let $n = 2i$. There are two cases to consider.

Case 1: $n = 0$ mod 4. Observe that in this case the variation of the middle-dimensional Betti number B^-_{i-1} and its dual B^-_i will vary by 0 if the vertex is of β-i type or by 1 otherwise.

Insert h^d_{i-1} vertices $h_{i-1} = 1$ of type $(i-1)$-d to the incoming edge of $G^- \cup \bigcup_{j=1}^{i-2} L^-_j$. Hence, the label on the incoming edge of the last vertex of this type inserted will be $B^-_{i-1} + h^d_{i-1}$. Subsequently, insert h^c_i vertices $h_i = 1$ of type $(i-1)$-c. The label on the incoming edge of $G^- \cup \bigcup_{j=1}^{i-1} L^-_j$ will be $B^-_{i-1} + h^d_{i-1} - h^c_i$. By duality the label B^-_i will be modified to $B^-_i + h^d_{i-1} - h^c_i$.

Similarly, insert h^c_{i+1} vertices $h_{i+1} = 1$ of type i-c to the outgoing edge of $G^+ \cup \bigcup_{j=1}^{i-2} L^+_j$. Hence, the label on the outgoing edge of the last vertex of this type inserted will be $B^+_i + h^c_{i+1}$. Subsequently, insert h^d_i vertices $h_i = 1$ of type i-d forming $G^+ \cup \bigcup_{j=1}^{i-1} L^+_j$ and the label on the outgoing edge of this graph will be $B^+_i - h^d_i + h^c_{i+1}$. By duality the label B^+_{i-1} has been modified to $B^+_{i-1} - h^d_i + h^c_{i+1}$.

Moreover, it is necessary that

$$B^-_{i-1} + h^d_{i-1} - h^c_i = B^+_{i-1} - h^d_i + h^c_{i+1}. \tag{25}$$

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By duality, if the above equation holds then the following is true:

$$B_i^- + h_{i-1}^d - h_i^c = B_i^+ - h_i^d + h_{i+1}^c.$$

Since the labels on the outgoing edge of one of the graphs now coincide entirely with the labels on the incoming edge of the other graph, they can be joined to form a connected graph.

Conclusion. Hence, at the end of this adjustment, (21), (22), (25), we are left with the following linear system that must be solved for $\{h_1^c, h_1^d, \ldots, \beta^i, \ldots, h_{2i-1}^c, h_{2i-1}^d\}$ in order for the algorithm to work:

$$\begin{cases} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d + \beta^i, \quad j = 1, \ldots, 2i-1, \beta^i = 0 \quad \text{if } j \neq i \\ e^+ - 1 - h_{2i-1}^d = 0 \\ \begin{cases} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ -(B_{i-1}^+ - B_{i-1}^-) + h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0. \end{cases} \end{cases} \tag{26}$$

Case 2: $n = 2$ *mod* 4. The adjustments in this case are identical to the previous case except for the fact that there are no vertices of type β-i.

In this case the linear system that must be solved for $\{h_1^c, h_1^d, \ldots, h_{2i-1}^c, h_{2i-1}^d\}$ is

$$\begin{cases} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d, \quad j = 1, \ldots, 2i-1 \\ e^+ - 1 - h_{2i-1}^d = 0 \\ \begin{cases} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ -(B_{i-1}^+ - B_{i-1}^-) + h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0. \end{cases} \end{cases} \tag{27}$$

Under the hypothesis of Theorem 1.1, i.e. that the vertex satisfies the Poincaré–Hopf inequalities (10), (11) and successively up to inequalities (12), (13) and (14) in the odd case, and inequalities (15) in the even case, it will be shown in §5 that the above systems always have non-negative integer solutions. In other words, the vertex can be exploded. Moreover, an upper bound on the number of explosions a given vertex can have is also determined.

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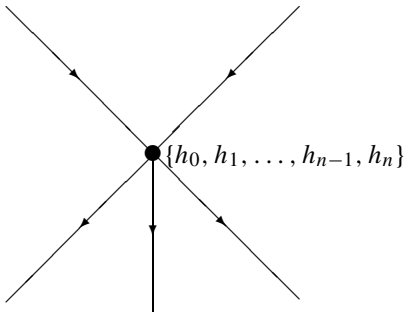


FIGURE 9. Generalized saddle vertex.

4.2. *General case.* In this section the generalized saddle-type vertex, repeller and attractor vertices are considered. For simplicity, the particular case of a saddle-type vertex with $h_0 = h_n = 0$ was considered in the previous section. However, the generalized saddle-type vertex does not impose this restriction, that is $h_0 \neq 0$ and $h_n \neq 0$ are permitted. Also, the attractor ($e^+ = 0$) and repeller vertices ($e^- = 0$) will be considered. The main body of the algorithm in these new cases is identical to the one in §4.1. Additional steps to adjust the incident edges to the graph must be performed before applying the algorithm.

4.2.1. *Case 1—generalized saddle-type vertex.* Given a generalized saddle vertex labeled with $\{h_0, h_1, \ldots, h_n\}$, see Figure 9, a partial explosion will be done in order to obtain a saddle vertex v labeled with $\{h_1, \ldots, h_{n-1}\}$.

This partial explosion consists of a preliminary step to the algorithm in §4. Before defining G^+ and G^-, as in the previous section, the graphs G_0^+ and G_0^- will be defined as follows:

(1) G_0^+ is formed with h_n vertices labeled with $h_n = 1$ and $h_n - 1$ vertices labeled with $h_{n-1} = 1$ of the type $(n-1)$-d and has one outgoing edge which connects to v;
(2) G_0^- is formed with h_0 vertices labeled with $h_0 = 1$ and $h_0 - 1$ vertices labeled with $h_1 = 1$ of the type 0-c and has one incoming edge which comes from v.

Now, the vertex v has $e^+ + 1$ incoming edges and $e^- + 1$ outgoing edges. Moreover, the vertex is now labeled with

$$\{h_n = 0, h_{n-1} - (h_n - 1), h_{n-2}, \ldots, h_3, h_2, h_1 - (h_0 - 1), h_0 = 0\},$$

see Figure 10.

Observe that by substituting the new labels for the partially exploded vertex, i.e. h_{n-1} by $h_{n-1} - (h_n - 1)$, h_1 by $h_1 - (h_0 - 1)$, e^+ by $e^+ + 1$ and e^- by $e^- + 1$, in (7), (10), (11) in the odd-dimensional case or (15), (16) in the even-dimensional case and successively up to inequalities (12), (13) and (14), we obtain the general Poincaré–Hopf inequalities (17), (18) and (20). Hence, the algorithm in §4.1 may be applied in order to complete the explosion.

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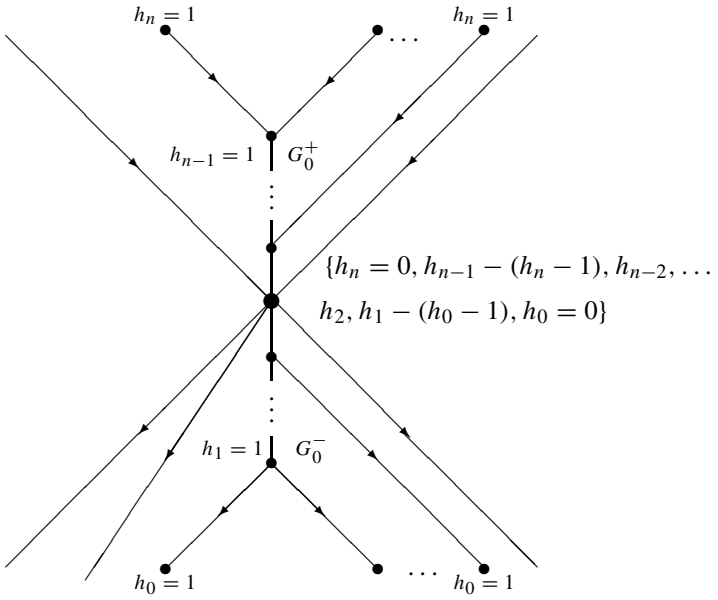


FIGURE 10. Partial explosion of a generalized saddle vertex.

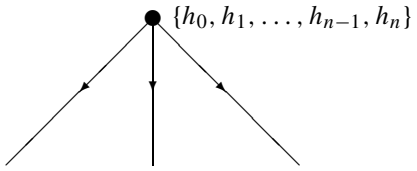


FIGURE 11. Repeller vertex.

4.2.2. *Case 2—attractor and repeller vertices.* Consider a repeller vertex, $e^+ = 0$, labeled with $\{h_0, h_1, \ldots, h_n\}$, see Figure 11, a partial explosion will be done in order to obtain a vertex v labeled with $\{h_1, \ldots, h_{n-1}\}$.

The graphs G_0^+ and G_0^- will be defined as in the previous case.

Now, vertex v has one incoming edge and $e^- + 1$ outgoing edges. Moreover, the vertex is now labeled with

$$\{h_n = 0, h_{n-1} - (h_n - 1), h_{n-2}, \ldots, h_3, h_2, h_1 - (h_0 - 1), h_0 = 0\},$$

see Figure 12.

Observe that by substituting the new labels for the partially exploded vertex, i.e. h_{n-1} by $h_{n-1} - (h_n - 1)$, h_1 by $h_1 - (h_0 - 1)$, e^+ by 1 and e^- by $e^- + 1$, in (7), (10), (11) in the odd-dimensional case or (15), (16) in the even-dimensional case and successively

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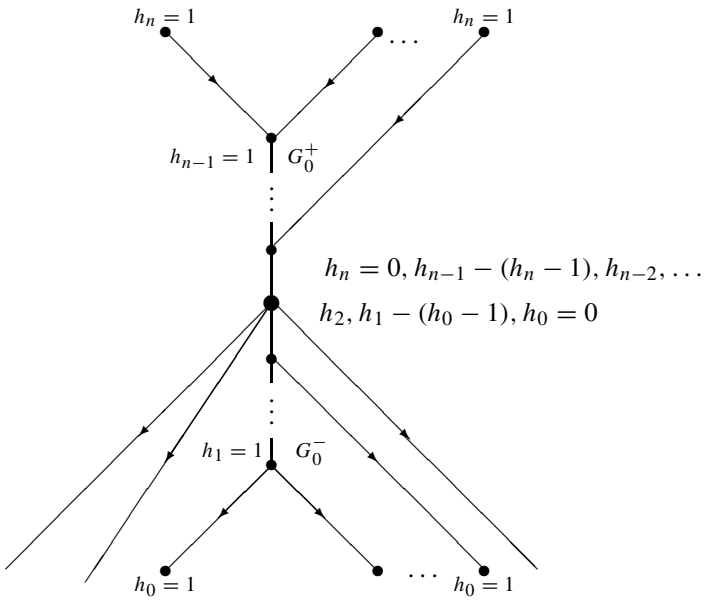


FIGURE 12. Partial explosion of a repeller vertex.

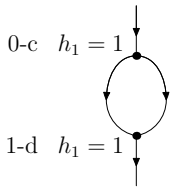


FIGURE 13. $\{h_1^d, h_1^c\}$ pair.

up to inequalities (12), (13) and (14), we obtain the general Poincaré–Hopf inequalities (17), (18) and (20). Hence, the algorithm in §4.1 may be applied in order to complete the explosion.

4.3. *Explosion algorithm for* $n = 2$. Let v be a generalized saddle vertex labeled with $\{h_0, h_1, h_2\}$ and e^+ incoming edges and e^- outgoing edges.

As in the previous cases the first step in the algorithm will realize the partial explosion as in §4.2 in order to obtain a saddle-type vertex labeled with $\{h_0 = 0, h_1 - ((h_2 - 1) + (h_0 - 1)), h_2 = 0\}$. This part of the algorithm uses (19) to guarantee that $h_1 - ((h_2 - 1) + (h_0 - 1)) \geq (e^+ + e^-) \geq 0$. In this partial explosion we have increased the number of incoming edges by one and the number of outgoing edges by one.

The next step will be Step 0 in §4.1, which adjusts the incident edges. In other words, we will use $((e^+ + 1) - 1)$ vertices labeled with $h_1 = 1$ of type 1-d and $((e^- + 1) - 1)$ vertices labeled with $h_1 = 1$ of type 0-c. This step is ensured by the Poincaré–Hopf inequality (19) which guarantees that $h_1 - ((h_2 - 1) + (h_0 - 1) + (e^+ + e^-)) \geq 0$.

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Let $\kappa = h_1 - ((h_2 - 1) + (h_0 - 1) + (e^+ + e^-))$. If $\kappa = 0$, the explosion has been completed. If $\kappa > 0$, the condition which requires κ to be even† permits us to conclude the explosion with pairs of $\{h_1^d, h_1^c\}$, see Figure 13, forming $\kappa/2$ cycles. Obviously this explosion is unique and represents a flow on an orientable block of genus $g_v = \kappa/2$ with e^+ entering boundary components and e^- exiting boundary components.

5. *Solutions to the linear systems*

There are four possibilities for n, namely $n = 0$, 1, 2 or 3 mod 4. The second and fourth possibilities correspond to the case where n is odd. This possibility is considered in §5.1. The first and third possibilities are analyzed in §5.2 and §5.3, respectively.

5.1. *Case n odd*

5.1.1. *The linear system.* Assume $n \geq 3$, $n = 2i + 1$. We want to find a non-negative integral solution of the linear system

$$\begin{matrix} \\ \text{1st set} \\ (2i+2)\text{ rows} \\ \\ \\ \\ \\ \\ \text{2nd set} \\ i\text{ rows} \\ \\ \\ \end{matrix}\!\!\begin{cases} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{cases}\!\!\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 1 & -1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots 1 \; -1 \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \vdots \\ \\ \\ x_{4i-2} \\ x_{4i-1} \\ x_{4i} \end{pmatrix} = \begin{pmatrix} \theta_1 \\ d_2 \\ \theta_3 \\ d_4 \\ \vdots \\ \theta_4 \\ d_3 \\ \theta_2 \\ d_1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_i \end{pmatrix} \tag{28}$$

where θ_j and d_j, for $j = 1, \ldots, i+1$, are positive integers and β_j, for $j = 1, \ldots, i$, are integers. The equations in (28) are partitioned into two sets, containing the first $2i + 2$ equations and the following i ones, respectively. The rows of the coefficient matrix $\overline{A}$ of (28) and its right-hand side vector $\overline{b}$ inherit this partition. The formulas for the jth row of the first group and the corresponding right-hand side element are

$$\overline{A}_j = \begin{cases} \boldsymbol{e}_1, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-3} + \boldsymbol{e}_{2j-2}, & \text{if } 2 \leq j \leq 2i+1 \\ \boldsymbol{e}_{4i}, & \text{if } j = 2i+2 \end{cases}$$

† If it is odd, the explosion may still be completed and will represent a flow on a non-orientable block.

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and

$$\overline{b}_j = \begin{cases} \theta_j, & \text{if } 1 \le j \le i+1,\ j \text{ odd} \\ d_j, & \text{if } 1 \le j \le i+1,\ \ j \text{ even} \\ \theta_{2i+3-j}, & \text{if } i+1 < j \le 2i+2,\ \ j \text{ odd} \\ d_{2i+3-j}, & \text{if } i+1 < j \le 2i+2,\ j \text{ even} \end{cases}$$

where $\boldsymbol{e}_j$ is the jth vector of $\mathbb{R}^{4i}$'s canonical basis. In the second group, the jth row and respective right-hand side are as follows

$$\overline{A}_{2i+2+j} = \begin{cases} \boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-2j} - \boldsymbol{e}_{4i-2j+1}, & \text{if } 1 \le j \le i-1 \\ \boldsymbol{e}_{2i} - \boldsymbol{e}_{2i+1}, & \text{if } j = i \end{cases}$$

and $\overline{b}_{2i+2+j} = \beta_j$, if $1 \le j \le i$.

Four variables in (28) have unique integer values that depend on certain elements of the right-hand side. Hence the non-negativity condition on the solution translates to a set of inequalities these values must satisfy. Eliminating these variables, we are left with a reduced system in the remaining variables—the boxes depicted in (28) contain the coefficient matrix of this reduced system. In §5.1.2 we show that the coefficient matrix of this new system is totally unimodular†, which implies that if the system has a non-negative solution, then it will have an non-negative integral solution. A set of constraints that is equivalent to the existence of non-negative solutions to the reduced system is constructed in §5.1.3.

The following equations may be used to eliminate some variables in (28):

(1) the first equation implies $x_1 = \theta_1$;
(2) the first and second equations imply $x_2 = d_2 - \theta_1$;
(3) the $(2i+2)$th equation implies $x_{4i} = d_1$; and
(4) the $(2i+2)$th and $(2i+1)$th equations imply $x_{4i-1} = \theta_2 - d_1$.

Therefore, using the fact that the vectors θ, d and β are integral, the following are necessary and sufficient conditions for the partial solution $(x_1, x_2, x_{4i}, x_{4i-1})$ to be a non-negative integral 4-tuple:

$$d_2 - \theta_1 \ge 0, \quad \theta_2 - d_1 \ge 0. \tag{29}$$

Having dealt with variables x_1, x_2, x_{4i-1} and x_{4i}, attention will be turned to the aforementioned reduced system. It remains to obtain necessary and sufficient conditions for the existence of non-negative integral solutions thereof. Note, however, that when $n = 3$, or, equivalently, $i = 1$, there is no reduced system left, since the original system contains only the four unknowns x_1, x_2, x_3 and x_4, having, therefore, a unique solution.

5.1.2. *Total unimodularity of the coefficient matrix of the reduced system.* We arrive at the reduced system (30) below after eliminating equations 1, 2, $2i+1$ and $2i+2$ and

† These matrices were considered by Poincaré [**9**], as early as 1900.

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variables x_1, x_2, x_{4i-1} and x_{4i} in (28):

$$\begin{matrix}\text{1st set}\\ (2i-2)\text{ rows}\\ \\ \\ \\ \text{2nd set}\\ i\text{ rows}\\ \\ \\ \end{matrix}\left\{\begin{matrix}\\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right.\!\!\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1 & 1\\ -1 & 0 & 0 & \cdots & 0 & 0 & 1\\ 0 & 1 & -1 & \cdots & 0 & -1 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots 1\ -1 \cdots & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3\\ \\ \vdots\\ \\ x_{4i-2}\end{pmatrix} = \begin{pmatrix}\theta_3\\ d_4\\ \vdots\\ \theta_4\\ d_3\\ \tilde{\beta}_1\\ \beta_2\\ \vdots\\ \beta_i\end{pmatrix}, \tag{30}$$

where $\tilde{\beta}_1 = \beta_1 - x_2 + x_{4i-1} = \beta_1 + \theta_1 + \theta_2 - d_1 - d_2$. This reduced system has $3i-2$ equations in $4i-4$ unknowns.

The rows of $\overline{\overline{A}}$ and $\overline{\overline{b}}$, the coefficient matrix and right-hand side vector of (30), inherit the partition established for their counterparts in (28). Hence the first set has $2i-2$ rows/elements (four equations were eliminated) while the second set maintains its i rows/elements. The formulas for the jth row and right-hand side element must be updated to account for the elimination of rows and columns (equations and variables). In the first group we have

$$\overline{\overline{A}}_j = \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}, \quad \text{if } 1 \le j \le 2i-2$$

and

$$\overline{\overline{b}}_j = \begin{cases} \theta_{j+2}, & \text{if } 1 \le j \le i-1,\ j \text{ odd}\\ d_{j+2}, & \text{if } 1 \le j \le i-1,\ j \text{ even}\\ \theta_{2i+1-j}, & \text{if } i-1 < j \le 2i-2,\ j \text{ odd}\\ d_{2i+1-j}, & \text{if } i-1 < j \le 2i-2,\ j \text{ even}\end{cases}$$

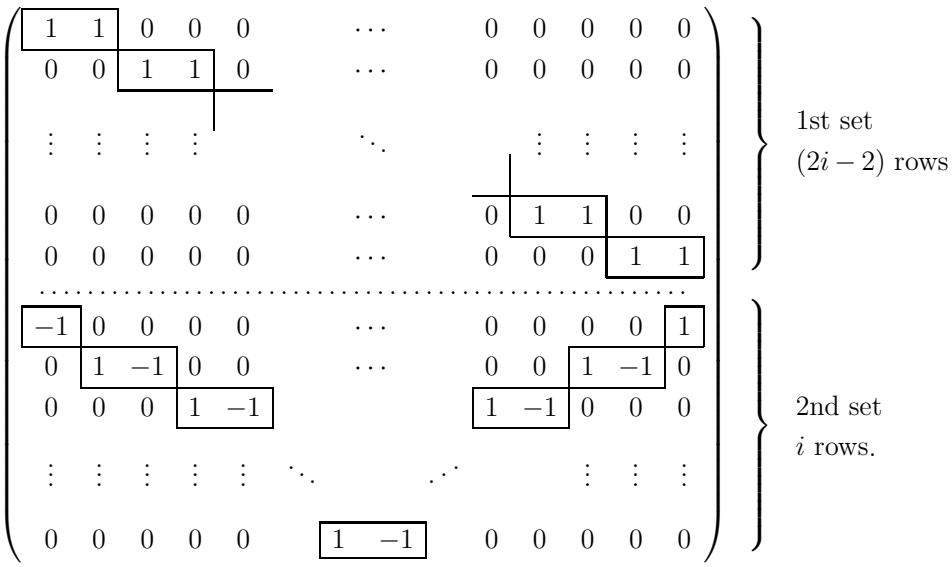
and, in the second set,

$$\overline{\overline{A}}_{2i-2+j} = \begin{cases} -\boldsymbol{e}_1 + \boldsymbol{e}_{4i-4}, & \text{if } j = 1\\ \boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-2-2j} - \boldsymbol{e}_{4i-1-2j}, & \text{if } 2 \le j \le i-1\\ \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}, & \text{if } j = i,\end{cases}$$

whereas $\overline{\overline{b}}_{2i-1} = \tilde{\beta}_1$ and $\overline{\overline{b}}_{2i-2+j} = \beta_j$, for $j = 2, \ldots, i$. Note that $\boldsymbol{e}_\ell$ is now the ℓth vector of $\mathbb{R}^{4i-4}$'s canonical basis. The following block diagram helps understanding $\overline{\overline{A}}$'s

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structure:



The first set of rows form a block diagonal matrix while in the second set there are two blocks per row, arranged symmetrically about the middle.

LEMMA 5.1. *Each column of $\overline{\overline{A}}$ contains exactly two non-null coefficients, one in the first set of rows and the other in the second set.*

Proof. Each entry in $\overline{\overline{A}}$ belongs to $\{0, 1, -1\}$. Thus it suffices to show that each column of $\tilde{A}$, the matrix obtained from $\overline{\overline{A}}$ by replacing each entry with its absolute value, contains two 1's, one in each set of rows.

In order to accomplish this, we calculate the sum of the rows in the first and second sets of $\tilde{A}$.

(1) Summing the rows in the first set; that is, rows 1 through $2i-2$ of $\tilde{A}$:

$$\begin{aligned}\sum_{j=1}^{2i-2} \tilde{A}_j = \sum_{j=1}^{2i-2} \overline{\overline{A}}_j &= \sum_{j=1}^{2i-2} (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) \\ &= \sum_{\substack{\ell \in \{1,\ldots,4i-4\} \\ \ell \text{ odd}}} \boldsymbol{e}_\ell + \sum_{\substack{\ell \in \{1,\ldots,4i-4\} \\ \ell \text{ even}}} \boldsymbol{e}_\ell \\ &= \sum_{\ell=1}^{4i-4} \boldsymbol{e}_\ell = \mathbf{1}_{4i-4},\end{aligned}$$

where $\mathbf{1}_m$ denotes the vector of ones in $\mathbb{R}^m$. Therefore, each column of $\tilde{A}$ contains exactly one non-null entry in the first set of rows, namely 1.

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(2) Summing the i rows in the second set:

$$\begin{aligned}\sum_{j=1}^{i}\tilde{A}_{2i-2+j} &= \boldsymbol{e}_1+\boldsymbol{e}_{4i-4}+\sum_{j=2}^{i-1}(\boldsymbol{e}_{2j-2}+\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{4i-2-2j}+\boldsymbol{e}_{4i-1-2j})\\ &\quad+\boldsymbol{e}_{2i-2}+\boldsymbol{e}_{2i-1}\\ &= \boldsymbol{e}_1+\boldsymbol{e}_{4i-4}+\sum_{\substack{\ell\in\{2,\dots,2i-3\}\\ \ell\text{ even}}}\boldsymbol{e}_\ell+\sum_{\substack{\ell\in\{2,\dots,2i-3\}\\ \ell\text{ odd}}}\boldsymbol{e}_\ell\\ &\quad+\sum_{\substack{\ell\in\{1,\dots,2i-4\}\\ \ell\text{ even}}}\boldsymbol{e}_{4i-4-\ell}+\sum_{\substack{\ell\in\{1,\dots,2i-4\}\\ \ell\text{ odd}}}\boldsymbol{e}_{4i-4-\ell}+\boldsymbol{e}_{2i-2}+\boldsymbol{e}_{2i-1}\\ &= \boldsymbol{e}_1+\boldsymbol{e}_{4i-4}+\sum_{\ell=2}^{2i-3}\boldsymbol{e}_\ell+\sum_{\ell=2i}^{4i-5}\boldsymbol{e}_\ell+\boldsymbol{e}_{2i-2}+\boldsymbol{e}_{2i-1}\\ &= \sum_{\ell=1}^{4i-4}\boldsymbol{e}_\ell=\mathbf{1}_{4i-4}.\end{aligned}$$

The above implies that each column of $\tilde{A}$ contains exactly one non-null entry, namely 1, in the second set of rows.

The previous two items conclude the proof. □

Multiplying by -1 the odd equations in (30), this system is transformed into the equivalent system:

$$\begin{array}{r}\text{1st set}\\ (2i-2)\text{ rows}\\ \\ \\ \\ \\ \text{2nd set}\\ i\text{ rows}\\ \\ \\ \\ \\ \end{array}\left\{\begin{array}{c}\left(\begin{array}{cccccccc}-1&-1&0&0&\cdots&0&0&0\\ 0&0&1&1&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots\\ 0&0&0&0&\cdots&0&1&1\end{array}\right)\\ \left(\begin{array}{cccccccc}1&0&0&0&\cdots&0&0&-1\\ 0&1&-1&0&\cdots&1&-1&0\\ \vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots\\ 0&0&0&0\cdots(-1)^i&(-1)^{i+1}\cdots&0&0&0\end{array}\right)\end{array}\right.\left(\begin{array}{c}x_3\\ \\ \vdots\\ \\ x_{4i-2}\end{array}\right)=\left(\begin{array}{c}-\theta_3\\ d_4\\ \vdots\\ -\theta_4\\ d_3\\ -\tilde{\beta}_1\\ \beta_2\\ \vdots\\ (-1)^i\beta_i\end{array}\right). \tag{31}$$

The matrix of coefficients A of (31) and its right-hand side vector b inherit the partition established for their predecessors in systems (28) and (30). The new formulas for the jth row and right-hand side element in the first set are

$$A_j=(-1)^j(\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{2j}),\quad \text{if } 1\le j\le 2i-2$$

and

$$b_j=\begin{cases}-\theta_{j+2}, & \text{if } 1\le j\le i-1,\ j \text{ odd}\\ d_{j+2}, & \text{if } 1\le j\le i-1,\ j \text{ even}\\ -\theta_{2i+1-j}, & \text{if } i-1<j\le 2i-2,\ j \text{ odd}\\ d_{2i+1-j}, & \text{if } i-1<j\le 2i-2,\ j \text{ even}\end{cases}$$

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and in the second set we have

$$A_{2i-2+j} = \begin{cases} -(-\boldsymbol{e}_1 + \boldsymbol{e}_{4i-4}), & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-2-2j} - \boldsymbol{e}_{4i-1-2j}), & \text{if } 2 \le j \le i-1 \\ (-1)^i(\boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}), & \text{if } j = i \end{cases}$$

$$b_{2i-2+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j\beta_j, & \text{if } 2 \le j \le i. \end{cases}$$

THEOREM 5.1. *Each column of A contains precisely two non-null entries:* 1 *and* -1.

Proof. Lemma 5.1 implies that A has two non-null entries of absolute value 1 per column. In order to conclude that this pair is $1, -1$, it suffices to show that summing all the rows produces the zero vector. We calculate separately the sums of the rows in the first and second sets:

$$\sum_{j=1}^{2i-2} A_j = \sum_{j=1}^{2i-2} (-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) \tag{32}$$

and

$$\begin{aligned} \sum_{j=1}^{i} A_{2i-2+j} &= \sum_{j=1}^{i} (-1)^j \overline{\overline{A}}_{2i-2+j} \\ &= -(-\boldsymbol{e}_1 + \boldsymbol{e}_{4i-4}) \\ &\quad + \sum_{j=2}^{i-1} (-1)^j(\boldsymbol{e}_{2(j-1)} - \boldsymbol{e}_{2(j-1)+1} + \boldsymbol{e}_{4i-4-2(j-1)} - \boldsymbol{e}_{4i-4-2(j-1)+1}) \\ &\quad + (-1)^i(\boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1}) \\ &= \sum_{\ell=1}^{i-2} (-1)^{\ell+1}(\boldsymbol{e}_{2\ell} + \boldsymbol{e}_{4i-4-2\ell}) + (-1)^i \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{4i-4} \\ &\quad + \boldsymbol{e}_1 + (-1)^{i+1}\boldsymbol{e}_{2i-1} + \sum_{j=2}^{i-1} (-1)^{j+1}(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2(j-1)+1}) \\ &= \sum_{\ell=1}^{i-2} (-1)^{\ell+1}\boldsymbol{e}_{2\ell} + (-1)^i \boldsymbol{e}_{2i-2} + \sum_{k=i}^{2i-3} (-1)^{k+1}\boldsymbol{e}_{2k} - \boldsymbol{e}_{4i-4} \\ &\quad + \boldsymbol{e}_1 + \sum_{j=2}^{i-1} (-1)^{j+1}\boldsymbol{e}_{2j-1} + (-1)^{i+1}\boldsymbol{e}_{2i-1} + \sum_{\ell=i+1}^{2i-2} (-1)^{\ell+1}\boldsymbol{e}_{2\ell-1} \\ &= \sum_{j=1}^{2i-2} (-1)^{j+1}\boldsymbol{e}_{2j} + \sum_{j=1}^{2i-2} (-1)^{j+1}\boldsymbol{e}_{2j-1}. \end{aligned} \tag{33}$$

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Equations (32) and (33) imply that

$$\begin{aligned}\sum_{j=1}^{3i-2} A_j &= \sum_{j=1}^{2i-2} A_j + \sum_{j=1}^{i} A_{2i-2+j} \\ &= \sum_{j=1}^{2i-2} (-1)^j (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) + \sum_{j=1}^{2i-2} (-1)^{j+1} (\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j-1}) \\ &= \mathbf{0}_{4i-4},\end{aligned}$$

where $\mathbf{0}_m$ is the null vector of $\mathbb{R}^m$. Hence the sum of rows of A is the zero vector of $\mathbb{R}^{4i-4}$, implying that the two non-zero entries in each column of A are 1 and -1. □

Theorem 5.1 implies (see p. 274 of [**10**]) that A (and thus also $\overline{\overline{A}}$) is a totally unimodular matrix. Furthermore, it has the structure of the incidence matrix of a digraph, which can be used to our advantage in exploring the solution set.

5.1.3. *Solutions of the reduced system.* It follows from Theorem 5.1 that A is totally unimodular and may be interpreted as the node-arc incidence matrix of a digraph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, with node set $\mathcal{N}$ and (directed) arc set $\mathcal{A}$. If the non-null entries 1 and -1 of column j belong to rows k and ℓ, respectively, then the corresponding arc leaves node k and enters node ℓ. The variable x_{j+2} associated with column j is interpreted as the flow in arc j (leaving node k and entering node ℓ), hence the ith equation may be read as

$$\text{flow that enters node } i - \text{flow that leaves node } i = b_i,$$

representing a flow balance constraint. Positive (respectively negative) b_i is interpreted as the demand (respectively $-$supply) of node i.

We seek a non-negative integral flow that satisfies the balance equations. Since A is totally unimodular and b is integral, it is well known—see, for instance, pp. 266–267 of [**10**]—that a non-negative integral flow exists if a non-negative one does.

The structure of $\mathcal{G}$ makes it possible to translate the existence of a non-negative flow into a set of constraints involving the demand/supply vector b. The following necessary condition for the existence of solutions to (31) is a simple consequence of Theorem 5.1, and has a natural statement in terms of the flow model described: supply must equal demand. Since the rows of A add up to the zero vector in $\mathbb{R}^{4i-4}$, the sum of the elements of b must also be zero:

$$\begin{aligned}0 &= -\sum_{j=3}^{i+1} \theta_j + \sum_{j=3}^{i+1} d_j + \sum_{j=1}^{i} b_{2i-2+j} \\ &= -\sum_{j=3}^{i+1} \theta_j + \sum_{j=3}^{i+1} d_j - \tilde{\beta}_1 + \sum_{j=2}^{i} (-1)^j \beta_j \\ &= -\sum_{j=3}^{i+1} \theta_j + \sum_{j=3}^{i+1} d_j - (\beta_1 + \theta_1 + \theta_2 - d_1 - d_2) + \sum_{j=2}^{i} (-1)^j \beta_j \\ &= -\sum_{j=1}^{i+1} \theta_j + \sum_{j=1}^{i+1} d_j + \sum_{j=1}^{i} (-1)^j \beta_j. \qquad (34)\end{aligned}$$

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TABLE 1. Demand/supply constants for nodes in $\mathcal{N}_j$.

	$1 \le j \le i-1$	
	j odd	j even
b_j	$-\theta_{j+2}$	d_{j+2}
b_{2i-1-j}	d_{j+2}	$-\theta_{j+2}$
b_{2i-2+j}	$-\tilde{\beta}_1$, if $j=1$ $-\beta_j$, if $j \ge 3$	β_j
b_{2i-1+j}	β_{j+1}	$-\beta_{j+1}$

THEOREM 5.2. *The set $\mathcal{A}$ may be partitioned into $i-1$ subsets, each corresponding to a non-directed cycle of $\mathcal{G}$. The jth cycle is given by the following sequence of nodes and arcs:*

- *if j is odd,*

$$\{j, (j, 2i-2+j), 2i-2+j, (2i-2+j, 2i-1-j), \\ 2i-1-j, (2i-1+j, 2i-1-j), 2i-1+j, (j, 2i-1+j), j\};$$

- *if j is even,*

$$\{2i-1-j, (2i-1-j, 2i-2+j), 2i-2+j, (2i-2+j, j), \\ j, (2i-1+j, j), 2i-1+j, (2i-1-j, 2i-1+j), 2i-1-j\}.$$

Proof. Since each cycle has four arcs, it suffices to show that the digraph contains the cycles described and that they are arc-disjoint, since then the union of these cycles will have $4(i-1)$ arcs, which is the cardinality of $\mathcal{A}$.

Denoting by $\mathcal{N}_j$ the node set $\{j, 2i-1-j, 2i-2+j, 2i-1+j\}$, the rows in the augmented matrix $A \mid b$ associated with nodes in $\mathcal{N}_j$ are

$$\left(\begin{array}{c|c} A_j & b_j \\ A_{2i-1-j} & b_{2i-1-j} \\ A_{2i-2+j} & b_{2i-2+j} \\ A_{2i-1+j} & b_{2i-1+j} \end{array}\right), \tag{35}$$

where the elements b_j, b_{2i-1-j}, b_{2i-2+j} and b_{2i-1+j} are given by Table 1.

Therefore, the elements of the right-hand side associated with nodes in $\mathcal{N}_j$, independently of the parity of j, are

$$-\theta_{j+2},\ (-1)^j \left\{ \begin{array}{ll} \tilde{\beta}_1, & \text{if } j=1 \\ \beta_j, & \text{if } 1 < j \le i-1 \end{array} \right\}, d_{j+2},\ (-1)^{j+1}\beta_{j+1},$$

and the intersection of the subgraphs $\mathcal{G}_j$ and $\mathcal{G}_\ell$ induced, respectively, by $\mathcal{N}_j$ and $\mathcal{N}_\ell$, for

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Obs.: the number written inside the node is the right-hand-side element of the associated equation.

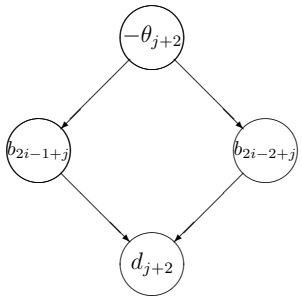


FIGURE 14. Cycle $\mathcal{C}_j$.

$1 \leq j, \ell \leq i-1$, is

$$\mathcal{N}_j \cap \mathcal{N}_\ell = \begin{cases} \{2i-1+j\}, & \text{if } \ell = j+1 \\ \{2i-2+j\}, & \text{if } \ell = j-1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

That is, the intersection contains at most one node, which means the subgraphs induced $\mathcal{G}_j$ and $\mathcal{G}_\ell$ are arc-disjoint. It remains to show that $\mathcal{G}_j$ consists of cycle $\mathcal{C}_j$ of Figure 14.

For $j = 1$, the submatrix in (35) reduces to

$$\left(\begin{array}{c|c} A_1 & -\theta_3 \\ A_{2i-2} & d_3 \\ A_{2i-1} & -\tilde{\beta}_1 \\ A_{2i} & \beta_2 \end{array}\right) = \left(\begin{array}{c|c} -\boldsymbol{e}_1 - \boldsymbol{e}_2 & -\theta_3 \\ \boldsymbol{e}_{4i-5} + \boldsymbol{e}_{4i-4} & d_3 \\ \boldsymbol{e}_1 - \boldsymbol{e}_{4i-4} & -\tilde{\beta}_1 \\ \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_{4i-6} - \boldsymbol{e}_{4i-5} & \beta_2 \end{array}\right). \tag{36}$$

The arcs in $\mathcal{G}_1$ are those that have both end nodes in $\mathcal{N}_1$. Thus the incidence matrix of $\mathcal{G}_1$ contains the columns of (36) that have both their non-null elements in the submatrix (36):

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 4i-5 & 4i-4 \end{array} \\ \left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -\theta_3 \\ 0 & 0 & 1 & 1 & d_3 \\ 1 & 0 & 0 & -1 & -\tilde{\beta}_1 \\ 0 & 1 & -1 & 0 & \beta_2 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2i-2 \\ 2i-1 \\ 2i \end{array}$$

which implies $\mathcal{G}_1$ consists of the non-directed cycle $\mathcal{C}_1$

$$\{1, (1, 2i-1), 2i-1, (2i-1, 2i-2), 2i-2, (2i, 2i-2), 2i, (1, 2i), 1\},$$

in accordance with the theorem's statement.

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If $2 \leq j < i-1$, the submatrix in (35) is, using the fact that $2i$ is even,

$$\left(\begin{array}{c|c} (-1)^j(\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{2j}) & b_j \\ (-1)^{2i-1-j}(\boldsymbol{e}_{4i-3-2j}+\boldsymbol{e}_{4i-2-2j}) & b_{2i-1-j} \\ (-1)^{2i-2+j}(\boldsymbol{e}_{2j-2}-\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{4i-2-2j}-\boldsymbol{e}_{4i-1-2j}) & (-1)^{2i-2+j}\beta_j \\ (-1)^{2i-1+j}(\boldsymbol{e}_{2j}-\boldsymbol{e}_{2j+1}+\boldsymbol{e}_{4i-4-2j}-\boldsymbol{e}_{4i-3-2j}) & (-1)^{2i-1+j}\beta_{j+1} \end{array}\right)$$
$$= (-1)^j \left(\begin{array}{c|c} \boldsymbol{e}_{2j-1}+\boldsymbol{e}_{2j} & (-1)^j b_j \\ -\boldsymbol{e}_{4i-3-2j}-\boldsymbol{e}_{4i-2-2j} & (-1)^j b_{2i-1-j} \\ \boldsymbol{e}_{2j-2}-\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{4i-2-2j}-\boldsymbol{e}_{4i-1-2j} & \beta_j \\ -\boldsymbol{e}_{2j}+\boldsymbol{e}_{2j+1}-\boldsymbol{e}_{4i-4-2j}+\boldsymbol{e}_{4i-3-2j} & -\beta_{j+1} \end{array}\right).$$

This implies that the incidence matrix of $\mathcal{G}_j$ is

$$\begin{array}{c} \\ (-1)^j \end{array} \begin{array}{c} \begin{array}{cccc|c} 2j-1 & 2j & 4i-3-2j & 4i-2-2j & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & (-1)^j b_j \\ 0 & 0 & -1 & -1 & (-1)^j b_{2i-1-j} \\ -1 & 0 & 0 & 1 & \beta_j \\ 0 & -1 & 1 & 0 & -\beta_{j+1} \end{array}\right) \end{array} \begin{array}{l} j \\ 2i-1-j \\ 2i-2+j \\ 2i-1+j. \end{array}$$

From this incidence matrix and the value of b_j and b_{2i-1-j} given in Table 1, we conclude that $\mathcal{G}_j$ is the cycle $\mathcal{C}_j$ depicted in Figure 14.

Finally, if $j = i-1$, the submatrix in (35) has the values

$$\left(\begin{array}{c|c} (-1)^{i-1}(\boldsymbol{e}_{2i-3}+\boldsymbol{e}_{2i-1}) & b_{i-1} \\ (-1)^{i}(\boldsymbol{e}_{2i-1}+\boldsymbol{e}_{2i}) & b_i \\ (-1)^{3i-3}(\boldsymbol{e}_{2i-4}-\boldsymbol{e}_{2i-3}+\boldsymbol{e}_{2i}-\boldsymbol{e}_{2i+1}) & (-1)^{3i-3}\beta_{i-1} \\ (-1)^{3i-2}(\boldsymbol{e}_{2i-2}-\boldsymbol{e}_{2i-1}) & (-1)^{3i-2}\beta_i \end{array}\right)$$
$$= (-1)^{i-1} \left(\begin{array}{c|c} \boldsymbol{e}_{2i-3}+\boldsymbol{e}_{2i-2} & (-1)^{i-1} b_{i-1} \\ -\boldsymbol{e}_{2i-1}-\boldsymbol{e}_{2i} & (-1)^{i-1} b_i \\ \boldsymbol{e}_{2i-4}-\boldsymbol{e}_{2i-3}+\boldsymbol{e}_{2i}-\boldsymbol{e}_{2i+1} & \beta_{i-1} \\ -\boldsymbol{e}_{2i-2}+\boldsymbol{e}_{2i-1} & -\beta_i \end{array}\right).$$

The subgraph $\mathcal{G}_{i-1}$ has, therefore, incidence matrix

$$\begin{array}{c} \\ (-1)^{i-1} \end{array} \begin{array}{c} \begin{array}{cccc|c} 2i-3 & 2i-2 & 2i-1 & 2i & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & (-1)^{i-1} b_{i-1} \\ 0 & 0 & -1 & -1 & (-1)^{i-1} b_i \\ -1 & 0 & 0 & 1 & \beta_{i-1} \\ 0 & -1 & 1 & 0 & -\beta_i \end{array}\right) \end{array} \begin{array}{l} i-1 \\ i \\ 3i-3 \\ 3i-2. \end{array}$$

This matrix and Table 1 imply that $\mathcal{G}_{i-1}$ consists of the $\mathcal{C}_{i-1}$, as desired. □

Theorem 5.2 implies that the digraph $\mathcal{G}$ has the structure depicted in Figure 15. That is, the graph obtained by removing the orientation from $\mathcal{G}$'s arcs may be decomposed in $i-1$ biconnected components, each of them a cycle with four nodes and arcs. Nodes with indices $2i, 2i+1, \ldots, 3i-3$ are cutset nodes, since their removal disconnects the graph. Note that these nodes correspond to rows in the second set of the partition. In fact all rows in the second set are associated with a cutset node, excepting the first and the last.

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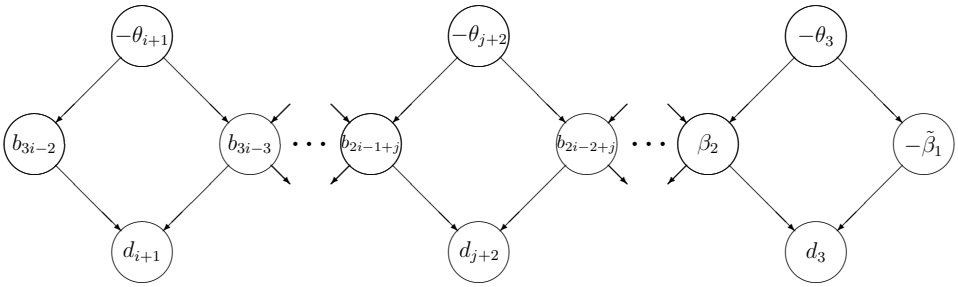


FIGURE 15. Structure of digraph $\mathcal{G}$.

THEOREM 5.3. *If equation (34) holds, the system (31) is equivalent to a set of* $i-1$ *independent linear systems. The* j*th system is given by*

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^j \\ w_2^j \\ w_3^j \\ w_4^j \end{pmatrix} = \begin{pmatrix} -\theta_{j+2} \\ d_{j+2} \\ \hat{\beta}_j \\ -\hat{\beta}_j + \theta_{j+2} - d_{j+2} \end{pmatrix}, \tag{37}$$

where

$$\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \qquad \text{for } j = 1, \ldots, i-1, \tag{38}$$

and

$$w^j = \begin{cases} (x_{2j+1}, x_{2j+2}, x_{4i-1-2j}, x_{4i-2j}), & \text{if } 1 \le j \le i-1 \text{ is odd} \\ (x_{4i-2j}, x_{4i-1-2j}, x_{2j+2}, x_{2j+1}), & \text{if } 1 \le j \le i-1 \text{ is even.} \end{cases} \tag{39}$$

Proof. The decomposition of a flow problem in a graph into an equivalent set of problems in the biconnected components of the graph is a well-known result, as mentioned in [**7**]. This decomposition involves the concept of the cut tree of the graph [**11**, pp. 90–92]. In the present instance, it is perhaps easier to verify this equivalence directly by means of a simple induction proof.

Consider equations 1, $2i-2$ and $2i-1$ of (31):

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_3 \\ -\tilde{\beta}_1 \end{pmatrix}.$$

Adding these equations to equation $2i$ we obtain

$$\begin{aligned} -x_5 + x_{4i-4} &= \beta_2 - \tilde{\beta}_1 - \theta_3 + d_3 \\ &= \beta_2 - (\beta_1 + \theta_1 + \theta_2 - d_1 - d_2) - \theta_3 + d_3 \\ &= \beta_2 - \beta_1 - \sum_{\ell=1}^{3} (\theta_\ell - d_\ell) = \hat{\beta}_2. \end{aligned}$$

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Replacing equation $2i$ of (31) with the above equation we arrive at an equivalent system, since only elementary row operations were performed. In the new system, variables x_3, x_4, x_{4i-3} and x_{4i-2} show up only in equations 1, $2i-2$, $2i-1$ and these equations contain no other variables. This new system may, therefore, be decomposed in the pair of independent linear systems:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_3 \\ -\tilde{\beta}_1 \end{pmatrix} \tag{40}$$

$$\begin{pmatrix} \tilde{A}_2 \\ \vdots \\ \tilde{A}_{2i-3} \\ \vdots \\ \tilde{A}_{2i} \\ \vdots \\ \tilde{A}_{3i-2} \end{pmatrix} \begin{pmatrix} x_5 \\ \vdots \\ x_{4i-4} \end{pmatrix} = \begin{pmatrix} d_4 \\ \vdots \\ -\theta_4 \\ \vdots \\ \hat{\beta}_2 \\ b_{2i} \\ \vdots \\ b_{3i-2} \end{pmatrix}, \tag{41}$$

where row $\tilde{A}_j$ is obtained from A_j by eliminating four columns: the first two and last two. Since A is the incidence matrix of the digraph depicted in Figure 15 and the columns and rows eliminated are associated with the arcs and nodes of biconnected component $\mathcal{C}_1$, with the exception of the cutset node, the matrix

$$\tilde{A} = \begin{pmatrix} \tilde{A}_2 \\ \vdots \\ \tilde{A}_{2i-3} \\ \tilde{A}_{2i} \\ \vdots \\ \tilde{A}_{3i-2} \end{pmatrix}$$

is the incidence matrix of the digraph given by the union of the remaining components: $\mathcal{C}_2, \ldots, \mathcal{C}_{i-1}$.

Furthermore, using the fact that $-\tilde{\beta}_1 = \hat{\beta}_1$, we have that the system (42) below is equivalent to system (40), since it is obtained from the latter by appending a redundant equation thereto:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_3 \\ \hat{\beta}_1 \\ -\hat{\beta}_1 + \theta_3 - d_3 \end{pmatrix}. \tag{42}$$

Note that (42) is the system (37), for $j = 1$, and the vector of unknowns is w^1.

Suppose by induction that, after k analogous steps, a set of $k+1$ systems, equivalent to (31), were obtained, as follows: k systems of type (37), for $j = 1, \ldots, k$, and the system

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$$\begin{pmatrix} \tilde{\tilde{A}}_{k+1} \\ \vdots \\ \tilde{\tilde{A}}_{2i-2-k} \\ \vdots \\ \tilde{\tilde{A}}_{2i-1+k} \\ \vdots \\ \tilde{\tilde{A}}_{3i-2} \end{pmatrix} \begin{pmatrix} x_{3+2k} \\ \vdots \\ x_{4i-2-2k} \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ \vdots \\ b_{2i-2-k} \\ \vdots \\ \hat{\beta}_{k+1} \\ b_{2i+k} \\ \vdots \\ b_{3i-2} \end{pmatrix}, \tag{43}$$

whose coefficient matrix $\tilde{\tilde{A}}$ is the incidence matrix of the digraph given by $\bigcup_{j=k+1}^{i-1} \mathcal{C}_j$, obtained by eliminating from A the columns and rows associated with the arcs and nodes in $\bigcup_{j=1}^{k} \mathcal{C}_j$, with the exception of the cutset node $\mathcal{C}_k \cap \mathcal{C}_{k+1}$.

The equations associated with the nodes in $\mathcal{C}_{k+1}$, with the exception of the cutset node $\mathcal{C}_{k+1} \cap \mathcal{C}_{k+2}$, are

$$(-1)^k \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_{2k+3} \\ x_{2k+4} \\ x_{4i-3-2k} \\ x_{4i-2-2k} \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ b_{2i-2-k} \\ \hat{\beta}_{k+1} \end{pmatrix}. \tag{44}$$

The equation associated with node $\mathcal{C}_{k+1} \cap \mathcal{C}_{k+2}$ is

$$(-1)^k (x_{2k+4} - x_{2k+5} + x_{4i-4-2k} - x_{4i-3-2k}) = b_{2i+k} = (-1)^{k+2} \beta_{k+2}. \tag{45}$$

Adding equations (44) to equation (45) and using Table 1 we have

$$\begin{aligned} (-1)^k (-x_{2k+5} + x_{4i-4-2k}) &= (-1)^{k+2} \beta_{k+2} - \theta_{k+3} + d_{k+3} + \hat{\beta}_{k+1} \\ &= \sum_{j=1}^{k+1} (-1)^j \beta_j - \sum_{j=1}^{k+3} (\theta_j - d_j) \\ &= \hat{\beta}_{k+2}. \end{aligned} \tag{46}$$

Hence (43) is equivalent to the system obtained by replacing equation (45) with equation (46). However, this substitution produces a system that may again be split in two: one of type (37) for $j = k + 1$ (in the vector of unknowns w^{k+1}) and the other whose coefficient matrix is the incidence matrix of the digraph $\bigcup_{j=k+2}^{i-1} \mathcal{C}_j$ and whose right-hand side vector is given by $b_{k+2}, \ldots, b_{2i-3-k}$, $\hat{\beta}_{k+2}$, $b_{2i+1+k}, \ldots, b_{3i-2}$. Therefore, by induction, this implies that (31) is equivalent to a set of $(i - 1)$ systems: $(i - 2)$ systems are of type (37), for $j = 1, \ldots, i - 2$, and the $(i - 1)$th system is

$$(-1)^i \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_{2i-1} \\ x_{2i} \\ x_{2i+1} \\ x_{2i+2} \end{pmatrix} = \begin{pmatrix} b_{i-1} \\ b_i \\ \hat{\beta}_{i-1} \\ (-1)^i \beta_i \end{pmatrix}. \tag{47}$$

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Using Table 1, adding the first three elements of the right-hand side of (47) we obtain

$$b_{i-1} + b_i + \hat{\beta}_{i-1} = -\theta_{i+1} + d_{i+1} + \sum_{j=1}^{i-1}(-1)^j\beta_1 - \sum_{j=1}^{i}(\theta_j - d_j)$$
$$= \sum_{j=1}^{i-1}(-1)^j\beta_1 - \sum_{j=1}^{i+1}(\theta_j - d_j).$$

Since we assume equation (34) holds, i.e.

$$(-1)^i\beta_i = -\left(\sum_{j=1}^{i-1}(-1)^j\beta_1 - \sum_{j=1}^{i+1}(\theta_j - d_j)\right),$$

Table 1 and equation (34) thus imply that (47) is also of the type (37) for $j = i - 1$. □

Theorem 5.3 implies that the solution to the original system (28) is the Cartesian product of 4-tuples of the type $q_j = (x_{2j-1}, x_{2j}, x_{4i+1-2j}, x_{4i+2-2j})$, for $j = 1, \ldots, i$. In §5.1.1 we see that q_1 has a unique value, determined by equations 1, 2, $2i + 2$ and $2i + 1$ of (28). The 4-tuple q_j, for $j = 2, \ldots, i$, is a reordering of a vector w that must satisfy a linear system of the type

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} -t \\ \delta \\ \xi \\ -\xi + t - \delta \end{pmatrix}. \tag{48}$$

LEMMA 5.2. *The general solution to system (48) is given by*

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} \xi \\ t - \xi \\ \delta \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \tag{49}$$

where $\alpha \in \mathbb{R}$.

Proof. Substituting, it can be easily verified that $(\xi, t - \xi, \delta, 0)$ is a solution to (48) and that $(1, -1, -1, 1)$ is a solution to the homogeneous version of (48). Since the null space of the coefficient matrix of (48) has dimension 1 and the difference between two solutions is a vector of this null space, formula (49) encompasses all solutions to (48). □

LEMMA 5.3. *If t, ξ and δ are integers, and t, δ are non-negative, the system (48) has a non-negative integral solution if and only if the following inequalities hold*

$$\delta \geq -\xi$$
$$t \geq \xi. \tag{50}$$

Furthermore, the number of non-negative integral solutions to (48) is equal to the number of integers in the interval

$$[\max\{0, -\xi\}, \min\{t - \xi, \delta\}].$$

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Proof. The matrix of coefficients of (48) is a submatrix of A and thus totally unimodular. Its right-hand side is integral. These facts imply that system (48) has a non-negative integral solution if and only if it has a non-negative solution. Lemma 5.2 implies that (48) has a non-negative solution if and only if the inequalities

$$\begin{aligned} \xi + \alpha &\geq 0 \\ t - \xi - \alpha &\geq 0 \\ \delta - \alpha &\geq 0 \\ \alpha &\geq 0 \end{aligned} \tag{51}$$

are satisfied. Applying the Fourier–Motzkin method (see [**1**, pp. 57–62]) to eliminate variable α in (51), the following inequalities are obtained:

$$\begin{aligned} t - \xi &\geq \alpha \\ \delta &\geq \alpha \\ \alpha &\geq -\xi \\ \alpha &\geq 0 \end{aligned} \quad \Leftrightarrow \quad \min\{t - \xi, \delta\} \geq \alpha \geq \max\{0, -\xi\}.$$

Lemma 5.2 establishes that the general solution to (48) is given by $(\xi, t - \xi, \delta, 0) + \alpha(1, -1, -1, 1)$. Therefore, for each integer value α in the interval $[\max\{0, -\xi\}, \min\{t - \xi, \delta\}]$ there is a non-negative integral solution to (48).

Finally, the interval $[\max\{0, -\xi\}, \min\{t - \xi, \delta\}]$ is non-empty if and only if the inequalities below are satisfied:

$$t - \xi \geq 0 \tag{52}$$

$$\delta \geq 0 \tag{53}$$

$$t - \xi \geq -\xi \tag{54}$$

$$\delta \geq -\xi. \tag{55}$$

Inequalities (53) and (54) are satisfied, since δ and t are assumed non-negative. The remaining two constitute system (50). □

We collect below the necessary and sufficient conditions for the existence of non-negative integral solutions to (28):

$$\begin{aligned} d_2 &\geq \theta_1 \\ \theta_2 &\geq d_1 \\ \sum_{j=1}^{i} (-1)^j \beta_j - \sum_{j=1}^{i+1} (\theta_j - d_j) &= 0 \\ \theta_{j+2} &\geq \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \quad \text{for } j = 1, \ldots, i-1 \\ d_{j+2} &\geq -\left(\sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell) \right), \quad \text{for } j = 1, \ldots, i-1. \end{aligned} \tag{56}$$

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The number of solutions is given by the product $\prod_{j=1}^{i-1} n_j$, where n_j is the number of integers in the interval $[\max\{0, -\hat{\beta}_j\}, \min\{\theta_{j+2} - \hat{\beta}_j, d_{j+2}\}]$ and

$$\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^{\ell} \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell).$$

5.2. *Case* $n = 0 \ mod \ 4$

5.2.1. *The linear system.* Let $4 \leq n = 2i$ be an integer multiple of four (or, equivalently, $2 \leq i$ is even). We want to find a non-negative integral solution of the linear system

$$\begin{array}{r} \\ \text{1st set} \\ (2i+1) \text{ rows} \\ \\ \\ \\ \\ \\ \\ \\ \text{2nd set} \\ i-1 \text{ rows} \\ \\ \end{array} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & \ddots & & & & & & & & & \\ 0 & & \cdots & & & 0 & 1 & 1 & 1 & 0 & & & \cdots & & 0 \\ & & & & & & & & & \ddots & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & & & \cdots & & & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & & & \cdots & & & 1 & -1 & 0 & 0 & 0 \\ & & & & & \ddots & & & & \ddots & & & & & \\ 0 & & \cdots & & & 1 & -1 & 0 & 1 & -1 & & & \cdots & & 0 \end{pmatrix} \right. \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{2i-1} \\ y \\ x_{2i} \\ \vdots \\ x_{4i-4} \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} \theta_1 \\ d_2 \\ \theta_3 \\ d_4 \\ \vdots \\ d_i \\ \Theta \\ \theta_i \\ \vdots \\ \theta_4 \\ d_3 \\ \theta_2 \\ d_1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{i-1} \end{pmatrix}, \tag{57}$$

where Θ, θ_j and d_j, for $j = 1, \ldots, i$, are positive integers and β_j, for $j = 1, \ldots, i-1$, are integers. The system has $3i$ equations and $4i - 1$ unknowns, coefficient matrix $\overline{A}$ and right-hand side vector $\overline{b}$. The equations, rows of $\overline{A}$ and $\overline{b}$ are partitioned into two sets, containing the first $2i + 1$ equations and the following $i - 1$ equations, respectively. The formulas for the jth row of the first group and the corresponding right-hand side element are

$$\overline{A}_j = \begin{cases} \boldsymbol{e}_1, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-3} + \boldsymbol{e}_{2j-2}, & \text{if } 2 \leq j \leq i \\ \boldsymbol{e}_{2i-1} + \boldsymbol{e}_{2i} + \boldsymbol{e}_{2i+1}, & \text{if } j = i+1 \\ \boldsymbol{e}_{2j-2} + \boldsymbol{e}_{2j-1}, & \text{if } i+2 \leq j \leq 2i \\ \boldsymbol{e}_{4i-1}, & \text{if } j = 2i+1 \end{cases}$$

and

$$\overline{b}_j = \begin{cases} \theta_j, & \text{if } 1 \leq j \leq i, \ j \text{ odd} \\ d_j, & \text{if } 1 \leq j \leq i, \ j \text{ even} \\ \Theta, & \text{if } j = i+1 \\ \theta_{2i+2-j}, & \text{if } i+2 \leq j \leq 2i+1, \ j \text{ even} \\ d_{2i+2-j}, & \text{if } i+2 \leq j \leq 2i+1, \ j \text{ odd}, \end{cases}$$

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where $\boldsymbol{e}_j$ is the jth vector of $\mathbb{R}^{4i-1}$'s canonical basis. In the second group, the jth row and respective right-hand side are as follows

$$\overline{A}_{2i+1+j} = \boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-1-2j} - \boldsymbol{e}_{4i-2j}, \quad \text{if } 1 \le j \le i-1$$

and

$$\overline{b}_{2i+1+j} = \beta_j, \quad \text{if } 1 \le j \le i-1.$$

As in the case of n odd, four variables in (57) are uniquely determined. The remaining variables must satisfy a reduced system. The coefficient matrix of the reduced system is not totally unimodular. Nevertheless, it is shown in §5.2.2 that, if we remove the column associated with variable x_{2i}, the remaining matrix is totally unimodular. We explore this fact in §5.2.3 to produce a set of conditions which is equivalent to the existence of a non-negative integral solution to the reduced system.

The conditions which may be used to eliminate the four variables in (57) are:

(1) the first equation implies $x_1 = \theta_1$;
(2) the first and second equations imply $x_2 = d_2 - \theta_1$;
(3) the $(2i+1)$th equation implies $x_{4i-2} = d_1$; and
(4) the $(2i+1)$th and $2i$th equations imply $x_{4i-3} = \theta_2 - d_1$.

Thus the inequalities (29), already obtained in §5.1.1, are necessary and sufficient conditions for the partial solution established in (1)–(4) above to be a non-negative integral vector.

The reduced system obtained with the removal of the 4-tuple $(x_1, x_2, x_{4i-3}, x_{4i-2})$ is examined next.

5.2.2. *The reduced system.* We arrive at the reduced system (58) below after eliminating equations 1, 2, $2i$ and $2i+1$ and variables x_1, x_2, x_{4i-3} and x_{4i-2} in (57):

$$\begin{array}{r} \\ \text{1st set} \\ (2i-3)\text{ rows} \\ \\ \\ \\ \\ \\ \text{2nd set} \\ (i-1)\text{ rows} \\ \\ \end{array}
\left\{\begin{array}{c}
\\
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & & & \\
0 & & \cdots & & & 0 & 1 & 1 & 1 & 0 & & & \cdots & & 0 \\
 & & & & & & & & & & \ddots & & & & \\
0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & & & \cdots & & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & & & \cdots & & & 1 & -1 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & & & \\
0 & & \cdots & & & 1 & -1 & 0 & 1 & -1 & & & \cdots & & 0
\end{pmatrix}
\\
\\
\end{array}\right.
\begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ y \\ x_{2i} \\ \vdots \\ x_{4i-5} \\ x_{4i-4} \end{pmatrix}
=
\begin{pmatrix} \theta_3 \\ d_4 \\ \vdots \\ d_i \\ \Theta \\ \theta_i \\ \vdots \\ \theta_4 \\ d_3 \\ \tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ \beta_{i-1} \end{pmatrix},
\tag{58}$$

where $\tilde{\beta}_1 = \beta_1 - x_2 + x_{4i-3} = \beta_1 + \theta_1 + \theta_2 - d_1 - d_2$. This reduced system has $3i-4$ equations in $4i-5$ unknowns.

The rows of $\overline{\overline{A}}$ and $\overline{\overline{b}}$, the coefficient matrix and right-hand side vector of (58), inherit the partition established for their counterparts in (57). Hence the two sets have $2i-3$ and $i-1$ rows/elements, respectively. The formulas for the jth row and right-hand side

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element must be updated to account for the elimination of rows and columns (equations and variables). In the first group we have

$$\overline{\overline{A}}_j = \begin{cases} \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}, & \text{if } 1 \le j \le i-2 \\ \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1}, & \text{if } j = i-1 \\ \boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1}, & \text{if } i \le j \le 2i-3 \end{cases}$$

and

$$\overline{\overline{b}}_j = \begin{cases} \theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ \Theta, & \text{if } j = i-1 \\ \theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd.} \end{cases}$$

In the second set, the formulas are

$$\overline{\overline{A}}_{2i-3+j} = \begin{cases} -\boldsymbol{e}_1 + \boldsymbol{e}_{4i-5}, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-3-2j} - \boldsymbol{e}_{4i-2-2j}, & \text{if } 2 \le j \le i-1 \end{cases}$$

and

$$\overline{\overline{b}}_{2i-3+j} = \begin{cases} \tilde{\beta}_1, & \text{if } j = 1 \\ \beta_j, & \text{if } 2 \le j \le i-1. \end{cases}$$

Note that $\boldsymbol{e}_j$ is now the jth vector of $\mathbb{R}^{4i-5}$'s canonical basis.

LEMMA 5.4. *Each column of $\overline{\overline{A}}$, except the (central) column $2i-2$, contains exactly two non-null coefficients, one in the first set of rows and the other in the second set. Column $2i-2$ contains only one non-null element, namely* 1, *in row $i-1$.*

Proof. Consider the matrix $\tilde{A}$ with each entry of $\overline{\overline{A}}$ replaced by its absolute value. Summing rows in the first set:

$$\begin{aligned} \sum_{j=1}^{2i-3} \tilde{A}_j &= \sum_{j=1}^{i-2} (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1} + \sum_{j=i}^{2i-3} (\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1}) \\ &= \sum_{\substack{j \in \{1,\ldots,2i-4\} \\ j \text{ odd}}} \boldsymbol{e}_j + \sum_{\substack{j \in \{1,\ldots,2i-4\} \\ j \text{ even}}} \boldsymbol{e}_j + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-2} + \boldsymbol{e}_{2i-1} \\ &\quad + \sum_{\substack{j \in \{2i,\ldots,4i-5\} \\ j \text{ even}}} \boldsymbol{e}_j + \sum_{\substack{j \in \{2i,\ldots,4i-5\} \\ j \text{ odd}}} \boldsymbol{e}_j \\ &= \mathbf{1}_{4i-5}. \end{aligned}$$

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Summing rows in the second set:

$$\begin{aligned}\sum_{j=1}^{i-1}\tilde{A}_{2i-3+j} &= \boldsymbol{e}_1+\boldsymbol{e}_{4i-5}+\sum_{j=2}^{i-1}(\boldsymbol{e}_{2j-2}+\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{4i-3-2j}+\boldsymbol{e}_{4i-2-2j})\\ &= \boldsymbol{e}_1+\sum_{\substack{j\in\{2,\dots,2i-3\}\\ j\ \text{even}}}\boldsymbol{e}_j+\sum_{\substack{j\in\{2,\dots,2i-3\}\\ j\ \text{odd}}}\boldsymbol{e}_j\\ &\quad+\sum_{\substack{j\in\{2i-1,\dots,4i-6\}\\ j\ \text{odd}}}\boldsymbol{e}_j+\sum_{\substack{j\in\{2i-1,\dots,4i-6\}\\ j\ \text{even}}}\boldsymbol{e}_j+\boldsymbol{e}_{4i-5}\\ &= \mathbf{1}_{4i-5}-\boldsymbol{e}_{2i-2}.\end{aligned}$$

The first sum implies that each column contains one non-null element, namely 1 (since rows in the first set have non-negative entries), in the first set of rows. Incidentally, row $i-1$ contributes with the non-null entry in column $2i-2$. The second sum implies each column, with the exception of column $2i-2$, contains one non-null element, with absolute value 1 in the second set of rows. The column $2i-2$ contains no non-null elements in the second set. Thus the lemma is proved. □

Multiply by -1 the jth equation, where we pick odd j, if $1\le j\le i-1$, and even j, if $i\le j\le 3i-4$. The resulting system is

$$\begin{matrix}\\ 2i-3\\ \\ \\ \\ \\ \\ \\ i-1\\ \\ \\ \\ \end{matrix}\!\!\begin{Bmatrix}\\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{Bmatrix}\!\!\begin{pmatrix} -1&-1&0&0&0&\cdots&0&0&0&0&0\\ 0&0&1&1&0&\cdots&0&0&0&0&0\\ &&&&\ddots&&&&&&\\ 0&\cdots&0&-1&-1&-1&0&\cdots&0&&\\ &&&&&&&\ddots&&&\\ 0&0&0&0&0&\cdots&0&-1&-1&0&0\\ 0&0&0&0&0&\cdots&0&0&0&1&1\\ 1&0&0&0&0&\cdots&0&0&0&0&-1\\ 0&1&-1&0&0&\cdots&0&0&1&-1&0\\ 0&0&0&-1&1&\cdots&-1&1&0&0&0\\ &&&&\ddots&&&\ddots&&&\\ 0&\cdots&-1&1&0&-1&1&\cdots&0&&\end{pmatrix}\begin{pmatrix}x_3\\ x_4\\ \vdots\\ x_{2i-1}\\ y\\ x_{2i}\\ \vdots\\ x_{4i-5}\\ x_{4i-4}\end{pmatrix}=\begin{pmatrix}-\theta_3\\ d_4\\ \vdots\\ d_i\\ -\Theta\\ -\theta_i\\ \vdots\\ -\theta_4\\ d_3\\ -\tilde{\beta}_1\\ \beta_2\\ \vdots\\ -\beta_{i-1}\end{pmatrix}. \tag{59}$$

The new system (59) has a coefficient matrix $\hat{A}$ and right-hand side vector $\hat{b}$. As before, they are partitioned into two sets. The jth row and right-hand side element in the first set are given by

$$\hat{A}_j=\begin{cases}(-1)^j(\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{2j}), & \text{if } 1\le j\le i-2\\ -\boldsymbol{e}_{2i-3}-\boldsymbol{e}_{2i-2}-\boldsymbol{e}_{2i-1}, & \text{if } j=i-1\\ (-1)^{j+1}(\boldsymbol{e}_{2j}+\boldsymbol{e}_{2j+1}), & \text{if } i\le j\le 2i-3,\end{cases}$$

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$$\hat{b}_j = \begin{cases} -\theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ -\Theta, & \text{if } j = i-1 \\ -\theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd.} \end{cases}$$

In the second set, the formulas are

$$\hat{A}_{2i-3+j} = \begin{cases} \boldsymbol{e}_1 - \boldsymbol{e}_{4i-5}, & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-3-2j} - \boldsymbol{e}_{4i-2-2j}), & \text{if } 2 \le j \le i-1, \end{cases}$$

$$\hat{b}_{2i-3+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j \beta_j, & \text{if } 2 \le j \le i-1. \end{cases}$$

THEOREM 5.4. *The jth column of $\hat{A}$ contains:*

- *two non-null entries, 1 and -1, if $j \neq 2i-2, 2i-1$;*
- *one non-null entry, -1, if $j = 2i-2$;*
- *two non-null entries, both -1, if $j = 2i-1$.*

Proof. By Lemma 5.4 and the fact that the entries of $\hat{A}$ belong to $\{0, -1, 1\}$, it suffices to show that the sum of the rows of $\hat{A}$ results in $-\boldsymbol{e}_{2i-2} - 2\boldsymbol{e}_{2i-1}$.

Adding the rows in the first set, we have:

$$\begin{aligned} \sum_{j=1}^{2i-3} \hat{A}_j &= \sum_{j=1}^{i-2}(-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}) - \boldsymbol{e}_{2i-3} - \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1} + \sum_{j=i}^{2i-3}(-1)^{j+1}(\boldsymbol{e}_{2j} + \boldsymbol{e}_{2j+1}) \\ &= \sum_{j=1}^{i-1}(-1)^j \boldsymbol{e}_{2j-1} + \sum_{j=1}^{i-2}(-1)^j \boldsymbol{e}_{2j} - \boldsymbol{e}_{2i-2} - \boldsymbol{e}_{2i-1} \\ &\quad + \sum_{j=i}^{2i-3}(-1)^{j+1}\boldsymbol{e}_{2j} + \sum_{j=i+1}^{2i-2}(-1)^j \boldsymbol{e}_{2j-1}. \end{aligned} \tag{60}$$

For the second set, we obtain

$$\begin{aligned} \sum_{j=1}^{i-1} \hat{A}_{2i-3+j} &= \boldsymbol{e}_1 - \boldsymbol{e}_{4i-5} + \sum_{j=2}^{i-1}(-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-3-2j} - \boldsymbol{e}_{4i-2-2j}) \\ &= \boldsymbol{e}_1 - \boldsymbol{e}_{4i-5} + \sum_{j=2}^{i-1}(-1)^j \boldsymbol{e}_{2j-2} + \sum_{j=2}^{i-1}(-1)^{j+1}\boldsymbol{e}_{2j-1} \\ &\quad + \sum_{j=2}^{i-1}(-1)^j \boldsymbol{e}_{4i-3-2j} + \sum_{j=2}^{i-1}(-1)^{j+1}\boldsymbol{e}_{4i-2-2j} \\ &= \sum_{j=1}^{i-2}(-1)^{j+1}\boldsymbol{e}_{2j} + \boldsymbol{e}_1 + \sum_{j=2}^{i-1}(-1)^{j+1}\boldsymbol{e}_{2j-1} \\ &\quad + \sum_{j=i}^{2i-3}(-1)^{j+1}\boldsymbol{e}_{2j-1} - \boldsymbol{e}_{4i-5} + \sum_{j=i}^{2i-3}(-1)^j \boldsymbol{e}_{2j} \end{aligned}$$

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$$= \sum_{j=1}^{i-2} (-1)^{j+1} \boldsymbol{e}_{2j} + \sum_{j=1}^{i-1} (-1)^{j+1} \boldsymbol{e}_{2j-1}$$
$$+ \sum_{j=i}^{2i-2} (-1)^{j+1} \boldsymbol{e}_{2j-1} + \sum_{j=i}^{2i-3} (-1)^{j} \boldsymbol{e}_{2j}. \tag{61}$$

Equations (60) and (61) imply the desired result. □

Theorem 5.4 implies that the matrix obtained from $\hat{A}$ by removing column $2i-1$ is totally unimodular. In order to exploit this fact, we consider the parametric linear system obtained by interpreting x_{2i} (the variable associated to column $2i-1$) as a parameter. Thus, for each fixed value p of this parameter we have a linear system with the right-hand side equal to $\hat{b}$ minus p times column $2i-1$ of column $\hat{A}$. Furthermore, if we add the redundant equation

$$\boldsymbol{e}_{2i-2} x = -\sum_{j=1}^{3i-4} \hat{b}_j - 2p,$$

the coefficient matrix of the parametric system will be the incidence matrix of a digraph.

5.2.3. *Solutions of the parametric system.* The parametric system, corresponding to $x_{2i} = p$, described in the previous section has the following structure:

$$\begin{matrix} \\ 2i-3 \\ \\ \\ \\ \\ \\ \\ i \\ \\ \\ \\ \\ \end{matrix}
\begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & & \cdots & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & \\
0 & & \cdots & & & 0 & -1 & -1 & 0 & & \cdots & & 0 \\
 & & & & & & & & & \ddots & & & \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & & \cdots & & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & & \cdots & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & & \cdots & & -1 & 1 & 0 & 0 & 0 \\
 & & & & & \ddots & & \ddots & & & & & \\
0 & & \cdots & & & -1 & 1 & 0 & 1 & & \cdots & & 0 \\
0 & & \cdots & & & & & 1 & & & \cdots & & 0
\end{pmatrix}
\begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ y \\ x_{2i+1} \\ \vdots \\ x_{4i-5} \\ x_{4i-4} \end{pmatrix}
=
\begin{pmatrix} -\theta_3 \\ d_4 \\ \vdots \\ d_i \\ -\Theta + p \\ -\theta_i \\ \vdots \\ -\theta_4 \\ d_3 \\ -\tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ -\beta_{i-1} + p \\ -\sum_{j=1}^{3i-4} b_j - 2p \end{pmatrix}. \tag{62}$$

The coefficient matrix A of (62) is $(3i-3) \times (4i-6)$. The first set of the partition is the same as before. We have, for the jth row of A and jth element of the right-hand side b of (62) in the first set

$$A_j = \begin{cases} (-1)^j (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } 1 \le j \le i-1 \\ (-1)^{j+1} (\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } i \le j \le 2i-3, \end{cases}$$

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$$b_j = \begin{cases} -\theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ -\Theta + p, & \text{if } j = i-1 \\ -\theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd.} \end{cases}$$

The redundant equation is added to the second set of the partition, thus we have

$$A_{2i-3+j} = \begin{cases} \boldsymbol{e}_1 - \boldsymbol{e}_{4i-6}, & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}), & \text{if } 2 \le j \le i-2 \\ -\boldsymbol{e}_{2i-4} + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-1}, & \text{if } j = i-1 \\ \boldsymbol{e}_{2i-2}, & \text{if } j = i, \end{cases}$$

$$b_{2i-3+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j \beta_j, & \text{if } 2 \le j < i-1 \\ -\beta_{i-1} + p, & \text{if } j = i-1 \\ -\sum_{j=1}^{3i-4} b_j - 2p, & \text{if } j = i. \end{cases}$$

Since the row vectors belong to $\mathbb{R}^{4i-6}$, we take $\boldsymbol{e}_k$ to be the kth vector of the canonical basis of this space.

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be the digraph whose node-arc incidence matrix is A. Equations (62) represent a parametric flow problem. As long as the parameter p is an integer, the problem will have non-negative integral solutions if it has non-negative solutions. The structure of $\mathcal{G}$ is unraveled in the next theorem. It will help to break up (62) into a set of smaller systems, as done in §5.1.3.

THEOREM 5.5. *The set $\mathcal{A}$ may be partitioned into $i-1$ subsets. Of those, $i-2$ correspond to non-directed cycles of length four in $\mathcal{G}$, and the remaining one corresponds to a non-directed path. The jth subset is the set of arcs $\mathcal{A}_j$ in the subgraph $\mathcal{G}_j = (\mathcal{N}_j, \mathcal{A}_j)$ induced by the nodes:*

- *if $j \le i-2$,*
$$\mathcal{N}_j = \{j, 2i-2-j, 2i-3+j, 2i-2+j\}$$
- *if $j = i-1$,*
$$\mathcal{N}_j = \{i-1, 3i-4, 3i-3\}.$$

Proof. It suffices to show that the subgraphs $\mathcal{G}_j$ have the structure described and are arc-disjoint, since the sum of the cardinalities of $\mathcal{A}_j$, for $j = 1, \ldots, i-1$, is $4(i-2)+2 = 4i-6$, the cardinality of $\mathcal{A}$. The right-hand side elements corresponding to nodes in $\mathcal{N}_j$ are given in Table 2.

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TABLE 2. Demand/supply constants for nodes in $\mathcal{N}_j$.

	$1 \le j \le i-2$	
	j odd	j even
b_j	$-\theta_{j+2}$	d_{j+2}
b_{2i-2-j}	d_{j+2}	$-\theta_{j+2}$
b_{2i-3+j}	$-\tilde{\beta}_1$, if $j=1$ $-\beta_j$, if $j \ge 3$	β_j
b_{2i-2+j}	β_{j+1}	$-\beta_{j+1}$, if $1 \le j < i-2$ $-\beta_{i-1}+p$, if $j=i-2$
b_{i-1}	$-\Theta+p$	
b_{3i-4}	$-\beta_{i-1}+p$	
b_{3i-3}	$-\sum_{j=1}^{3i-4} b_j - 2p$	

From the table it is clear that the subgraphs $\mathcal{G}_j$ are arc-disjoint since

$$\mathcal{N}_\ell \cap \mathcal{N}_j = \begin{cases} \{2i-3+j\}, & \text{if } j = \ell+1 \\ \{2i-2+j\}, & \text{if } j = \ell-1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

The submatrix of $A \mid b$ associated with nodes in $\mathcal{N}_j$ is

$$\left(\begin{array}{c|c} A_j & b_j \\ A_{2i-2-j} & b_{2i-2-j} \\ A_{2i-3+j} & b_{2i-3+j} \\ A_{2i-2+j} & b_{2i-2+j} \end{array}\right). \tag{63}$$

For $j=1$ the submatrix (63) becomes

$$\left(\begin{array}{c|c} -\boldsymbol{e}_1 - \boldsymbol{e}_2 & -\theta_3 \\ \boldsymbol{e}_{4i-7} + \boldsymbol{e}_{4i-6} & d_3 \\ \boldsymbol{e}_1 - \boldsymbol{e}_{4i-6} & -\tilde{\beta}_1 \\ \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_{4i-8} - \boldsymbol{e}_{4i-7} & \beta_2 \end{array}\right). \tag{64}$$

Therefore $\mathcal{G}_j$, the subgraph induced by $\mathcal{N}_j$ has the incidence matrix

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 4i-7 & 4i-6 \end{array} \\ \left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -\theta_3 \\ 0 & 0 & 1 & 1 & d_3 \\ 1 & 0 & 0 & -1 & -\tilde{\beta}_1 \\ 0 & 1 & -1 & 0 & \beta_2 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2i-3 \\ 2i-2 \\ 2i-1. \end{array}$$

If we suppose $j=1$ in Figure 16, we obtain subgraph $\mathcal{G}_1$. Notice the similarity to Figure 14. We show that all subgraphs, except the last, have this structure.

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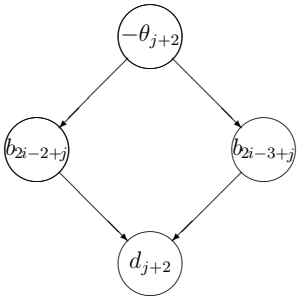
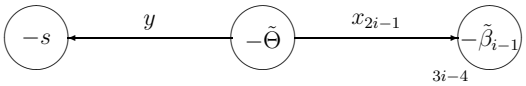


FIGURE 16. Subgraph $\mathcal{G}_j$ consists of non-directed cycle $\mathcal{C}_j$, for $j = 1, \ldots, i-2$.



Where $\tilde{\Theta} = \Theta - p$, $\tilde{\beta}_{i-1} = \beta_{i-1} - p$ and $s = \sum_{j=1}^{3i-4} b_j + 2p$

FIGURE 17. Subgraph $\mathcal{G}_{i-1}$.

The submatrix (63), for $2 \leq j \leq i-2$, is given by

$$\left(\begin{array}{c|c} (-1)^j(\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{2j}) & b_j \\ (-1)^{2i-1-j}(\boldsymbol{e}_{4i-5-2j}+\boldsymbol{e}_{4i-4-2j}) & b_{2i-2-j} \\ (-1)^j(\boldsymbol{e}_{2j-2}-\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{4i-4-2j}-\boldsymbol{e}_{4i-3-2j}) & (-1)^j\beta_j \\ (-1)^{j+1}(\boldsymbol{e}_{2j}-\boldsymbol{e}_{2j+1}+\boldsymbol{e}_{4i-6-2j}-\boldsymbol{e}_{4i-5-2j}) & \left\{\begin{array}{ll}(-1)^{j+1}\beta_{j+1}, & \text{if } j<i-2\\ -\beta_{i-1}+p, & \text{if } j=i-2\end{array}\right\} \end{array}\right)$$

$$= (-1)^j \left(\begin{array}{c|c} \boldsymbol{e}_{2j-1}+\boldsymbol{e}_{2j} & (-1)^j b_j \\ -\boldsymbol{e}_{4i-5-2j}-\boldsymbol{e}_{4i-4-2j} & (-1)^j b_{2i-2-j} \\ \boldsymbol{e}_{2j-2}-\boldsymbol{e}_{2j-1}+\boldsymbol{e}_{4i-4-2j}-\boldsymbol{e}_{4i-3-2j} & \beta_j \\ -\boldsymbol{e}_{2j}+\boldsymbol{e}_{2j+1}-\boldsymbol{e}_{4i-6-2j}+\boldsymbol{e}_{4i-5-2j} & \left\{\begin{array}{ll}-\beta_{j+1}, & \text{if } j<i-2\\ -\beta_{i-1}+p, & \text{if } j=i-2\end{array}\right\} \end{array}\right).$$

Thus the incidence matrix of $\mathcal{G}_j$, for $j = 2, \ldots, i-2$, is

$$(-1)^j \begin{array}{c} \begin{array}{cccc} 2j-1 & 2j & 4i-5-2j & 4i-4-2j \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & (-1)^j b_j \\ 0 & 0 & -1 & -1 & (-1)^j b_{2i-2-j} \\ -1 & 0 & 0 & 1 & \beta_j \\ 0 & -1 & 1 & 0 & \left\{\begin{array}{ll}-\beta_{j+1}, & \text{if } j<i-2\\ -\beta_{i-1}+p, & \text{if } j=i-2\end{array}\right\} \end{array}\right) \end{array} \begin{array}{l} i \\ 2i-2-j \\ 2i-3+j \\ \\ 2i-2+j. \end{array}$$

Using Table 2 and the above matrix we conclude that $\mathcal{G}_j$ has the desired structure.

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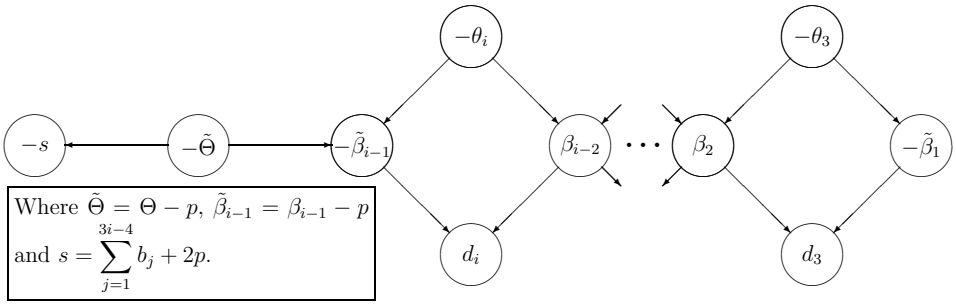


FIGURE 18. Structure of digraph $\mathcal{G}$.

Finally, we consider the submatrix of rows associated with nodes in $\mathcal{N}_{i-1}$:

$$\left(\begin{array}{c|c} -\boldsymbol{e}_{2i-3} - \boldsymbol{e}_{2i-2} & -\Theta + p \\ -\boldsymbol{e}_{2i-4} + \boldsymbol{e}_{2i-3} + \boldsymbol{e}_{2i-1} & -\beta_{i-1} + p \\ \boldsymbol{e}_{2i-2} & -\sum_{j=1}^{3i-4} b_j - 2p \end{array}\right)$$

$$= \begin{array}{c} \begin{array}{cccc} 2i-4 & 2i-3 & 2i-2 & 2i-1 \end{array} \\ \left(\begin{array}{cccc|c} 0 & -1 & -1 & 0 & -\Theta + p \\ -1 & 1 & 0 & 1 & -\beta_{i-1} + p \\ 0 & 0 & 1 & 0 & -\sum_{j=1}^{3i-4} b_j - 2p \end{array}\right) \end{array} \begin{array}{l} i-1 \\ 3i-4 \\ 3i-3. \end{array}$$

Hence the subgraph $\mathcal{G}_{i-1}$, depicted in Figure 17, contains only two arcs, associated with columns $2i-3$ and $2i-2$ of matrix A given in (62). □

The digraph $\mathcal{G}$ is the union of the subgraphs $\mathcal{G}_j$ for $j = 1, \ldots, i-1$. Figure 18 gives an idea of the whole digraph. The nodes associated with the second set of equations $2i-1, \ldots, 3i-4$, can be used to split the linear system (62) into an equivalent set of smaller linear systems, as done in §5.1.3.

THEOREM 5.6. *The linear system (62) is equivalent to a set of $i-1$ systems. The jth system in this set, for $1 \le j \le i-2$, is given by*

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^j \\ w_2^j \\ w_3^j \\ w_4^j \end{pmatrix} = \begin{pmatrix} -\theta_{j+2} \\ d_{j+2} \\ \hat{\beta}_j \\ -\hat{\beta}_j + \theta_{j+2} - d_{j+2} \end{pmatrix}, \tag{65}$$

where

$$\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \quad \textit{for } j = 1, \ldots, i-2, \tag{66}$$

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and

$$w^j = \begin{cases} (x_{2j+1}, x_{2j+2}, x_{4i-3-2j}, x_{4i-2-2j}), & \text{if } 1 \le j \le i-2 \text{ is odd} \\ (x_{4i-2-2j}, x_{4i-3-2j}, x_{2j+2}, x_{2j+1}), & \text{if } 1 \le j \le i-2 \text{ is even.} \end{cases} \tag{67}$$

The $(i-1)$*th system is*

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{2i-1} \\ y \end{pmatrix} = \begin{pmatrix} -\Theta + p \\ \hat{\beta}_{i-1} + p \\ -\sum_{j=1}^{3i-4} b_j - 2p \end{pmatrix}, \tag{68}$$

where

$$\hat{\beta}_{i-1} = -\beta_{i-1} + \hat{\beta}_{i-2} - \theta_i + d_i. \tag{69}$$

Proof. Digraph $\mathcal{G}$ is the union of $\mathcal{G}_1, \ldots, \mathcal{G}_{i-2}, \mathcal{G}_{i-1}$, where the first $i-2$ are cycles like the one in Figure 16 and the last one is depicted in Figure 17. The splitting procedure used in the proof given for Theorem 5.3 may be repeated, with analogous results up to and including the $(i-3)$th splitting.

That is, in the first step cutset node $2i-1$ (with node constant β_2) is used to 'separate' subgraph $\mathcal{G}_1$ from the rest of $\mathcal{G}$. Algebraically speaking, the linear system (62) is split into a pair of systems, one associated with the subgraph $\mathcal{G}_1$ and the other with $\bigcup_{j=2}^{i-1} \mathcal{G}_j$. The system associated with $\mathcal{G}_1$ is given by (65), for $j = 1$. After $i-3$ such splittings, we will have produced a set with $i-3$ linear systems given by (65) with $j = 1, \ldots, i-3$, plus one linear system associated with digraph $\mathcal{G}_{i-2} \cup \mathcal{G}_{i-1}$, where the constant associated with cutset node $\mathcal{G}_{i-3} \cap \mathcal{G}_{i-2}$ is $\hat{\beta}_{i-2}$, satisfying (66). Thus the remaining linear system is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{2i-3} \\ x_{2i-2} \\ x_{2i-1} \\ y \\ x_{2i+1} \\ x_{2i+2} \end{pmatrix} = \begin{pmatrix} d_i \\ -\Theta + p \\ -\theta_i \\ \hat{\beta}_{i-2} \\ -\beta_{i-1} + p \\ -\sum_{j=1}^{3i-4} b_j - 2p \end{pmatrix}. \tag{70}$$

Adding the first, third and fourth equations of (70) to the fifth one, and rearranging rows and columns, we obtain the following equivalent system:

$$\left(\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x_{2i-3} \\ x_{2i-2} \\ x_{2i+1} \\ x_{2i+2} \\ x_{2i-1} \\ y \end{pmatrix} = \begin{pmatrix} d_i \\ -\theta_i \\ \hat{\beta}_{i-2} \\ -\Theta + p \\ -\beta_{i-1} + p + \hat{\beta}_{i-2} - \theta_i + d_i \\ -\sum_{j=1}^{3i-4} b_j - 2p \end{pmatrix}.$$

The system above is clearly decomposable along the lines shown into two equivalent ones: the first can be seen to be equivalent to (65) for $j = i-2$ (notice that the last equation in (65) is redundant) and the second is precisely (68), given the formulas (66) and (69) for $\hat{\beta}_{i-2}$ and $\hat{\beta}_{i-1}$, respectively. □

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It is important to note that only two components in the vector of unknowns of the parametric system (62), namely x_{2i-1} and y, will depend on the parameter p (the value of x_{2i}), since this parameter shows up only in subsystem (68). Hence the solution to (57) has been broken into $(i-1)$ 4-tuples and one 3-tuple. The first 4-tuple, $q_1 = (x_1, x_2, x_{4i-3}, x_{4i-2})$, has the unique value $(\theta_1, d_2 - \theta_1, \theta_2 - d_1, d_1)$. The jth 4-tuple, for $2 \le j \le i-1$, $q_j = (x_{2j-1}, x_{2j}, x_{4i-1-2j}, x_{4i-2j})$, is the appropriate reordering of the solution of the linear system

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^{j-1} \\ w_2^{j-1} \\ w_3^{j-1} \\ w_4^{j-1} \end{pmatrix} = \begin{pmatrix} -\theta_{j+1} \\ d_{j+1} \\ \hat{\beta}_{j-1} \\ -\hat{\beta}_{j-1} + \theta_{j+1} - d_{j+1} \end{pmatrix}.$$

By Lemma 5.2, $w^{j-1} = (\hat{\beta}_{j-1}, \theta_{j+1} - \hat{\beta}_{j-1}, d_{j+1}, 0) + \alpha(1, -1, -1, 1)$, for $j = 2, \ldots, i-1$, and Lemma 5.3 gives necessary and sufficient conditions for such a solution to be a non-negative integral, as well as the number of solutions. The last piece of the puzzle is $q_i = (x_{2i-1}, y, x_{2i})$. System (68) has a solution if and only if

$$-\Theta + p + \hat{\beta}_{i-1} + p - \sum_{j=1}^{3i-4} b_j - 2p = 0.$$

However, as the above condition is by construction always satisfied, we conclude that

$$q_i = \left(\hat{\beta}_{i-1} + p, -\sum_{j=1}^{3i-4} b_j - 2p, p\right).$$

This last vector must also satisfy non-negativity and integrality conditions. Now q_i is integral if and only if p is integral, since all the data of the problem is integral. Non-negativity conditions are equivalent to the three inequalities:

$$p \ge -\hat{\beta}_{i-1}$$
$$\frac{-\sum_{j=1}^{3i-4} b_j}{2} = \frac{\Theta - \hat{\beta}_{i-1}}{2} \ge p$$
$$p \ge 0.$$

Eliminating p with the Fourier–Motzkin method we arrive at

$$\Theta \ge -\hat{\beta}_{i-1}$$
$$\Theta \ge \hat{\beta}_{i-1}.$$

Thus (57) has a non-negative integral solution if and only if the following conditions are satisfied:

$$d_2 \ge \theta_1$$
$$\theta_2 \ge d_1$$
$$\theta_{j+1} \ge \hat{\beta}_{j-1} = \sum_{\ell=1}^{j-1} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j} (\theta_\ell - d_\ell), \qquad \text{for } j = 2, \ldots, i-1$$

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$$d_{j+1} \geq -\hat{\beta}_{j-1} = -\sum_{\ell=1}^{j-1}(-1)^{\ell}\beta_{\ell} + \sum_{\ell=1}^{j}(\theta_{\ell} - d_{\ell}), \quad \text{for } j = 2, \ldots, i-1$$

$$\Theta \geq -\sum_{\ell=1}^{i-1}(-1)^{\ell}\beta_{\ell} + \sum_{\ell=1}^{i}(\theta_{\ell} - d_{\ell})$$

$$\Theta \geq \sum_{\ell=1}^{i-1}(-1)^{\ell}\beta_{\ell} - \sum_{\ell=1}^{i}(\theta_{\ell} - d_{\ell}). \tag{71}$$

The number of solutions is given by the product $\prod_{j=2}^{i} n_j$, where n_j is the number of integers in the interval $[\max\{0, -\hat{\beta}_{j-1}\}, \min\{\theta_{j+1} - \hat{\beta}_{j-1}, d_{j+1}\}]$, for $j = 2, \ldots, i-1$, and n_i is the number of integers in the interval $[\max\{0, -\hat{\beta}_{i-1}\}, (\Theta - \hat{\beta}_{i-1})/2]$.

5.3. *Case $n = 2$ mod 4*

5.3.1. *The linear system.* Now we assume $n = 2i$, where $i \geq 3$ is odd. The problem is to find a non-negative integral solution of the linear system

$$\begin{array}{r} \left.\begin{array}{l}\\\\\\\\\\\\\\\\\end{array}\right\} \text{1st set } (2i+1) \text{ rows} \\ \left.\begin{array}{l}\\\\\end{array}\right\} \text{2nd set } (i-1) \text{ rows} \end{array} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & \ddots & & & & \\ 0 & \cdots & & 1\ 0\ 0\ 0 & \cdots & & 0 \\ 0 & \cdots & & 0\ 1\ 1\ 0 & \cdots & & 0 \\ 0 & \cdots & & 0\ 0\ 0\ 1 & \cdots & & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & -1 & \cdots & 1 & -1 & 0 \\ & & \ddots & & & & \\ 0 & 0 & 0 \cdots & 1\ -1\ 1\ -1 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{2i-1} \\ x_{2i} \\ \vdots \\ x_{4i-4} \\ x_{4i-3} \\ x_{4i-2} \end{pmatrix} = \begin{pmatrix} \theta_1 \\ d_2 \\ \theta_3 \\ d_4 \\ \vdots \\ \theta_i \\ \Theta \\ d_i \\ \vdots \\ \theta_4 \\ d_3 \\ \theta_2 \\ d_1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{i-1} \end{pmatrix}, \tag{72}$$

where Θ, θ_j and d_j, for $j = 1, \ldots, i$, are positive integers and β_j, for $j = 1, \ldots, i-1$, are integers. The coefficient matrix $\overline{A}$ of (72) is $3i \times (4i-2)$, and its rows are partitioned into two sets, the first containing rows 1 through $2i+1$ and, the second, rows $2i+2$ through $3i$. The right-hand side $\overline{b}$ and the equations are partitioned in the same way. The formulas for the jth row of the first group and the corresponding right-hand side element are

$$\overline{A}_j = \begin{cases} \boldsymbol{e}_1, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-3} + \boldsymbol{e}_{2j-2}, & \text{if } 2 \leq j \leq 2i \\ \boldsymbol{e}_{2j-4}, & \text{if } j = 2i+1 \end{cases}$$

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and

$$\overline{b}_j = \begin{cases} \theta_j, & \text{if } 1 \le j \le i,\ j \text{ odd} \\ d_j, & \text{if } 1 \le j \le i,\ j \text{ even} \\ \Theta, & \text{if } j = i+1 \\ \theta_{2i+2-j}, & \text{if } i+2 \le j \le 2i+1,\ j \text{ even} \\ d_{2i+2-j}, & \text{if } i+2 \le j \le 2i+1,\ j \text{ odd,} \end{cases}$$

where $\boldsymbol{e}_j$ is the jth vector of $\mathbb{R}^{4i-2}$'s canonical basis. In the second group, the jth row and respective right-hand side are as follows:

$$\overline{A}_{2i+1+j} = \boldsymbol{e}_{2j} - \boldsymbol{e}_{2j+1} + \boldsymbol{e}_{4i-2-2j} - \boldsymbol{e}_{4i-1-2j}, \quad \text{if } 1 \le j \le i-1$$

and

$$\overline{b}_{2i+1+j} = \beta_j, \quad \text{if } 1 \le j \le i-1.$$

We will follow the approach adopted in the previous cases studied. Our exposition will be succinct, since this is already the third time this approach is used. The first step is to eliminate variables x_1, x_2, x_{4i-2} and x_{4i-3}, in order to reduce the system. The equations used and values obtained are the same as in §5.2.1. They lead once more to the set of inequalities (29), obtained initially in §5.1.1, necessary and sufficient conditions for the partial solution $(x_1, x_2, x_{4i-3}, x_{4i-2}) = (\theta_1, d_2 - \theta_1, \theta_2 - d_1, d_1)$ to be non-negative. The next section deals with the reduced system.

5.3.2. *The reduced system and its solutions.* Eliminating variables x_1, x_2, x_{4i-3} and x_{4i-2} and equations 1, 2, $2i+1$ and $2i$, the reduced system (73) is obtained from (72):

$$\begin{matrix} \\ \text{1st set} \\ (2i-3)\ \text{rows} \\ \\ \\ \\ \\ \\ \\ \text{2nd set} \\ (i-1)\ \text{rows} \\ \\ \\ \\ \end{matrix} \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & & \ddots & & & & & & & \\ 0 & \cdots & & & & 1\ 0\ 0\ 0 & & \cdots & & & 0 \\ 0 & \cdots & & & & 0\ 1\ 1\ 0 & & \cdots & & & 0 \\ 0 & \cdots & & & & 0\ 0\ 0\ 1 & & \cdots & & & 0 \\ & & & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 1 & -1 & 0 & 0 & 0 \\ & & & \ddots & & & & \ddots & & & \\ 0 & \cdots & & & & 1\ -1\ 1\ -1 & & \cdots & & & 0 \end{pmatrix} \right. \begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ x_{2i} \\ \vdots \\ x_{4i-5} \\ x_{4i-4} \end{pmatrix} = \begin{pmatrix} \theta_3 \\ d_4 \\ \vdots \\ \theta_i \\ \Theta \\ d_i \\ \vdots \\ \theta_4 \\ d_3 \\ \tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ \beta_{i-1} \end{pmatrix}, \tag{73}$$

where $\tilde{\beta}_1 = \beta_1 + \theta_1 + \theta_2 - d_1 - d_2$. This reduced system has $3i-4$ equations in $4i-6$ unknowns.

System (73) has a coefficient matrix $\overline{\overline{A}}$ and right-hand side $\overline{\overline{b}}$, which inherit the two-set partition of $\overline{A}$ and $\overline{b}$. The jth row and right-hand side element in the first set is

$$\overline{\overline{A}}_j = \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}, \quad \text{for } j = 1, \ldots, 2i-3$$

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and

$$\overline{\overline{b}}_j = \begin{cases} \theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ \Theta, & \text{if } j = i-1 \\ \theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd.} \end{cases}$$

In the second set, the formulas are

$$\overline{\overline{A}}_{2i-3+j} = \begin{cases} -\boldsymbol{e}_1 + \boldsymbol{e}_{4i-6}, & \text{if } j = 1 \\ \boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}, & \text{if } 2 \le j \le i-1 \end{cases}$$

and

$$\overline{\overline{b}}_{2i-3+j} = \begin{cases} \tilde{\beta}_1, & \text{if } j = 1 \\ \beta_j, & \text{if } 2 \le j \le i-1. \end{cases}$$

Note that $\boldsymbol{e}_j$ is now the jth vector of $\mathbb{R}^{4i-6}$'s canonical basis.

LEMMA 5.5. *Each column of $\overline{\overline{A}}$, contains exactly two non-null coefficients, one in the first set of rows and the other in the second set.*

Proof. The proof is analogous to those of Lemmas 5.1 and 5.4. □

Multiplying by -1 the jth equation of (73), where $1 \le j \le i-2$ is odd and $i-1 \le j \le 3i-4$ is even, we obtain the equivalent system

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & & \ddots & & & & & & & \\ 0 & \cdots & & & & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & & & & 0 & -1 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & & & & 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & \cdots & -1 & 1 & 0 & 0 & 0 \\ & & & \ddots & & & & & \ddots & & \\ 0 & \cdots & & & & 1 & -1 & 1 & -1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_{2i-1} \\ x_{2i} \\ \vdots \\ x_{4i-5} \\ x_{4i-4} \end{pmatrix} = \begin{pmatrix} -\theta_3 \\ d_4 \\ \vdots \\ -\theta_i \\ -\Theta \\ d_i \\ \vdots \\ -\theta_4 \\ d_3 \\ -\tilde{\beta}_1 \\ \beta_2 \\ \vdots \\ \beta_{i-1} \end{pmatrix}. \tag{74}$$

Formulas for the jth row and right-hand side element must be adapted accordingly. Maintaining the partition and letting A and b be the coefficient matrix and right-hand side

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vector of (74), we have, for the first set

$$A_j = \begin{cases} (-1)^j(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } 1 \le j \le i-2 \\ (-1)^{j+1}(\boldsymbol{e}_{2j-1} + \boldsymbol{e}_{2j}), & \text{if } i-1 \le j \le 2i-3, \end{cases}$$

$$b_j = \begin{cases} -\theta_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ odd} \\ d_{j+2}, & \text{if } 1 \le j \le i-2,\ j \text{ even} \\ -\Theta, & \text{if } j = i-1 \\ -\theta_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ even} \\ d_{2i-j}, & \text{if } i \le j \le 2i-3,\ j \text{ odd.} \end{cases}$$

In the second set, the formulas are

$$A_{2i-3+j} = \begin{cases} \boldsymbol{e}_1 - \boldsymbol{e}_{4i-6}, & \text{if } j = 1 \\ (-1)^j(\boldsymbol{e}_{2j-2} - \boldsymbol{e}_{2j-1} + \boldsymbol{e}_{4i-4-2j} - \boldsymbol{e}_{4i-3-2j}), & \text{if } 2 \le j \le i-1, \end{cases}$$

$$b_{2i-3+j} = \begin{cases} -\tilde{\beta}_1, & \text{if } j = 1 \\ (-1)^j \beta_j, & \text{if } 2 \le j \le i-1. \end{cases}$$

THEOREM 5.7. *System (74) is equivalent to a set of $i-1$ smaller systems. The jth system, for $1 \le j \le i-2$, is given by*

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w_1^j \\ w_2^j \\ w_3^j \\ w_4^j \end{pmatrix} = \begin{pmatrix} -\theta_{j+2} \\ d_{j+2} \\ \hat{\beta}_j \\ -\hat{\beta}_j + \theta_{j+2} - d_{j+2} \end{pmatrix}, \tag{75}$$

where

$$\hat{\beta}_j = \sum_{\ell=1}^{j} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{j+1} (\theta_\ell - d_\ell), \quad \text{for } j = 1, \ldots, i-2, \tag{76}$$

and

$$w^j = \begin{cases} (x_{2j+1}, x_{2j+2}, x_{4i-3-2j}, x_{4i-2-2j}), & \text{if } 1 \le j \le i-2 \text{ is odd} \\ (x_{4i-2-2j}, x_{4i-3-2j}, x_{2j+2}, x_{2j+1}), & \text{if } 1 \le j \le i-2 \text{ is even.} \end{cases} \tag{77}$$

System $i-1$, the last one, is given by

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} -\Theta \\ \hat{\beta}_{i-1} \end{pmatrix} \tag{78}$$

where

$$\hat{\beta}_{i-1} = \sum_{\ell=1}^{i-1} (-1)^\ell \beta_\ell - \sum_{\ell=1}^{i} (\theta_\ell - d_\ell).$$

Proof. Applying precisely the same decomposition that was employed in the proofs of Theorems 5.3 and 5.6 to (74), we obtain $(i-2)$ flow problems in non-directed cycles of length four of type (75). The remaining system is (78). □

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System (78) has the unique solution $(x_{2i-1}, x_{2i}) = ((\Theta - \hat{\beta}_{i-1})/2, (\Theta + \hat{\beta}_{i-1})/2)$. This subvector of x is non-negative and integral if and only if the following conditions hold:

$$\begin{aligned} \Theta - \hat{\beta}_{i-1} &\text{ is even} \\ \Theta &\geq \hat{\beta}_{i-1} \\ \Theta &\geq -\hat{\beta}_{i-1}. \end{aligned} \tag{79}$$

The decomposition afforded by Theorem 5.7 immediately leads to a set of conditions for the existence of non-negative integral values for the remaining components of x. The set of all conditions results in system (71) together with the restriction that $\Theta - \hat{\beta}_{i-1}$ be even. The number of solutions is given by the product $\prod_{j=2}^{i-1} n_j$, where n_j is the number of integers in the interval $[\max\{0, -\hat{\beta}_{j-1}\}, \min\{\theta_{j+1} - \hat{\beta}_{j-1}, d_{j+1}\}]$, for $j = 2, \ldots, i-1$.

6. *Conclusion*

In this section the main theorem is proved, by establishing the correspondence between the linear systems obtained from the algorithms in §4 and the algebra in §5.

Proof of Theorem 1.1. It is easy to see that the linear systems (24), (26) and (27) obtained from the algorithm in §4 are represented by the following matrices.

Odd-dimensional case.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 1 & -1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots 1 \; -1 \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_1^c \\ h_1^d \\ h_2^c \\ h_2^d \\ h_3^c \\ h_3^d \\ \vdots \\ h_{2i}^c \\ h_{2i}^d \end{pmatrix} = \begin{pmatrix} e^- - 1 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{2i} \\ e^+ - 1 \\ B_1^+ - B_1^- \\ B_2^+ - B_2^- \\ \vdots \\ \dfrac{B_i^+ - B_i^-}{2} \end{pmatrix}.$$

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Even-dimensional case $n = 0$ *mod* 4.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & & & \\
0 & & \cdots & & & 0 & 1 & 1 & 1 & 0 & & & \cdots & & 0 \\
 & & & & & & & & & & \ddots & & & & \\
0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & & & \cdots & & & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & & & \cdots & & & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & & & \cdots & & & 1 & -1 & 0 & 0 & 0 \\
 & & & & & \ddots & & & & & & & & & \\
0 & & \cdots & & & 1 & -1 & 0 & 1 & -1 & & & \cdots & & 0
\end{pmatrix}
\begin{pmatrix}
h_1^c \\ h_1^d \\ h_2^c \\ h_2^d \\ h_3^c \\ h_3^d \\ \vdots \\ h_i^c \\ \beta^i \\ h_i^d \\ \vdots \\ h_{2i-1}^c \\ h_{2i-1}^d
\end{pmatrix}
=
\begin{pmatrix}
e^- - 1 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_i \\ \vdots \\ h_{2i-1} \\ e^+ - 1 \\ B_1^+ - B_1^- \\ B_2^+ - B_2^- \\ \vdots \\ B_{i-1}^+ - B_{i-1}^-
\end{pmatrix}$$

Even-dimensional case $n = 2$ *mod* 4.

$$\begin{pmatrix}
1 & 0 & 0 & & & & & & & & & 0 & 0 & 0 \\
1 & 1 & 0 & & & & & & & & & 0 & 0 & 0 \\
 & & & \ddots & & & & & & & & & & \\
0 & \cdots & & & 1 & 0 & 0 & 0 & & & \cdots & & & 0 \\
0 & \cdots & & & 0 & 1 & 1 & 0 & & & \cdots & & & 0 \\
0 & \cdots & & & 0 & 0 & 0 & 1 & & & \cdots & & & 0 \\
 & & & & & & & & \ddots & & & & & \\
0 & 0 & 0 & & & & & & & & & 0 & 1 & 1 \\
0 & 0 & 0 & & & & & & & & & 0 & 0 & 1 \\
0 & 1 & -1 & & & & & & & & & 1 & -1 & 0 \\
 & & & \ddots & & & & & & & & & & \\
0 & 0 & 0 & \cdots & 1 & -1 & 1 & -1 & \cdots & & & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
h_1^c \\ h_1^d \\ h_2^c \\ h_2^d \\ h_3^c \\ h_3^d \\ \vdots \\ h_i^c \\ h_i^d \\ \vdots \\ h_{2i-1}^c \\ h_{2i-1}^d
\end{pmatrix}
=
\begin{pmatrix}
e^- - 1 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_i \\ \vdots \\ h_{2i-1} \\ e^+ - 1 \\ B_1^+ - B_1^- \\ B_2^+ - B_2^- \\ \vdots \\ B_{i-1}^+ - B_{i-1}^-
\end{pmatrix}.$$

These matrices correspond to (28), (57) and (72), respectively, in §5. In that section, necessary and sufficient conditions were proved for the existence of non-negative integer solutions for these systems. These conditions are summarized in §5 in the odd-dimensional case in (56), in the case 0 mod 4 in (71) and in the case 2 mod 4 in (71) and (79). These conditions correspond to the Poincaré–Hopf inequalities (17), (18) and (20) where h_n and h_0 are taken to be zero.

Hence the Poincaré–Hopf inequalities (17), (18) and (20) are necessary and sufficient conditions in order for a vertex to explode. The continuation of the abstract Lyapunov graph will occur if this condition is verified for each vertex. □

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