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Pullback measure attractors for stochastic p-Laplacian parabolic equations on thin domains

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We investigate the pullback measure attractors for non-autonomous stochastic p-Laplacian equations driven by nonlinear noise on thin domains. The concept of complete orbits for such systems is presented to establish the structures of pullback measure attractors. We first present some essential uniform estimates, as well as the existence and uniqueness of pullback measure attractors. A novel technical proof method is shown to overcome the difficulty of the estimates of the solutions in $W^{1,p}$ on thin domains. Then, we prove the upper semicontinuity of these measure attractors as the $(n+1)$-dimensional thin domains collapse onto the lower n-dimensional space.

Keywords: measure attractors; nonlinear noise; stochastic p-Laplacian equation; thin domains

2010 *Mathematics Subject Classification:* 35B40, 37L55, 37L30, 60H15

1. Introduction

The dynamical behaviour of partial differential equations (PDEs) has always aroused high attention because of its wide application in physics, engineering sciences, biological systems modelling, and so on. As is well known, many applications involve PDEs over regions in which the size in one direction is much larger than the size in the other direction. In such cases, it is natural to try to determine PDEs on a lower-dimensional domain which could reflect all of the dynamics of the related systems. The systematic research on the asymptotic behaviour of PDEs on thin domains was first initiated in [9–11], they presented the case of domain of the form as follows

$$\mathcal{O}_\varepsilon = \{x = (x^*, x_{n+1}) \,|x^* = (x_1, \ldots, x_n) \in \mathcal{Q} \text{ and } 0 < x_{n+1} < \varepsilon g(x^*)\},$$

where $\mathcal{Q}$ is a smooth bounded domain in $\mathbb{R}^n$, g is a smooth positive function defined on $\mathcal{Q}$, the classical reaction-diffusion problem is considered:

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$$\begin{cases} du^\varepsilon - \Delta u^\varepsilon dt + \lambda u^\varepsilon dt = f(u^\varepsilon)dt, & x \in \mathcal{O}_\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial \nu_\varepsilon} = 0, & x \in \partial\mathcal{O}_\varepsilon. \end{cases}$$

When $\varepsilon \to 0$, they compare the dynamics of the above model with the dynamics of the equation defined on $\mathcal{Q}$:

$$\begin{cases} du - \frac{1}{g}\mathrm{div}(g\nabla u)dt + \lambda u dt = f(u)dt, & x \in \mathcal{Q}, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\mathcal{Q}. \end{cases}$$

Especially they derived the convergence of the family of global attractors of the two systems as $\varepsilon \to 0$. Since then, these results inspired many works of studying dynamics on the thin domain, see [1–8, 20, 23] for deterministic differential equations and the references therein. Recently, these results have been extended to the stochastic case in [12, 13, 15, 21, 22, 26].

It is very interesting to extend the concept of measure attractors to thin domains for study. Measure attractors come from the mathematical theory and chaos theory of dynamical systems, which involves the relationship between the basin of attraction and measures. By combining the concepts of measure and attractor to study the measure attractor, it can describe the dynamical behaviour of complex systems, and predict their long-term evolution trends more precisely, especially stability analysis of control systems in engineering, the dynamic evolution process of economic systems in economics, etc. The concept of measure attractor for stochastic PDEs was first presented in [25]. Very recently, Li and Wang [14] present the existence and uniqueness of pullback measure attractors for the inhomogeneous Markov process associated with the stochastic reaction-diffusion equations, including the concept of complete orbits. More details about the concept of measure attractors can be found in [17, 18, 24]. It is also worth noting the recent development of the related work on invariant measures of locally monotone and generally coercive SPDEs [28, 29].

In this paper, we focus on the limiting behaviour of measure attractors of the stochastic evolution equation governed by the p-Laplacian operator defined on $\mathcal{O}_\varepsilon$:

$$\begin{cases} d\hat{u}^\varepsilon(t) - \mathrm{div}(|\nabla \hat{u}^\varepsilon|^{p-2}\nabla \hat{u}^\varepsilon)dt + \lambda \hat{u}^\varepsilon(t)dt = \left(f\left(t, x, \hat{u}^\varepsilon(t)\right) + G\left(t, x\right)\right)dt \\ \qquad\qquad + \sum_{k=1}^{\infty}\left(\sigma_k(x) + \rho_k(\hat{u}^\varepsilon(t))\right)dW_k(t), \quad x \in \mathcal{O}_\varepsilon,\ t > \tau, \\ \frac{\partial \hat{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, \quad x \in \partial\mathcal{O}_\varepsilon, \end{cases} \tag{1.1}$$

with initial condition

$$\hat{u}^\varepsilon(\tau, x) = \hat{u}^\varepsilon_\tau(x), \quad x \in \mathcal{O}_\varepsilon, \tag{1.2}$$

where $p > 2$, $\lambda > 0$, f is a nonlinear function defined on $\widetilde{\mathcal{O}} \times \mathbb{R}$, G and σ_k are defined on $\widetilde{\mathcal{O}}$, the nonlinear function $\rho_k : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function of u. For every $k \in \mathbb{N}$, $\{W\}_{k=1}^{\infty}$ is a sequence of independent real-valued standard Wiener processes on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$, and ν_ε is the unit outward normal vector for $\partial\mathcal{O}_\varepsilon$.

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Notice that the domain $\mathcal{O}_\varepsilon$ collapses to the lower dimensional space $\mathcal{Q}$ as $\varepsilon \to 0$, then the limit equation of (1.1) is given by

$$\begin{aligned} du^0(t) - \tfrac{1}{g}\mathrm{div}(g|\nabla u^0|^{p-2}\nabla u^0)dt + \lambda u^0 dt = \left(f\left(t,(y^*,0),u^0\right) + G\left(t,y^*,0\right)\right)dt \\ + \sum_{k=1}^{\infty}\left(\sigma_k(y^*,0) + \rho_k(u^0(t))\right)dW_k(t), \quad t > \tau, \end{aligned} \tag{1.3}$$

with initial condition

$$u^0(\tau, y^*) = u^0_\tau(y^*), \tag{1.4}$$

where $y^* = (y_1, \ldots, y_n) \in \mathcal{Q}$.

To the best of our knowledge, this paper is the first work on studying the limiting behaviour for stochastic p-Laplacian problems with nonlinear noise on thin domains. The target of our work is to present the existence and uniqueness of pullback measure attractors for (1.1) and its limit system (1.3). The structure of the pullback measure attractor is characterized by the complete solutions of dynamical systems generated by (1.1). It is worth emphasizing that we used a novel technique proof method to derive the estimates of solutions of (1.1) in $W^{1,p}(\mathcal{O})$ with higher regularity in the sense of thin domains. Then the pullback measure attractors can be shown based on some uniform estimates, the existence of pullback absorbing sets, and the pullback asymptotic compactness. Furthermore, we will discuss the limiting behaviour of the pullback measure attractors of (1.1) with $\varepsilon \to 0$.

This paper is organized as follows. In §2, some basic concepts and results of pullback measure attractors for non-autonomous dynamical systems are provided. In §3, we give some notions and make a transformation from the varying thin domain to a fixed domain. In §4, we derive some essential uniform estimates of the solutions for (1.1) and (1.3). Finally, we present the existence, uniqueness, and upper semi-continuity of pullback measure attractors of the stochastic equations.

2. Preliminaries

In this section, we introduce the useful results with the basic concepts of pullback measure attractors.

Let X be a separable Banach space with norm $||\cdot||_X$ and $\mathcal{P}(X)$ be the set of probability measures on $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Given $p > 0$, let $\mathcal{P}_p(X)$ be the subset of $\mathcal{P}(X)$ define by

$$\mathcal{P}_p(X) = \{\mu \in \mathcal{P}(X) : \int_X ||x||_X^p \mu(dx) < \infty\}.$$

Given $r > 0$, define

$$B_{\mathcal{P}_p(X)}(r) = \{\mu \in \mathcal{P}_p(X) : \left(\int_X ||x||_X^p \mu(dx)\right)^{\frac{1}{p}} < r\}.$$

Definition 2.1. *Let $\mathcal{D}$ a collection of some families of nonempty subsets of $\mathcal{P}_p(X)$, the set $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for a*

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continuous non-autonomous dynamical system S if for any $\tau \in \mathbb{R}$, $D \in \mathcal{D}$, there exists $T = T(\tau, D) > 0$,

$$S(t, \tau - t)D(\tau - t) \subseteq K(\tau), \ t \geqslant T.$$

Definition 2.2. *The non-autonomous dynamical system S is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{P}_p(X)$ if for any $\tau \in \mathbb{R}$, $\{S(t_n, \tau - t_n)\mu_n\}_{n=1}^{\infty}$ has a convergent subsequence in $\mathcal{P}_p(X)$ as $t_n \to +\infty$ and $\mu_n \in D(\tau - t_n)$ with $D \in \mathcal{D}$.*

Definition 2.3. *A family $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback measure attractor for S if the following conditions hold,*

(i) *$\mathcal{A}(\tau)$ is compact in $\mathcal{P}_p(X)$, $\forall \tau \in \mathbb{R}$;*

(ii) *$\mathcal{A}$ is invariant : $S(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(\tau + t)$, $\forall \tau \in \mathbb{R}$, $\forall t \in \mathbb{R}^+$;*

(iii) *$\mathcal{A}$ attracts all members of $\mathcal{D}$: for every $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$,*

$$\lim_{t \to \infty} d_X(S(t, \tau - t)D(\tau - t), \mathcal{A}(\tau)) = 0,$$

where d_X is the Hausdorff semidistance in X.

Definition 2.4. *A mapping $\psi : \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(X)$ is called a complete orbit of S if for any $s \in \mathbb{R}, t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$,*

$$S(t, s + \tau)\psi(s, \tau) = \psi(t + s, \tau).$$

If, in addition, there exists $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ such that $\psi(t, \tau) \in D(\tau + t)$, $\forall t \in \mathbb{R}$, $\forall \tau \in \mathbb{R}$, then ψ is called a $\mathcal{D}$-complete orbit of S.

Proposition 2.5. *Let $\mathcal{D}$ be a neighbourhood-closed collection of families of subsets of $\mathcal{P}_p(X)$ and S be a continuous non-autonomous dynamical system on $\mathcal{P}_p(X)$. If* S *has a closed $\mathcal{D}$-pullback absorbing set $K \in \mathcal{D}$ and S is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{P}_p(X)$, then S has a unique $\mathcal{D}$-pullback measure attractor $\mathcal{A}$ in $\mathcal{P}_p(X)$:*

$$\mathcal{A}(\tau) = \omega(K, \tau) = \{\psi(0, \tau) : \psi \text{ is a } \mathcal{D}\text{-complete orbit of } S\}, \ \tau \in \mathbb{R},$$

where $\omega(K, \tau) = \{\nu \in \mathcal{P}_p(X) : \text{ there exist } t_n \to \infty, \mu_n \in K(\tau - t_n) \text{ such that } \nu = \lim_{n \to \infty} S(t_n, \tau - t_n)\mu_n\}$.

3. Abstract formulation of stochastic equations

In this section, we present some assumptions on the nonlinear terms and reformulate (1.1) in the fixed domain $\mathcal{O}$.

Let $g \in C^2(\bar{\mathcal{Q}}, (0, +\infty))$ and $0 < \varepsilon \leqslant 1$, then there exist two positive constants g_1 and g_2 such that

$$g_1 \leqslant g(x^*) \leqslant g_2, \quad \forall x^* \in \bar{\mathcal{Q}}. \tag{3.1}$$

Denote $\mathcal{O} = \mathcal{Q} \times (0, 1)$ and $\widetilde{\mathcal{O}} = \mathcal{Q} \times [0, g_2]$ which contains $\mathcal{O}_\varepsilon$ for $0 < \varepsilon \leqslant 1$. We assume nonlinear drift term f satisfies some conditions such that for any $x \in \widetilde{\mathcal{O}}$ and $t, s \in \mathbb{R}$, there exist some positive constants α_1, α_2 and α_3,

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$$f(t,x,s)\,s \leqslant -\alpha_1 |s|^q + \psi_1(t,x), \tag{3.2}$$

$$|f(t,x,s)| \leqslant \alpha_2 |s|^{q-1} + \psi_2(t,x), \tag{3.3}$$

$$\frac{\partial f(t,x,s)}{\partial s} \leqslant \alpha_3, \tag{3.4}$$

$$\left|\frac{\partial f(t,x,s)}{\partial x}\right| \leqslant \psi_3(t,x), \tag{3.5}$$

where $q > p$, $\psi_1, \psi_2, \psi_3 \in L^\infty(\widetilde{\mathcal{O}})$. We assume that for $x = (x^*, x_{n+1}) \in \widetilde{\mathcal{O}}$,

$$\sum_{k=1}^{\infty} ||\sigma_k||^2_{L^\infty(\widetilde{\mathcal{O}})} < \infty, \tag{3.6}$$

$$\left|\nabla \sigma_k(x)\right| < \sigma_k^*(x), \tag{3.7}$$

where $\sigma_k^* \in L^\infty(\widetilde{\mathcal{O}})$ with $\sum\limits_{k=1}^{\infty} ||\sigma_k^*||^2_{L^\infty(\widetilde{\mathcal{O}})} < \infty$. For each $k \in \mathbb{N}$, there exist positive numbers L_k and γ_k such that for all $s, s_1, s_2 \in \mathbb{R}$,

$$|\rho_k(s_1) - \rho_k(s_2)| \leqslant L_k |s_1 - s_2|, \tag{3.8}$$

$$|\rho_k(s)| \leqslant \beta_k + \gamma_k |s|, \tag{3.9}$$

where $\beta_k = \rho_k(0)$,$\gamma_k = L_k$.

$$\sum_{k=1}^{\infty} (\beta_k^2 + \gamma_k^2 + L_k^2 + \rho_k^2(0)) < \infty, \tag{3.10}$$

$$\sum_{k=1}^{\infty} \gamma_k^2 < \frac{1}{4}\lambda. \tag{3.11}$$

The following results will be used later:

$$||u||^p_{L^p(\mathcal{O})} \leqslant \frac{q-p}{q-2} ||u||^2_{L^2(\mathcal{O})} + \frac{p-2}{q-2} ||u||^q_{L^q(\mathcal{O})}, \tag{3.12}$$

where $2 < p < q$.

$$\begin{aligned} \int_{-\infty}^{\tau} e^{\gamma t} (&\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \\ &+ \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}) ds < \infty. \end{aligned} \tag{3.13}$$

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We denote a transformation $T_\varepsilon : \mathcal{O}_\varepsilon \to \mathcal{O}$ by $T_\varepsilon(x^*, x_{n+1}) = \left(x^*, \frac{x_{n+1}}{\varepsilon g(x^*)}\right)$ for $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$. Let $y = (y^*, y_{n+1}) = T_\varepsilon(x^*, x_{n+1})$. Then we have

$$x^* = y^*, \qquad x_{n+1} = \varepsilon g\left(y^*\right) y_{n+1}.$$

The Jacobian matrix of T_ε is given by

$$\Lambda = \frac{\partial(y_1, \ldots, y_{n+1})}{\partial(x_1, \ldots, x_{n+1})} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ -\frac{y_{n+1}}{g} g_{y_1} & -\frac{y_{n+1}}{g} g_{y_2} & \cdots & -\frac{y_{n+1}}{g} g_{y_n} & \frac{1}{\varepsilon g(y^*)} \end{pmatrix}.$$

The determinant of Λ is $|\Lambda| = \frac{1}{\varepsilon g(y^*)}$. Let Λ^* be the transport of Λ, we get

$$\Lambda\Lambda^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\frac{y_{n+1}}{g} g_{y_1} \\ 0 & 1 & \cdots & 0 & -\frac{y_{n+1}}{g} g_{y_2} \\ & & \vdots & & \\ 0 & 0 & \cdots & 1 & -\frac{y_{n+1}}{g} g_{y_n} \\ -\frac{y_{n+1}}{g} g_{y_1} & -\frac{y_{n+1}}{g} g_{y_2} & \cdots & -\frac{y_{n+1}}{g} g_{y_n} & \sum_{i=1}^{n} \left(\frac{y_{n+1}}{g} g_{y_i}\right)^2 + \left(\frac{1}{\varepsilon g(y^*)}\right)^2 \end{pmatrix}.$$

In [16], the gradient operator and the gradient operator in $x \in \mathcal{O}_\varepsilon$ and $y \in \mathcal{O}$ are described by

$$\nabla_x u(x) = \Lambda^* \nabla_y u(y) \doteq \mathcal{R}_\varepsilon u(y)$$
$$\mathrm{div}_x \left(|\nabla u\left(x\right)|^{p-2} \nabla u\left(x\right) \right) = \frac{1}{\mathrm{g}} \mathrm{div}_y \left(|\mathcal{R}_\varepsilon u\left(y\right)|^{p-2} P_\varepsilon u\left(y\right) \right),$$

where ∇_x and div_x are the gradient operator and the divergence operator in $x \in \mathcal{O}_\varepsilon$, respectively, ∇_y and div_y are the gradient operator and the divergence operator in $y \in \mathcal{O}$, respectively, $\mathcal{R}_\varepsilon u(y) = \left(u_{y_1} - \frac{g_{y_1}}{g} y_{n+1} u_{y_{n+1}}, \ldots, u_{y_n} - \frac{g_{y_n}}{g} y_{n+1} u_{y_{n+1}}, \frac{1}{\varepsilon g} u_{y_{n+1}}\right)$. The operator P_ε is given by

$$P_\varepsilon u(y) = \begin{pmatrix} g u_{y_1} - g_{y_1} y_{n+1} u_{y_{n+1}} \\ \vdots \\ g u_{y_n} - g_{y_n} y_{n+1} u_{y_{n+1}} \\ -\sum_{i=1}^{n} y_{n+1} g_{y_i} u_{y_i} + \frac{1}{\varepsilon^2 g} \left(1 + \sum_{i=1}^{n} \left(\varepsilon y_{n+1} g_{y_i}\right)^2\right) u_{y_{n+1}} \end{pmatrix}.$$

In this paper, for $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$, $y = (y^*, y_{n+1}) \in \mathcal{O}$, and $t, s \in \mathbb{R}$, we define

$$f_\varepsilon(t, y^*, y_{n+1}, s) = f(y^*, \varepsilon g(y^*) y_{n+1}, s), \quad f_0(t, y^*, s) = f(y^*, 0, s),$$

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$$G_\varepsilon(t,y^*,y_{n+1},s)=G(y^*,\varepsilon g(y^*)y_{n+1},s),\quad G_0(t,y^*,s)=f(y^*,0,s),$$
$$\sigma_{k,\varepsilon}(y^*,y_{n+1})=\sigma_k(y^*,\varepsilon g(y^*)y_{n+1}),\quad \sigma_{k,0}(y^*)=\sigma_k(y^*,0).$$

Then the system (1.1) is equivalent to:

$$\begin{cases} du^\varepsilon-(\frac{1}{\mathrm{g}}\mathrm{div}_y\left(\left|\mathcal{R}_\varepsilon u\left(y\right)\right|^{p-2}P_\varepsilon u\left(y\right)\right)-\lambda u^\varepsilon)dt=\left(f_\varepsilon\left(t,y,u^\varepsilon\right)+G_\varepsilon\left(t,y\right)\right)dt \\ \qquad\qquad\qquad\qquad +\sum\limits_{k=1}^{\infty}(\sigma_{k,\varepsilon}(y)+\rho_k(u^\varepsilon(t)))dW_k(t),\quad t>0, \\ P_\varepsilon u^\varepsilon\cdot\nu=0,\quad y\in\partial\mathcal{O}, \end{cases} \tag{3.14}$$

with initial condition

$$u^\varepsilon(\tau,y)=u^\varepsilon_\tau(y)=\hat{u}^\varepsilon_\tau(T_\varepsilon^{-1}(y)),\quad y\in\mathcal{O}, \tag{3.15}$$

where ν is the unit outward normal vector to $\partial\mathcal{O}$.

We define the inner product $(\cdot,\cdot)_{L^2_g(\mathcal{O})}$ on $L^2(\mathcal{O})$

$$(u,v)_{L^2_g(\mathcal{O})}=\int_{\mathcal{O}}guvdy,\quad \text{for all } u,v\in L^2(\mathcal{O})$$

and denote by $L^2_g(\mathcal{O})$ the space equipped with this inner product. It is obvious that $L^2_g(\mathcal{O})$ is a Hilbert space with norm equivalent to the natural norm of $L^2(\mathcal{O})$ by the continuity of g and (3.1).

We introduce equivalent norms in $L^r(\mathcal{O})$, $r\geq 2$, and in $W^{1,r}(\mathcal{O})$ denoted by

$$\|u\|_{L^r_g(\mathcal{O})}=\left(\int_{\mathcal{O}}g\left|u\right|^r dy\right)^{\frac{1}{r}}$$

and

$$\|u\|_{H^r_\varepsilon(\mathcal{O})}=\left(\int_{\mathcal{O}}g\left(\left|\nabla_{y^*}u\right|^r+\frac{1}{\varepsilon^r}\left|u_{y+1}\right|^r+\left|u\right|^r\right)dy\right)^{\frac{1}{r}},$$

respectively.

Let $\|u\|_{\dot{H}^r_\varepsilon(\mathcal{O})}=\left(\int_{\mathcal{O}}g\left(\left|\nabla_{y^*}u\right|^r+\frac{1}{\varepsilon^r}\left|u_{y+1}\right|^r\right)dy\right)^{\frac{1}{r}}$. Then there exist positive constants ε_0, η_1 and η_2, for any $0<\varepsilon<\varepsilon_0$ and $u\in W^{1,r}(\mathcal{O})$,

$$\eta_1\|u\|^r_{\dot{H}^r_\varepsilon(\mathcal{O})}\leq\|\mathcal{R}_\varepsilon u\|^r_{L^r_g(\mathcal{O})}\leq\eta_2\|u\|^r_{\dot{H}^r_\varepsilon(\mathcal{O})} \tag{3.16}$$

and

$$\eta_1\|u\|^r_{H^r_\varepsilon(\mathcal{O})}\leq\|\mathcal{R}_\varepsilon u\|^r_{L^r_g(\mathcal{O})}+\|u\|^r_{L^r(\mathcal{O})}\leq\eta_2\|u\|^r_{H^r_\varepsilon(\mathcal{O})}. \tag{3.17}$$

Let A_ε be unbounded operator on $L^2_g(\mathcal{O})$ with

$$D\left(A_\varepsilon\right)=\left\{u\in W^{1,p}\left(\mathcal{O}\right),P_\varepsilon u\cdot\nu=0\text{ on }\partial\mathcal{O}\right\}$$

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denoted by

$$A_\varepsilon u = -\frac{1}{\mathrm{g}}\mathrm{div}_y\left(\left|\mathcal{R}_\varepsilon u\right|^{p-2}P_\varepsilon u\right).$$

Then we have

$$\left(\left|\mathcal{R}_\varepsilon u\right|^{p-2}\mathcal{R}_\varepsilon u, \mathcal{R}_\varepsilon v\right)_{L_g^2(\mathcal{O})} = \left(A_\varepsilon u, v\right)_{L_g^2(\mathcal{O})}, \quad \forall\, u \in D\left(A_\varepsilon\right), \forall\, v \in H^1(\mathcal{O}). \tag{3.18}$$

The system (3.14) and (3.15) is equivalent to

$$\begin{cases} du^\varepsilon + A_\varepsilon u^\varepsilon dt + \lambda u^\varepsilon dt = f_\varepsilon\left(t, y, u^\varepsilon\right)dt + G_\varepsilon\left(t, y\right)dt \\ \qquad\qquad + \sum\limits_{k=1}^{\infty}\left(\sigma_{k,\varepsilon}(y) + \rho_k(u^\varepsilon(t))\right)dW_k(t),\ y \in \mathcal{O},\, t > \tau, \\ u^\varepsilon\left(\tau\right) = u_\tau^\varepsilon. \end{cases} \tag{3.19}$$

To reformulate system (1.3) and (1.4), we introduce an inner product $(\cdot,\cdot)_{L_g^2(\mathcal{Q})}$ on $L^2(\mathcal{Q})$ given by

$$(u, v)_{L_g^2(\mathcal{Q})} = \int_{\mathcal{Q}} guv dy^*, \quad \text{for all } u, v \in L^2(\mathcal{Q}),$$

and denote by $L_g^2(\mathcal{Q})$ the space equipped with this inner product.

Let A_0 be unbounded operator on $L_g^2(\mathcal{Q})$ with

$$D\left(A_0\right) = \left\{u \in W^{1,p}\left(\mathcal{Q}\right), \frac{\partial u}{\partial \nu_0} = 0 \text{ on } \partial\mathcal{Q}\right\}$$

given by

$$A_0 u = -\frac{1}{g}\mathrm{div}_{y^*}(g|\nabla u|^{p-2}\nabla u).$$

Then we have

$$\left(|\nabla u|^{p-2}\nabla u, \nabla v\right)_{L_g^2(\mathcal{Q})} = \left(A_0 u, v\right)_{L_g^2(\mathcal{Q})}, \quad \forall\, u \in D\left(A_0\right), \forall\, v \in H^1(\mathcal{Q}).$$

The system (1.3) and (1.4) can be written as

$$\begin{cases} du^0 + A_0 u^0 dt + \lambda u^0 dt = f_0\left(t, y^*, u^0\right)dt + G_0\left(t, y^*\right)dt \\ \qquad\qquad + \sum\limits_{k=1}^{\infty}\left(\sigma_{k,0}(y^*) + \rho_k(u^0(t))\right)dW_k(t),\ y^* \in \mathcal{Q},\, t > \tau, \\ u^0\left(\tau\right) = u_\tau^0. \end{cases} \tag{3.20}$$

Given $\tau \in \mathbb{R}$, we use $L_{\mathcal{F}_\tau}^2\left(\Omega, L^2(\mathcal{O})\right)$ to denote the space of all $\mathcal{F}_\tau$-measurable, $L^2(\mathcal{O})$-valued random variables φ with $\mathbb{E}(||\varphi||_{L^2(\mathcal{O})}^2) < \infty$. By theorem 2.4 in [29], one can verify that if (3.2) and (3.11) hold, then for every $u_\tau^\varepsilon \in L_{\mathcal{F}_\tau}^2(\Omega, L^2(\mathcal{O}))$,

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system (3.19) has a unique solution $u^\varepsilon(t,\tau,u_\tau^\varepsilon)$ in the sense of Definition 2.3 in [29]. In particular, $u^\varepsilon(t,\tau,u_\tau^\varepsilon)$, $t>\tau$, is a continuous $L^2(\mathcal{O})$-valued $\mathcal{F}_\tau$-adapted stochastic process, then for every $T>0$,

$$\begin{aligned} u^\varepsilon \in & L^2(\Omega, C([\tau,\tau+T],L^2(\mathcal{O}))) \bigcap L^q(\Omega, L^q(\tau,\tau+T;L^q(\mathcal{O}))) \\ & \bigcap L^p(\Omega, L^p(\tau,\tau+T;W^{1,p}(\mathcal{O}))). \end{aligned}$$

Similarly, for every $u_\tau^0 \in L^2_{\mathcal{F}_\tau}(\Omega, L^2(\mathcal{Q}))$, system (3.20) has a unique solution $u^0(t,\tau,u_\tau^0)$ which is a continuous $L^2(\mathcal{Q})$-valued $\mathcal{F}_\tau$-adapted stochastic process, for every $T>0$,

$$\begin{aligned} u^0 \in & L^2(\Omega, C([\tau,\tau+T],L^2(\mathcal{Q}))) \bigcap L^q(\Omega, L^q(\tau,\tau+T;L^q(\mathcal{Q}))) \\ & \bigcap L^2(\Omega, L^2(\tau,\tau+T;W^{1,p}(\mathcal{Q}))). \end{aligned}$$

4. Uniform estimates

In this section, in order to present the existence and convergence of pullback measure attractors, we first give some essential uniform estimates of solutions of (3.19) and (3.20). Here, we define $\mathcal{L}(\xi)$ as the distribution law of a random variable ξ.

LEMMA 4.1. *Assume that* (3.2)-(3.11) *and* (3.13) *hold. Then for every* $\tau\in\mathbb{R}$ *and* $D_1=\{D_1(t):t\in\mathbb{R}\}\in\mathcal{D}_1$, *there exists* $T=T(\tau,D_1)>0$, *independent of* ε, *such that for all* $t\geq T$ *and* $0<\varepsilon<\varepsilon_0$, *the solution* u^ε *of* (3.19) *satisfies*

$$\begin{aligned} &\mathbb{E}\Big(\|u^\varepsilon(\tau,\tau-t,u_\tau^\varepsilon)\|^2_{L^2_g(\mathcal{O})}\Big) \\ &\leq M_1\Big(1+\int_{-\infty}^{\tau} e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})}+\|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds\Big) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)}\mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)||^p_{L^p_g(\mathcal{O})}\Big)ds \\ &+\int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)}\mathbb{E}\Big(||u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)||^q_{L^q_g(\mathcal{O})}\Big)ds \\ &\leq M_1\Big(1+\int_{-\infty}^{\tau} e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})}+\|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds\Big), \end{aligned} \tag{4.2}$$

where $u_\tau^\varepsilon\in L^2_{\mathcal{F}_{\tau-t}}(\Omega,L^2(\mathcal{O}))$ *with* $\mathcal{L}(u_\tau^\varepsilon)\in D_1(\tau-t)$, M_1 *is a positive constant independent of* τ, ε *and* D_1.

Proof. By (3.19) and Ito's formula, for $t\geqslant\tau$

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$$\begin{aligned}
&\|u^{\varepsilon}(t)\|^2_{L^2_g(\mathcal{O})} + 2\int_{\tau}^{t} ||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||^p_{L^p_g(\mathcal{O})}ds + 2\lambda\int_{\tau}^{t}\|u^{\varepsilon}(s)\|^2_{L^2_g(\mathcal{O})}ds \\
&= \|u^{\varepsilon}_{\tau}\|^2_{L^2_g(\mathcal{O})} + 2\int_{\tau}^{t}\left(f_{\varepsilon}\left(t,y,u^{\varepsilon}(s)\right),u^{\varepsilon}(s)\right)_{L^2_g(\mathcal{O})}ds + 2\int_{\tau}^{t}\left(G_{\varepsilon},u^{\varepsilon}(s)\right)_{L^2_g(\mathcal{O})}ds \\
&+ \sum_{k=1}^{\infty}\int_{\tau}^{t}||\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(s))||^2_{L^2_g(\mathcal{O})}ds + 2\sum_{k=1}^{\infty}\int_{\tau}^{t}(\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(s)),u^{\varepsilon}(s))_{L^2_g(\mathcal{O})}dW_k(s),
\end{aligned} \tag{4.3}$$

and we further have

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\Big(\|u^{\varepsilon}(t)\|^2_{L^2_g(\mathcal{O})}\Big) + 2\mathbb{E}\Big(||\mathcal{R}_{\varepsilon}u^{\varepsilon}(t)||^p_{L^p_g(\mathcal{O})}\Big) + 2\lambda\mathbb{E}\Big(\|u^{\varepsilon}(t)\|^2_{L^2_g(\mathcal{O})}\Big) \\
&= 2\mathbb{E}\Big(\left(f_{\varepsilon}\left(t,y,u^{\varepsilon}(t)\right),u^{\varepsilon}(t)\right)_{L^2_g(\mathcal{O})}\Big) + 2\mathbb{E}\Big(\left(G_{\varepsilon},u^{\varepsilon}(t)\right)_{L^2_g(\mathcal{O})}\Big) \\
&+ \sum_{k=1}^{\infty}\mathbb{E}\Big(||\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(t))||^2_{L^2_g(\mathcal{O})}\Big).
\end{aligned} \tag{4.4}$$

For the nonlinear term, by (3.2), we obtain

$$\begin{aligned}
&\left(f_{\varepsilon}\left(y,u^{\varepsilon}(t)\right),u^{\varepsilon}(t)\right)_{L^2_g(\mathcal{O})} \\
&= \int_{\mathcal{O}} gf\left(t,y^*,\varepsilon g(y^*)y_{n+1},u^{\varepsilon}(t)\right)u^{\varepsilon}(t)dy \\
&\leqslant -\alpha_1\int_{\mathcal{O}} g\,|u^{\varepsilon}(t)|^q dy + \int_{\mathcal{O}} g\psi_1\left(t,y^*,\varepsilon g(y^*)y_{n+1}\right)dy \\
&\leqslant -\alpha_1\int_{\mathcal{O}} g\,|u^{\varepsilon}(t)|^q dy + g_2|\mathcal{O}|\|\psi_1(t,\cdot)\|_{L^{\infty}(\widetilde{\mathcal{O}})},
\end{aligned} \tag{4.5}$$

where $|\mathcal{O}|$ is the Lebesgue measure of $\mathcal{O}$. By Young's inequality,

$$\begin{aligned}
&\left(G_{\varepsilon},u^{\varepsilon}(t)\right)_{L^2_g(\mathcal{O})} \\
&\leqslant ||G_{\varepsilon}||_{L^2_g(\mathcal{O})}||u^{\varepsilon}(t)||_{L^2_g(\mathcal{O})} \\
&\leqslant \frac{\lambda}{4}\|u^{\varepsilon}(t)\|^2_{L^2_g(\mathcal{O})} + \frac{g_2}{\lambda}|\mathcal{O}|\|G(t,\cdot)\|^2_{L^{\infty}(\widetilde{\mathcal{O}})}.
\end{aligned} \tag{4.6}$$

By (3.7) and (3.11) we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty}||\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(t))||^2_{L^2_g(\mathcal{O})} \\
&\leqslant 2\sum_{k=1}^{\infty}||\sigma_{k,\varepsilon}||^2_{L^2_g(\mathcal{O})} + 2\sum_{k=1}^{\infty}|||\beta_k(\cdot)+\gamma_k|u^{\varepsilon}(t)|||^2_{L^2_g(\mathcal{O})} \\
&\leqslant 2g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^2_{L^{\infty}(\widetilde{\mathcal{O}})} + 4g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2 + 4||u^{\varepsilon}(t)||^2_{L^2_g(\mathcal{O})}\sum_{k=1}^{\infty}\gamma_k^2.
\end{aligned} \tag{4.7}$$

Then it follows from (4.4) and (4.7) that

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$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\Big(\|u^\varepsilon(t)\|^2_{L^2_g(\mathcal{O})}\Big)+2\mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(t)||^p_{L^p_g(\mathcal{O})}\Big)+(\lambda-4\sum_{k=1}^{\infty}\gamma_k^2)\mathbb{E}\Big(\|u^\varepsilon(t)\|^2_{L^2_g(\mathcal{O})}\Big)\\
&+2\alpha_1 g_1\mathbb{E}\Big(||u^\varepsilon(t)||^q_{L^q_g(\mathcal{O})}\Big)\\
&\leqslant 2g_2|\mathcal{O}|\|\psi_1(t,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})}+\frac{2g_2}{\lambda}|\mathcal{O}|\|G(t,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})}+2g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^2_{L^\infty(\tilde{\mathcal{O}})}\\
&+4g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2. \qquad (4.8)
\end{aligned}$$

Multiplying (4.8) by $e^{\gamma t}$ and then integrating on $(\tau-t,\tau)$, we get

$$\begin{aligned}
&\mathbb{E}\Big(\|u^\varepsilon(\tau,\tau-t,u^\varepsilon_\tau)\|^2_{L^2_g(\mathcal{O})}\Big)+2\int_{\tau-t}^{\tau}e^{-\gamma(\tau-s)}\mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^p_{L^p_g(\mathcal{O})}\Big)ds\\
&+2\alpha_1 g_1\int_{\tau-t}^{\tau}e^{-\gamma(\tau-s)}\mathbb{E}\Big(||u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^q_{L^q_g(\mathcal{O})}\Big)ds\\
&\leqslant e^{-\gamma t}\mathbb{E}\Big(||u^\varepsilon_\tau||^2_{L^2_g(\mathcal{O})}\Big)+2g_2|\mathcal{O}|\int_{\tau-t}^{\tau}e^{-\gamma(\tau-s)}\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})}ds\\
&+\frac{2g_2}{\lambda}|\mathcal{O}|\int_{\tau-t}^{\tau}e^{-\gamma(\tau-s)}\|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})}ds\\
&+\frac{1}{\gamma}\Big(2g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^2_{L^\infty(\tilde{\mathcal{O}})}+4g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2\Big)\\
&\leqslant e^{-\gamma t}\mathbb{E}\Big(||u^\varepsilon_\tau||^2_{L^2_g(\mathcal{O})}\Big)+2g_2|\mathcal{O}|\int_{-\infty}^{\tau}e^{-\gamma(\tau-s)}\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})}ds\\
&+\frac{2g_2}{\lambda}|\mathcal{O}|\int_{-\infty}^{\tau}e^{-\gamma(\tau-s)}\|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})}ds\\
&+\frac{1}{\gamma}\Big(2g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^2_{L^\infty(\tilde{\mathcal{O}})}+4g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2\Big), \qquad (4.9)
\end{aligned}$$

where $\gamma=\lambda-4\sum\limits_{k=1}^{\infty}\gamma_k^2$. Since $\mathcal{L}(u^\varepsilon_\tau)\in D_1(\tau-t)$ we have

$$\begin{aligned}
e^{-\gamma t}\mathbb{E}\Big(||u^\varepsilon_\tau||^2_{L^2_g(\mathcal{O})}\Big)&\leqslant g_2e^{-\gamma t}\mathbb{E}\Big(||u^\varepsilon_\tau||^2_{L^2(\mathcal{O})}\Big)\\
&\leqslant g_2e^{-\gamma t}||D_1(\tau-t)||^2_{L^2(\mathcal{O})}\to 0,\ as\ t\to 0,
\end{aligned}$$

then there exists $T=T(\tau,D_1)>0$, for all $t\geq T$,

$$e^{-\gamma t}\mathbb{E}\Big(||u^\varepsilon_\tau||^2_{L^2_g(\mathcal{O})}\Big)\leqslant\frac{1}{\gamma}(2g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^2_{L^\infty(\tilde{\mathcal{O}})}+4g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2).$$

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This proof is completed. □

By lemma 4.1, we have the following result.

LEMMA 4.2. *Assume that (3.2)-(3.11) and (3.13) hold. Then for every $\tau \in \mathbb{R}$ and $D_1 = \{D_1(t) : t \in \mathbb{R}\} \in D_1$, there exists $T = T(\tau, D_1) > 1$, independent of ε, such that for all $t \geq T$ and $0 < \varepsilon < \varepsilon_0$, the solution u^ε of (3.19) satisfies*

$$\begin{aligned}&\int_{\tau-1}^{\tau} \mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(s, \tau - t, u_\tau^\varepsilon)||^p_{L^p_g(\mathcal{O})}\Big)ds\\ &\leqslant M_2(1 + \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})})ds),\end{aligned} \tag{4.10}$$

$$\begin{aligned}&\int_{\tau-1}^{\tau} \mathbb{E}\Big(||u^\varepsilon(s, \tau - t, u_\tau^\varepsilon)||^q_{L^q_g(\mathcal{O})}\Big)ds\\ &\leqslant M_2(1 + \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})})ds),\end{aligned} \tag{4.11}$$

where $u_\tau^\varepsilon \in L^2_{\mathcal{F}_{\tau-t}}(\Omega, L^2(\mathcal{O}))$ with $\mathcal{L}(u_\tau^\varepsilon) \in D_1(\tau - t)$, M_2 is a positive constant independent of τ, ε and D_1.

LEMMA 4.3. *Assume that (3.2)-(3.11) and (3.13) hold. Then for every $\tau \in \mathbb{R}$, $T > 0$, $u_\tau^\varepsilon \in L^2_{\mathcal{F}_\tau}(\Omega, L^2(\mathcal{O}))$ and $0 < \varepsilon < \varepsilon_0$, for all $t \in [\tau, \tau + T]$, the solution u^ε of (3.19) satisfies*

$$\begin{aligned}&\int_{\tau}^{t} \mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(s, \tau - t, u_\tau^\varepsilon)||^p_{L^p_g(\mathcal{O})}\Big)ds\\ &\leqslant M_3(1 + \mathbb{E}\Big(||u_\tau^\varepsilon||^2_{L^2_g(\mathcal{O})}\Big) + \int_{\tau}^{\tau+T} (\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})})ds),\end{aligned} \tag{4.12}$$

$$\begin{aligned}&\int_{\tau}^{t} \mathbb{E}\Big(||u^\varepsilon(s, \tau - t, u_\tau^\varepsilon)||^q_{L^q_g(\mathcal{O})}\Big)ds\\ &\leqslant M_3(1 + \mathbb{E}\Big(||u_\tau^\varepsilon||^2_{L^2_g(\mathcal{O})}\Big) + \int_{\tau}^{\tau+T} (\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})})ds),\end{aligned} \tag{4.13}$$

where M_3 is a positive constant independent of τ, T and ε.

Similar to lemma 4.3, we have the following uniform estimates of solutions of (3.20).

LEMMA 4.4. *Assume that (3.2)-(3.11) and (3.13) hold. Then for every $\tau \in \mathbb{R}$, $T > 0$, $u_\tau^0 \in L^2_{\mathcal{F}_\tau}(\Omega, L^2(\mathcal{Q}))$, for all $t \in [\tau, \tau + T]$, the solution u^0 of (3.20) satisfies*

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$$\int_\tau^t \mathbb{E}\Big(||\nabla_{y^*} u^0(s,\tau-t,u^0_\tau)||^p_{L^p_g(\mathcal{Q})} + ||u^0(s,\tau-t,u^0_\tau)||^p_{L^p_g(\mathcal{Q})}\Big)ds$$
$$\leqslant M_4(1+\mathbb{E}\Big(||u^0_\tau||^2_{L^2_g(\mathcal{Q})}\Big) + \int_\tau^{\tau+T}(\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds), \quad (4.14)$$

$$\int_\tau^t \mathbb{E}\Big(||u^0(s,\tau-t,u^0_\tau)||^q_{L^q_g(\mathcal{Q})}\Big)ds$$
$$\leqslant M_4(1+\mathbb{E}\Big(||u^0_\tau||^2_{L^2_g(\mathcal{Q})}\Big) + \int_\tau^{\tau+T}(\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds), \quad (4.15)$$

where M_4 *is a positive constants independent of* τ*,* T *and* ε*.*

In order to derive the uniform estimates of solutions in $H^p_\varepsilon(\mathcal{O})$, we need the following conclusion.

LEMMA 4.5. *Assume that* (3.2)-(3.11) *and* (3.13) *hold. Then for every* $\tau\in\mathbb{R}$ *and* $D_1=\{D_1(t):t\in\mathbb{R}\}\in D_1$*, there exists* $T=T(\tau,D_1)>1$*, independent of* ε*, such that for all* $t\geq T$ *and* $0<\varepsilon<\varepsilon_0$*, the solution* u^ε *of* (3.19) *satisfies*

$$\mathbb{E}\Big(||u^\varepsilon(\tau,\tau-t,u^\varepsilon_\tau)||^q_{L^q_g(\mathcal{O})}\Big)$$
$$\leqslant M_5(1+\int_{-\infty}^\tau e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds), \quad (4.16)$$

and

$$\int_{\tau-1}^\tau \mathbb{E}\Big(\|u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)\|^{2q-2}_{L^{2q-2}(\mathcal{O})}\Big)ds$$
$$\leqslant M_5(1+\int_{-\infty}^\tau e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds), \quad (4.17)$$

where M_5 *is a positive constant independent of* τ*,* D_1 *and* ε*.*

Proof. By Ito's formula, for $t>1$ and $\sigma\in(\tau-1,\tau)$,

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$$
\begin{aligned}
&\int_{\mathcal{O}} g|(u^{\varepsilon}(\tau,\tau-t,u_{\tau}^{\varepsilon}))^2|^{\frac{q}{2}}dy+\lambda q\int_{\sigma}^{\tau}||u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})||_{L_g^q(\mathcal{O})}^q ds\\
&+q\int_{\sigma}^{\tau}(A_{\varepsilon}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}),|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2}\,u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}))_{L_g^2(\mathcal{O})}ds\\
&=\int_{\mathcal{O}} g|(u^{\varepsilon}(\sigma))^2|^{\frac{q}{2}}dy+q\int_{\sigma}^{\tau}\left(f_{\varepsilon}\left(s,\cdot,u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})\right),|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2}\right.\\
&\left.\times\,u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})\right)_{L_g^2(\mathcal{O})}ds\\
&+q\int_{\sigma}^{\tau}\left(G_{\varepsilon}(s,\cdot),|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})\right)_{L_g^2(\mathcal{O})}ds\\
&+\frac{q(q-1)}{2}\sum_{k=1}^{\infty}\int_{\sigma}^{\tau}(|\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}))|^2,|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2})_{L_g^2(\mathcal{O})}ds\\
&+q\sum_{k=1}^{\infty}\int_{\sigma}^{\tau}(\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})),|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}))_{L_g^2(\mathcal{O})}\\
&\times dW_k(s).
\end{aligned}
\tag{4.18}
$$

Note that

$$
\begin{aligned}
&\int_{\sigma}^{\tau}(A_{\varepsilon}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}),|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2}\,u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}))_{L_g^2(\mathcal{O})}ds\\
&=(q-1)\int_{\sigma}^{\tau}\left(|\mathcal{R}_{\varepsilon}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{p-2}\,\mathcal{R}_{\varepsilon}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}),|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2}\right.\\
&\left.\times\,\mathcal{R}_{\varepsilon}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})\right)_{L_g^2(\mathcal{O})}ds\\
&\geqslant 0.
\end{aligned}
\tag{4.19}
$$

By (3.2) and Young's inequality,

$$
\begin{aligned}
&q\int_{\sigma}^{\tau}(f_{\varepsilon}(t,\cdot,u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})),|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})|^{q-2}\,u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}))_{L_g^2(\mathcal{O})}ds\\
&\leq -q\alpha_1 g_1\int_{\sigma}^{\tau}\|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})\|_{L^{2q-2}(\mathcal{O})}^{2q-2}\,ds+\frac{3}{8}\lambda q\int_{\sigma}^{\tau}\|u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})\|_{L_g^q(\mathcal{O})}^q ds\\
&+\frac{\left(\frac{3\lambda q}{4(q-2)}\right)^{\frac{q-2}{2}}}{q}g_2|\mathcal{O}|\int_{\sigma}^{\tau}\|\psi_1(s,\cdot)\|_{L^{\infty}(\widetilde{\mathcal{O}})}^{\frac{q}{2}}ds.
\end{aligned}
\tag{4.20}
$$

Similarly, the last two terms are bounded by

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$$
\begin{aligned}
&\frac{q(q-1)}{2}\int_\sigma^\tau \sum_{k=1}^{\infty}(|\sigma_{k,\varepsilon}+\rho_k(u^\varepsilon(s))|^2,|u^\varepsilon(s)|^{q-2})_{L_g^2(\mathcal{O})}ds\\
&\leqslant \frac{q(q-1)}{2}\int_\sigma^\tau\Big(\frac{\lambda}{8(q-1)}||u^\varepsilon(s)||^q_{L_g^q(\mathcal{O})}+\frac{2(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\\
&\times\Big(\sum_{k=1}^{\infty}(||\sigma_{k,\varepsilon}+\rho_k(u^\varepsilon(s))||^2_{L_g^2(\mathcal{O})})^{\frac{q}{2}}\Big)ds\\
&\leqslant \frac{q(q-1)}{2}\int_\sigma^\tau\Big(\frac{\lambda}{8(q-1)}||u^\varepsilon(s)||^q_{L_g^q(\mathcal{O})}+\frac{2^q(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\Big(\sum_{k=1}^{\infty}||\sigma_{k,\varepsilon}||^2_{L_g^2(\mathcal{O})}\Big)^{\frac{q}{2}}\\
&+\frac{2^q(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\Big(\sum_{k=1}^{\infty}||\beta_k+\gamma_k|u^\varepsilon(s)|||^2_{L_g^2(\mathcal{O})}\Big)^{\frac{q}{2}}\Big)ds\\
&\leqslant \frac{q(q-1)}{2}\int_\sigma^\tau\Big(\frac{\lambda}{8(q-1)}||u^\varepsilon(s)||^q_{L_g^q(\mathcal{O})}+\frac{2^q(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\Big(\sum_{k=1}^{\infty}||\sigma_{k,\varepsilon}||^2_{L_g^2(\mathcal{O})}\Big)^{\frac{q}{2}}\\
&+\frac{2^{2q-1}(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\Big[\Big(g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2\Big)^{\frac{q}{2}}+||u^\varepsilon(s)||^q_{L_g^q(\mathcal{O})}\Big(\sum_{k=1}^{\infty}\gamma_k^2\Big)^{\frac{q}{2}}\Big]\Big)ds\\
&\leqslant \frac{1}{16}\lambda q\int_\sigma^\tau||u^\varepsilon(s)||^q_{L_g^q(\mathcal{O})}ds+\frac{2^{q-1}q(q-1)(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\Big(g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)^{\frac{q}{2}}\\
&\times(\tau-\sigma)+\frac{2^{2q-2}q(q-1)(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\Big(g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2\Big)^{\frac{q}{2}}(\tau-\sigma)\\
&+\frac{2^{2q-2}q(q-1)(\frac{\lambda q}{8(q-1)(q-2)})^{\frac{q-2}{2}}}{q}\Big(\sum_{k=1}^{\infty}\gamma_k^2\Big)^{\frac{q}{2}}\int_\sigma^\tau||u^\varepsilon(s)||^q_{L_g^q(\mathcal{O})}ds
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
&q\int_\sigma^\tau(G_\varepsilon(s,\cdot),|u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)|^{q-2}u^\varepsilon(s,\tau-t,u^\varepsilon_\tau))_{L_g^2(\mathcal{O})}ds\\
&\leqslant\frac{q\alpha_1 g_1}{2}\int_\sigma^\tau\|u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)\|^{2q-2}_{L^{2q-2}(\mathcal{O})}\,ds+\frac{qg_2}{2\alpha_1}|\mathcal{O}|\int_\sigma^\tau\|G\,(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}ds.
\end{aligned}
\tag{4.22}
$$

Combining (4.18) and (4.22), we get

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$$\begin{aligned}
&\mathbb{E}\Big(||u^{\varepsilon}(\tau,\tau-t,u^{\varepsilon}_{\tau})||^{q}_{L^{q}_{g}(\mathcal{O})}\Big)+q\alpha_1 g_1\int_{\sigma}^{\tau}\mathbb{E}\Big(\|u^{\varepsilon}(s,\tau-t,u^{\varepsilon}_{\tau})\|^{2q-2}_{L^{2q-2}(\mathcal{O})}\Big)ds\\
&\leqslant \mathbb{E}\Big(||u^{\varepsilon}(\sigma)||^{q}_{L^{q}_{g}(\mathcal{O})}\Big)+C_1\Big(\int_{\sigma}^{\tau}\mathbb{E}\Big(||u^{\varepsilon}(s,\tau-t,u^{\varepsilon}_{\tau})||^{q}_{L^{q}_{g}(\mathcal{O})}\Big)ds\\
&+g_2|\mathcal{O}|\int_{\sigma}^{\tau}\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^{\infty}(\widetilde{\mathcal{O}})}ds+g_2|\mathcal{O}|\int_{\sigma}^{\tau}\|G(s,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}ds\\
&+\Big(g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big)^{\frac{q}{2}}(\tau-\sigma)+\Big(g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2\Big)^{\frac{q}{2}}(\tau-\sigma)\Big).
\end{aligned}\tag{4.23}$$

Now integrating the above with respect to σ over $(\tau-1,\tau)$,

$$\begin{aligned}
&\mathbb{E}\Big(||u^{\varepsilon}(\tau,\tau-t,u^{\varepsilon}_{\tau})||^{q}_{L^{q}_{g}(\mathcal{O})}\Big)+q\alpha_1 g_1\int_{\tau-1}^{\tau}\mathbb{E}\Big(\|u^{\varepsilon}(s,\tau-t,u^{\varepsilon}_{\tau})\|^{2q-2}_{L^{2q-2}(\mathcal{O})}\Big)ds\\
&\leqslant (1+C_1)\int_{\tau-1}^{\tau}\mathbb{E}\Big(||u^{\varepsilon}(s,\tau-t,u^{\varepsilon}_{\tau})||^{q}_{L^{q}_{g}(\mathcal{O})}\Big)ds+C_1\Big(g_2|\mathcal{O}|\int_{\tau-1}^{\tau}\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^{\infty}(\widetilde{\mathcal{O}})}ds\\
&+(g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^{2}_{L^{\infty}(\widetilde{\mathcal{O}})})^{\frac{q}{2}}(\tau-\sigma)+(g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2)^{\frac{q}{2}}(\tau-\sigma)\Big)\\
&\leq (1+C_1)\int_{\tau-1}^{\tau}e^{-\gamma(\tau-s)}\mathbb{E}\Big(||u^{\varepsilon}(s,\tau-t,u^{\varepsilon}_{\tau})||^{q}_{L^{q}_{g}(\mathcal{O})}\Big)ds\\
&+C_1\Big(g_2|\mathcal{O}|\int_{\tau-1}^{\tau}e^{-\gamma(\tau-s)}\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^{\infty}(\widetilde{\mathcal{O}})}ds+g_2|\mathcal{O}|\int_{\tau-1}^{\tau}e^{-\gamma(\tau-s)}\|G(s,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}ds\\
&+\Big(g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k||^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big)^{\frac{q}{2}}(\tau-\sigma)+(g_2|\mathcal{O}|\sum_{k=1}^{\infty}\beta_k^2)^{\frac{q}{2}}\Big).
\end{aligned}\tag{4.24}$$

which together with lemma 4.2 completes the proof. □

LEMMA 4.6. *Assume that (3.2)-(3.11) and (3.13) hold. Then for every $\tau\in\mathbb{R}$ and $D_1=\{D_1(t):t\in\mathbb{R}\}\in D_1$, there exists $T=T(\tau,D_1)>1$, independent of ε, such that for all $t\geq T$ and $0<\varepsilon<\varepsilon_0$, the solution u^{ε} of (3.19) satisfies*

$$\begin{aligned}
&\mathbb{E}\Big(\|u^{\varepsilon}(\tau,\tau-t,u^{\varepsilon}_{\tau})\|^{p}_{H^{p}_{\varepsilon}(\mathcal{O})}\Big)\\
&\leqslant M_6\Big(1+\int_{-\infty}^{\tau}e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^{\infty}(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|_{L^{\infty}(\widetilde{\mathcal{O}})}\\
&+\|\psi_2(s,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}+\|G(s,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})})ds\Big),
\end{aligned}\tag{4.25}$$

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where M_6 *is a positive constant independent of* τ, D_1 *and* ε.

Proof. From (3.19) and Ito's formula, for $t>1$ and $\sigma \in (\tau-1,\tau)$,

$$\begin{aligned}
&\int_{\mathcal{O}} g|(\mathcal{R}_\varepsilon u^\varepsilon(\tau,\tau-t,u_\tau^\varepsilon))^2|^{\frac{p}{2}}dy + p\int_\sigma^\tau ||A_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)||^2_{L^2_g(\mathcal{O})}ds\\
&+\lambda p\int_\sigma^\tau ||\mathcal{R}_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)||^p_{L^p_g(\mathcal{O})}ds\\
&\leqslant \int_{\mathcal{O}} g|(\mathcal{R}_\varepsilon u^\varepsilon(\sigma))^2|^{\frac{p}{2}}dy + p\int_\sigma^\tau \left(f_\varepsilon\left(s,\cdot,u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)\right),A_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)\right)_{L^2_g(\mathcal{O})}\\
&\times ds\\
&+\frac{p(p-1)}{2}\sum_{k=1}^\infty\int_\sigma^\tau \Big(\mathcal{R}_\varepsilon(\sigma_{k,\varepsilon}+\rho_k(u^\varepsilon(s,\tau-t,u_\tau^\varepsilon))),|\mathcal{R}_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)|^{p-2}\\
&\times \mathcal{R}_\varepsilon(\sigma_{k,\varepsilon}+\rho_k(u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)))\Big)_{L^2_g(\mathcal{O})}ds\\
&+p\int_\sigma^\tau (G_\varepsilon(s,\cdot),A_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon))_{L^2_g(\mathcal{O})}\,ds\\
&+p\sum_{k=1}^\infty\int_\sigma^\tau (\sigma_{k,\varepsilon}+\rho_k(u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)),A_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon))_{L^2_g(\mathcal{O})}dW_k(s).
\end{aligned} \tag{4.26}$$

For the second term on the right-hand side of (4.26), by (3.3), we have

$$\begin{aligned}
&p\int_\sigma^\tau \left(f_\varepsilon\left(s,\cdot,u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)\right),A_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)\right)_{L^2_g(\mathcal{O})}ds\\
&\leqslant pg_2\alpha_2^2\int_\sigma^\tau ||u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)||^{2q-2}_{L^{2q-2}(\mathcal{O})}ds + p|\mathcal{O}|\int_\sigma^\tau \|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}ds
\end{aligned} \tag{4.27}$$

$$+\frac{p}{2}\int_\sigma^\tau ||A_\varepsilon u^\varepsilon(s,\tau-t,u_\tau^\varepsilon)||^2_{L^2_g(\mathcal{O})}ds. \tag{4.28}$$

The third term on the right-hand side of (4.26) is bounded by

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$$
\begin{aligned}
&\frac{p(p-1)}{2}\sum_{k=1}^{\infty}\int_{\sigma}^{\tau}\Big(\mathcal{R}_{\varepsilon}(\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(s))),|\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)|^{p-2}\mathcal{R}_{\varepsilon}(\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(s)))\Big)_{L_g^2(\mathcal{O})}ds\\
&\leqslant\int_{\sigma}^{\tau}\Big(\frac{\lambda p}{16}||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||_{L_g^p(\mathcal{O})}^p+\frac{p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{q}\\
&\times\Big(\sum_{k=1}^{\infty}(||\mathcal{R}_{\varepsilon}(\sigma_{k,\varepsilon}+\rho_k(u^{\varepsilon}(s)))||_{L_g^2(\mathcal{O})}^2)^{\frac{p}{2}}\Big)ds\\
&\leqslant\int_{\sigma}^{\tau}\Big(\frac{\lambda p}{16}||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||_{L_g^p(\mathcal{O})}^p+\frac{2^{p-1}p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{p}\Big(\sum_{k=1}^{\infty}||\sigma_{k,\varepsilon}^*||_{L_g^2(\mathcal{O})}^2\Big)^{\frac{p}{2}}\\
&+\frac{2^{p-1}p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{p}\Big(\sum_{k=1}^{\infty}L_k^2\Big)^{\frac{p}{2}}\int_{\sigma}^{\tau}||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||_{L_g^p(\mathcal{O})}^p ds\\
&+\frac{2^{p-1}p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{p}\Big(\sum_{k=1}^{\infty}L_k^2||\mathcal{R}_{\varepsilon}|u^{\varepsilon}(s)|||_{L_g^2(\mathcal{O})}^2)^{\frac{p}{2}}\Big)ds\\
&\leqslant\int_{\sigma}^{\tau}\Big(\frac{\lambda p}{16}||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||_{L_g^p(\mathcal{O})}^p+\frac{2^{p-1}p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{p}\Big(\sum_{k=1}^{\infty}||\sigma_{k,\varepsilon}^*||_{L_g^2(\mathcal{O})}^2\Big)^{\frac{p}{2}}\\
&+\frac{2^{p-1}p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{p}||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||_{L_g^p(\mathcal{O})}^p\Big(\sum_{k=1}^{\infty}L_k^2)^{\frac{p}{2}}\Big)ds\\
&\leqslant\frac{1}{16}\lambda p\int_{\tau}^{t+1}||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||_{L_g^p(\mathcal{O})}^p ds\\
&+\frac{2^{p-1}p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{p}\Big(g_2|\mathcal{O}|\sum_{k=1}^{\infty}||\sigma_k^*||_{L^{\infty}(\tilde{\mathcal{O}})}^2\Big)^{\frac{p}{2}}(\tau-\sigma)\\
&+\frac{2^{p-1}p(p-1)(\frac{\lambda p}{8(p-1)(p-2)})^{\frac{p-2}{2}}}{p}\Big(\sum_{k=1}^{\infty}L_k^2\Big)^{\frac{p}{2}}\int_{\sigma}^{\tau}||\mathcal{R}_{\varepsilon}u^{\varepsilon}(s)||_{L_g^p(\mathcal{O})}^p ds. \qquad (4.29)
\end{aligned}
$$

For the fourth term on the right-hand side of (4.26),

$$
\begin{aligned}
&2p\int_{\sigma}^{\tau}(G_{\varepsilon}(s,\cdot),A_{\varepsilon}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon}))_{L_g^2(\mathcal{O})}\,ds\\
&\leqslant p\int_{\sigma}^{\tau}||A_{\varepsilon}u^{\varepsilon}(s,\tau-t,u_{\tau}^{\varepsilon})||_{L_g^2(\mathcal{O})}^2 ds+pg_2|\mathcal{O}|\int_{\sigma}^{\tau}|G(s,\cdot)||_{L^{\infty}(\tilde{\mathcal{O}})}^2 ds. \qquad (4.30)
\end{aligned}
$$

By (4.23) and (4.26)-(4.29),

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$$\begin{aligned}
&\mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(\tau,\tau-t,u^\varepsilon_\tau)||^p_{L^p_g(\mathcal{O})}\Big)\\
&\leq \mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(\sigma)||^p_{L^p_g(\mathcal{O})}\Big)+C_2\Big(\int_\sigma^\tau \mathbb{E}\Big(||u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^{2q-2}_{L^{2q-2}(\mathcal{O})}\Big)ds\\
&+\int_\sigma^\tau \mathbb{E}\Big(\mathcal{R}_\varepsilon||u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^p_{L^p_g(\mathcal{O})}+\int_\sigma^\tau |G(s,\cdot)||^2_{L^\infty(\widetilde{\mathcal{O}})}ds\Big)\\
&+\int_\sigma^\tau \|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}ds+\Big(\sum_{k=1}^\infty ||\sigma_k^*||^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)^{\frac{p}{2}}(\tau-\sigma)\Big).
\end{aligned}\tag{4.31}$$

Integrating (4.31) with respect to σ over $(\tau-1,\tau)$,

$$\begin{aligned}
&\mathbb{E}\Big(||\mathcal{R}_\varepsilon u^\varepsilon(\tau,\tau-t,u^\varepsilon_\tau)||^p_{L^p_g(\mathcal{O})}\Big)\\
&\leqslant (1+C_2)\Big(\int_{\tau-1}^\tau \mathbb{E}\Big(\mathcal{R}_\varepsilon||u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^p_{L^p_g(\mathcal{O})}\Big)\\
&+C_2\Big(\int_{\tau-1}^\tau \mathbb{E}\Big(||u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^{2q-2}_{L^{2q-2}(\mathcal{O})}\Big)ds+\int_{\tau-1}^\tau |G(s,\cdot)||^2_{L^\infty(\widetilde{\mathcal{O}})}ds\Big)\\
&+\int_{\tau-1}^\tau \|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}ds+\Big(\sum_{k=1}^\infty ||\sigma_k^*||^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)^{\frac{p}{2}}\Big)\\
&\leq (1+C_2)\Big(\int_{\tau-1}^\tau \mathbb{E}\Big(\mathcal{R}_\varepsilon||u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^p_{L^p_g(\mathcal{O})}\Big)\\
&+C_2\Big(\int_{\tau-1}^\tau \mathbb{E}\Big(||u^\varepsilon(s,\tau-t,u^\varepsilon_\tau)||^{2q-2}_{L^{2q-2}(\mathcal{O})}\Big)ds+\int_{\tau-1}^\tau e^{-\gamma(\tau-s)}|G(s,\cdot)||^2_{L^\infty(\widetilde{\mathcal{O}})}ds\Big)\\
&+\int_{\tau-1}^\tau e^{-\gamma(\tau-s)}\|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}ds+\Big(\sum_{k=1}^\infty ||\sigma_k^*||^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)^{\frac{p}{2}}\Big)
\end{aligned}\tag{4.32}$$

where C_2 is a positive constant. It remains to give the bound of $\mathbb{E}(||u^\varepsilon||^p_{L^p(\mathcal{O})})$. Note that (3.12), lemmas 4.1 and 4.5, we see that

$$\begin{aligned}
&\mathbb{E}\Big(||u^\varepsilon(t)||^p_{L^p_g(\mathcal{O})}\Big)\\
&\leqslant M_6(1+\int_{-\infty}^\tau e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|^{\frac{q}{2}}_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}+\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})})ds),
\end{aligned}\tag{4.33}$$

which along with (3.17), (4.32), lemmas 4.2 and 4.5 completes the proof. □

5. Existence of pullback measure attractors

In this section, we present the existence and uniqueness of $\mathcal{D}_1$-pullback measure attractors of (3.19) and $\mathcal{D}_0$-pullback measure attractors of (3.20).

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As usual, if $\phi : L^2(\mathcal{O}) \to \mathbb{R}$ is a bounded Borel function, then for $s \le t$ and $u_0^\varepsilon \in L^2(\mathcal{O})$, we set

$$(p_{s,t}^\varepsilon \phi)(u_0^\varepsilon) = \mathbb{E}\Big(\phi(u^\varepsilon(t,s,u_0^\varepsilon))\Big).$$

As usual, for $\Gamma \in \mathcal{B}(L^2(\mathcal{O}))$, $0 \le s \le t$ and $u_0^\varepsilon \in L^2(\mathcal{O})$, we set

$$p^\varepsilon(s,u_0^\varepsilon;t,\Gamma) = (p_{s,t}^\varepsilon \mathbf{1}_\Gamma)(u_0^\varepsilon) = P\{\omega \in \Omega : u^\varepsilon(t,s,u_0^\varepsilon) \in \Gamma\},$$

where $\mathbf{1}_\Gamma$ denotes the characteristic function of Γ.

Under (3.2) and (3.11), the Feller property of $p_{s,t}$ are verified for $0 \le s \le t$: if $\phi : L^2(\mathcal{O}) \to \mathbb{R}$ is bounded and continuous, the function $p_{s,t}^\varepsilon \phi : L^2(\mathcal{O}) \to \mathbb{R}$ is also bounded and continuous. According to the result in ([19], p.250), the solution $u^\varepsilon(t,s,u_0^\varepsilon)$ of the problem (3.19) is a $L^2(\mathcal{O})$-valued Markov process: if $\phi : L^2(\mathcal{O}) \to \mathbb{R}$ is a bounded Borel function, then for any $0 \le s \le t, P-a.s.$,

$$(p_{s,t}^\varepsilon \phi)(u_0^\varepsilon) = p_{s,r}^\varepsilon (p_{r,t}^\varepsilon \phi)(u_0^\varepsilon).$$

Given $t \ge s$ and $\mu \in \mathcal{P}_2(L^2(\mathcal{O}))$, define

$$p_*^\varepsilon(t,s)\mu(\cdot) = \int_{L^2(\mathcal{O})} P^\varepsilon(s,\xi;t,\cdot)\mu(d\xi),$$

thus, the operator $p_*^\varepsilon(t,s) : \mathcal{P}_2(L^2(\mathcal{O})) \to \mathcal{P}_2(L^2(\mathcal{O}))$ is the dual operator of $p^\varepsilon(t,s)$. Similarly, we can define the operator $p_0^\varepsilon(t,s) : \mathcal{P}_2(L^2(\mathcal{Q})) \to \mathcal{P}_2(L^2(\mathcal{Q}))$ associated with (3.20).

For $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $0 < \varepsilon < \varepsilon_0$, denote $S^\varepsilon(t,\tau) : \mathcal{P}_2(L^2(\mathcal{O})) \to \mathcal{P}_2(L^2(\mathcal{O}))$ by

$$S^\varepsilon(t,\tau)\mu = p_*^\varepsilon(\tau+t,\tau)\mu, \quad \mu \in \mathcal{P}_2(L^2(\mathcal{O})), \quad t \ge \tau.$$

Similarly, For $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $0 < \varepsilon < \varepsilon_0$, denote $S^0(t,\tau) : \mathcal{P}_2(L^2(\mathcal{Q})) \to \mathcal{P}_2(L^2(\mathcal{Q}))$ by

$$S^0(t,\tau)\mu = p_*^0(\tau+t,\tau)\mu, \quad \mu \in \mathcal{P}_2(L^2(\mathcal{Q})), \quad t \ge \tau.$$

By lemma 4.1, the $\mathcal{D}_1$-pullback absorbing set for S^ε can be obtained below.

LEMMA 5.1. *Assume that* (3.2)-(3.11) *and* (3.13) *hold. For* $\tau \in \mathbb{R}$, $0 < \varepsilon < \varepsilon_0$, *define by*

$$K(\tau) = B_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(L_\varepsilon^{\frac{1}{2}}(\tau)\Big), \tag{5.1}$$

where

$$L_\varepsilon = \frac{M_1}{g_1}\Big(1 + \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)}(\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})} + \|G(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})}^2)ds\Big). \tag{5.2}$$

Then $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_1$ *is a closed* $\mathcal{D}_1$*-pullback absorbing set of* S^ε.

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Proof. According to lemma 4.1 and (5.1), we have

$$S^{\varepsilon}(t,\tau-t)D_1(\tau-t)\subseteq K(\tau). \tag{5.3}$$

By (5.1)-(5.2) and (3.13),

$$\begin{aligned} &e^{\gamma\tau}||K(\tau)||^2_{\mathcal{P}_2(L^2(\mathcal{O}))}=e^{\gamma\tau}L_{\varepsilon}(\tau)\\ &=\frac{M_1}{g_1}\Big(e^{\gamma\tau}+\int_{-\infty}^{\tau}e^{\gamma s}(\|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})}+\|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds\Big)\to 0 \text{ as } \tau\to-\infty, \end{aligned}$$

then $K=\{K(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}_1$, which together with (5.3) completes the proof. □

We now establish the existence and uniqueness of $\mathcal{D}_1$-pullback measure attractors of (3.19) in $\mathcal{P}_2(L^2(\mathcal{O}))$.

THEOREM 5.2. *Assume that (3.2)-(3.11) and (3.13) hold. Then the dynamical system* S^{ε} *has a unique* $\mathcal{D}_1$*-pullback measure attractor* $A_{\varepsilon}=\{A_{\varepsilon}(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}_1$ *in* $\mathcal{P}_2(L^2(\mathcal{O}))$*, that is for each* $\tau\in\mathbb{R}$*,*

$$\begin{aligned} A_{\varepsilon}(\tau)&=\omega(K,\tau)=\{\psi(0,\tau):\psi \text{ is a } \mathcal{D}_1-\text{complete orbit of } S^{\varepsilon}\}\\ &=\{u^{\varepsilon}(\tau):u^{\varepsilon} \text{ is a } \mathcal{D}_1-\text{complete solution of } S^{\varepsilon}\}, \end{aligned}$$

where $K=\{K(\tau):\tau\in\mathbb{R}\}$ *is the* $\mathcal{D}_1$*-pullback absorbing set defined in lemma 5.1.*

Proof. By [14] and Proposition 3.6 in [27], the system (3.19) generated a continuous non-autonomous dynamical system. From lemma 4.6, the tightness of the distributions of $\{u^{\varepsilon}(\tau,\tau-t,u^{\varepsilon}_{\tau}):\mathcal{L}(u^{\varepsilon}_{\tau})\in D_1(\tau-t)\}_{t\geq T}$ on $L^2(\mathcal{O})$ can be verified by the compactness of the embedding $H^{1,p}(\mathcal{O})\hookrightarrow L^2(\mathcal{O})$, which along with lemma 5.1, Proposition 3.6 in [27] and Proposition 2.10 in [14] completes the proof. □

A similar result also holds for the solutions to the problem (3.20).

THEOREM 5.3. *Assume that (3.2)-(3.11) and (3.13) hold. Then the dynamical system* S^0 *has a unique* $\mathcal{D}_0$*-pullback measure attractor* $A_0=\{A_0(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}_0$ *in* $\mathcal{P}_2(L^2(\mathcal{Q}))$*, that is for each* $\tau\in\mathbb{R}$*,*

$$\begin{aligned} A_0(\tau)&=\omega(K_0,\tau)=\{\psi(0,\tau):\psi \text{ is a } \mathcal{D}_0-\text{complete orbit of } S^0\}\\ &=\{u^0(\tau):u^0 \text{ is a } \mathcal{D}_0-\text{complete solution of } S^0\}, \end{aligned}$$

where $K_0=\{K_0(\tau):\tau\in\mathbb{R}\}$ *is the* $\mathcal{D}_0$*-pullback absorbing set of* S^0*.*

6. Limits of pullback measure attractors

In this section, we discuss the limiting behaviour of pullback measure attractors of the problem (3.19) as $\varepsilon\to 0$. We first give some abstract Lemmas, and then we apply the abstract results to discuss the limiting behaviours of the stochastic system (3.19).

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Define the average function $\mathcal{M} : L^2(\mathcal{O}) \to L^2(\mathcal{Q})$ by:

$$\mathcal{M}\varphi(y^*) = \int_0^1 \varphi\,(y^*, y_{n+1})dy_{n+1} \quad for \quad \varphi \in L^2(\mathcal{O}),\ y^* \in \mathcal{Q}.$$

Let $\mathcal{I} : L^2(\mathcal{Q}) \to L^2(\mathcal{O})$ be the operator, for every $\varphi \in L^2(\mathcal{Q})$, define

$$\mathcal{I}\varphi(y) = \varphi(y^*),\ y = (y^*, y_{n+1}) \in \mathcal{O}.$$

The following Lemma on the average function can be found in [9].

LEMMA 6.1. *If $u \in H^1(\mathcal{O})$, then $\mathcal{M}u \in H^1(\mathcal{Q})$ and*

$$||u - \mathcal{M}u||_{L^2_g(\mathcal{O})} \le c\varepsilon||u||_{H^2_\varepsilon(\mathcal{O})},$$

where c *is a constant, independent of* ε.

We assume that there exist $\kappa_1, \kappa_2 \in L^2_{loc}(\mathbb{R})$ and $\varrho_k \in \mathbb{N}$ such that for all $s, t \in \mathbb{R}$,

$$||f_\varepsilon(t,\cdot,s) - f_0(t,\cdot,s)||_{L^2(\mathcal{O})} \le \varepsilon\kappa_1(t), \tag{6.1}$$

$$||G_\varepsilon(t,\cdot) - G_0(t,\cdot)||_{L^2(\mathcal{O})} \le \varepsilon\kappa_2(t), \tag{6.2}$$

$$||\sigma_{k,\varepsilon}(\cdot) - \sigma_{k,0}(\cdot)||_{L^2(\mathcal{O})} \le \varepsilon\varrho_k, \tag{6.3}$$

where $\sum_{k=1}^{\infty} \varrho_k^2 < \infty$.

LEMMA 6.2. *Assume that (3.2)-(3.11), (3.13) and (6.1)-(6.3) hold. Given $\tau \in \mathbb{R}$, $R > 0$, for all $t \ge 0$,*

$$\lim_{\varepsilon\to 0} \sup_{\mu^\varepsilon \in B^\varepsilon(R)} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(S^\varepsilon(t,\tau)\mu^\varepsilon, (S^0(t,\tau)(\mu^\varepsilon \circ \mathcal{M}^{-1})) \circ \mathcal{I}^{-1}\Big) = 0,$$

where $B^\varepsilon(R) = \{\mu \in \mathcal{P}_2(L^2(\mathcal{O})) : \int_{H^2_\varepsilon(\mathcal{O})} ||\xi||^2_{H^2_\varepsilon(\mathcal{O})}\mu(d\xi) \le R\}$.

Proof. Denote $u^\varepsilon(t) = u^\varepsilon(t,\tau,u^\varepsilon_\tau)$ and $u^0(t) = u^0(t,\tau,\mathcal{M}u^\varepsilon_\tau)$. From (3.19)-(3.20), for all $t \ge 0$,

$$\begin{aligned} &d(u^\varepsilon(t) - u^0(t)) + (A_\varepsilon u^\varepsilon(t) - A_0 u^0(t))dt + \lambda(u^\varepsilon(t) - u^0(t))dt \\ &= (f_\varepsilon(t,y,u^\varepsilon(t)) - f_0(t,y^*,u^0(t)))dt + (G_\varepsilon(t,y) - G_0(t,y^*))dt \\ &+ \sum_{k=1}^{\infty}(\sigma_{k,\varepsilon}(y) - \sigma_{k,0}(y^*) + \rho_k(u^\varepsilon(t)) - \rho_k(u^0(t)))dW_k(t). \end{aligned}$$

By Ito's formula, for $t \ge \tau$,

$$||u^\varepsilon(t) - u^0(t)||^2_{L^2_g(\mathcal{O})} + 2\lambda\int_\tau^t ||u^\varepsilon(t) - u^0(t)||^2_{L^2_g(\mathcal{O})}ds$$

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$$
\begin{aligned}
&= ||u_\tau^\varepsilon - u_\tau^0||^2_{L^2_g(\mathcal{O})} - 2\int_\tau^t (A_\varepsilon u^\varepsilon(s) - A_0 u^0(s), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds \\
&+ 2\int_\tau^t (f_\varepsilon(s,\cdot,u^\varepsilon(s)) - f_0(s,\cdot,u^0(s)), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds \\
&+ 2\int_\tau^t (G_\varepsilon(s,\cdot) - G_0(s,\cdot), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds \\
&+ \sum_{k=1}^\infty \int_\tau^t ||\sigma_{k,\varepsilon}(\cdot) - \sigma_{k,0}(\cdot) + \rho_k(u^\varepsilon(s)) - \rho_k(u^0(s))||^2_{L^2_g(\mathcal{O})} ds \\
&+ 2\sum_{k=1}^\infty \int_\tau^t (\sigma_{k,\varepsilon}(\cdot) - \sigma_{k,0}(\cdot), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} dW_k(s) \\
&+ 2\sum_{k=1}^\infty \int_\tau^t (\rho_k(u^\varepsilon(s)) - \rho_k(u^0(s)), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} dW_k(s). \qquad (6.4)
\end{aligned}
$$

For the second term on the right-hand side of (6.4),

$$
\begin{aligned}
&- 2\int_\tau^t (A_\varepsilon u^\varepsilon(s) - A_0 u^0(s), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds \\
&= -2\int_\tau^t (A_\varepsilon u^\varepsilon(s), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds + 2\int_\tau^t (A_0 u^0(s), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds \\
&= -2\int_\tau^t (|\mathcal{R}_\varepsilon u^\varepsilon(s)|^{p-2}\mathcal{R}_\varepsilon u^\varepsilon(s), \mathcal{R}_\varepsilon u^\varepsilon(s) - \mathcal{R}_\varepsilon u^0(s))_{L^2_g(\mathcal{O})} ds \\
&+ 2\sum_{k=1}^n \int_0^t (|u^0_{y_i}(s)|^{p-2} u^0_{y_i}(s), (u^\varepsilon(s) - u^0(s))_{y_i})_{L^2_g(\mathcal{O})} ds \\
&= -2\int_\tau^t (|\mathcal{R}_\varepsilon u^\varepsilon(s)|^{p-2}\mathcal{R}_\varepsilon u^\varepsilon(s), \mathcal{R}_\varepsilon u^\varepsilon(s) - \mathcal{R}_\varepsilon u^0(s))_{L^2_g(\mathcal{O})} ds \\
&+ 2\int_\tau^t (|\mathcal{R}_\varepsilon u^0(s)|^{p-2}\mathcal{R}_\varepsilon u^0(s), \mathcal{L}_\varepsilon u^\varepsilon(s) - \mathcal{R}_\varepsilon u^0(s))_{L^2_g(\mathcal{O})} ds \\
&+ 2\sum_{k=1}^n \int_0^t \Big(\frac{g_{y_i}}{g}|\mathcal{R}_\varepsilon u^0_{y_i}(s)|^{p-2} u^0_{y_i}(s), y_{n+1}\big(u^\varepsilon_{y_{n+1}}(s) - u^0_{y_{n+1}}(s)\big)\Big)_{L^2_g(\mathcal{O})} ds \\
&= -2\int_\tau^t (|\mathcal{R}_\varepsilon u^\varepsilon(s)|^{p-2}\mathcal{R}_\varepsilon u^\varepsilon(s) - |\mathcal{R}_\varepsilon u^0(s)|^{p-2}\mathcal{R}_\varepsilon u^0(s), \mathcal{R}_\varepsilon u^\varepsilon(s) - \mathcal{R}_\varepsilon u^0(s))_{L^2_g(\mathcal{O})} ds \\
&+ 2\sum_{k=1}^n \int_0^t \Big(\frac{g_{y_i}}{g}|\mathcal{R}_\varepsilon u^0(s)|^{p-2} u^0_{y_i}(s), y_{n+1}\big(u^\varepsilon_{y_{n+1}}(s) - u^0_{y_{n+1}}(s)\big)\Big)_{L^2_g(\mathcal{O})} ds. \qquad (6.5)
\end{aligned}
$$

By (3.4) and (6.1),

$$
\begin{aligned}
&2\int_\tau^t (f_\varepsilon(s,\cdot,u^\varepsilon(s)) - f_0(s,\cdot,u^0(s)), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds \\
&= 2\int_\tau^t (f(s,y^*,\varepsilon g(y^*) y_{n+1}, u^\varepsilon(s)) - f(s,y^*,0,u^0(s)), u^\varepsilon(s) - u^0(s))_{L^2_g(\mathcal{O})} ds
\end{aligned}
$$

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$$
\begin{aligned}
&= 2\int_\tau^t (f(s,y^*,\varepsilon g(y^*)y_{n+1},u^\varepsilon(s)) - f(s,y^*,\varepsilon g(y^*)y_{n+1},u^0(s)),u^\varepsilon(s)-u^0(s))_{L_g^2(\mathcal{O})}\\
&\times ds\\
&+ 2\int_\tau^t (f(s,y^*,\varepsilon g(y^*)y_{n+1},u^0(s)) - f(s,y^*,0,u^0(s)),u^\varepsilon(s)-u^0(s))_{L_g^2(\mathcal{O})}ds\\
&\leqslant 2\alpha_3\int_0^t ||u^\varepsilon(s)-u^0(s)||^2_{L_g^2(\mathcal{O})}ds\\
&+ 2\int_\tau^t ||f(s,y^*,\varepsilon g(y^*)y_{n+1},u^0(s)) - f(s,y^*,0,u^0(s))||_{L_g^2(\mathcal{O})}||u^\varepsilon(s)-u^0(s)||_{L_g^2(\mathcal{O})}ds\\
&\leqslant 2\alpha_3\int_\tau^t ||u^\varepsilon(s)-u^0(s)||^2_{L_g^2(\mathcal{O})}ds + 2\varepsilon g_2\int_\tau^t \kappa_1(s)||u^\varepsilon(s)-u^0(s)||_{L_g^2(\mathcal{O})}ds
\end{aligned}
$$

$$
\leqslant 2\alpha_3\int_\tau^t ||u^\varepsilon(s)-u^0(s)||^2_{L_g^2(\mathcal{O})}ds + \varepsilon g_2\int_\tau^t \kappa_1^2(s) + ||u^\varepsilon(s)-u^0(s)||_{L_g^2(\mathcal{O})}ds^2 ds. \tag{6.6}
$$

By (6.2),

$$
\begin{aligned}
&2\int_\tau^t (G_\varepsilon(s,\cdot)-G_0(s,\cdot),u^\varepsilon(s)-u^0(s))_{L_g^2(\mathcal{O})}ds\\
&\leqslant 2\int_\tau^t ||G_\varepsilon(s,\cdot)-G_0(s,\cdot)||^2_{L_g^2(\mathcal{O})}||u^\varepsilon(s)-u^0(s)||^2_{L_g^2(\mathcal{O})}ds\\
&\leqslant g_2\varepsilon\int_\tau^t \kappa_2^2(t) + ||u^\varepsilon(s)-u^0(s)||^2_{L_g^2(\mathcal{O})}ds.
\end{aligned} \tag{6.7}
$$

It follows from (3.8) and (6.3), we have

$$
\begin{aligned}
&\sum_{k=1}^\infty \int_\tau^t ||\sigma_{k,\varepsilon}(\cdot)-\sigma_{k,0}(\cdot)+\rho_k(u^\varepsilon(s))-\rho_k(u^0(s))||^2_{L_g^2(\mathcal{O})}ds\\
&\leqslant 2\sum_{k=1}^\infty \int_\tau^t ||\sigma_{k,\varepsilon}-\sigma_{k,0}||^2_{L_g^2(\mathcal{O})} + 2\sum_{k=1}^\infty \int_\tau^t ||\rho_k(u^\varepsilon(s))-\rho_k(u^0(s))||^2_{L_g^2(\mathcal{O})}\\
&\leqslant 2\varepsilon^2(t-\tau)\sum_{k=1}^\infty \varrho_k + 2\sum_{k=1}^\infty L_k^2\int_\tau^t ||u^\varepsilon(s)-u^0(s)||^2_{L_g^2(\mathcal{O})}ds.
\end{aligned} \tag{6.8}
$$

Finally, by (3.17), we obtain

$$
\begin{aligned}
&2\sum_{k=1}^n \int_\tau^t \left(\frac{g_{y_i}}{g}|\mathcal{R}_\varepsilon u^0(s)|^{p-2}u^0_{y_i}(s), y_{n+1}\left(u^\varepsilon_{y_{n+1}}(s)-u^0_{y_{n+1}}(s)\right)\right)_{L_g^2(\mathcal{O})}ds\\
&= 2\sum_{i=1}^n \int_\tau^t \left(g_{y_i}\left|\nabla u^0(s)\right|^{p-2}v^0_{y_i}, y_{n+1}\left(u^\varepsilon_{y_{n+1}}(s)-u^0_{y_{n+1}}(s)\right)\right)_{L^2(\mathcal{O})}ds
\end{aligned}
$$

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$$\begin{aligned}
&\leqslant c_2 \sum_{i=1}^{n} \int_{\tau}^{t} \left(\left|\nabla u^0\right|^{p-2} u^0_{y_i}, \left(u^{\varepsilon}_{y_{n+1}}(s) - u^0_{y_{n+1}}(s) \right) \right)_{L^2(\mathcal{O})} ds \\
&\leqslant c_3 \int_{\tau}^{t} \left\|\nabla v^0\right\|^{p-1}_{L^p(\mathcal{Q})} \|u^{\varepsilon}_{y_{n+1}} - u^0_{y_{n+1}}\|_{L^p(\mathcal{O})} ds \\
&\leqslant c_4 \varepsilon \int_{\tau}^{t} \left\|\nabla u^0\right\|^{p-1}_{L^p(\mathcal{Q})} \left\|u^{\varepsilon}(s) - u^0(s)\right\|_{H^p_{\varepsilon}(\mathcal{O})} ds \\
&\leqslant c_5 \varepsilon \int_{\tau}^{t} \left(\|u^{\varepsilon}(s)\|^p_{H^p_{\varepsilon}(\mathcal{O})} + \left\|\nabla u^0(s)\right\|^p_{L^p(\mathcal{Q})} \right) ds. \qquad (6.9)
\end{aligned}$$

Taking the expectation of (6.4) and by (6.5)-(6.9), then for all $t \geq \tau$,

$$\begin{aligned}
&\mathbb{E}\Big(||u^{\varepsilon}(t) - u^0(t)||^2_{L^2_g(\mathcal{O})} \Big) \\
&\leq ||u^{\varepsilon}_{\tau} - \mathcal{M}u^{\varepsilon}_{\tau}||^2_{L^2_g(\mathcal{O})} + c_6 \Big(\int_{\tau}^{t} \mathbb{E}\Big(||u^{\varepsilon}(s) - u^0(s)||^2_{L^2_g(\mathcal{O})} \Big) \\
&+ \varepsilon \int_{\tau}^{t} \mathbb{E}\Big(\|u^{\varepsilon}(s)\|^p_{H^p_{\varepsilon}(\mathcal{O})} + \left\|\nabla u^0(s)\right\|^p_{L^p(\mathcal{Q})} \Big) ds + \varepsilon \int_{\tau}^{t} (k_1^2(s) + k_2^2(s)) ds + \varepsilon^2 (t - \tau) \Big). \qquad (6.10)
\end{aligned}$$

By lemmas 4.4 and 4.6 and (6.10), for every $T > 0$, $\tau \leq t \leq \tau + T$, $\varepsilon \in (0, \varepsilon_0)$ and $u^{\varepsilon}_{\tau} \in L^2_{\mathcal{F}_{\tau}}(\Omega, L^2(\Omega))$ with $\mathbb{E}(||u^{\varepsilon}_{\tau}||_{H^2_{\varepsilon}(\mathcal{O})}) \leq R$,

$$\begin{aligned}
&\mathbb{E}\Big(||u^{\varepsilon}(t) - u^0(t)||^2_{L^2_g(\mathcal{O})} \Big) \\
&\leq ||u^{\varepsilon}_{\tau} - \mathcal{M}u^{\varepsilon}_{\tau}||^2_{L^2_g(\mathcal{O})} + c_7 \Big(\int_{\tau}^{t} \mathbb{E}\Big(||u^{\varepsilon}(s) - u^0(s)||^2_{L^2_g(\mathcal{O})} \Big) + \varepsilon + \varepsilon^2 T \Big). \qquad (6.11)
\end{aligned}$$

By Gronwall's inequality and lemma 6.1, for every $T > 0$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}
\mathbb{E}\Big(||u^{\varepsilon}(t) - u^0(t)||^2_{L^2_g(\mathcal{O})} \Big) &\leqslant (\mathbb{E}(||u^{\varepsilon}_{\tau} - \mathcal{M}u^{\varepsilon}_{\tau}||^2_{L^2_g(\mathcal{O})}) + c_8(\varepsilon + \varepsilon^2 T)) e^{c_7(t-\tau)} \\
&\leqslant (c_9 \varepsilon \mathbb{E}(||u^{\varepsilon}_{\tau}||^2_{H^2_{\varepsilon}(\mathcal{O})}) + c_8(\varepsilon + \varepsilon^2 T)) e^{c_7 T} \\
&\leqslant (c_9 \varepsilon R + c_8(\varepsilon + \varepsilon^2 T)) e^{c_7 T} \qquad (6.12)
\end{aligned}$$

By simple computations, we have

$$\lim_{\varepsilon \to 0} \sup_{\mathbb{E}(||u^{\varepsilon}_{\tau}||_{H^2_{\varepsilon}(\mathcal{O})}) \leqslant R} \mathbb{E}\Big(||u^{\varepsilon}(t, \tau, u^{\varepsilon}_{\tau}) - u^0(t, \tau, \mathcal{M}u^{\varepsilon}_{\tau})||^2_{L^2(\mathcal{O})} \Big) = 0 \qquad (6.13)$$

Notice that

$$\begin{aligned}
&\sup_{\mathbb{E}(||u^{\varepsilon}_{\tau}||_{H^2_{\varepsilon}(\mathcal{O})}) \leq R} \sup_{\varphi \in L_b(L^2(\mathcal{O})), ||\varphi||_L \leq 1} \left| \mathbb{E}\Big(\varphi(u^{\varepsilon}(t, \tau, u^{\varepsilon}_{\tau})) \Big) - \mathbb{E}\Big(\varphi(u^0(t, \tau, \mathcal{M}u^{\varepsilon}_{\tau})) \Big) \right| \\
&\leqslant \sup_{\mathbb{E}(||u^{\varepsilon}_{\tau}||_{H^2_{\varepsilon}(\mathcal{O})}) \leq R} \sup_{\varphi \in L_b(L^2(\mathcal{O})), ||\varphi||_L \leq 1} \mathbb{E}\Big(\left| \varphi(u^{\varepsilon}(t, \tau, u^{\varepsilon}_{\tau})) - \varphi(u^0(t, \tau, \mathcal{M}u^{\varepsilon}_{\tau})) \right| \Big)
\end{aligned}$$

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$$\leqslant \Big(\sup_{\mathbb{E}(||u_\tau^\varepsilon||_{H_\varepsilon^2(\mathcal{O})})\leq R} \mathbb{E}\Big(||u^\varepsilon(t,\tau,u_\tau^\varepsilon) - u^0(t,\tau,\mathcal{M}u_\tau^\varepsilon)||^2_{L^2(\mathcal{O})} \Big) \Big)^{\frac{1}{2}},$$

then we have

$$\lim_{\varepsilon\to 0} \sup_{\mathbb{E}(||u_\tau^\varepsilon||_{H_\varepsilon^2(\mathcal{O})})\leq R} \sup_{\varphi\in L_b(L^2(\mathcal{O})),||\varphi||_L\leq 1} \Big| \mathbb{E}\Big(\varphi(u^\varepsilon(t,\tau,u_\tau^\varepsilon))\Big) - \mathbb{E}\Big(\varphi(u^0(t,\tau,\mathcal{M}u_\tau^\varepsilon))\Big) \Big| = 0.$$

□

Finally, we present the main result of this section.

LEMMA 6.3. *Assume that* (3.2)-(3.11), (3.13) *and* (6.1)-(6.3) *hold. Then the* $\mathcal{D}_1$*-pullback measure attractors* $\mathcal{A}_\varepsilon$ *are upper semi-continuous as* $\varepsilon \to 0$*, that is , for every* $\tau \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(\mathcal{A}_\varepsilon(\tau), \mathcal{A}_0(\tau)\circ\mathcal{I}^{-1}\Big) = 0.$$

Proof. By lemma 4.6, one can verify that for all $\varepsilon \in (0,\varepsilon_0)$, there exists $C_2 = C_2(\tau) > 0$,

$$\int_{H_\varepsilon^2(\mathcal{O})} ||\xi||^2_{H_\varepsilon^2(\mathcal{O})}\mu(d\xi) \leq C_2(\tau). \tag{6.14}$$

Let $K = \{K(\tau) : \tau \in \mathbb{R}\}$ be the $\mathcal{D}_1$-pullback absorbing set of S^ε defined in lemma 5.1. Denote $K_0 = \{K_0(\tau) : \tau \in \mathbb{R}\}$ with $K_0(\tau) = \{\mu\circ\mathcal{M}^{-1} : \mu \in K(\tau)\}$. Then $K_0 \in \mathcal{D}_0$. Let $\delta > 0$, then there exists $T = T(\delta,\tau) \geq 1$,

$$d_{\mathcal{P}_2(L^2(\mathcal{Q}))}\Big(S^0(T,\tau-T)K_0(\tau-T), \mathcal{A}_0(\tau)\Big) \leq \frac{1}{2}\delta. \tag{6.15}$$

In addition, by lemma 6.2 and (6.14),

$$\lim_{\varepsilon\to 0} \sup_{\eta\in A_\varepsilon(\tau-T)} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(S^\varepsilon(T,\tau-T)\eta, (S^0(T,\tau-T)(\eta\circ\mathcal{M}^{-1}))\circ\mathcal{I}^{-1}\Big) = 0.$$

Then there exists $\varepsilon_1 \in (0,\varepsilon_0)$, for all $0 < \varepsilon < \varepsilon_1$,

$$\sup_{\eta\in A_\varepsilon(\tau-T)} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(S^\varepsilon(T,\tau-T)\eta, (S^0(T,\tau-T)(\eta\circ\mathcal{M}^{-1}))\circ\mathcal{I}^{-1}\Big) < \frac{1}{2}\delta. \tag{6.16}$$

Let $\eta \in A_\varepsilon(\tau-T)$, since $A_\varepsilon(\tau-T) \subseteq K(\tau-T)$, note that $\eta\circ\mathcal{M}^{-1} \in K_0(\tau-T)$, by (6.15),

$$\sup_{\eta\in A_\varepsilon(\tau-T)} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(S^0(T,\tau-T)(\eta\circ\mathcal{M}^{-1}), \mathcal{A}_0(\tau)\Big) < \frac{1}{2}\delta,$$

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and we further have

$$\sup_{\eta\in A_\varepsilon(\tau-T)} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big((S^0(T,\tau-T)(\eta\circ\mathcal{M}^{-1}))\circ\mathcal{I}^{-1},\mathcal{A}_0(\tau)\circ\mathcal{I}^{-1}\Big)<\frac{1}{2}\delta,$$

then we get

$$\sup_{\eta\in A_\varepsilon(\tau-T)} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(S^\varepsilon(\tau,\tau-T)\eta,\mathcal{A}_0(\tau)\circ\mathcal{I}^{-1}\Big)<\delta, \tag{6.17}$$

By the invariance of $\mathcal{A}_\varepsilon$, for any $\mu_\varepsilon\in A_\varepsilon(\tau)$, there exists $\eta\in A_\varepsilon(\tau-T)$ such that

$$\mu_\varepsilon=S^\varepsilon(T,\tau-T)\eta. \tag{6.18}$$

By (6.17) and (6.18), for $\varepsilon\in(0,\varepsilon_1)$,

$$\sup_{\mu_\varepsilon\in A_\varepsilon(\tau)} d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(\mu_\varepsilon,\mathcal{A}_0(\tau)\circ\mathcal{I}^{-1}\Big)<\delta,$$

thus, we have that

$$d_{\mathcal{P}_2(L^2(\mathcal{O}))}\Big(\mathcal{A}_\varepsilon(\tau),\mathcal{A}_0(\tau)\circ\mathcal{I}^{-1}\Big)<\delta.$$

The proof is completed. □

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Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest
We have no conflicts of interest/competing interests to disclose.

Ethical approval
This declaration is not applicable.

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