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RECURSIVE EQUILIBRIA IN ECONOMIES WITH INCOMPLETE MARKETS

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We examine minimal sufficient state spaces for equilibria in a Lucas asset pricing model with heterogeneous agents and incomplete markets. It is clear that even if all fundamentals of the economy follow a first-order Markov process, equilibrium prices and allocations generally will depend not only on the current exogenous shock but also on the distribution of wealth among the heterogeneous agents. The main contribution of this paper is to give an example of an infinite-horizon economy with Markovian fundamentals, where the joint process of equilibrium asset holdings and exogenous shocks does not constitute a sufficient state space either.

Keywords: Incomplete Markets, Heterogeneous Agents, Recursive Equilibria, Markovian Shocks, Sufficient State Space

1. INTRODUCTION

The Lucas (1978) asset-pricing model has been very influential in macroeconomics and finance. One of the attractive features of the model is that equilibria are dynamically simple in the sense that equilibrium prices are a function of the exogenous shock alone. Equilibria can be easily approximated numerically [see Judd (1998)] and one can explore the quantitative predictions of the model. However, in doing so, one readily notices that the model fares very poorly in explaining observed security prices [see, e.g., Mehra and Prescott (1985)]. It is therefore an important question to determine whether extensions of the Lucas model that incorporate incomplete consumption insurance and heterogeneous agents enrich its pricing implications [see, e.g., Constantinides and Duffie (1996) or Heaton and Lucas (1996)]. However, in these more sophisticated models with heterogeneous agents, the exogenous shock generally does not constitute a sufficient state space, and so, it is not straightforward to compute equilibrium prices. To approximate equilibria

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numerically, various papers [see, e.g., Telmer (1993), Lucas (1994), Heaton and Lucas (1996), or Zhang (1997)] assume that agents' portfolio holdings along with the current exogenous shock provide sufficient statistics for the future evolution of equilibrium prices and portfolio holdings. However, when there are finitely many agents or when there is aggregate uncertainty, it is not clear whether such dynamically simple equilibria exist.[1] Duffie et al. (1994) investigate the existence of stationary equilibria and they are only able to show existence of an ergodic equilibrium for a much larger state space that also includes past prices and consumptions.

In this paper, we examine the existence of recursive equilibria for a variety of different state spaces. We consider a model with only one-period assets where (subject to regularity conditions on preferences) equilibria always exist. We show that when markets are incomplete, the exogenous shock alone generically does not suffice to describe equilibrium. This result is not surprising because the distribution of wealth at the beginning of each period presumably influences prices and allocations in that period. In fact, intuitively, one would expect that the wealth distribution constitutes a sufficient endogenous state space. The argument would be that the initial distribution of wealth is the only endogenous variable that influences the equilibrium behavior of the economy. However, we give an example that proves that the wealth distribution alone does not always constitute a sufficient endogenous state space. The reason lies in the fact that equilibrium decisions at time t also must be consistent with expectations at time $t-1$ and that these expectations cannot always be summarized in the wealth distribution. In this example, multiple equilibria are consistent with one wealth distribution and one needs additional information to determine which equilibrium was expected in the previous period. The fact that, in this example, agents' portfolio holdings do constitute a sufficient state space confirms this reasoning. There are several assets and the multidimensional portfolio holding contains more information about last-period expectations than the summary statistic wealth.

Subsequently, we give another example that shows that even portfolio holdings do not always constitute a sufficient statistic for the future evolution of the system. In this example, last period's exogenous shocks (in addition to portfolio holdings and current shock) are needed to determine equilibrium prices and portfolio holdings.

This example is somewhat disturbing because it demonstrates that current computational techniques cannot be expected to be successful for all possible incomplete-market economies. We argue that a certain form of multiplicity of equilibria is the main reason for this phenomenon. If the equilibrium was unique for all possible subtrees of the economy (in a sense that we make precise later), wealth would always constitute a sufficient minimal endogenous state space.

In the context of stochastic overlapping generation models with multiple goods, results similar to the ones in this paper have been obtained by Spear (1985), who shows that stochastic steady-state equilibria generically fail to exist, and by Hellwig (1982), who provides an example of a sequence economy in which the unique rational expectations equilibrium is not Markovian.

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The paper is organized as follows: In Section 2, we describe the economic model. In Section 3, we show that the exogenous shock alone generically does not constitute a sufficient state space when there are fewer assets than states minus one. We also discuss how weakly recursive equilibria in which the wealth distribution does constitute a sufficient minimal state space can be approximated numerically. In Section 4, we give an example of a complete-markets economy, which has multiple equilibria, and we use this example to construct economies with infinitely many sunspot equilibria. In Section 5, we present one example that shows that the wealth distribution does not always constitute a sufficient state space and another example that shows that neither do portfolio holdings.

2. MODEL

We consider a standard infinite-horizon pure exchange economy. We assume that endowment and dividend shocks follow a Markov chain and that all households have separable time-invariant utility functions.

Physical economy. Time is indexed by $t \in \mathbf{N}_0$. At each period $1 \leq t \leq \infty$ one of S possible shocks $s \in \mathcal{S} = \{1, \ldots, S\}$ occurs. We represent the resolution of uncertainty by an event tree Σ. The root of the tree σ_0 is given by a fixed state $s_0 \in \mathcal{S}$, $\sigma_0 = (s_0)$. Each node of the tree is characterized by a history of shocks $\sigma = (s_0 \cdots s_t) = \sigma_t$. There are S successors of any node σ_t, namely, $\sigma_t s = (s_0 \cdots s_t s)$ for each $s \in \mathcal{S}$. Furthermore, each node σ_t, $t > 1$, has a unique predecessor $\sigma^* = (s_0 \cdots s_{t-1})$. To simplify notation we sometimes refer to the root node's predecessor σ_0^* and include it in the event tree Σ. We collect all possible nodes at time t in a set $\mathcal{N}_t$. At each note $\sigma \in \Sigma$, for each set of shocks $\mathcal{T} \subset \mathcal{S}$ we define a new event tree $\Xi_{\mathcal{T}}(\sigma) \subset \Sigma$, which consists of σ and all nodes succeeding σ whose histories after σ only involve shocks in $\mathcal{T}$, i.e., $\Xi_{\mathcal{T}}(\sigma)$ is defined recursively by

$$\sigma \in \Xi_{\mathcal{T}}(\sigma) \quad \text{and} \quad \tau \in \Xi_{\mathcal{T}}(\sigma) \quad \text{if and only if} \quad \tau = (\tau^* s)$$
$$\text{with} \quad \tau^* \in \Xi_{\mathcal{T}}(\sigma) \quad \text{and} \quad s \in \mathcal{T}.$$

We assume that shocks $s \in \mathcal{S}$ follow a first-order Markov chain with transition probabilities $\pi(s, s')$ for all $s, s' \in \mathcal{S}$.

There is a finite number of infinitely lived agents indexed by $h \in \mathcal{H}$ and there is a single perishable consumption good at each node. Agent h's individual endowment at any node $(\sigma s) \in \Sigma$ is given by

$$e^h(\sigma s) = \bar{e}^h(s) \in I\!R_{++}.$$

Note that the function $\bar{e}^h \colon \mathcal{S} \to I\!R_+$ is time-invariant and depends on the exogenous shock alone. Each agent $h \in \mathcal{H}$ has separable utility functions over consumption processes $c = [c(\sigma)]_{\sigma \in \Sigma}$,

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$$U_h(c) = \sum_{t=0}^{\infty} \beta^t \sum_{(\sigma s)\in\mathcal{N}_t} \rho(\sigma s) u_s^h(c(\sigma s)).$$

We assume that the functions $u_s(.)$ are shock-dependent utilities $u_s^h : \mathbb{R}_{++} \to \mathbb{R}$, which are assumed to be strictly monotone, C^2, strictly concave and to possess the Inada property, that is, $\lim_{x\to 0}(u_s^h)'(x) = \infty$. The assumption of shock-dependent utility is needed for our examples below. We argue that our conclusions do not depend crucially on this assumption.

We assume identical probabilities and discount factors. We assume that the discount factors $\beta \in (0, 1)$ and that the probabilities are objective; that is, for all $\sigma = (\sigma^* s) \in \Sigma$ and all $s' \in \mathcal{S}$, $\rho(\sigma s') = \rho(\sigma)\pi(s' \mid s)$. To simplify notation, we frequently use $E(.)$ to denote the expectations under the probability measure induced by π and write c_t for the random variable $[c(\sigma)]_{\sigma\in\mathcal{N}_t}$.

Assets. At each node $\sigma \in \Sigma$, there are J securities $j \in \mathcal{J}$. To abstract from problems that might arise from long-lived assets, we assume that each asset j traded at node σ is a one-period asset and only pays a dividend at direct successor nodes $(\sigma s)_{s\in\mathcal{S}}$. We assume that dividends are nonnegative functions of the shock alone, $d_j(\sigma s) = \bar{d}_j(s)$ for all $\sigma \in \Sigma$, $s \in \mathcal{S}$ with $\bar{d}_j : \mathcal{S} \to \mathbb{R}_+$, $\bar{d}_j(s) \geq 0$ for all $s \in \mathcal{S}$. We denote the price of security j at node σ by $q_j(\sigma)$. The security is in zero net supply and we denote agent h's position in j at node σ by $\theta_j^h(\sigma) \in \mathbb{R}$. We define household h's wealth at node σ by $\omega^h(\sigma) = e^h(\sigma) + \theta^h(\sigma^*) \cdot d(\sigma)$ and we write $\omega(\sigma) = [\omega^1(\sigma), \ldots, \omega^H(\sigma)]$.

Competitive equilibrium. Let

$$\boldsymbol{e} = \begin{pmatrix} \bar{e}^1(1) & \cdots & \bar{e}^H(1) \\ \vdots & \ddots & \vdots \\ \bar{e}^1(S) & \cdots & \bar{e}^H(S) \end{pmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} \bar{d}_1(1) & \ldots & \bar{d}_1(S) \\ \vdots & \ddots & \vdots \\ \bar{d}_J(S) & \ldots & \bar{d}_J(S) \end{pmatrix}$$

denote the matrix of possible individual endowments, **e**, and the matrix of possible dividends, **d**, respectively. We assume throughout the paper that the payoff matrix **d** has full row rank J. The primitives of the economy can then be summarized as

$$\mathcal{E} = \left[\mathbf{e}, \mathbf{d}, (U^h)_{h\in\mathcal{H}}\right].$$

An equilibrium is defined as a collection of portfolio holdings $[\theta^h(\sigma)]_{\sigma\in\Sigma}^{h\in\mathcal{H}}$ and asset prices $[q_j(\sigma)]_{\sigma\in\Sigma}^{j\in\mathcal{J}}$ such that security markets clear and all agents maximize their utility over their budget sets, given the prices. This economy is a special case of the model considered by Magill and Quinzii (1994). To prove existence of an equilibrium, one has to make an assumption on the agents' budget sets to rule out Ponzi schemes (the indefinite postponement of debt). The conventional transversality condition from representative-agent models cannot be used because

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with incomplete markets the expected present value of future wealth is not unambiguously defined. To close the model, we therefore impose an implicit debt constraint as an additional requirement of equilibrium: each agent h's portfolio process (θ^h) is required to satisfy

$$\sup_{\sigma \in \Sigma} |\theta^h(\sigma) \cdot q(\sigma)| < \infty.$$

Note that this constraint does not constitute a market imperfection—it is just needed to ensure existence of a solution to the agents' optimization problem and in equilibrium it will never be binding. This point is emphasized by Levine and Zame (1996).

DEFINITION 1. *A competitive equilibrium for an economy $\mathcal{E}$ is a collection of consumptions $[c^h(\sigma)]_{\sigma\in\Sigma}^{h\in\mathcal{H}}$, portfolio holdings $[\theta^h(\sigma)]_{\sigma\in\Sigma}^{h\in\mathcal{H}}$, and asset prices $[q(\sigma)]_{\sigma\in\Sigma}$ such that*

(*i*) *For all $\sigma \in \Sigma$,*

$$\sum_{h\in\mathcal{H}} \theta^h(\sigma) = 0.$$

(*ii*) *For each agent h, $[\theta^h(\sigma), c^h(\sigma)]_{\sigma\in\Sigma} \in \arg\max U_h(c)$ s.t. for all $\sigma \in \Sigma$,*

$$c(\sigma) = e^h(\sigma) + \theta(\sigma^*) \cdot d(\sigma) - \theta(\sigma) \cdot q(\sigma) \quad \text{and}$$
$$\sup_{\sigma\in\Sigma} |\theta(\sigma)q(\sigma)| < \infty.$$

Under the additional assumptions that utility functions are continuous on $\mathbb{R}_+$, that all probabilities are strictly positive, and that there is a riskless bond, an equilibrium always exists; for a proof of existence, see Magill and Quinzii (1994). Some of the examples we present in this paper will not satisfy these additional assumptions; however, in these examples it is always clear that an equilibrium exists.

For our discussion later, it is important to emphasize that the agents' Euler equations together with market clearing fully characterize a competitive equilibrium. Lemma 1 follows from our assumptions on utilities and the discussion by Constantinides and Duffie (1996).

LEMMA 1. *Consumptions $[c^h(\sigma)]_{\sigma\in\Sigma}^{h\in\mathcal{H}}$, portfolio holdings $[\theta^h(\sigma)]_{\sigma\in\Sigma}^{h\in\mathcal{H}}$, and asset prices $[q(\sigma)]_{\sigma\in\Sigma}$, which satisfy*

$$\lim_{t\to\infty} \beta^t E u^{h\prime}\left(c_t^h\right) = 0 \text{ for all } h \in \mathcal{H} \tag{1}$$

and $\sup_{\sigma\in\Sigma} |\theta^h(\sigma) \cdot q(\sigma)| < \infty$ constitute a competitive equilibrium if and only if for all $\sigma = (\sigma^* s) \in \Sigma$ the following Euler equations and market-clearing equations are satisfied. For all $h \in \mathcal{H}$,

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$$
\begin{aligned}
c^h(\sigma) &= e^h(\sigma) + \theta^h(\sigma^*) \cdot d(\sigma) - \theta^h(\sigma) q(\sigma) \\
q(\sigma)\left(u_s^h\right)'(c^h(\sigma)) - \beta \sum_{s' \in \mathcal{S}} \pi(s' \mid s) d(\sigma s')\left(u_{s'}^h\right)'(c^h(\sigma s')) &= 0 \qquad (2) \\
\sum_{h \in \mathcal{H}} \theta^h(\sigma) &= 0.
\end{aligned}
$$

Proof. The proposed processes obviously satisfy market clearing. Furthermore, it is well known that equation (2) is necessary for agents' optimality. To verify that it is also sufficient, we follow the methods of Constantinides and Duffie (1996) and show that for all $(\delta_\sigma)_{\sigma \in \Sigma}$, which can be obtained by a self-financing trading strategy satisfying the implicit debt constraint, $U^h(c) - U^h(c + \delta) \geq 0$ if c satisfies Equation (2). With our specification of preferences and by concavity of u^h, this inequality is implied by

$$
\Delta := E_0 \sum_{t=0}^{\infty} \beta^t u'(c_t) \delta_t \leq 0.
$$

For all $\sigma \in \Sigma$, we can write $\delta(\sigma) = d(\sigma) \cdot \phi(\sigma^*) - q(\sigma) \cdot \phi(\sigma)$, for a trading strategy $[\phi(\sigma)]_{\sigma \in \Sigma}$ with $\phi(\sigma) \in \mathbb{R}^J$ and $\phi(\sigma_0^*) = 0$. Then, we have that

$$
\begin{aligned}
E \sum_{t=0}^{T} \beta^t u'(c_t) \delta_t = E \sum_{t=0}^{T-1} \beta^t [u'(c_t) d_t \cdot \phi_{t-1} - \phi_t \cdot \beta E_t u'(c_{t+1}) d_{t+1}] \\
+ \beta^T E[u'(c_T)(\phi_{T-1} \cdot d_T - \phi_T \cdot q_T].
\end{aligned}
$$

Therefore,

$$
\Delta = \lim_{t \to \infty} -E\left[\beta^T u'(c_T) q_T \cdot \phi_T\right] = 0.
$$

The latter equality follows from our assumption on the consumption process (Requirement 1) and the implicit debt constraint. ■

Note that Requirement 1 is somewhat nonstandard. It is satisfied by any equilibrium, given the additional conditions of Magill and Quinzii (1994), and is a very innocuous additional assumption that is also made by Constantinides and Duffie (1996). In the following, we never consider the possibility that there is an equilibrium that does not satisfy (1).

Remark. Since we have separable utility functions, we can construct a new economy $\mathcal{E}_{\mathcal{T}}(\sigma)$ for each subtree $\Xi_{\mathcal{T}}(\sigma)$ from $\mathcal{E}$ by restricting utility to be summed over nodes in $\Xi_{\mathcal{T}}(\sigma)$. If for all $s' \in \mathcal{T}$ and all $s \in \mathcal{S}$, $s \notin \mathcal{T}$, there are no assets j with $\pi(s \mid s') d_j(s) > 0$ in the original economy $\mathcal{E}$, any equilibrium to $\mathcal{E}$ will also be an equilibrium to $\mathcal{E}_{\mathcal{T}}(\sigma)$. On the other hand, an equilibrium to $\mathcal{E}_{\mathcal{T}}(\sigma)$ will be part of an equilibrium of $\mathcal{E}$ if it is unique.

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2.1. Recursive Equilibria

To compute an equilibrium for an infinite-horizon model, it is necessary to focus on equilibria that are dynamically simple in the sense that one can choose a simple state space such that the current state is a sufficient statistic for the future evolution of the system and that this evolution can be approximated by a finite number of parameters. Lucas (1978) examines the easiest case of a recursive equilibrium in which the future evolution of the economy depends on the exogenous shock alone.

DEFINITION 2. *A competitive equilibrium for an economic $\mathcal{E}$ with consumption $[c^h(\sigma)]$, portfolio holdings $[\theta^h(\sigma)]$, and asset prices $[q(\sigma)]$ is called strongly recursive if there exist $(\bar{q}(s), [\bar{c}^h(s), \bar{\theta}^h(s)]_{h\in\mathcal{H}})_{s\in\mathcal{S}}$ such that for all $(\sigma s)\in\Sigma$, $q(\sigma s)=\bar{q}(s)$, $\theta^h(\sigma s)=\bar{\theta}^h(s)$, and $c^h(\sigma s)=\bar{c}^h(s)$.*

We show later that, in economies with incomplete markets and heterogeneous agents, the current exogenous state generally does not constitute such a sufficient statistic. In this case the state space also includes endogenous variables such as the distribution of wealth or the agents' portfolio holdings.

DEFINITION 3. *An equilibrium is called (weakly) recursive if for all $s\in\mathcal{S}$, there exist continuous functions $f_s^h : \mathbb{R}^{JH}\to\mathbb{R}^J$ for all $h\in\mathcal{H}$ and $g_s^j : \mathbb{R}^{JH}\to\mathbb{R}_+$ for all $j\in\mathcal{J}$ such that for all $\sigma\in\Sigma$ and all $s\in\mathcal{S}$, $q_j(\sigma s)=g_s^j([\theta^h(\sigma)]_{h\in\mathcal{H}})$ and $\theta^h(\sigma s)=f_s^h([\theta^h(\sigma)]_{h\in\mathcal{H}})$. For simplicity, we refer to weakly recursive equilibria just as recursive equilibria.*

We say an equilibrium is wealth recursive if it is recursive and if there exist continuous functions $\tilde{f}_s^h : \mathbb{R}^H\to\mathbb{R}^J$ for all $h\in\mathcal{H}$ and $\tilde{g}_s^j\colon \mathbb{R}^H\to\mathbb{R}_+^J$ for all $j\in\mathcal{J}$ such that

$$\tilde{g}_s^j(\omega(\sigma)) = g_s^j\left([\theta^h(\sigma)]_{h\in\mathcal{H}}\right)$$

and

$$\tilde{f}_s^h(\omega(\sigma)) = f_s^h\left([\theta^h(\sigma)]_{h\in\mathcal{H}}\right).$$

Clearly, the existence of wealth-recursive equilibria would greatly facilitate the computation of equilibria. As we argued in the introduction, the wealth distribution seems to be the natural endogenous state space for a model with only short-lived assets. However, we show later that wealth-recursive equilibria do not always exist. We present an example in which there exists a recursive equilibrium that cannot be represented as a wealth-recursive equilibrium. The main contribution of this paper is to show that even weakly recursive equilibria do not always exist. Duffie et al. (1994) examine stationary economies and introduce the concept of time-homogeneous Markov equilibria. These equilibria always exist in our framework but are not always recursive according to our definition. In these equilibria, last-period prices and shock must be included in the state space. We therefore define a stationary equilibrium that always exits as follows.

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DEFINITION 4. *An equilibrium is called DGMM recursive if, for all* $s \in \mathcal{S}$, *there exist functions* $f_s^h : \mathbb{R}^{S+H+J+JH} \to \mathbb{R}^J$ *for all* $h \in \mathcal{H}$ *and* $g_s^j : \mathbb{R}^{S+H+J+JH} \to \mathbb{R}_+^J$ *for all* $j \in \mathcal{J}$ *such that for all* $\sigma = (\sigma^* s') \in \Sigma$, *it holds that*

$$q_j(\sigma s) = g_s^j\left(s', [c^h(\sigma)]_{h\in\mathcal{H}}, q(\sigma), [\theta^h(\sigma)]_{h\in\mathcal{H}}\right)$$

and

$$\theta^h(\sigma s) = f_s^h\left(s', [c^h(\sigma)]_{h\in\mathcal{H}}, q(\sigma), [(\theta^h(\sigma)]_{h\in\mathcal{H}}\right).$$

Although we show later that not every equilibrium is DGMM recursive, it is easy to see that DGMM-recursive equilibria usually exist.

LEMMA 2. *Each economy* $\mathcal{E}$ *that has a competitive equilibrium with*

$$\lim_{t\to\infty} \beta^t E u^{h\prime}\left(c_t^h\right) = 0 \text{ for all } h \in \mathcal{H}$$

also possesses a DGMM-recursive equilibrium.

Proof. From Lemma 1, we know that an equilibrium can be characterized by the Euler equations for optimality. Suppose the equilibrium is not DGMM recursive. Then there must be nodes $\sigma = (\sigma^{**} s s')$ and $\tilde{\sigma} = (\tilde{\sigma}^{**} s s')$ such that

$$\left([c^h(\sigma^*), \theta^h(\sigma^*)]^{h\in\mathcal{H}}, q(\sigma^*)\right) = \left([c^h(\tilde{\sigma}^*), \theta^h(\tilde{\sigma}^*)]^{h\in\mathcal{H}}, q(\tilde{\sigma}^*)\right)$$

$$\text{but} \quad \left([\theta^h(\sigma)]^{h\in\mathcal{H}}, q(\sigma)\right) \neq \left([\theta^h(\tilde{\sigma})]^{h\in\mathcal{H}}, q(\tilde{\sigma})\right). \tag{3}$$

However, without loss of generality, we can assume that σ does not succeed $\tilde{\sigma}$ and replace $q(\tau), \theta(\tau), c(\tau)$ for all $\tau \in \Xi_{\mathcal{S}}(\tilde{\sigma})$ with the associated $q(\tau), \theta(\tau), c(\tau)$ for all $\tau \in \Xi_{\mathcal{S}}(\sigma)$. The resulting allocation, portfolio holdings, and prices still must satisfy all first-order conditions and therefore constitute an equilibrium. We can repeat this substitution for all $\sigma = (\sigma^{**} s s')$ and $\tilde{\sigma} = (\sigma^{**} s s')$, satisfying (3), and the resulting equilibrium will be DGMM recursive. ■

Duffie et al. (1994) prove much more than the existence of a DGMM-recursive equilibrium because they actually prove the existence of ergodic equilibria. Current computational techniques cannot take advantage of this fact. We argue that it is not feasible to compute an abstract DGMM-recursive equilibrium. Therefore, nonexistence of a weakly recursive equilibrium poses serious challenges to existing computational methods.

3. GENERIC NON-EXISTENCE OF STRONGLY RECURSIVE EQUILIBRIA

It is well known that for economies where agents have identical CRRA utility functions and individual endowments are spanned by the securities' dividends, there exists a Pareto-efficient equilibrium even when markets are incomplete. In this equilibrium the exogenous income state alone constitutes a sufficient state [see Lucas (1978)]. A strongly recursive equilibrium as in Definition 2 exists.

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This special case restricts both preferences and endowments, resulting economically in only remotely interesting models. Generally, strongly stationary equilibria do not exist.

THEOREM 1. *Suppose we have an economy $\mathcal{E}$ with $\pi(s \mid s') > 0$ for all $s, s' \in \mathcal{S}$, with $J < S - 1$ and with $\bar{d}_j \gg 0$ for at least one $j \in \mathcal{J}$. Then, strongly stationary equilibria exist at most for a set of individual endowments that has zero Lebesgue measure.*

To prove the theorem, we use the following crucial lemma, which is easy to prove but somewhat surprising.

LEMMA 3. *Suppose we have an economy $\mathcal{E}$ with $\pi(s \mid s') > 0$ for all $s, s' \in \mathcal{S}$. Then, in every strongly stationary equilibrium, agents have constant portfolios, that is, $\bar{\theta}^h(s) = \bar{\theta}^h(s')$ for all $s, s' \in \mathcal{S}$, $h \in \mathcal{H}$.*

Proof. From the budget constraints of agent h,

$$\bar{c}^h(s) = \bar{e}^h(s) + \bar{\theta}^h(s') \cdot \bar{d}(s) - \bar{\theta}^h(s) \cdot \bar{q}(s) \quad \text{for all} \quad s, s' \in \mathcal{S},$$

we obtain the following equations:

$$[\bar{\theta}^h(y) - \bar{\theta}^h(z)] \cdot \bar{d}(s) = 0 \quad \text{for all} \quad y, z, s \in \mathcal{S}.$$

This system of equations is equivalent to

$$[\bar{\theta}^h(s) - \bar{\theta}^h(s')] \cdot \mathbf{d} = 0 \quad \text{for all} \quad s, s' \in \mathcal{S}.$$

Since the asset payoff matrix $\mathbf{d}$ has full row rank J, it follows that $\bar{\theta}^h(s) = \bar{\theta}^h(s')$ for all $s, s' \in \mathcal{S}$. ∎

The lemma implies that for strongly recursive equilibria we can drop the state subscript and write $\theta^h = \bar{\theta}^h(s)$ for all $s \in \mathcal{S}$.

Proof of Theorem 1. We first show that, in any strongly stationary equilibrium, all agents' gradients must be collinear. Then we show that this can only occur for a set of endowments that has measure zero. Suppose we have a strongly stationary equilibrium. Define a matrix

$$A = \begin{pmatrix} \dfrac{\bar{q}_j(1)}{\beta} - \pi(1 \mid 1)\bar{d}_j(1) & -\pi(2 \mid 1)\bar{d}_j(2) & \cdots & -\pi(S \mid 1)\bar{d}_j(S) \\ -\pi(1 \mid 2)\bar{d}_j(1) & \dfrac{\bar{q}_j(2)}{\beta} - \pi(2 \mid 2)\bar{d}_j(2) & \cdots & -\pi(S \mid 2)\bar{d}_j(S) \\ \vdots & & \ddots & \vdots \\ -\pi(1 \mid S)\bar{d}_j(1) & -\pi(2 \mid S)\bar{d}_j(2) & \cdots & \dfrac{\bar{q}_j(S)}{\beta} - \pi(S \mid S)\bar{d}_j(S) \end{pmatrix}$$

Observe that $Ax = 0$ is exactly the system of Euler equations for each agent. We show that $Ax = 0$ cannot have two solutions that are strictly positive and not collinear. If there existed x and x' both strictly greater than zero, then there exists

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an $\alpha > 0$ such that $z = x - ax' > 0$ but $z_i = 0$ for at least one i and such that $Az = 0$. The latter is impossible because the only nonnegative elements in A are on the diagonal. Therefore, all agents' gradients must be collinear.

This collinearity can happen only for a measure zero set of endowments. Consider the following system of equations:

$$\bar{q}_j(s)\left(u_s^1\right)'(\bar{c}^1(s)) - \beta \sum_{y \in \mathcal{S}} \pi(y \mid s)\bar{d}_j(y)\left(u_y^1\right)'(\bar{c}^1(y)) = 0, \tag{4}$$

$$\bar{c}^h(s) - \bar{e}^h(s) - \theta^h \bar{d}(s) + \theta^h \bar{q}(s) = 0, \tag{5}$$

$$\sum_{h=1}^{H} \theta^h = 0, \tag{6}$$

$$\frac{\left(u_s^1\right)'(\bar{c}^1(s))}{\left(u_1^1\right)'(\bar{c}^1(1))} - \frac{\left(u_s^h\right)'(\bar{c}^h(s))}{\left(u_1^h\right)'(\bar{c}^h(1))} = 0. \tag{7}$$

This system of equations has the $HS + HJ + JS$ unknowns $\bar{c}^h(s)$, θ^h, and $\bar{q}_j(s)$ and $JS + HS + J + (H-1)(S-1)$ equations. We now argue that the matrix of derivatives taken with respect to $\bar{e}^h(s)$, $\bar{c}^h(s)$, θ^h, and $\bar{q}_j(s)$ has full row rank. The endowments $\bar{e}^h(s)$ only appear in the budget equations. The derivatives of the left-hand sides of these constraints with respect to $\bar{e}^h(s)$ clearly have full row rank. Thus, these equations are independent and also independent from all the other equations. A similar argument using the asset prices $\bar{q}_j(s)$ shows that the Euler equations of agent 1 are independent. Taking derivatives of the market-clearing conditions with respect to θ^1 shows that these are independent. Finally, taking derivatives in Equations (7) with respect to $\bar{c}^h(s)$ shows that these equations are also independent. In summary, the Jacobian of the system (4)–(7) with respect to $\bar{e}^h(s)$, $\bar{c}^h(s)$, θ^h, and $\bar{q}_j(s)$ has rank $JS + HS + J + H(S-1)$. It now follows from the transversality theorem and the preimage theorem that, for all endowments in a set of Lebesgue measure one, the solution set of the system (4)–(7) has dimension $HS + HJ + JS - [JS + HS + J + (H-1)(S-1)] = (H-1)[J - (S-1)]$. Hence, for $J < S - 1$, the set of solutions to the system is empty. ■

3.1. Economies with Weakly Recursive Equilibria

We now present an example of an economy in which a strongly recursive equilibrium does not exist but which possesses a weakly recursive equilibrium. Assume that there are $H = 2$ agents, $S = 3$ states, and $J = 1$ asset. Let the dividends of the asset be $\bar{d}(1) = 1$ and $\bar{d}(2) = \bar{d}(3) = 0$. Assume that both agents have log-utility with probabilities $\pi^h(s \mid s') = 1/3$ for all s, $s' \in \mathcal{S}$ and discount factors $\beta = 0.9$. Assume that individual endowments are $\bar{e}^1(1) = 3$, $\bar{e}^1(2) = 5$, $\bar{e}^1(3) = 3$, and $\bar{e}^2(1) = 5$, $\bar{e}^2(2) = 3$, $\bar{e}^2(3) = 5$. It is easy to verify that there is no strongly recursive equilibrium in this economy. Moreover, the following policy rules describe a weakly stationary equilibrium:

$$\tilde{g}_s(\omega) = 0.3 \quad \text{for } s = 1, \ldots, 3 \quad \text{and for all} \quad \omega \in I\!R^2.$$

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For state 1,

$$\tilde{f}_1^1(3,5) = -\tilde{f}_1^2(3,5) = 0, \qquad \tilde{f}_1^1(1,7) = -\tilde{f}_1^2(1,7) = 2$$

and for the other states $\tilde{f}_2^1(\omega) = -\tilde{f}_2^2(\omega) = 2$ and $\tilde{f}_3^1(\omega) = \tilde{f}_3^2(\omega) = 0$ for all $\omega \in I\!R^2$.

These rules just follow from the first-order conditions. Notice that the policy rules do not satisfy the first-order conditions for all $\omega \in I\!R^2$ because that is not necessary given our definition of a recursive equilibrium.

In this example the prices are independent of the wealth distribution and depend just on the shock s. Equilibrium portfolio holdings do depend on the last-period portfolio holdings and therefore on the distribution of wealth.

Computation of weakly recursive equilibria. As we pointed out in the introduction, infinite-horizon models are commonly used to examine the effects of incomplete markets on asset prices and the wealth distribution. One way to proceed is to calibrate the model and to simulate equilibrium prices. To compute the necessary policy functions, one has to assume that a weakly recursive equilibrium exists. Then, one can approximate the equilibrium functions g and f numerically in a fashion that ensures that all agents' Euler equations hold for all possible values of the endogenous state space. For the above example, a wealth-recursive equilibrium exists and we use the algorithm described by Judd et al. (2001), which can be amended easily to compute equilibria with short-lived assets, to compute the equilibrium functions. The results are depicted in Figure 1.

Note that there exists a continuum of policy functions that all describe the recursive equilibrium. The particular policy function found by the algorithm has the property that the Euler equations for agents' optimality are satisfied at all wealth points, not only on the equilibrium path. The existence of a wealth-recursive equilibrium makes it feasible to compute equilibria for economies with a relatively large number of exogenous shocks, a large number of short-lived assets, and three agents because there will be only $H - 1 = 2$ endogenous state variables; Judd et al. (2001) show that it is feasible to compute equilibria for a two-dimensional endogenous state space. This model has a significant advantage compared to economies where only portfolio-recursive equilibria exist, in these latter economies the dimension of the endogenous state space increases with the number of assets and it is currently only feasible to compute equilibria for economies with two agents and two assets or for models with a continuum of agents and two assets.[2]

Unfortunately, even weakly recursive equilibria do not always exist. In these cases, equilibria cannot be approximated numerically. For a DGMM-recursive equilibrium, the endogenous state space naturally will be more than a two-dimensional state space.

We want to argue in this paper that the main reason for non-existence is multiplicity of equilibria. To make this argument, we present an example of an economy with complete markets and multiple equilibria, and from this example, we show how one can construct economies with sunspot equilibria and economies without

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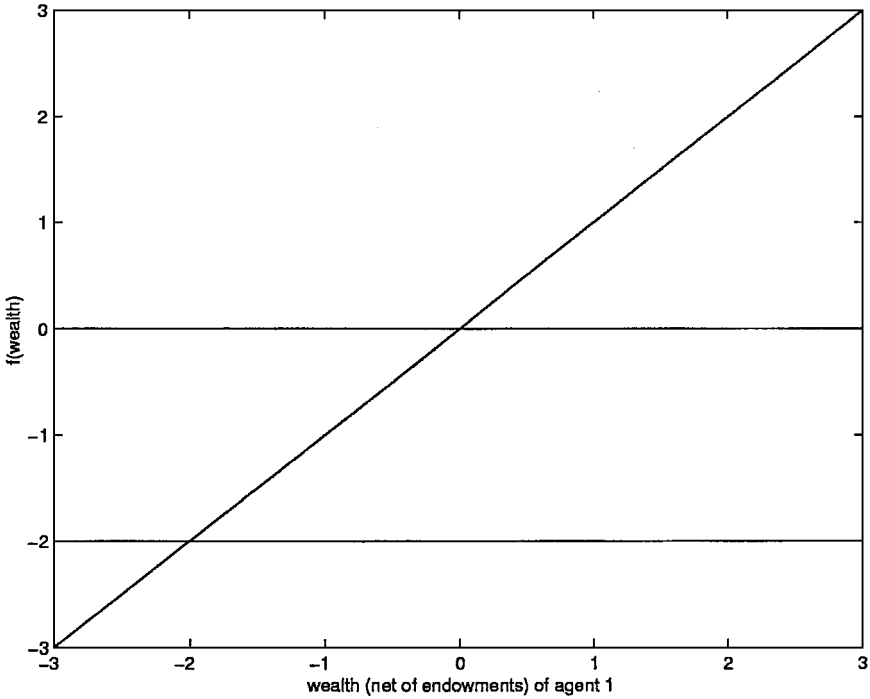


FIGURE 1. Portfolio policy functions.

recursive equilibria. (For the remainder of this paper, we refer to our example by subsection numbers; for example, the example in Section 4.1 will be called Example 4.1.)

4. MULTIPLICITY

We will show in the next section that weakly recursive equilibria do not always exist if the equilibrium is not unique. To provide an example of this phenomenon, we first give an example of multiple equilibria in complete markets (where it is easy to check that there are exactly three equilibria) and then use this example to construct an example of multiplicity of equilibria in incomplete financial markets. An immediate consequence of this second example is then the existence of an infinite number of sunspot equilibria. This result will play an important role in our counterexamples to the existence of weakly stationary equilibria.

4.1. Multiple Equilibria in Complete Markets

We consider a dynamic exchange economy with complete markets which has been constructed by extending an example of an Arrow–Debreu model with multiple equilibria [see Kehoe (1991)] to an infinite-horizon model. There are $H = 2$ agents

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with the common discount factor of $\beta = 0.75$. There are $S = 3$ exogenous shocks in which the two agents have individual endowments of the consumption good of $e^1 = [\bar{e}^1(1), \bar{e}^1(2), \bar{e}^1(3)] = (4, 12, 1)$ and $e^2 = (4, 1, 12)$, respectively. The agents' state-dependent Bernoulli functions are of the form $u_s^h(c) = a_s^h(c^{1-\gamma}/1 - \gamma)$. The multipliers for the two agents are $(a_1^1, a_2^1, a_3^1) = (1, 1024, 1)$ and $(a_1^2, a_2^2, a_3^2) = (1, 1, 1024)$, respectively. Both agents have a coefficient of relative risk aversion of $\gamma = 5$. Given any node (σs), all shocks are equally likely in the next period, and so, the transition probabilities are $\pi(s' \mid s) \equiv \frac{1}{3}$ for all $s' \in \mathcal{S}$. The economy starts in state $s_0 = 1$.

To fix ideas, we first assume that there are three Arrow securities, one for each shock, which ensure that the complete-markets allocation will be obtained through dynamic trading. The first asset pays one unit of the consumption good, contingent on the occurrence of shock 1, and so, its dividend vector equals $\boldsymbol{d}_1 = [\bar{d}_1(1), \bar{d}_1(2), \bar{d}_1(3)] = (1, 0, 0)$. Similarly, the second asset in a shock-contingent contract for state 2, and thus, $d_2 = (0, 1, 0)$; the third asset is a shock-contingent contract for state 3, and thus $d_3 = (0, 0, 1)$. Prior to time $t = 0$, both agents have zero endowments of the three financial securities. We can easily compute equilibria using a Negishi approach [see Judd et al. (2000)]. The economy has three equilibria:

- First equilibrium:
  Consumption:

$$c^1 = [\bar{c}^1(1), \bar{c}^1(2), \bar{c}^1(3)] = (4, 10.4, 2.6) \quad \text{and} \quad c^2 = (4, 2.6, 10.4).$$

  Asset prices:

$$q_1 = [\bar{q}_1(1), \bar{q}_1(2), \bar{q}_1(3)] = (0.25, 0.029007, 0.029007),$$
$$q_2 = (2.154633, 0.25, 0.25), \quad \text{and}$$
$$q_3 = (2.154633, 0.25, 0.25).$$

  Portfolio holdings (shock independent):

$$\theta_1^1 = 0, \quad \theta_2^1 = -1.6, \quad \theta_3^1 = 1.6.$$

- Second equilibrium:
  Consumption:

$$c^1 = (4.712541, 11.069487, 3.429724) \quad \text{and}$$
$$c^2 = (3.287459, 1.930513, 9.570276).$$

  Asset prices:

$$q_1 = (0.25, 0.017458, 0.051046),$$
$$q_2 = (3.57996, 0.25, 0.730986), \quad \text{and}$$
$$q_3 = (1.224389, 0.085503, 0.25).$$

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Portfolio holdings:

$$\theta_1^1 = 0, \quad \theta_2^1 = -0.980272, \quad \theta_3^1 = 2.284235.$$

- Third equilibrium:
  Consumption:

$$c^1 = (3.287459, 9.570276, 1.930513) \quad \text{and}$$
$$c^2 = (4.712541, 3.429724, 11.069487).$$

  Asset prices:

$$q_1 = (0.25, 0.051046, 0.017458),$$
$$q_2 = (1.224389, 0.25, 0.085503), \quad \text{and}$$
$$q_3 = (3.57996, 0.730986, 0.25).$$

  Portfolio holdings:

$$\theta_1^1 = 0, \quad \theta_2^1 = -2.284235, \quad \theta_3^1 = 0.980272.$$

We now show that there are no other equilibria in this economy. By the first welfare theorem, all equilibrium allocations will be Pareto-efficient. Since the set of all efficient allocations can be parameterized by an H-dimensional vector of Negishi weights, we can compute all efficient consumption allocations for the two agents by varying these weights.[3] The resulting allocations are competitive equilibria if and only if all agents' budget constraints are satisfied, given the supporting prices.

Because of Walras' law, it suffices to consider agent 1's budget constraint, and so, we can plot a one-dimensional graph whose zeros are equivalent to competitive equilibria of a complete-markets infinite-horizon economy. Figure 2 shows this plot. It depicts agent 1's excess expenditure, that is, the difference between his present value of consumption and his present value of endowment at time $t = 0$, as a function of the Negishi weight. The agent's budget constraint is satisfied exactly three times, namely at the three Negishi weights corresponding to the three reported equilibria.[4] Note that the map also indicates which initial wealth redistributions, [other than (0, 0)] lead to multiplicity of equilibria.

4.2. Multiple Equilibria in Incomplete Financial Markets

Now we change the structure of the economy in the preceding section. The agents are no longer allowed to trade in complete markets, but instead they are restricted to trade two short-lived assets, the Arrow security for state 2 and the Arrow security for state 3. Note that markets are incomplete, since in every period there are three possible states, so $S = 3 > J = 2$. As before, the economy starts with $s_0 = 1$.

The previous three equilibria from the complete-markets economy did not involve any trade in the first asset and are therefore also equilibria for this incomplete-markets economy. These three equilibria are strongly recursive (recursive equilibria exist for a set of endowments that has positive measure because the conditions of

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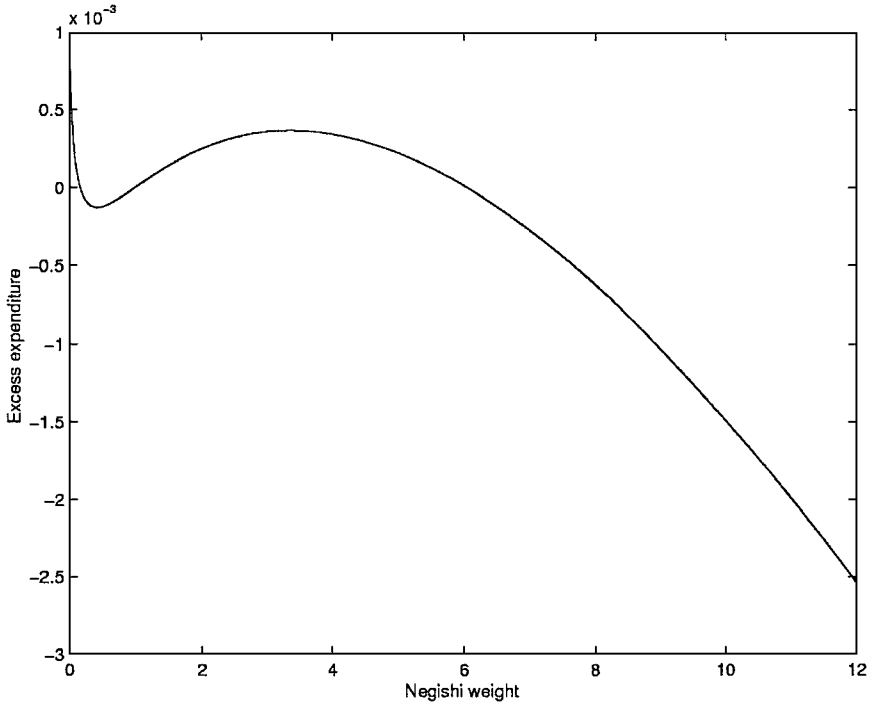


FIGURE 2. Negishi map.

Theorem 1 are not satisfied since $J = S - 1$). At any time t, the prices of the two assets depend only on the exogenous shock s. Note that this economy has many other, nonrecursive, equilibria.

4.3. Sunspot Equilibria

The following two observations make the existence of sunspot equilibria in the economy of the previous example readily apparent:

1. Neither of the two assets pays off in state 1. As a consequence, whenever the economy enters state $s = 1$ the agents have the same wealth as when the economy starts at time 0, namely just their initial endowments $\bar{e}^h(1)$.
2. A second consequence of the zero payoffs of the assets in state 1 is that the agents' marginal utilities of consumption in state 1 never enter the Euler equations as a future discounted term.[5] Therefore, the Euler equations place no restrictions on the agents' consumption in state 1 at time $t + 1$ through the consumption at the predecessor s^* at time t.

These observations imply that the history of shocks yields no restriction for the agents' behavior whenever the economy enters state 1. Put differently, the economy effectively "starts over" whenever it enters state 1. Thus, after entering state 1, the economy can switch from one of the strongly recursive equilibria into

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another one. Hence, the behavior of the economy exhibits endogenous fluctuations that cannot be explained by the economy's fundamentals. We call these equilibria sunspot equilibria. Note that in particular there now exist equilibria that are not DGMM recursive.

5. AN ECONOMY WITHOUT WEAKLY RECURSIVE EQUILIBRIA

In this section, we use the above analysis and construct an elaborate example of an economy without a weakly recursive equilibrium.

5.1. No Wealth-Recursive Equilibrium

We first show that the existence of a weakly recursive equilibrium does not always imply the existence of a wealth-recursive equilibrium. We take Example 4.1 above and add two additional states. The objective probabilities of the five states are now assumed to be given by $\pi(1 \mid s) = \pi(2 \mid s) = \pi(3 \mid s) = 0.3$, $\pi(4 \mid s) = \pi(5 \mid s) = 0.05$ for $s = 1, \ldots, 4$ and $\pi(1 \mid 5) = \pi(2 \mid 5) = \pi(5 \mid 5) = 0.3$, $\pi(4 \mid 5) = \pi(3 \mid 5) = 0.05$. There are three Arrow securities for shocks 1–3. To ensure that the prices and consumptions from Section 4.1 constitute equilibria for all economies $\mathcal{E}_{\{1,2,3\}}(\sigma 1)$, we assume that the discount factor is now given by $\beta = 5/6$. Endowments in the fourth state are given by $\bar{e}^1(4) = \bar{e}^2(4) = 10$; endowments in the fifth state are given by $\bar{e}^1(5) = 8.690655$ and $\bar{e}^2(5) = 11.309345$. Agents 1's utility functions for shocks $s = 4, 5$ are given by $u_s^1(x) = -1/2x^{-2}$, and agent 2's are $u_4^2(x) = -1/2x^{-2}$ and $u_5^2(x) = -6.053049 \cdot 1/2x^{-2}$. The economy starts with shock 4, that is, $s_0 = 4$; we refer to this economy as $\mathcal{E}$. We define $\Theta_1 = (\theta^1; \theta^2) = (0, -1.6, 1.6; 0, 1.6. -1.6)$ and $\Theta_2 = (0, -0.980272, 2.284235; 0, 0.980272, -2.284235)$. It is then easy to verify that the following constitutes a weakly recursive equilibrium. Prices (depending on the shock and last-period portfolio holdings) are given by

$$
\begin{aligned}
&q(1, \Theta_1) = (0.25, 2.154633, 2.154633)\\
&q(1, \Theta_2) = (0, 25, 3.57996, 1.224389)\\
&\hline
&q(2, \Theta_1) = (0.029007, 0.25, 0.25)\\
&q(2, \Theta_2) = (0.017458, 0.25, 0.085503)\\
&\hline
&q(3, \Theta_1) = (0.029007, 0.25, 0.25)\\
&q(3, \Theta_2) = (0.051046, 0.730968, 0.25)\\
&\hline
&q(4, \Theta_1) = q(4, \Theta_2) = (0.244141, 2.104133, 2.104133)\\
&q(5, \Theta_1) = q(5, \Theta_2) = (0.107563, 1.540290, 0.087799)
\end{aligned}
$$

with portfolio holdings

$$
\begin{aligned}
\theta^1(1, \Theta_1) = \theta^1(2, \Theta_1) = \theta^1(3, \Theta_1) &= \Theta_1\\
\theta^1(1, \Theta_2) = \theta^1(2, \Theta_2) = \theta^1(3, \Theta_2) &= \Theta_2\\
\theta^1(4, \Theta_1) = \theta^1(4, \Theta_2) &= \Theta_1\\
\theta^1(5, \Theta_1) = \theta^1(5, \Theta_2) &= \Theta_2.
\end{aligned}
$$

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Equilibrium consumption in states 4 and 5 turns out to be 10 for both agents. This equilibrium is not wealth recursive since $\omega(1, \Theta_1) = \omega(1, \Theta_2)$. In the first state the wealth distribution is no longer a sufficient state variable.

Of course, our example would be of limited interest if there were other equilibria in this economy that are wealth recursive. We prove that such is not the case. Our argument proceeds in several steps. First, we show that, for all $\sigma \in \Sigma$, the equilibrium conditions for an economy $\mathcal{E}_{\{1,2,3\}}(\sigma 4)$ imply the equilibrium conditions for another economy, with complete markets, which is identical to $\mathcal{E}_{\{1,2,3,4\}}(\sigma 4)$ except that it also contains an Arrow security for shock 4. Then, we show that the complete-markets economy has a unique equilibrium by tracing out agent 1's excess expenditure for all Negishi weights. Finally, we argue that this implies that $\mathcal{E}_{\{1,2,3,4\}}(\sigma 4)$ must have a unique equilibrium. We proceed similarly for all economies $\mathcal{E}_{\{1,2,3,5\}}(\sigma 5)$ and therefore show that $\mathcal{E}$ must have a unique equilibrium.

At any node $\sigma \in \Sigma$ with $\sigma = (\sigma^* 4)$ the following conditions must be satisfied in any equilibrium for $\mathcal{E}$:

$$q_s(\sigma)\left(u_4^h\right)'(c^h(\sigma)) - \beta\pi(s \mid 4)\left(u_s^h\right)'(c^h(\sigma s)) = 0 \quad \text{for} \quad h = 1, 2, s = 1, 2, 3,$$
$$c^h(\sigma) - \bar{e}^h(4) + \theta^h(\sigma) \cdot q(\sigma) = 0 \quad \text{for} \quad h = 1, 2,$$
$$\theta^1(\sigma) + \theta^2(\sigma) = 0,$$

and, for all $\tau = (\tau^* s)$ with $\tau^* \in \Xi_{\{1,2,3\}}(\sigma)$ and $s = 1, 2, 3$,

$$q_j(\tau)\left(u_s^h\right)'(c^h(\tau)) - \beta\pi(j \mid s)\left(u_j^h\right)'(c^h(\tau j)) = 0 \quad \text{for} \quad h = 1, 2, j = 1, 2, 3,$$
$$c^h(\tau) - \bar{e}^h(s) - \theta_s^h(\tau^*) + \theta^h(\tau) \cdot q(\tau) = 0 \quad \text{for} \quad h = 1, 2,$$
$$\theta^1(\tau) + \theta^2(\tau) = 0.$$

The first-order conditions with respect to the three Arrow securities together with the Euler equations for node σ imply that the marginal utilities of the agents are collinear for all nodes $\tau \in \Xi_{\{1,2,3\}}(\sigma)$. Therefore, consumption allocations at this node must be a function of the current shock alone. There must exist $q_4(\tau)$ for all nodes $\tau \in \Xi_{\{1,2,3\}}(\sigma)$, such that the following additional equations are satisfied:

$$q_4(\tau)\left(u_s^h\right)'(c^h(\tau)) - \beta\pi(4 \mid s)\left(u_4^h\right)'(c^h(\sigma)) = 0 \quad \text{for} \quad h = 1, 2.$$

However, if all of these conditions are satisfied, we must have a strongly recursive equilibrium for a complete-markets economy that starts at node σ and involves shocks 1, 2, 3, 4 because the necessary and sufficient Euler equations will be satisfied for all $\tau \in \Xi_{\{1,2,3,4\}}(\sigma)$. Of course, this recursive equilibrium will have $\theta_4 = 0$. We can compute all equilibria of this artificial complete-markets economy using the Negishi approach from Example 4.1.

Figure 3 shows that there is a unique equilibrium for this economy (we again traced out the excess expenditure much further than shown in the figure). If there is only one equilibrium for the complete-markets economy, then there also must be a

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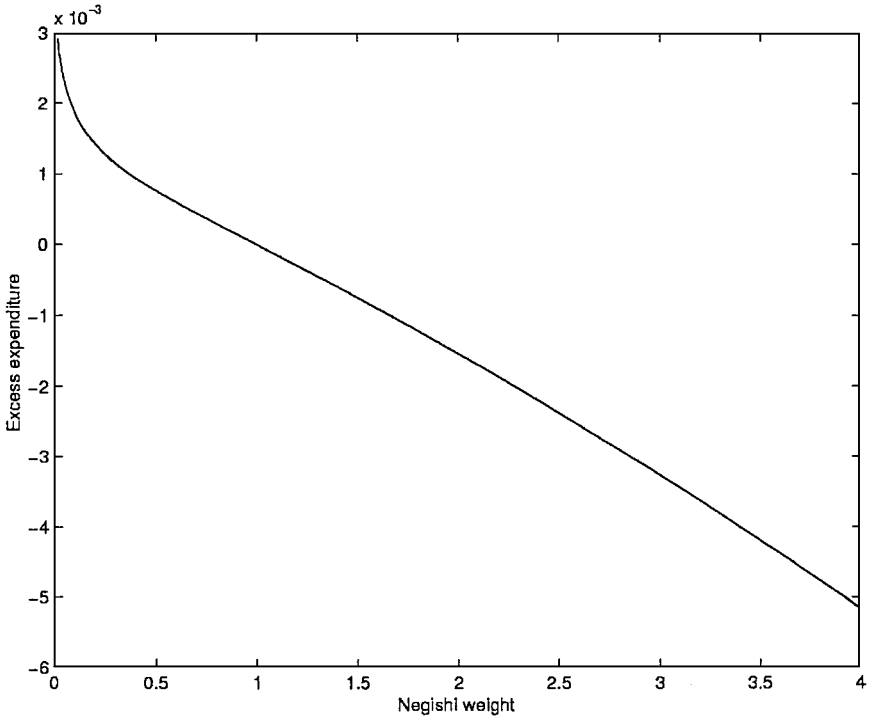


FIGURE 3. Unique equilibrium in state 4.

unique equilibrium for each economy $\mathcal{E}_{\{1,2,3,4\}}(\sigma 4)$, $\sigma \in \Sigma$. This statement is true because the initial conditions given shock 4 are always the same and because the above argument proves that there is a unique solution for the first-order conditions in shock 4.

We can repeat this exercise for the fifth state and also find a unique solution. Since initial conditions are also always the same given shock 5, this proves that there is only one equilibrium for each economy $\mathcal{E}_{\{1,2,3,5\}}(\sigma 5)$, $\sigma \in \Sigma$. Since the original economy $\mathcal{E}$ starts in state 4 and since there are no assets paying in states 4 or 5, only one equilibrium exists for $\mathcal{E}$ and therefore a wealth-recursive equilibrium does not exist.

5.2. No Portfolio-Stationary Equilibrium

We now argue that a slight variation of the above example results in an economy where there is no weakly recursive equilibrium at all, that is, even portfolio holdings no longer constitute a sufficient state variable.

Assume that there are $S = 5$ states. The first three states are taken from the above examples. The objective probabilities are now given by

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$$\pi(1\,|\,s) = \pi(2\,|\,s) = \pi(3\,|\,s) = 0.3, \quad \pi(4\,|\,s) = 0.1, \quad \pi(5\,|\,s) = 0 \quad \text{for } s = 1, \ldots, 3$$

$$\pi(1\,|\,4) = \pi(2\,|\,4) = \pi(3\,|\,4) = \pi(4\,|\,4) = \pi(5\,|\,4) = 0.2$$

and

$$\pi(1\,|\,5) = \pi(2\,|\,5) = \pi(3\,|\,5) = \pi(5\,|\,5) = 0, \quad \pi(4\,|\,5) = 1$$

The first security has payoffs $\bar{d}^1(s) = 1$ for $s = 1, 5$, and $\bar{d}^1(s) = 0$ for $s = 2, 3, 4$. In addition, there are two Arrow securities for shocks 2 and 3. To ensure that the prices and consumptions from Example 4.1 satisfy the Euler equations along the equilibrium paths where shocks 4 and 5 do not occur, we again assume that the discount factor is given by $\beta = 5/6$. Endowments in the fourth and fifth state are given by

$$\bar{e}^1(4) = \bar{e}^2(4) = 10 \quad \text{and} \quad \bar{e}^1(5) = 6.8044587, \quad \bar{e}^2(5) = 10,$$

respectively. Utilities in these states are

$$u_4^h(x) = u_5^h(x) = -1/2x^{-2}.$$

We define Θ_1 and Θ_2 as above.

The following is a DGMM-recursive equilibrium, depending on current shock, last-period portfolio holdings, as well as on last period's shock:

$q(1, \Theta_1, s_-) = (0.25, 2.154633, 2.154633)$	for $s_- = 1, 2, 3$
$q(1, \Theta_1, s_-) = (0.25, 3.57996, 1.224389)$	for $s_- = 4$
$q(1, \Theta_2, s_-) = (0.25, 3.57996, 1.224389)$	for $s_- = 1, 2, 3, 4$
$q(2, \Theta_1, s_-) = (0.029007, 0.25, 0.25)$	for $s_- = 1, 2, 3, 4$
$q(2, \Theta_2, s_-) = (0.017458, 0.25, 0.085503)$	for $s_- = 1, 2, 3, 4$
$q(3, \Theta_1, s_-) = (0.029007, 0.25, 0.25)$	for $s_- = 1, 2, 3, 4$
$q(3, \Theta_2, s_-) = (0.051046, 0.730986, 0.25)$	for $s_- = 1, 2, 3, 4$
$q(4, \Theta_1, s_-) = q(4, \Theta_2, s_-) = (0.600724, 1.402776, 1.402776)$	for $s_- = 1, 2, 3, 4$
$q(5, \Theta_1, s_-) = q(5, \Theta_2, s_-) = (0, 0, 0)$	for $s_- = 4$

with portfolio holdings

$\theta^1(1, \Theta_1, s_-) = \Theta_1$	for $s_- = 1, 2, 3$
$\theta^1(1, \Theta_1, 4) = \Theta_2$	
$\theta^1(1, \Theta_2, s_-) = \Theta_2$	for $s_- = 1, 2, 3, 4$
$\theta^1(1, \Theta_1, s_-) = \theta^1(3, \Theta_1, s_-) = \Theta_1$	for $s_- = 1, 2, 3, 4$
$\theta^1(2, \Theta_2, s_-) = \theta^1(3, \Theta_2, s_-) = \Theta_2$	for $s_- = 1, 2, 3, 4$
$\theta^1(4, \Theta_1, s_-) = \theta^1(4, \Theta_2, s_-) = \Theta_1$	for $s_- = 1, 2, 3, 4$

It is easy to verify that these values describe an equilibrium: At nodes where the sequence of shocks (41) did not occur, we have consumptions at nodes 1–3,

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which are the same as in the first complete-markets equilibrium; that is, we in particular have $\bar{c}^h(1) = \bar{e}^h(1)$ for both agents $h = 1, 2$. In state 4, however, the agents' first-order conditions will not be satisfied if they expect to consume $\bar{e}(1)$ when shock 1 occurs. A different consumption is necessary because suddenly state 5 is possible and the first-order condition with respect to the first asset cannot be satisfied. For the Euler equations to be satisfied at shock 4, agent 1 must consume more than his endowments in case shock 1 occurs; in this case he will consume $\bar{e}^1(1) + 0.712542$.

It is evident that this equilibrium is not weakly recursive according to Definition 3: Given shock 1, portfolio holdings alone do not constitute a sufficient state variable. The equilibrium is not weakly recursive because the agents achieve this different consumption without altering their portfolio holdings at shock 4; instead, they change their portfolio decision at shock 1 depending on whether shock 1 immediately succeeds shock 4 or whether it does not. For simplicity, we require that, after shock 5, shock 4 will occur with probability 1, which allows us to fix state-5 consumption to individual endowments.

Last, there are no other equilibria in this economy that could possibly be recursive equilibria. We show this fact in the following way: Whenever shock 4 occurs [at any node $\sigma = (\sigma^* 4) \in \Sigma$], agents' portfolio holding must be equal to Θ_1 because the system of Euler equations for all nodes $\Xi_{\{1,2,3\}}(\sigma)$ does not have any other solutions. The argument goes as follows.

Any equilibrium allocation for the economy $\mathcal{E}_S(\sigma 1)$ must constitute an optimal allocation for $\mathcal{E}_{\{1,2,3\}}(\sigma 1)$. All Pareto-efficient allocations for the economy $\mathcal{E}_{\{1,2,3\}}(\sigma 1)$ can be parameterized by a one-dimensional Negishi weight. These allocations can be implemented as competitive equilibria of $\mathcal{E}_{\{1,2,3\}}(\sigma 1)$, given an appropriate transfer at node $(\sigma 1)$. This transfer can be implemented with a unique choice of $\theta_1^h(\sigma)$, resulting in a unique consumption at node $(\sigma 5)$ and, for the Euler equations at node σ to be satisfied for both agents, in a unique $c^h(\sigma)$. Now, the Euler equations at σ and the equilibrium conditions for the economies $\mathcal{E}_{\{1,2,3\}}(\sigma 2)$ and $\mathcal{E}_{\{1,2,3\}}(\sigma 3)$ yield unique $\theta_2(\sigma)$ and $\theta_3(\sigma)$ as well as prices $q(\sigma)$. With all of these variables computed, we can show that the budget constraint at σ is only satisfied for $\theta_1^h(\sigma) = 0$.

We plot $c^1(\sigma) - e^1(\sigma) + q(\sigma) \cdot \theta^1(\sigma)$ as a function of the Negishi weight, parameterizing all efficient allocations for $\mathcal{E}_{\{1,2,3\}}(\sigma 1)$. Figure 4 shows this graph and, because the graph only has one zero (as always we traced it out much further than shown in the figure), the Euler equations imply that, for each equilibrium, $\theta_1(\sigma) = 0$. Therefore, a recursive equilibrium does not exist for this economy.

5.3. Generality?

While the above example is a special case and one needs some nonstandard assumptions to make it work (shock-dependent utility and zero probabilities), we argue that the non-existence of weakly recursive equilibria is a phenomenon that can occur whenever there is multiplicity of equilibria in some subeconomy. Furthermore,

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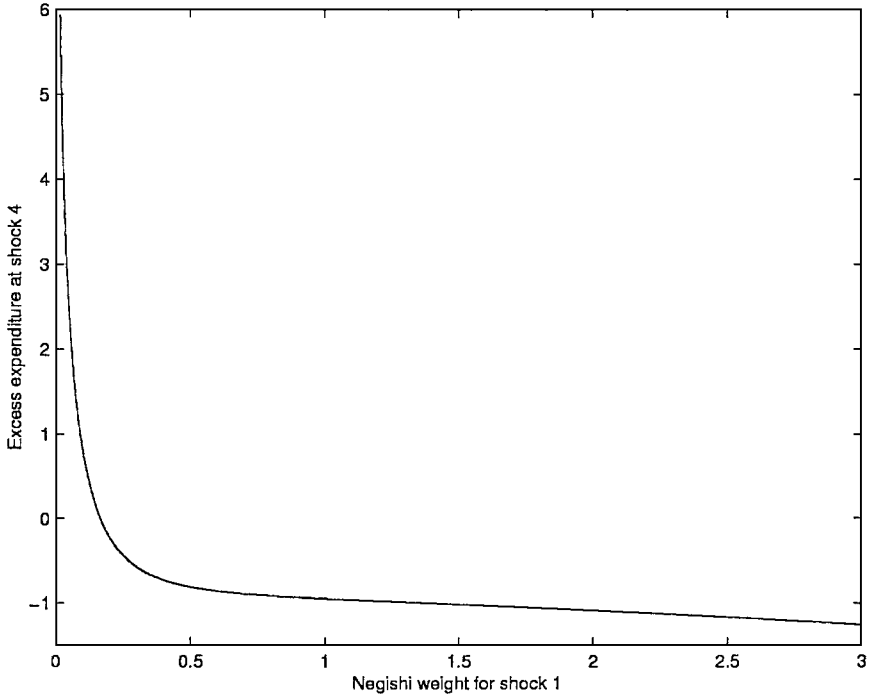


FIGURE 4. Excess expenditure in shock 4.

multiplicity of equilibrium is very likely to lead to non-existence of wealth-recursive equilibria. If, on the contrary, the equilibrium is globally unique for all possible wealth distributions and for all exogenous shocks, a wealth-recursive equilibrium does exist.

THEOREM 2. *Given an economy $\mathcal{E}$ for which there exists a competitive equilibrium with*

$$\lim_{t\to\infty} \beta^t E u^{h\prime}\left(c_t^h\right) = 0 \text{ for all } h \in \mathcal{H}.$$

Suppose that for all $\sigma \in \Sigma$ and all $\omega(\sigma) \in \mathbb{R}^H$ with $\sum_{h\in\mathcal{H}}[\omega^h(\sigma) - e^h(\sigma)] = 0$ there exists at most one equilibrium for the economy $\mathcal{E}_S(\sigma)$. Then there exists a wealth-recursive equilibrium for this economy.

Problems arise as soon as there are multiple equilibria.

Multiplicity in models with complete and incomplete markets. There are no known conditions that ensure global uniqueness in the infinite-horizon incomplete-markets model under consideration. For the case of complete markets, Dana (1993) gives sufficient conditions for uniqueness. These conditions essentially restrict the coefficients of relative risk aversion to lie below 1. We conjecture that multiple

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equilibria can exist even when utilities are not shock dependent and that the conditions of Theorem 2 are very strong and generally can not be verified from the fundamentals of the economy. Voss (1999) has recently shown (for shock-dependent utilities) that, for two-period models, the number of equilibria depends on the market structure. If such a result can be shown for simple models, one would expect similar examples in infinite-horizon models. At this time the problem of finding more general conditions for uniqueness of equilibria in infinite-horizon models with a Markovian structure remains an open research question.

General structure of economies without weakly recursive equilibria. Of course, global uniqueness of equilibria for all possible values of the endogenous state space is not a necessary condition for the existence of weakly recursive equilibrium or even for the existence of a wealth-recursive equilibrium. There is an additional crucial characteristic of the equilibrium in Examples 5.1 and 5.2. There must be two different nodes σ and $\tilde{\sigma}$ such that along the equilibrium path the value of the endogenous state variable is the same at σ and at $\tilde{\sigma}$ and such that there exist more than one equilibrium for both the economies $\mathcal{E}_{\mathcal{S}}(\sigma)$ and $\mathcal{E}_{\mathcal{S}}(\tilde{\sigma})$. In addition, σ^* and $\tilde{\sigma}^*$ have to be sufficiently heterogeneous, in the sense that asset prices and consumptions there support two different consumptions at σ and $\tilde{\sigma}$.[6] The requirement of sufficient heterogeneity of σ^* and $\tilde{\sigma}^*$ does not seem to be the crucial one. It is crucial, and somewhat nongeneric, that, in addition, agents make the same (at least with respect to one state in 5.1) portfolio choice in both σ^* and $\tilde{\sigma}^*$. In fact, in the examples above a slight perturbation in individual endowments will restore the existence of a weakly recursive equilibrium and/or will lead to the existence of a wealth-recursive equilibrium. If agents choose any other portfolio than Θ_1, say $\tilde{\Theta}_1$ with $\tilde{\theta}_1$ being different from Θ_1 in all elements at shock 4, a weakly recursive equilibrium will exist; getting to shocks 2 or 3, the agents will then change their portfolio position in asset 1 and therefore state $(1, \Theta_1)$ will never be possible after the occurrence of shock 4.

NOTES

1. For the case of a continuum of ex ante identical agents and no aggregate uncertainty, Aiyagari (1994) shows existence. His arguments do not extend to economies with aggregate uncertainty; see Krusell and Smith (1998). In this paper, we focus on the case of finitely many classes of ex ante heterogeneous agents.

2. See Krusell and Smith (1997), Storesletten et al. (1999), and also den Haan (2001) for models with two agents or a continuum of agents and one bond.

3. This computation can be easily done using a homotopy with the Negishi weight as the homotopy parameter. Note that with two agents we can normalize one of the two weights to be equal to one and can therefore parameterize all efficient allocations by a single number.

4. Although it is not shown in the figure, we actually traced out his excess expenditure up to a Negishi weight of 100; at that point, it is hugely negative and there are no other equilibria—the figure proves that there are exactly three equilibria in this economy.

5. Zero asset payoffs in a state have the same impact in the Euler equations as a zero transition probability for that state.

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6. In Example 5.2, this was guaranteed by introducing an additional shock 5, which is possible only after shock 4. Note that this construction is obviously just a shortcut for tractability; in a richer model the phenomenon can occur even if all transition probabilities are strictly positive.

REFERENCES

Aiyagari, S.R. (1994) Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics* 109, 659–684.

Constantinides, G.M. & D. Duffie (1996) Asset pricing with heterogeneous consumers. *Journal of Political Economy* 104, 219–240.

Dana, R.-A. (1993) Existence, uniqueness and determinary of Arrow-Debreu equilibria in finance models. *Journal of Mathematical Economics* 22, 563–579.

Duffie, D., J. Geanakoplos, A. Mas-Colell, & A. McLennan (1994) Stationary Markov equilibria. *Econometrica* 62, 745–781.

den Haan, W.J. (2001) The importance of the number of different agents in a heterogeneous asset-pricing model. *Journal of Economic Dynamic & Control* 25, 721–746.

Heaton, J. & D.J. Lucas (1996) Evaluating the effects of incomplete markets on risk sharing and asset pricing. *Journal of Political Economy* 104, 443–487.

Hellwig, M.F. (1982) Rational expectations and the markov property of temporary equilibrium processes. *Journal of Mathematical Economics* 9, 135–144.

Judd, K.L. (1998) *Numerical Methods in Economics*. Cambridge, MA: MIT Press.

Judd, K.L., F. Kubler, & K. Schmedders (2000). Asset Trading Volume with Dynamically Complete Markets and Heterogeneous Agents. Working Paper.

Judd, K.L., F. Kubler, & K. Schmedders (2001). A Solution Method for Incomplete Asset Markets with Heterogeneous Agents. Working paper.

Kehoe, T. (1991) Computation and multiplicity of equilibria. In W. Hildenbrand & H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, Vol. IV, pp. 2049–2143. Amsterdam: North-Holland.

Krusell, P. & A.A. Smith, Jr. (1997) Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns. *Macroeconomic Dynamics* 1, 387–422.

Krusell, P. & A.A. Smith, Jr. (1998) Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economics* 106, 867–896.

Levine, D. & W. Zame (1996) Debt constraint and equilibrium in infinite horizon economies with incomplete markets. *Journal of Mathematical Economics* 26, 103–131.

Lucas, D.J. (1994) Asset pricing with undiversifiable income risk and short-sale constraints. Deepening the equity premium puzzle. *Journal of Monetary Economics* 34, 325–241.

Lucas, R.E., Jr. (1978) Prices in an exchange economy. *Econometrica* 46, 1429–1445.

Magill, M. & M. Quinzii (1994) Infinite horizon incomplete markets. *Econometrica* 62, 853–880.

Mehra, R. & E.C. Prescott (1985) The equity premium: A puzzle. *Journal of Monetary Economics* 15, 145–161.

Spear, S.E. (1985) Rational expectations in the overlapping generations model. *Journal of Economic Theory* 38, 251–275.

Storesletten, K., C.I. Telmer, & A. Yaron (1999) Asset Pricing with Idiosyncratic Risk and Overlapping Generations. Working paper.

Telmer, C.I. (1993) Asset-pricing puzzles and incomplete markets. *Journal of Finance* 48, 1803–1832.

Voss, B. (1999) Non-Equivalence of Uniqueness in Complete and Incomplete Market Models. Working paper.

Zhang, H.H. (1997) Endogeneous borrowing constraints with incomplete markets. *Journal of Finance* 52, 2187–2209.

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