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SOME MONOCHROMATIC PATTERNS IN NATURAL NUMBERS

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Abstract

The set of sums of two squares plays a significant role in number theory. We establish the existence of several rich monochromatic configurations in the natural numbers by exploiting algebraic structures induced by the set of sums of two squares. The proofs rely on algebraic properties arising from the induced structures on the Stone–Čech compactification of the natural numbers.

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1. Introduction

The search for monochromatic patterns within colourings of algebraic structures has been a longstanding theme in Ramsey theory. Among the fundamental results in this area, the celebrated *van der Waerden theorem* guarantees the existence of monochromatic arithmetic progressions for any finite partition of the natural numbers. This theorem illustrates that the family of arithmetic progressions is *partition regular*. The van der Waerden theorem is a finitary Ramsey-type result relying solely on the additive structure of $\mathbb{N}$. Its infinitary counterpart, *Hindman's theorem*, asserts that for any finite colouring of $\mathbb{N}$, there exists an injective sequence whose finite sums are all monochromatic.

A natural direction in Ramsey theory is the transition from additive to multiplicative structures. One such transition can be achieved via the transformation $n \mapsto p^n$ for a prime p, leading to the *multiplicative van der Waerden theorem*, which states that for any finite partition of $\mathbb{N}$, a monochromatic geometric progression of arbitrary length exists.

Patterns that intertwine the additive and multiplicative structures of natural numbers present greater challenges. The pioneering work of Bergelson [5] and Hindman [15] independently established the monochromaticity of the configuration a, b, c, d where

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$a + b = c \cdot d$. Further progress was made by Moreira [18], who demonstrated that the pattern $a, a + b, a \cdot b$ is monochromatic. Subsequently, Barrett *et al.* [1] extended this result by proving the monochromaticity of $a, a + b, a + b + a \cdot b$. In [19], Sahasrabudhe proved rich exponential patterns in $\mathbb{N}$, showing in particular that the sets $\{a, b, a^b\}$ and $\{a, b, ab, a^b\}$ are monochromatic where $b > 1$. Subsequently, Goswami in [13] established the result that $\{a, b, ab, (a + 1)b\}$ is monochromatic. However, the question of whether the set $\{a, b, a + b, a \cdot b\}$ is always monochromatic remains an open problem.

Recently, in [12], Di Nasso found many new additive and multiplicative monochromatic patterns by introducing the notions of symmetric polynomials. Polynomial extension of some important classical results, particularly the polynomial van der Waerden theorem and the polynomial Deuber theorem for symmetric polynomials, have been explored by Chakraborty and Goswami in [10].

Given these advances, it is natural to examine the Ramsey-theoretic properties of well-structured subsets of natural numbers. Define $S(m, n)$ as the set

$$S(m, n) := \{x_1^m + \cdots + x_n^m : x_1, \ldots, x_m \in \mathbb{N}_0\}.$$

From elementary number theory, it is well known that $S(2, 4) = \mathbb{N}_0$, while $S(2, 3)$ consists of natural numbers that are not of the form $4^a(8b + 7)$ for some $a, b \in \mathbb{N}_0$. The set $S(2, 3)$ lacks multiplicative closure, whereas $S(2, 4)$ is multiplicatively closed but trivial in the sense that it encompasses all natural numbers. Thus, our focus is on $\Sigma = S(2, 2)$, the set of all natural numbers whose prime factors of the form $4k + 3$ (if any) have even exponents.

We introduce a new algebraic structure on the natural numbers derived from the set of sums of two squares and, employing the Stone–Čech compactification, establish analogues of several classical results, including Hindman's theorem, Deuber's theorem, Brauer's theorem, Milliken–Taylor's theorem, the existence of geo-arithmetic progressions and a polynomial variant of van der Waerden's theorem.

2. A new structure and Stone–Čech compactification

Let $\Sigma = \{a^2 + b^2 : a, b \in \mathbb{N}_0\} \subseteq \mathbb{N}_0$ denote the set of nonnegative integers expressible as sums of two squares. Under standard multiplication, Σ forms a *commutative semigroup*, as the product of two elements in Σ remains in Σ by the identity

$$(a^2 + b^2)(c^2 + d^2) = (ad - bc)^2 + (ac + bd)^2.$$

We order Σ increasingly as

$$\Sigma = \{s_0 < s_1 < s_2 < s_3 < \cdots\};$$

the first few elements are $s_0 = 0$, $s_1 = 1$, $s_2 = 2$, $s_3 = 4$, $s_4 = 5$, $s_5 = 8$, $s_6 = 9$, $s_7 = 10$, $s_8 = 13$, $s_9 = 16$, $s_{10} = 17$, $s_{11} = 18$, $s_{12} = 20$, $s_{13} = 25$, $s_{14} = 26$, $s_{15} = 29$, $s_{16} = 32$.

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For $s \in \Sigma$, define the *predecessor set* $\Sigma_{<s} = \{y \in \Sigma : y < s\}$. Let $g : \Sigma \to \mathbb{N}_0$ be the function

$$g(s) = |\Sigma_{<s}|,$$

which counts the number of elements in Σ strictly less than s; g is a bijection, with inverse $f : \mathbb{N}_0 \to \Sigma$ given by $f(n) = s_n$, the nth term in the ordered sequence of Σ.

DEFINITION 2.1 (Induced operation). The operation $*_f$ on $\mathbb{N}_0$ is defined by

$$m *_f n = g(f(m) \cdot f(n)) = |\Sigma_{<s_m \cdot s_n}|.$$

EXAMPLE 2.2. Let $m = 2$ and $n = 5$. Then, $s_m = 2$ and $s_n = 8$. Their product is $s_m \cdot s_n = 16 = s_9$. The set $\Sigma_{<16}$ contains the first nine elements of Σ (that is, $s_0, s_1, \ldots, s_8$), so $|\Sigma_{<16}| = 9$. Hence, $2 *_f 5 = 9$.

REMARK 2.3. Note that Σ is multiplicatively closed, that is, the product $s_i \cdot s_j = s_k$ is an element of Σ for some $k \in \mathbb{N} \cup \{0\}$. The value of k depends on the ordering (based on the numerical value) of Σ, which is not uniquely determined by i and j alone.

PROPOSITION 2.4. *$(\mathbb{N}_0, *_f)$ is a commutative semigroup with identity element 1 and $f : (\mathbb{N}_0, *_f) \to (\Sigma, \cdot)$ is a semigroup isomorphism.*

PROOF. The operation $*_f$ inherits associativity and commutativity from the semigroup $(\Sigma, \cdot)$ via the bijection f. Moreover:

- for all $n \in \mathbb{N}_0$,

$$1 *_f n = g(f(1) \cdot f(n)) = g(s_1 \cdot s_n) = g(s_n) = |\Sigma_{<s_n}| = n,$$

  since $s_1 = 1$ is the multiplicative identity in Σ;
- by definition,

$$f(m *_f n) = f(g(s_m \cdot s_n)) = s_{g(s_m \cdot s_n)} = s_m \cdot s_n = f(m) \cdot f(n),$$

  confirming f is a homomorphism; as f is bijective, it is an isomorphism. □

Define the *power* $x^{(n)}$ in $(\mathbb{N}_0, *_f)$ recursively by:

- $x^{(0)} = 1$;
- $x^{(n)} = x *_f x^{(n-1)} = x *_f x *_f \cdots *_f x = |\Sigma_{<s_x^n}|$.

We briefly recall the algebraic structure of the *Stone–Čech compactification* βS for a discrete semigroup $(S, \cdot)$. The elements of βS are the *ultrafilters* on S. By identifying the principal ultrafilters with the points of S, we may regard S as a subset of βS, that is, $S \subseteq \beta S$.

For a subset $A \subseteq S$, define

$$\overline{A} = \{p \in \beta S \mid A \in p\}.$$

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The collection $\{\overline{A} \mid A \subseteq S\}$ forms a basis for a topology on βS. The semigroup operation $\cdot$ on S can be extended to βS, making $(\beta S, \cdot)$ a compact right topological semigroup. This means:

- for any $p \in \beta S$, the right translation map $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous;
- the semigroup S is contained in the topological centre of βS, meaning that for any $x \in S$, the left translation map $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous.

The extended operation on βS is characterised as follows: for $p, q \in \beta S$ and $A \subseteq S$,

$$A \in p \cdot q \quad \text{if and only if} \quad \{x \in S \mid x^{-1}A \in q\} \in p,$$

where $x^{-1}A = \{y \in S \mid x \cdot y \in A\}$. A fundamental result due to Ellis states that every compact right topological semigroup contains an idempotent element. Thus, there exists an idempotent ultrafilter $p \in \beta S$ satisfying $p \cdot p = p$.

For a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $(S, \cdot)$, the set of finite products is defined by

$$FP(\langle x_n \rangle_{n=1}^{\infty}) = \{x_{n_1} \cdot x_{n_2} \cdots x_{n_k} \mid n_1 < n_2 < \cdots < n_k\}.$$

Idempotent ultrafilters play a crucial role in Ramsey theory. This is exemplified by Galvin's theorem, which states that for any semigroup $(S, \cdot)$ and an idempotent ultrafilter $p = p \cdot p$ in $(\beta S, \cdot)$, every set $A \in p$ contains the set of finite products $FP(\langle x_n \rangle_{n=1}^{\infty})$ for some sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S. For further details on the Stone–Čech compactification and its applications in Ramsey theory, see [16, 21].

Since $(\mathbb{N}_0, *_f)$ is a discrete commutative semigroup, its Stone–Čech compactification $(\mathbb{N}_0, *_f)$ gives rise to numerous Ramsey-type results in $\mathbb{N}_0$ induced by the operation $*_f$.

3. Monochromatic configurations

In this section, we introduce new monochromatic configurations within the semigroup $(\mathbb{N}, *_f)$, which encodes information about sums of two squares. We begin with the following definition.

For any infinite sequence of natural numbers $\langle x_n \rangle_{n=1}^{\infty}$, the corresponding set of finite sums is denoted and defined by

$$FS(\langle x_n \rangle_{n=1}^{\infty}) = \{x_{n_1} + x_{n_2} + \cdots + x_{n_k} : n_1 < n_2 < \cdots < n_k\}.$$

A fundamental result in arithmetic Ramsey theory due to Hindman guarantees the existence of infinite monochromatic patterns within such sets of finite sums.

THEOREM 3.1 (Hindman's finite sum theorem, [14]). *Let $r \geq 1$. For every r-colouring of the natural numbers, $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, there exists a colour C_i and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C_i$.*

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A similar result holds when considering finite products instead of finite sums and, more generally, in any semigroup [16, Theorem 5.8]. Analogously, for the semigroup $(\mathbb{N}_0, *_f)$, we define the set of finite $*_f$-operations by

$$\begin{aligned}\mathfrak{FP}_f(\langle x_n\rangle_{n=1}^{\infty}) &= \{x_{n_1} *_f x_{n_2} *_f \cdots *_f x_{n_k} : n_1 < n_2 < \cdots < n_k\}\\ &= \{|\Sigma_{<s_{x_{n_1}} s_{x_{n_2}} \cdots s_{x_{n_k}}}| : n_1 < n_2 < \cdots < n_k\},\end{aligned}$$

where Σ denotes the set of sums of two squares.

THEOREM 3.2. *For any $r \geq 1$ and any finite r-colouring of the nonnegative integers, $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$, there exist a colour C_i and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

$$\mathfrak{FP}_f(\langle x_n\rangle_{n=1}^{\infty}) = |\Sigma_{<s_{x_{n_1}} s_{x_{n_2}} \cdots s_{x_{n_k}}}| : n_1 < n_2 < \cdots < n_k\} \subseteq C_i.$$

PROOF. The proof is verbatim from the proof of the Hindman's theorem [16, Theorem 5.8], replacing the usual addition with the associative operation $*_f$. □

A cornerstone result in arithmetic Ramsey theory is van der Waerden's theorem (1927), which guarantees the existence of arbitrarily long monochromatic arithmetic progressions. A year later, Brauer strengthened this result by showing that one can also ensure that the common difference belongs to the same colour class as the elements of the progression.

THEOREM 3.3 (van der Waerden's theorem, [22]). *For every finite colouring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ and for every $L \in \mathbb{N}$, there exists a monochromatic arithmetic progression of length L; that is, there exist a colour C_i and elements $a, b \in \mathbb{N}$ such that $a, a+b, a+2b, \ldots, a+Lb \in C_i$.*

THEOREM 3.4 (Brauer's theorem, [9]). *For every finite colouring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ and for every $k \in \mathbb{N}$, there exist a colour C_i and elements $a, b \in \mathbb{N}$ such that $\{a, b, a+b, a+2b, \ldots, a+kb\} \subseteq C_i$.*

We establish an analogous version of Brauer's theorem in our setting.

THEOREM 3.5. *For every finite colouring $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$ and for every $k \in \mathbb{N}$, there exist a colour C_i and elements $x, z \in \mathbb{N}$ such that*

$$\{x, z, |\Sigma_{<s_x^j s_z}| : j = 1, 2, \ldots, k\} \subseteq C_i.$$

Deuber established a result concerning the generalised partition regularity of homogeneous systems of linear Diophantine equations. In particular, he demonstrated the partition regularity of the so-called (m, p, c)-sets.

THEOREM 3.6 (Deuber's theorem, [11]). *For every $m, p, c \in \mathbb{N}$ and for every finite colouring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, there exists a monochromatic (m, p, c)-set. That is, there exists a colour C_i and elements $a_0, a_1, \ldots, a_m \in C_i$ such that $ca_j + \sum_{s=0}^{j-1} n_s a_s \in C_i$ for every $j \in \{1, 2, \ldots, m\}$ and for all $n_0, n_1, \ldots, n_{j-1} \in \{-p, \ldots, p\}$,*

We present an analogous version of Deuber's theorem in our context.

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THEOREM 3.7. *Let $m, p, r \in \mathbb{N}$. For every r-colouring $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$, there exists a colour C_i and elements $x_0, x_1, \ldots, x_m \in C_i$ such that*

$$\{|\Sigma_{<s_{x_0}^{n_0} s_{x_1}^{n_1} \cdots s_{x_{j-1}}^{n_{j-1}} s_{x_j}}| : n_0, n_1, \ldots, n_{j-1} \in \{0, 1, \ldots, p\} \text{ and } j = 1, 2, \ldots, m\} \subseteq C_i.$$

PROOF. We prove this theorem using a generalisation of Deuber's theorem for commutative semirings, recently established by Bergelson *et al.* in [7, Corollary 3.7]. This result can be stated as follows.

Let $(S, *)$ be a commutative semigroup and, for each $j = 1, 2, \ldots, m$, let $\mathfrak{F}_j$ be a finite set of endomorphisms $\psi : S^j \to S$. Then, for every r-colouring $S = C_1 \cup C_2 \cup \cdots \cup C_r$, there exist a colour C_i and elements $x_0, x_1, \ldots, x_m$ distinct from the identity such that $x_0 \in C_i$ and

$$\psi(x_0, x_1, \ldots, x_{j-1}) * x_j \in C_i \quad \text{for every } j = 1, 2, \ldots, m \text{ and for every } \psi \in \mathfrak{F}_j.$$

This statement generalises Theorem 3.6 for the case $c = 1$. We apply the Bergelson–Johnson–Moreira result with $(S, *) = (\mathbb{N}_0, *_f)$. For each j-tuple $\bar{n} = (n_0, n_1, \ldots, n_{j-1}) \in (\mathbb{N}_0)^j$, define

$$\psi_{\bar{n}} : (x_0, x_1, \ldots, x_{j-1}) \mapsto {x_0}^{(n_0)} *_f {x_1}^{(n_1)} *_f \cdots *_f {x_j}^{(n_j)}.$$

Since $(\mathbb{N}_0, *_f)$ is a commutative semigroup, each $\psi_{\bar{n}}$ is a semigroup homomorphism. Define

$$\mathfrak{F}_j = \{\psi_{\bar{n}} : \mathbb{N}^j \to \mathbb{N} : \bar{n} = (n_0, n_1, \ldots, n_{j-1}) \in \{0, 1, 2, \ldots, p\}^j\},$$

which consists of homomorphisms for each $j = 1, 2, \ldots, m$. By the Bergelson–Johnson–Moreira theorem, for every finite colouring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, there exist a colour C_i and elements $x_0, x_1, \ldots, x_m$ distinct from the identity such that:

- $x_0 \in C_i$;
- $\psi_{\bar{n}}(x_0, x_1, \ldots, x_{j-1}) *_f x_j \in C_i$ for every $j = 1, 2, \ldots, m$ and all $\bar{n} \in \{0, 1, 2, \ldots, p\}^j$.

From the second condition, we deduce that

$$\psi_{\bar{n}}(x_0, x_1, \ldots, x_{j-1}) *_f x_j = {x_0}^{(n_0)} *_f {x_1}^{(n_1)} *_f \cdots *_f {x_j}^{(n_j)} *_f x_j \in C_i.$$

Therefore, for all $(n_0, n_1, \ldots, n_{j-1}) \in \{0, 1, \ldots, p\}^j$ and $j = 1, 2, \ldots, m$, we obtain

$$|\Sigma_{<s_{x_0}^{n_0} s_{x_1}^{n_1} \cdots s_{x_{j-1}}^{n_{j-1}} s_{x_j}}| \in C_i.$$

Hence, the result follows. □

PROOF OF THEOREM 3.5. The proof follows from the Bergelson–Johnson–Moreira theorem, as in Theorem 3.7. For $j = 0, 1, 2, \ldots, k+1$, let $\psi_j : (\mathbb{N}, *_f) \to (\mathbb{N}, *_f)$ be the endomorphism defined by $\psi_j(x) = x^{(j)}$. Then, for every r-colouring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, there exist a colour C_i and elements $x, y \neq 1$ such that

$$\{x, \psi_0(x) *_f y, \psi_1(x) *_f y, \psi_2(x) *_f y, \ldots, \psi_{k+1}(x) *_f y\} \subseteq C_i.$$

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Setting $z := x *_f y$, we obtain

$$\{x, z, x *_f z, x^{(2)} *_f z, x^{(3)} *_f z, \dots, x^{(k)} *_f z\} = \{x, z, |\Sigma_{<s^j_x s_z}| : j = 1, 2, \dots, k\} \subseteq C_i.$$

Hence, the result follows. □

4. Milliken–Taylor theorem

The Milliken–Taylor theorem simultaneously generalises Hindman's finite sums theorem and Ramsey's theorem. It has been widely applied in combinatorial number theory, including extensions of Szemerédi's theorem on arithmetic progressions.

To state the theorem, we first establish some notation. Let $\mathcal{P}_f(\mathbb{N})$ denote the set of all finite nonempty subsets of $\mathbb{N}$. For $m \in \mathbb{N}$, define $[\mathbb{N}]^m = \{A \subseteq \mathbb{N} : |A| = m\}$. For $F, G \in \mathcal{P}_f(\mathbb{N})$, we write $F < G$ if $\max F < \min G$.

THEOREM 4.1 (Milliken–Taylor theorem [17, 20]). *Let $r, m \in \mathbb{N}$ and let $[\mathbb{N}]^m = C_1 \cup \cdots \cup C_r$ be an r-colouring. There exist an injective sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and a colour C_i such that*

$$\{(x_{F_1}, \dots, x_{F_m}) : F_1 < \cdots < F_m\} \subseteq C_i,$$

where for $F = \{n_1 < \cdots < n_k\} \in \mathcal{P}_f(\mathbb{N})$, we define $x_F = \sum_{i=1}^{k} x_{n_i}$.

REMARK 4.2. When $m = 1$, this reduces to Hindman's theorem. If each F_i is a singleton, we recover Ramsey's theorem.

4.1. Ultrafilter formulation. The theorem admits an ultrafilter-theoretic formulation using tensor products. Let $(S_i, \cdot)$ be semigroups and $p_i \in \beta S_i$ ultrafilters. The *tensor product* $\bigotimes_{i=1}^{k} p_i \in \beta(\times_{i=1}^{k} S_i)$ is defined inductively:

- for $k = 1$, $\bigotimes_{i=1}^{1} p_i = p_1$;
- for $k \geq 1$, $A \in \bigotimes_{i=1}^{k+1} p_i$ if and only if

$$\{(x_1, \dots, x_k) \in \times_{i=1}^{k} S_i : \{x_{k+1} \in S_{k+1} : (x_1, \dots, x_{k+1}) \in A\} \in p_{k+1}\} \in \bigotimes_{i=1}^{k} p_i.$$

THEOREM 4.3 (Bergelson–Hindman–Williams [6]). *Let S be a semigroup, $m \in \mathbb{N}$ and $A \subseteq \times_{i=1}^{m} S$. The following statements are equivalent.*

- *There exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S with $\{(x_{F_1}, \dots, x_{F_m}) : F_1 < \cdots < F_m\} \subseteq A$.*
- *There exists an idempotent ultrafilter $p \in \beta S$ such that $A \in \bigotimes_{i=1}^{m} p$.*

For a function $\phi : X \to Y$, the induced map $\phi_* : \beta X \to \beta Y$ is defined by $\phi_*(p) = \{B \subseteq Y : \phi^{-1}(B) \in p\}$. Note that $A \in p$ implies $\phi(A) \in \phi_*(p)$, but the converse fails.

COROLLARY 4.4. *Let $(S, \cdot)$ be a semigroup, $m \in \mathbb{N}$, $\phi : S^m \to S$ and $B \subseteq S$. The following statements are equivalent.*

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- *There exists a sequence* $\langle x_n\rangle_{n=1}^{\infty}$ *in* S *with* $\{\phi(x_{F_1},\dots,x_{F_m}) : F_1 < \cdots < F_m\} \subseteq B$.
- $B \in \phi_*(\bigotimes_{i=1}^{m} p)$ *for some idempotent* $p \in \beta S$.

PROOF. Apply Theorem 4.3 with $A = \phi^{-1}(B)$. For details, see [12, Corollary 5.4]. □

THEOREM 4.5. *Let* $r, m \in \mathbb{N}$, $\phi : (\mathbb{N}_0)^m \to \mathbb{N}_0$ *and* $\mathbb{N}_0 = C_1 \cup \cdots \cup C_r$ *be an r-colouring. There exists a sequence* $\langle x_n\rangle_{n=1}^{\infty}$ *in* $\mathbb{N}$ *and a colour* C_i *such that*

$$\{\phi(|\Sigma_{<F_1}|,\dots,|\Sigma_{<F_m}|) : F_1 < \cdots < F_m\} \subseteq C_i,$$

where for $F_j = \{n_{j_1} < \cdots < n_{j_{k_j}}\} \in \mathcal{P}_f(\mathbb{N})$, *we define* $\Sigma_{<F_j} = \Sigma_{<s_{x_{n_{j_1}}}\cdots s_{x_{n_{j_{k_j}}}}}$.

PROOF. Let $(S,\cdot) = (\mathbb{N}_0, *_f)$, where $*_f$ is defined by $x *_f y = |\Sigma_{<s_x s_y}|$. Let $p \in \beta\mathbb{N}_0$ be an idempotent ultrafilter. By Corollary 4.4, with $q = \phi_*(\bigotimes_{i=1}^{m} p)$, there exist $C_i \in q$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that

$$\{\phi(x_{F_1},\dots,x_{F_m}) : F_1 < \cdots < F_m\} \subseteq C_i.$$

For $F_j = \{n_{j_1} < \cdots < n_{j_{k_j}}\}$, compute

$$x_{F_j} = x_{n_{j_1}} *_f \cdots *_f x_{n_{j_{k_j}}} = |\Sigma_{<s_{x_{n_{j_1}}}\cdots s_{x_{n_{j_{k_j}}}}}| = |\Sigma_{<F_j}|.$$

Substituting into ϕ yields the result. □

5. Application of the Hales–Jewett theorem

We start this section with the definition of a *partial semigroup*.

DEFINITION 5.1. A *partial semigroup* is a triple $(S, X \subseteq S\times S, *)$ of a set S, a subset $X \subseteq S\times S$ and an operation $*$ defined on X, satisfying

$$(x * y) * z = x * (y * z) \quad \text{for all } x, y, z \in G,$$

in the sense that if either side is defined, so is the other and they are equal.

Let $\mathcal{A}$ be a nonempty, finite set called the alphabet and $v \notin \mathcal{A}$ be a symbol called a variable. A located word in the alphabet $\mathcal{A}$ is a finitely supported function $b : \mathrm{Dom}(b) \to \mathcal{A}$, where $\mathrm{Dom}(b)$ is a (possibly empty) finite subset of $\mathbb{N}_0$. Similarly, a *located variable word* in the alphabet $\mathcal{A}$ for a variable v is a finitely supported function $b : \mathrm{Dom}(b) \to \mathcal{A}\cup\{v\}$ whose range contains v. Let $L(\mathcal{A})$ be the set of *located words* in $\mathcal{A}$ and let $L(\mathcal{A}v)$ be the set of located variable words in $\mathcal{A}$ for the variable v. Then, $S := L(\mathcal{A}) \cup L(\mathcal{A}v)$ has a natural partial semigroup operation, obtained by letting $b_0 * b_1$ be defined whenever the domains of b_0 and b_1 are disjoint. In such a case, $b_0 * b_1$ is just $b_0 \cup b_1$.

THEOREM 5.2 (Hales–Jewett theorem). *Let* $L(\mathcal{A})$ *be finitely coloured. Then, there exist* $\alpha \in L(\mathcal{A})$ *and* $\gamma \in \mathcal{P}_f(\mathbb{N})$ *such that* $\mathrm{Dom}(\alpha) \cap \gamma = \emptyset$ *and* $\{\alpha \cup (\gamma \times \{s\}) : s \in \mathcal{A}\}$ *is monochromatic.*

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In 2008, M. Beiglböck extended the Hales–Jewett theorem, obtaining a result stronger than the one above. His extension involves partition regular families of $\mathbb{N}$. A *partition regular family* $\mathcal{F}$ of $\mathbb{N}$ is a subset of $\mathcal{P}_f(\mathbb{N})$ such that for any partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exists an $i \in \{1, \ldots, r\}$ such that $C_i \in \mathcal{F}$.

THEOREM 5.3 (Beiglböck, 2008). *Let $\mathcal{F}$ be a partition regular family of finite subsets of $\mathbb{N}$ that contains no singletons and let $\mathcal{A}$ be a finite alphabet set. For any finite colouring of $L(\mathcal{A})$, there exist $\alpha \in L(\mathcal{A})$, $\gamma \in \mathcal{P}_f(\mathbb{N})$ and $F \in \mathcal{F}$ such that* $\mathrm{Dom}(\alpha)$, γ *and F are pairwise disjoint sets, and*

$$\{\alpha \cup (\gamma \cup \{t\} \times \{s\}) : s \in \mathcal{A}, t \in F\}$$

is monochromatic.

5.1. Geo-arithmetic progressions. Bergelson proved that for any finite colouring of $\mathbb{Z}$, there exists a monochromatic geo-arithmetic progression of arbitrary length. This result can be viewed as a combined extension of the additive and multiplicative versions of van der Waerden's theorem. Bergelson initially proved this property using ergodic theory [4]. Later, Beiglböck *et al.* [3] provided an alternative proof using the algebra of the Stone–Čech compactification.

THEOREM 5.4 (Geo-arithmetic progression). *If $n, r \in \mathbb{N}$ and $\mathbb{Z}$ is r-coloured, then there exist $a, b, d \in \mathbb{N}$ such that the set $\{a \cdot (b + i \cdot d)^j : 0 \le i, j \le n\}$ is monochromatic.*

In [2], Beiglböck demonstrated that the extension of the Hales–Jewett theorem is sufficiently strong to imply Theorem 5.4. We will use this variant of the Hales–Jewett theorem (Theorem 5.3) to prove a geo-arithmetic result in our structure.

THEOREM 5.5. *If $k, r \in \mathbb{N}$ and $\mathbb{N}$ is r-coloured, then there exist $a, b, d \in \mathbb{N}$ and $\gamma \in \mathcal{P}_f(\mathbb{N})$ such that the set*

$$\{|\Sigma_{<s_b(\prod_{t\in\gamma} s_t \cdot s_{a+id})^j}| : i, j = 0, 1, 2, \ldots, k\}$$

is monochromatic, where Σ is the set of sums of two squares.

PROOF. Assume that $\mathbb{N}$ is finitely coloured. Fix $k \in \mathbb{N}$ and consider the set $\mathcal{F} = \{\{a, a + d, \ldots, a + kd\} : a, d \in \mathbb{N}\}$ of all $(k + 1)$-term arithmetic progressions. Let $\mathcal{A} = \{0, 1, \ldots, k\}$ and define the function

$$h : L(\mathcal{A}) \to \mathbb{N} \quad \text{by} \quad h(\alpha) = \underset{t\in \mathrm{Dom}(\alpha)}{*_f} t^{(\alpha(t))}.$$

We colour each $\alpha \in L(\mathcal{A})$ with the colour of $h(\alpha)$, and choose α, γ and $F = \{a, a + d, \ldots, a + kd\}$.

By Theorem 5.3, the set

$$\{h(\alpha \cup (\gamma \cup \{a + id\}) \times \{j\}) : i, j \in \{0, 1, \ldots, k\}\}$$

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is monochromatic. Expanding h, we obtain

$$h(\alpha \cup (\gamma \cup \{a+id\}) \times \{j\}) = \Big(\underset{t\in \mathrm{Dom}(\alpha)}{*_f} t^{(\alpha(t))} \Big) *_f \Big(\underset{t\in\gamma}{*_f} t^{(j)} \Big) *_f (a+id)^{(j)}$$
$$= |\Sigma_{<(\prod_{t\in \mathrm{Dom}(\alpha)} s_t^{\alpha(t)})(\prod_{t\in\gamma} s_t \cdot s_{a+id})^j}|.$$

Let $s_b = \prod_{t\in \mathrm{Dom}(\alpha)} s_t^{\alpha(t)}$. Then, the set

$$\{|\Sigma_{<s_b(\prod_{t\in\gamma} s_t\cdot s_{a+id})^j}| : i,j \in \{0,1,\dots,k\}\}$$

is monochromatic, as required. □

5.2. Polynomial van der Waerden's theorem. The polynomial extension of van der Waerden's theorem relies on the polynomial version of the Hales–Jewett theorem. In 1988, Bergelson and Liebman proved the polynomial extension of the Hales–Jewett theorem by introducing and developing the apparatus of set-polynomials (polynomials whose coefficients are finite sets), and applying the methods of topological dynamics [8]. Later, Walters provided short and purely combinatorial proofs of those results in [23]. Let us begin with the statement of the polynomial Hales–Jewett theorem, along with some relevant notation.

For fixed positive numbers q, N, d, consider the set $X(q,N,d) = \prod_{i=1}^{d} [q]^{N^i}$, where $[q] = \{1,\dots,q\}$. An element $x \in X(q,N,d)$ is of the form $(\mathbf{b}_1,\dots,\mathbf{b}_d)$, where $\mathbf{b}_i : [N]^i \to [q]$. For $a = (\mathbf{a}_1,\dots,\mathbf{a}_d)$, $\gamma \subseteq [N]$ and $(x_1,\dots,x_d) \in [q]^d$, define an element $x = a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d$ as follows: if $x = (\mathbf{b}_1,\dots,\mathbf{b}_d)$, then

$$\mathbf{b}_j((i_1,\dots,i_j)) = \begin{cases} x_j & \text{if } (i_1,\dots,i_j) \in \gamma^j, \\ \mathbf{a}_j((i_1,\dots,i_j)) & \text{otherwise.} \end{cases}$$

THEOREM 5.6 (Polynomial Hales–Jewett (PHJ) theorem). *For any $q,k,d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $X(q,N,d)$ is k-coloured, there exist $a \in X(q,N,d)$ and $\gamma \subseteq [N]$ such that the set*

$$\{a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d : (x_1,\dots,x_d) \in [q]^d\}$$

is monochromatic.

We will use Theorem 5.6 to derive our version of the polynomial van der Waerden theorem, which is stated next.

THEOREM 5.7. *Let $d,k,\ell \in \mathbb{N}$ and $\{F_1,\dots,F_\ell\} \subset \mathcal{P}_f(\mathbb{N})$ with $F_i = \{a_{i1},\dots,a_{id}\}$ for all $i \in \{1,\dots,\ell\}$. Then, for any k-colouring of $\mathbb{N}$, there exist $b,c \in \mathbb{N}$ such that the set*

$$\{|\Sigma_{<s_b s_{a_{i1}}^c \cdots s_{a_{id}}^{c^d}}| : i = 1,\dots,\ell\}$$

is monochromatic, where Σ is the set of sums of two squares.

PROOF. Let $q,k,d \in \mathbb{N}$ be as in the statement of the polynomial Hales–Jewett theorem. Let $N = N(q,k,d)$ be the natural number given by that theorem. Suppose that

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$\chi : (\mathbb{N}, *_f) \to [k]$ is a k-colouring of $\mathbb{N}$, and let $m : X(q, N, d) \to (\mathbb{N}, *_f)$ be the canonical map defined by

$$m((\mathbf{b}_1, \dots, \mathbf{b}_d)) = \mathop{*_f}_{j=1}^{d}\Big(\mathop{*_f}_{(i_1,\dots,i_j)\in[N]^j} \mathbf{b}_j((i_1, \dots, i_j))\Big).$$

The composite $\chi \circ m$ is a k-colouring of $X(q, N, d)$. By the polynomial Hales–Jewett theorem, there exist $a \in X(q, N, d)$ and $\gamma \subseteq [N]$ such that the set

$$\{a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d : (x_1, \dots, x_d) \in [q]^d\}$$

is monochromatic. Therefore, the image

$$m(\{a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d : (x_1, \dots, x_d) \in [q]^d\})$$

is monochromatic for the colouring χ of $\mathbb{N}$. Note that

$$\begin{aligned} &m(a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d) \\ &\quad = \mathop{*_f}_{j=1}^{d}\Big(\mathop{*_f}_{(i_1,\dots,i_j)\in\gamma^j} \mathbf{b}_j((i_1, \dots, i_j))\Big) *_f \Big(\mathop{*_f}_{j=1}^{d}\Big(\mathop{*_f}_{(i_1,\dots,i_j)\in[N]^j\setminus\gamma^j} \mathbf{b}_j((i_1, \dots, i_j))\Big)\Big). \end{aligned}$$

This can be rewritten as

$$x_1^{(c)} *_f \cdots *_f x_d^{(c^d)} *_f b = |\Sigma_{<s_b s_{x_1}^c \cdots s_{x_d}^{c^d}}|,$$

where $c = |\gamma|$ and

$$b = \mathop{*_f}_{j=1}^{d}\Big(\mathop{*_f}_{(i_1,\dots,i_j)\in[N]^j\setminus\gamma^j} \mathbf{a}_j((i_1, \dots, i_j))\Big). \qquad \square$$

REMARK 5.8. Let $(P, <)$ be a countably infinite poset. Then, there is a canonical bijection

$$\phi : P \to \mathbb{N} \quad \text{given by} \quad \phi(x) = |\{y \in P : y < x\}|,$$

with an inverse ψ given by the $\psi(n) = (n + 1)$st smallest element of P. Since ψ does not preserve the multiplication of $\mathbb{N}$, if P is a countable multiplicative subgroup of $\mathbb{N}$, then the pullback ψ is not a semigroup homomorphism with respect to the usual multiplication. Thus, we obtain a new operation on $\mathbb{N}$ via

$$m *_\psi n = \phi(\psi(m) \cdot \psi(n))$$

(as we did before for the case of Σ). Note that $(\mathcal{P}_f(\mathbb{N}), <)$ is a countable ordered poset and, for the map $\sigma : \mathcal{P}_f(\mathbb{N}) \to \mathbb{N}$ given by $\sigma(F) = \sum_{n\in F} 2^n$, this map is not of the above form, although it induces operations on $\mathcal{P}_f(\mathbb{N})$.

REMARK 5.9. The set $\Sigma = \{a^2 + b^2 : a, b \in \mathbb{N}_0\}$ is symmetric with respect to a, b, so there is a canonical bijection between the set $T := \{(a, b) \in \mathbb{N}_0^2 : a \le b\}$ and Σ. Then, the map $\psi : T \to \mathbb{N}$ given by

$$\psi(m, 2n) = (n + 1)^2 - m \quad \text{and} \quad \psi(m, 2n + 1) = (n + 1)(n + 2) - m$$

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is a bijection that induces a new operation on T. Hence, one may find many different monochromatic configurations in $\mathbb{N}$.

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