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Topological entropy of m-fold maps on trees

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Abstract. We establish the analogue for maps on trees of the result established by Bobok (*Studia Math.* **152** (2002), 249–261 and *Studia Math.* **166** (2005), 11–27) for interval maps, that a continuous self-map for which all but countably many points have at least m preimages (and none have less than two) has topological entropy bounded below by $\log m$.

1. *Introduction*

This paper continues the investigation in [**Bob02, Bob05, BN05**] of the relation between the topological entropy $h_{\text{top}}(f)$ of a continuous map $f : X \to X$ of a compact space to itself and the number of preimages of points under f.

A continuous map $f : X \to X$ is *m-fold* on a subset $Y \subset X$ if every point in Y has at least m preimages in X. We drop the reference to the subset Y when it is the whole space.

An argument of Misiurewicz and Przytycki [**MP77**] shows that any C^1 map on any compact manifold which is m-fold on the set of its regular (non-critical) values satisfies the estimate

$$h_{\text{top}}(f) \geq \log m. \tag{1}$$

This argument is detailed in [**BN05**]. Coven [**Cov94**] conjectured that the differentiability condition could be replaced by continuity for a two-fold map of the interval; this was established in [**Bob02**].

A simple example [**BN05**] shows that for a continuous map of the interval, failure of the two-fold condition at a single point can allow zero entropy; the same example can be adapted to create an m-fold map on the circle with zero entropy. However, in [**Bob05**] it was shown that for a map of the interval, once the two-fold condition is assumed to hold everywhere, the estimate (1) is guaranteed with any higher value of m as soon as the m-fold condition holds on the complement of a countable set. We call a continuous map $f : X \to X$ *cocountably m-fold* if it is two-fold on X and m-fold on a cocountable set $Y \subset X$ (i.e. the complement $X \setminus Y$ has at most countably many points).

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In [**BN05**] it was shown that equation (1) holds for a cocountably m-fold self-map of the circle provided there is a positive lower bound on the diameter of all preimage sets of points.

In this paper, we establish the following extension to trees of the result established by the second author [**Bob05**] for maps of the interval.

THEOREM 1.1. *Suppose $f : T \to T$ is a cocountably m-fold map of a (finite) tree to itself. Then equation (1) holds:*

$$h_{\text{top}}(f) \geq \log m.$$

Our strategy follows that of [**Bob02, Bob05**], with some modifications to adapt it to more general trees. In §2 we describe a modification of [**BN05**, Theorem 4.8], which systematizes the strategy of [**Bob02, Bob05**] as an abstract set of conditions on a kind of weak 'horseshoe' (which we refer to as a *shift system*) that guarantees the estimate (1). This is general symbolic dynamics, and is not specific to tree maps. Then in §3 we formulate a scheme for linearly ordering the points of a tree, which takes the place of the linear ordering on an interval in our arguments. The hypotheses of the symbolic result are established for any m-fold map on a tree in §§4 and 5, and everything is combined to prove our main result in §6.

We close this section with an example that shows that the set of points where the m-fold condition fails in the hypotheses of Theorem 1.1 cannot be allowed to be uncountable, even if it is nowhere dense (at least not without further conditions).

PROPOSITION 1.2. *For each integer $m > 0$ there exists a map $f : I \to I$ of the interval $I = [0, 1]$ to itself such that:*

(1) *f is globally two-fold;*
(2) *f is m-fold on a set $Y = I \setminus K$, where K is a nowhere dense, closed (uncountable) set;*
(3) $h_{\text{top}}(f) = \log 2$.

Proof. We begin with a 'flattened tent map' $g : I \to I$, taking both endpoints to 0, taking a central interval $[a, b]$ to the right endpoint, and affine (or even just strictly monotone) on each of the complementary intervals. For example, taking $a = \frac{1}{3}$, $b = \frac{2}{3}$, we can define g by

$$g(x) := \begin{cases} 3x & \text{for } 0 \leq x \leq \frac{1}{3}, \\ 1 & \text{for } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ 3(1-x) & \text{for } \frac{2}{3} \leq x \leq 1. \end{cases}$$

Clearly, $h_{\text{top}}(g) = \log 2$.

Now, the iterated preimages of the central interval (a, b) consist of disjoint open intervals

$$g^{-k}[(a, b)] = \bigcup_{n=1}^{2^k} (a_n^k, b_n^k), \quad k = 0, 1, 2, \ldots,$$

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whose union is a dense open set

$$Y = \bigcup_{k=0}^{\infty} g^{-k}[(a, b)];$$

the complement of Y is a Cantor-like set $K \subset I$ (with $a = \frac{1}{3}$ and $b = \frac{2}{3}$, K is the classical middle-third Cantor set). We modify g on each of the intervals (a_n^k, b_n^k), $k > 0$, to a map $f : I \to I$ mapping (a_n^k, b_n^k) and its mirror image, $(1 - b_n^k, 1 - a_n^k)$, onto $g((a_n^k, b_n^k))$ in an m-fold manner.

Then clearly f is m-fold on the set Y, but since $f^{k+1}((a_n^k, b_n^k)) = \{1\}$, each of the intervals (a_n^k, b_n^k) is wandering; thus

$$h_{\text{top}}(f) = h_{\text{top}}(f|K) = h_{\text{top}}(g|K) = \log 2. \qquad \square$$

2. *Entropy via shift systems*

The material of this section is not specific to trees, and closely follows [**BN05**, §§3 and 4]. However, a modification of the definition of a 'locally dividing' set was needed here. We sketch many of the arguments, referring the reader to the exposition in [**BN05**] for details, but provide more detailed proofs where the modification mentioned above requires them.

An *m-shift system* for a map $f : X \to X$ is a collection

$$\mathcal{H} = \{H_1, \ldots, H_m\}$$

of m non-empty (but not necessarily closed or disjoint) sets $H_i \subset X$ satisfying

$$f(H_i) \supseteq \mathbb{H} := H_1 \cup \cdots \cup H_m \quad \text{for } i = 1, \ldots, m. \tag{2}$$

The *address set* of $x \in X$ is

$$\alpha(x) := \{a \in \{1, \ldots, m\} \mid x \in H_a\}$$

and its cardinality $\eta(x)$ is the *multiplicity* of $\mathcal{H}$ at x. The set of points with positive multiplicity is precisely $\mathbb{H}$; we define the *kernel* (respectively *center*) of $\mathcal{H}$ to be the set of points with multiplicity greater than one (respectively equal to m),

$$\mathfrak{K}(\mathcal{H}) := \{x \mid \eta(x) > 1\} = \bigcup_{i \neq j} H_i \cap H_j,$$

$$\mathfrak{Z}(\mathcal{H}) := \{x \mid \eta(x) = m\} = \bigcap_{i=1}^{m} H_i.$$

We also define the *core* of $\mathcal{H}$ as the set of points whose orbit remains in $\mathfrak{Z}(\mathcal{H})$ for all time:

$$\mathfrak{Z}_0(\mathcal{H}) := \{x \in X \mid f^k(x) \in \mathfrak{Z}(\mathcal{H}) \text{ for all } k\} = \bigcap_{k=0}^{\infty} f^{-k}[\mathfrak{Z}(\mathcal{H})].$$

Obviously, $\mathfrak{Z}_0(\mathcal{H}) \subseteq \mathfrak{Z}(\mathcal{H}) \subseteq \mathfrak{K}(\mathcal{H}) \subseteq \mathbb{H}$. We will call the shift system $\mathcal{H}$ *non-trivial* if $H_i \setminus \mathfrak{Z}_0(\mathcal{H}) \neq \emptyset$ for all i, and *closed* if each H_i is a closed subset of X.

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A closed m-shift system with an *empty* kernel ($\mathfrak{K}(\mathcal{H}) = \emptyset$) is usually called a horseshoe in the context of maps of the interval [**ALM00**]; the *itinerary* of a point with respect to a horseshoe is the sequence $a_0 \dots$ of addresses of its iterates, defined by

$$f^i(x) \in H_{a_i}. \tag{3}$$

However, when $\mathfrak{K}(\mathcal{H}) \neq \emptyset$, equation (3) need not define a unique itinerary for a point $x \in X$. Let $\Omega_m(n)$ denote the set of n-tuples (or 'n-words') $a_0 \dots a_{n-1}$ with $a_i \in \{1, \dots, m\}$. For $n \in \mathbb{N}$, the *n-itinerary set* of $x \in X$ is the subset of $\Omega_m(n)$ defined via equation (3):

$$\Omega(x)(n) := \{a_0 \dots a_{n-1} \in \Omega_m(n) \mid f^i(x) \in H_{a_i} \text{ for } i < n\} = \bigtimes_{i=0}^{n-1} \alpha(f^i(x)).$$

The set of points for which $\Omega(x)(n) \neq \emptyset$

$$\mathbb{D}_n := \bigcap_{i=0}^{n-1} f^{-i}[\mathbb{H}]$$

is the union of the sets

$$\Pi(w) := \{x \in X \mid w \in \Omega(x)(n)\} = \bigcap_{i=0}^{n-1} f^{-i}[H_{w_i}] \tag{4}$$

as $w = w_0 \dots w_{n-1}$ ranges over the finite collection $\Omega_m(n)$ of n-words. The intersection

$$\mathbb{D} := \bigcap_{i=0}^{\infty} f^{-i}[\mathbb{H}] = \bigcap_{n=1}^{\infty} \mathbb{D}_n$$

is the set of points whose whole forward orbit is contained in $\mathbb{H}$; we refer to it as the *domain* of $\mathcal{H}$.

For a horseshoe, the assignment of an itinerary to each point of $\mathbb{D}$ is a continuous map onto the m-shift space Ω_m, and hence provides a semiconjugacy from the $\mathbb{D}$-restriction of f to the one-sided shift map on Ω_m; the entropy estimate (1) on is an immediate consequence in this case.

The continuity of itineraries with respect to a horseshoe has a semicontinuity analogue for a general closed shift system. Recall that for any sequence of sets A_i, $i = 1, 2, \dots$,

$$\limsup A_i := \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i.$$

LEMMA 2.1. *Suppose $\mathcal{H}$ is a closed m-shift system.*

(1) *For each $x \in X$, $\Omega(x)$ is a closed subset of Ω_m.*
(2) *For each non-empty (closed) set $A \subset \Omega_m(n)$, $n \in \mathbb{N}$ ($n = \infty$), $\Pi(A)$ is a non-empty closed subset of X.*
(3) *For $n \in \mathbb{N}$, if $x_i \in \mathbb{D}_n$ for $i = 1, 2, \dots$, then $\limsup \Omega(x_i)(n) \neq \emptyset$.*
(4) *The set-valued maps $x \mapsto \Omega(x)(n)$, $n \in \mathbb{N} \cup \{\infty\}$, are upper semicontinuous: if $x_i \to x$ in X, then $\limsup \Omega(x_i)(n) \subset \Omega(x)(n)$.*

This is Lemma 3.3 of [**BN05**]; we refer the reader there for a proof.

To obtain entropy estimates from a shift system with a non-empty kernel, we require some further assumptions.

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Definition 2.2. We say a set $W \subset X$ locally divides the m-shift system $\mathcal{H}$ if the following conditions are satisfied.

(1) $f(W) \subset W$.

(2) $H_i \setminus W \neq \emptyset$ for $i = 1, \ldots, m$.

(3) There exists a closed shift-invariant set $\Lambda \subset \Omega_m$ such that

$$h_{\text{top}}(\Lambda) < \log m$$

and a neighborhood V of W in X such that if $f^i(x) \in V \setminus W$ for $i = 0, \ldots, n-1$ then

$$\Omega(x)(n) \subset \Lambda(n)$$

where $\Lambda(n)$ denotes the set of initial words of length n for sequences in Λ.

We note that condition (3) is somewhat different from that in the definition of local division in [**BN05**], but contains it as a special case by [**BN05**, Lemma 4.2].

Remark 2.3. Let $\mathcal{H}$ be a closed m-shift system, and $w \in \Omega_m(n)$ any finite word. Then:

(1) $\Pi(w)$ is a closed non-empty set in X;

(2) if W locally divides $\mathcal{H}$, then $\Pi(w)$ is not contained in W.

Condition (1) is a consequence of equation (4), while equation (2) is a consequence of conditions (1) and (2) in Definition 2.2.

LEMMA 2.4. *If W_j, $j = 1, \ldots, n$, are sets that locally divide a closed m-shift system $\mathcal{H}$, then their union $W := \bigcup_{j=1}^{n} W_j$ also locally divides $\mathcal{H}$, provided that $H_i \setminus W \neq \emptyset$ for $i = 1, \ldots, m$.*

Proof. The only condition from Definition 2.2 which is not immediate is (3). Let Λ_j and V_j be the shift-invariant set and neighborhood specified in Definition 2.2(3) for W_j, and set

$$\Lambda' := \bigcup_{j=1}^{n} \Lambda_j.$$

Clearly $h_{\text{top}}(\Lambda') = \max h_{\text{top}}(\Lambda_j) < \log m$, and we can choose a word $w = w_0 \ldots w_{k-1}$ of (some) length k that does not appear in any sequence in Λ':

$$w \in \Omega_m(k) \setminus \Lambda'(k).$$

Pick $i \neq w_{k-1} \in \{1, \ldots, m\}$.

Since each W_j is invariant, we can find a neighborhood G_j of W_j with

$$\bigcup_{i=0}^{k-1} f^i(G_j) \subset V_j.$$

Then

$$V = \bigcup_{j=1}^{n} G_j$$

is a neighborhood of W. For $\ell = 1, \ldots,$ set

$$F_\ell := \bigcap_{i=0}^{\ell-1} f^{-i}[V \setminus W]$$

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and define $\tilde{F}_\ell \subset \Omega_m$ to consist of all words of the form $\alpha iii\ldots$, where α belongs to the ℓ-itinerary set of some point in F_ℓ.

Now the set

$$\Lambda := \{iii\ldots\} \cup \operatorname{clos} \bigcup_{\ell \geq 1} \tilde{F}_\ell$$

is clearly a closed shift-invariant subset of Ω_m. Moreover, if $f^j(x) \in V \setminus W$ for $i = 0, \ldots, \ell - 1$, then $\Omega(x)(\ell) \subset \Lambda(\ell)$. To complete the proof of the lemma it suffices to show $w \notin \Lambda(k)$, which implies that $h_{\text{top}}(\Lambda) < \log m$.

If $w \in \Lambda(k)$, then it belongs to $\Omega(F_\ell)(k)$ for some ℓ. However, by the choice of V, w belongs to $\Omega(x)(k)$ for some point x with $f^i(x) \in V_j \setminus W_j$, $i = 0, \ldots, k$, for some j. Hence $w \in \Lambda_j(k) \subset \Lambda'(k)$, contrary to our choice of w. □

The following is a modification of [**BN05**, Lemma 4.4] to fit our more general definition of local division.

LEMMA 2.5. *Suppose $\mathcal{H}$ is a closed m-shift system and W is a set which locally divides $\mathcal{H}$ and contains all minimal sets in the core $\mathfrak{Z}_0(\mathcal{H})$ (i.e. which are contained in the center $\mathfrak{Z}(\mathcal{H})$).*

Then there exists $\zeta \in \mathbb{N}$ such that any orbit segment of length ζ which is contained in $\mathfrak{Z}(\mathcal{H})$ terminates in W.

Proof. Suppose $\{f^j(x_n)\}_{j=0}^{n-1}$ are (arbitrarily long) orbit segments contained in $\mathfrak{Z}(\mathcal{H}) \setminus W$ and (passing to a subsequence if necessary) assume $x_n \to x$. Then the orbit of x is contained in $\mathfrak{Z}(\mathcal{H})$ and so there is a minimal set $M \subset \omega(x) \cap W$. In particular, the continuity of f implies that we can find orbit segments $\{f^j(x)\}_{j=k_n}^{k_n+n-1}$ of x with increasing length contained in V, and hence we can find a subsequence $\{x_{l_n}\}$ of our original points whose orbit segments $\{f^j(x_{l_n})\}_{j=k_n}^{k_n+n-1}$ are contained in $\mathfrak{Z}(\mathcal{H}) \cap (V \setminus W)$. However, this means that $\Omega(f^{k_n}(x_{l_n}))(n) \subset \Lambda(n)$ consists of all words of length n, contradicting condition (3) of Definition 2.2. □

Given a closed m-shift system $\mathcal{H}$ for $f : X \to X$, we can associate to any closed f-invariant set $S \subset \mathbb{D}$ two different 'entropies': the topological entropy of the restriction of f to S, $\text{ent}(S) := h_{\text{top}}(f|S)$, and the topological entropy of the restriction of the shift map to the itinerary set of S, $\text{ent}(\Omega(S))$, which we refer to as the *virtual entropy* of S. These are not related in any *a priori* way—in particular the virtual entropy of f on a periodic orbit, unlike the topological entropy, need not be zero. However, we can sometimes obtain an *a priori* bound on it.

PROPOSITION 2.6. *Under the conditions of Lemma 2.5, there exists $\beta < \log m$ such that any periodic orbit $P \subset \mathbb{D}$ which is not contained in the set W (and hence is disjoint from the core) has virtual entropy bounded by β:*

$$\text{ent}(\Omega(P)) \leq \beta.$$

The proof of this is the same as that of [**BN05**, Proposition 4.5], with the center $\mathfrak{Z}(\mathcal{H})$ replaced by the set W.

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LEMMA 2.7. *Suppose $\mathcal{H}$ is a closed m-shift system whose kernel is* eventually countable*:*

$$f^j(\mathfrak{K}(\mathcal{H})) \text{ is (at most) countable for some } j.$$

Then for any infinite minimal set $M \subset X$,

$$\text{ent}(\Omega(M)) \leq h_{\text{top}}(M).$$

The proof of this is the same as that of [**BN05**, Lemma 4.7].

From this, we have the following analogue of [**BN05**, Theorem 4.8].

THEOREM 2.8. *Suppose $f : X \to X$ has a closed, non-trivial m-shift system $\mathcal{H}$ for which:*

(1) *the kernel is eventually countable;*
(2) *there exists a set $W \subset X$ such that*
   (a) *W contains all minimal sets in the core $\mathfrak{Z}_0(\mathcal{H})$, and*
   (b) *W locally divides $\mathcal{H}$.*

Then

$$h_{\text{top}}(f) \geq \log m.$$

Note that if the kernel is eventually countable, then every minimal set in the core $\mathfrak{Z}_0(\mathcal{H})$ is a periodic orbit.

To prove Theorem 2.8, we first establish some preliminary results.

Remark 2.9. Suppose $x \in X$ and $A \subset \Omega_m$ satisfy, for some $j, k \in \mathbb{N}$,

$$\Omega(f^j(x))(k) \cap \sigma^j[A](k) = \emptyset.$$

Then

$$\Omega(x)(k+j) \cap A(k+j) = \emptyset.$$

The proof of this is the same as that of [**BN05**, Remark 4.9].

LEMMA 2.10. *Suppose $A \subset \Omega_m$ is closed and shift-invariant, and $y \in \omega(x) \subset X$ with $\Omega(y) \cap A = \emptyset$.*

Then there exists a neighborhood U of x and $k \in \mathbb{N}$ such that

$$\Omega(x')(k) \cap A(k) = \emptyset$$

for every $x' \in U$.

This is the same as [**BN05**, Lemma 4.10].

PROPOSITION 2.11. *Suppose $\mathcal{H}$ satisfies the hypotheses of Theorem 2.8, and that $\Gamma \subset \Omega_m$ is a shift-minimal set such that, for every f-minimal set M disjoint from W,*

$$\text{ent}(\Gamma) > \max\{\text{ent}(\Lambda).\, \text{ent}(\Omega(M))\}$$

where Λ is as in Definition 2.2.

Then there exists $k \in \mathbb{N}$ such that every point $x \in X$ with $\Omega(x)(k) \cap \Gamma(k) \neq \emptyset$ satisfies $f^{k-1}(x) \in W$.

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Proof. We construct for every $x \in X$ a neighborhood $U(x)$ and an associated integer $k(x)$ such that every point $x' \in W$ with $f^{k(x)-1}(x') \notin W$ has $\Omega(x')(k(x)) \cap \Gamma(k(x)) = \emptyset$. We consider three cases; even though the second and third need not be mutually exclusive, this presents no problem.

(1) If $x \notin \mathbb{D}$, pick $k(x)$ so that $f^{k(x)}(x) \notin \mathbb{H}$, and a neighborhood $U(x)$ of x for which $f^{k(x)}(U(x)) \cap \mathbb{H} = \emptyset$. Then $\Omega(x')(k(x)) = \emptyset$ for all $x' \in U(x)$.

(2) If $\omega(x)$ contains a minimal set M which is not contained in W, then since $\mathrm{ent}(\Omega(M)) < \mathrm{ent}(\Gamma)$ and Γ is minimal, $\Omega(M)$ is disjoint from Γ. It follows by Lemma 2.10 with y any element of M and $A = \Gamma$ that we can find $U(x)$ and $k(x)$ so that $\Omega(x')(k(x)) \cap \Gamma(k(x)) = \emptyset$ for all $x' \in U(x)$.

(3) If $M \subset \omega(x) \cap W$, pick Λ and V associated to W in Definition 2.2. Since $\mathrm{ent}(\Lambda) < \mathrm{ent}(\Gamma)$, Λ is disjoint from Γ, and hence $\Lambda(k_0) \cap \Gamma(k_0) = \emptyset$ for some $k_0 \in \mathbb{N}$. However, since $M \subset \omega(x)$, there exists $k_1 \in \mathbb{N}$ so that $f^{k_1+j}(x) \in V$ for $0 \leq j < k_0$, and a neighborhood $U(x)$ so that the same holds true for every $x' \in U(x)$. Let $k(x) = k_0 + k_1$. For any $x' \in U(x)$ with $f^{k(x)-1}(x') \notin W$ we have $f^{k_1+j}(x') \in V \setminus W$ for $0 \leq j < k_0$, and hence $\Omega(x')(k(x))$ is disjoint from $\Gamma(k(x))$ by Lemma 2.10.

Since $\omega(x)$ always contains some minimal set, these cases are exhaustive, and so $\{U(x) \mid x \in X\}$ forms an open cover of X. Let $\{U(x_i) \mid i = 1, \ldots, N\}$ be a finite subcover, and set

$$k = \max_{i=1,\ldots,N} k(x_i).$$

Then we clearly have the desired conclusion with this value of k. □

Proof of Theorem 2.8. Let β as in Proposition 2.6. We will show that for $0 < \varepsilon < \log m - \max\{\mathrm{ent}(\Lambda), \beta\}$, f has minimal sets M with

$$h_{\mathrm{top}}(M) \geq \log m - \varepsilon.$$

By [**Gri73**], Ω_m contains shift-minimal sets with entropy arbitrarily near $\log m$, so we can find Γ_ε minimal with

$$\mathrm{ent}(\Gamma_\varepsilon) > \log m - \varepsilon > \max\{\mathrm{ent}(\Lambda), \beta\}.$$

If some $M \subset \mathbb{H} \setminus W$ has $h_{\mathrm{top}}(M) < \log m - \varepsilon$, then by Proposition 2.6 and Lemma 2.7, $\mathrm{ent}(\Omega(M)) < \log m - \varepsilon$. Thus, if no minimal set M has $\mathrm{ent}(\Omega(M)) \geq \log m - \varepsilon$, then Proposition 2.11 says that for some $k \in \mathbb{N}$, $\Omega(x)(k) \cap \Gamma_\varepsilon(k) = \emptyset$ whenever $f^{k-1}(x) \notin W$. However, Remark 2.3(3) says that every $w \in \Gamma_\varepsilon(k)$ belongs to some $\Omega(x)(k)$ for a point with $f^{k-1}(x) \notin W$, a contradiction.

This establishes the existence of minimal sets satisfying $h_{\mathrm{top}}(M) \geq \log m - \varepsilon$. Thus $h_{\mathrm{top}}(f) \geq \log m - \varepsilon$, and since $\varepsilon > 0$ can be chosen arbitrarily small, the conclusion follows. □

3. *Trees*

Topologically, a (finite) *tree* is a uniquely arcwise connected Hausdorff space, which is a union of (finitely many) closed intervals. The complement $T \setminus \{x\}$ of a point $x \in T$ has

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finitely many components, called the *branches* of T at x; a *closed branch* at x is the union of $\{x\}$ with a branch at x. The *valence* of x is the number of branches at x; a point of valence one (respectively valence > 2) is called an *endpoint* (respectively a *branchpoint*) of T; the set of all branchpoints of T is denoted $\mathsf{B}(T)$.

We endow T with further combinatorial structure, first by distinguishing a finite set $\mathsf{V}(T)$ of *vertices* that includes all endpoints and branchpoints (and perhaps some valence two points), and then distinguishing one vertex v_0 as the *root* of T. An *edge* of T is the closure of a component of $T \setminus \mathsf{V}(T)$. Each closed branch of T at $v \in \mathsf{V}(T)$ contains a unique edge with endpoint v; for $v \neq v_0$, the *incoming* branch (respectively incoming edge) at v is the branch containing v_0 (respectively the unique edge at v contained in the convex hull $\langle v_0, v\rangle$), and the other branches at v—as well as edges at v contained in their closures—are *outgoing* at v. We direct each edge of T so that v is the terminal or right (respectively initial or left) endpoint of the incoming (respectively any outgoing) edge at v. We use interval notation, denoting the edge with left endpoint u and right endpoint v by $[u, v]$, and adapt the notation of open and half-open intervals to denote edges missing one or both endpoints. The number of outgoing branches (equivalently edges) at $v \in \mathsf{V}(T)$ is its *outdegree* (clearly, for $v \neq v_0$ this is one less than the valence). The *level* of a vertex $v \in \mathsf{V}(T)$ is the number of edges contained in $\langle v_0, v\rangle$ (so the root is at level zero).

Now, we wish to define a linear ordering $\prec$ on the points of T. We begin by numbering the outgoing edges at each vertex $v \in \mathsf{V}(T)$; this induces a numbering of the outgoing branches, $B_i(v)$, $i = 1, \ldots, \nu$, where ν is the outdegree of v; for the moment, this numbering is arbitrary, but we will impose a further condition on it in §5. For $v \neq v_0$, the incoming branch is numbered zero, so

$$T \setminus \{v\} = \bigcup_{i=0}^{\nu} B_i(v)$$

(for $v = v_0$, the only difference is that there is no $B_0(v_0)$).

Given this numbering, we assign to each point $x \neq v_0$ an address as follows. There is a unique simple path $\gamma(x)$ from v_0 to x; let $V(x) = (v_0, \ldots, v_k)$ be the sequence of vertices occurring along $\gamma(x)$—if $x \in \mathsf{V}(T)$ then v_k is the last vertex along $\gamma(x)$ before x. For each $j = 0, \ldots, k$, there is precisely one outgoing branch at v_j, say $B_{i_j}(v_j)$, containing x. The sequence $\alpha(x) := (i_0, \ldots, i_k)$ is the *address* of x in T. Two points have the same address in T precisely if they belong to the same left-open edge $(v_k, v_{k+1}]$. The linear ordering $\prec$ is then defined by lexicographic comparison of addresses and, within an edge, the direction from left to right. More precisely, we make the following definition.

Definition 3.1. (The linear ordering $\prec$ on T)

(1) $v_0 \prec x$ for every $x \neq v_0$ in T.

(2) Given $x \neq x'$ in T with $\alpha(x) = (i_0, \ldots, i_k)$ and $\alpha(x') = (i'_0, \ldots, i'_{k'})$:
   (a) if $\alpha(x) = \alpha(x')$, then x and x' both belong to a common left-open edge $(v_k, v_{k+1}]$ (we write $x \prec x'$ if $x \in \langle v_k, x'\rangle$);
   (b) if $i_j = i'_j$ for $j = 0, \ldots, k$ (and $k' > k$) then $x \prec x'$;
   (c) if $j_0 := \min\{j \mid i_j \neq i'_j\}$ then $x \prec x'$ if and only if $i_{j_0} < i'_{j_0}$.

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For $v \in \mathsf{V}(T)$ with outdegree ν and any $i \in \{1, \dots, \nu\}$, set

$$\mathcal{B}_v^-(i) := \bigcup_{0<i'<i} B_{i'}(v),$$
$$\mathcal{B}_v^+(i) := \bigcup_{i'>i} B_{i'}(v).$$

Remark 3.2. (Topological interpretation of the ordering $\prec$ on T)

(1) If $x' \in \operatorname{int}\langle v_0, x\rangle$ then $v_0 \prec x' \prec x$.

(2) For any vertex $v \in \mathsf{V}(T)$ with $V(v) = (v_0, \dots, v_k)$ and $\alpha(v) = (i_0, \dots, i_k)$,

$$\{x \mid x \succ v\} = \bigcup_{j=0}^{k} \mathcal{B}_{v_j}^+(i_j) \cup \bigcup_{i>0} B_i(v),$$
$$\{x \mid x \prec v\} = \bigcup_{j=0}^{k} \mathcal{B}_{v_j}^-(i_j) \cup \langle v_0, v\rangle \setminus \{v\}.$$

(3) For $x \in (v_k, v_{k+1})$ with $V(x) = (v_0, \dots, v_k)$ and $\alpha(x) = (i_0, \dots, i_k)$,

$$\begin{aligned}\{x' \mid x' \succ x\} &= (x, v_{k+1}] \cup \{x' \mid x' \succ v_{k+1}\} \\ &= (x, v_{k+1}] \cup \bigcup_{j=0}^{k} \mathcal{B}_{v_j}^+(i_j) \cup \bigcup_{i>0} B_i(v_{k+1}),\end{aligned}$$
$$\begin{aligned}\{x' \mid x' \prec x\} &= [v_k, x) \cup \mathcal{B}_{v_k}^-(i_k) \cup \{x' \mid x' \prec v_k\} \\ &= \bigcup_{j=0}^{k} \mathcal{B}_{v_j}^-(i_j) \cup \langle v_0, x\rangle \setminus \{x\}.\end{aligned}$$

Note in particular that for any $v \in \mathsf{V}(T)$ the *outgoing* branches at v are comparable: for $0 < i < i'$ if $x \in B_i(v)$ and $x' \in B_{i'}(v)$ then $x \prec x'$; however, this is in general false if $i = 0$.

A key property of the linear ordering on the real line is its continuity: that if two convergent sequences $x_i \to x$ and $x_i' \to x'$ satisfy $x_i \le x_i'$ for all i, then their limits satisfy the same inequality: $x \le x'$. This is false for our ordering: if v_j is a vertex belonging to $V(x')$ and i_j is the corresponding element of $\alpha(x)$, then a sequence $x_i' \in \mathcal{B}_{v_j}^+(i_j)$ will satisfy $x \prec x_i'$, but if it converges to $x' = v_j$ then $\lim x_i' \prec x$. However, this can only happen when the limit is a vertex.

LEMMA 3.3. *Suppose $x_i \to x$ and $x_i' \to x'$ are convergent sequences in T with*

$$x_i \prec x_i', \quad i = 1, 2, \dots .$$

If $x, x' \notin \mathsf{V}(T)$, then either $x = x'$ or

$$x \prec x'.$$

Proof. If x and x' are interior to the same edge of T, the conclusion is trivial. If this is not the case then they are interior to distinct edges of T, and these are comparable. Since the convergent sequences are eventually interior to the corresponding edges, the conclusion is immediate. □

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4. *The kernel*

In this section, we show, given a cocountably m-fold map $f : T \to T$ on the tree T, how to construct an m-shift system $\mathcal{H}$ with eventually countable kernel (in fact, with $f(\mathcal{K}(\mathcal{H}))$ countable), thus fulfilling the first condition of Theorem 2.8. This is based on the idea of *regular m-sections* from [**BN05**].

4.1. *Regular values.* Suppose $f : T \to T$ is continuous and $x \in T$ with $y = f(x) \in T \setminus \mathsf{V}(T)$. Then a small neighborhood of y is an interval disjoint from $\mathsf{B}(T)$, so that it makes sense to talk about points being on one or the other side of y. We say that x is a *non-minimal* (respectively *non-maximal*) preimage of y if there exist points x' arbitrarily near x with $f(x') < y$ (respectively $f(x') > y$).

Given $m \in \mathbb{N}$, we say that $y \in T \setminus \mathsf{V}(T)$ is a *left m-regular* (respectively *right m-regular*) value for $f : T \to T$ if y has at least m non-minimal (respectively non-maximal) preimages. The set of all left m-regular (respectively right m-regular) values of f will be denoted $\mathfrak{C}_m(f, \ell)$ (respectively $\mathfrak{C}_m(f, r)$), and their intersection, the set of *m-regular values*, will be denoted $\mathfrak{C}_m(f)$.

In [**BN05**, §5.1] it is shown that $\mathfrak{C}_m(f)$ is an open cocountable set for any cocountably m-fold map of the circle; the analogous result in our case, $f : T \to T$, T a (finite) tree, follows from an analogous argument.

PROPOSITION 4.1. *For any cocountably m-fold map $f : T \to T$ on a finite tree, $\mathfrak{C}_m(f)$ is an open cocountable set.*

Proof. The argument in [**BN05**] has four steps; we summarize these and indicate any adjustments to make each step work in our setting.

(1) [**BN05**, Lemma 5.1] *If I and $Y = f(I)$ are closed intervals, then every $y \in \operatorname{int} Y$ has at least one non-minimal and at least one non-maximal preimage in I.*

This is because the components of $I \setminus f^{-1}[y]$ containing the endpoints of I must map onto one-sided neighborhoods of y.

(2) [**BN05**, Lemma 5.2] *If y is left (respectively right) m-regular, then the interior of some left (respectively right) neighborhood of y is contained in $\mathfrak{C}_m(f)$.*

This carries over, since we can separate m non-minimal (respectively non-maximal) preimages of y with neighborhoods disjoint from $\mathsf{B}(T)$.

(3) [**BN05**, Lemma 5.3] *If $f^{-1}[y]$ has at least m components, then there exists a non-trivial interval Y with y an endpoint such that $\operatorname{int} Y \subset \mathfrak{C}_m(f)$.*

This carries over provided $y \notin f(\mathsf{V}(T))$, since we can find $2m$ intervals, each contained in a single edge of T, with one endpoint an endpoint of a component of $f^{-1}[y]$, and each mapping onto a one-sided neighborhood of y; at least m of these intervals map onto the same side of y, and we can apply the preceding result.

(4) [**BN05**, Proposition 5.4] $\mathfrak{C}_m(f)$ is open by the second result above, and the points y for which $f^{-1}[y]$ has a non-empty interior is at most countable. Throwing these out as well as the (finite) set $f(\mathsf{V}(T))$, we have a cocountable set $Y \subset T$ for which the third result above says that each $y \in Y$ is either an element of $\mathfrak{C}_m(f)$ or an endpoint of a component of $\mathfrak{C}_m(f)$. Since $\mathfrak{C}_m(f)$ is open, it has (at most) countably many components, so throwing away their endpoints (from Y) we obtain a cocountable subset of the open set $\mathfrak{C}_m(f)$. □

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4.2. *Regular sections.* If $f : T \to T$ is m-fold on the subset Y, then an *m-section* for f on Y is a choice for each $y \in Y$ of a collection of m distinct preimages $\psi_j(y) \in f^{-1}[y]$, $j = 1, \dots, m$, which we combine as a map

$$\psi : Y \to \overset{m}{\underset{i=1}{\times}} T := T \times \cdots \times T.$$

We say ψ is a *left-regular section* (respectively *right-regular section*) if $\psi_i(y)$ is a non-minimal (respectively non-maximal) preimage of y for every $y \in Y$. Clearly, a left-regular (respectively right-regular) section can only be defined on a subset of $\mathfrak{C}_m(f, \ell)$ (respectively $\mathfrak{C}_m(f, r)$), but even if $Y \subset \mathfrak{C}_m(f)$ it need not be possible to define a section on Y that is both left-regular and right-regular.

We define several calibrations of the 'spread' of an m-section ψ on $Y \subset T$. Given $y \in Y$, set

$$\delta(\psi, y) := \min_{1 \le j < j' \le m} \operatorname{dist}(\psi_j(y), \psi_{j'}(y))$$

and then the *mesh* of ψ on the set $U \subset Y$ is

$$\Delta(\psi, U) := \inf\{\delta(\psi, y) \mid y \in U\}.$$

Also, the distance on the product $\times_{i=1}^{m} T$ defines a distance between the values of ψ at two points $y, y' \in Y$

$$\operatorname{mdist}(\psi(y), \psi(y')) := \max_{j=1,\dots,m} \operatorname{dist}(\psi_j(y), \psi_j(y'))$$

and the *variation* of ψ across a set $U \subset Y$ is

$$\|\psi\|_U := \sup\{\operatorname{mdist}(\psi(y), \psi(y')) \mid y, y' \in U\}.$$

We refer the reader to [**BN05**] for the proof of the following result.

PROPOSITION 4.2. [**BN05**, Proposition 5.8] *Suppose $f : T \to T$ is a cocountably m-fold map. Then there is a cocountable open subset $Y \subset \mathfrak{C}_m(f)$ and a left-regular (respectively right-regular) m-section ψ of f on Y such that, for every component Y_i of Y,*

$$\Delta(\psi, Y_i) > 0, \tag{5}$$

$$\|\psi\|_{Y_i} < \tfrac{1}{2}\Delta(\psi, Y_i). \tag{6}$$

The condition (6) has several useful consequences.

LEMMA 4.3. *Suppose that, as in Proposition 4.2, ψ is an m-section of f on the cocountable open set $Y \subset T$ such that every component Y_i of Y satisfies equation (6).*

If for some sequence $y_i \in Y$ and some sequence $j_i \in Z$ we have $\psi_{j_i}(y_i)$ converging to a point $x \in f^{-1}[Y]$, then j_i is eventually constant.

Proof. Suppose $j_1 \neq j_2$ and $y_i, y_i' \in Y$ satisfy

$$u = \lim \psi_{j_1}(y_i) = \lim \psi_{j_2}(y_i')$$

with $f(u) \in Y$. By continuity of f,

$$y := f(u) = \lim f(\psi_{j_1}(y_i)) = \lim y_i = \lim y_i'.$$

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We can assume that y, y_i and y_i' all belong to the same component U of Y. For i large,

$$\operatorname{dist}(\psi_{j_1}(y_i), x) < \tfrac{1}{4}\Delta(\psi, U) \quad \text{and} \quad \operatorname{dist}(\psi_{j_2}(y_i'), x) < \tfrac{1}{4}\Delta(\psi, U).$$

However, then

$$\begin{aligned}\Delta(\psi, U) &\le \operatorname{dist}(\psi_{j_1}(y_i), \psi_{j_2}(y_i)) \\ &\le \operatorname{dist}(\psi_{j_1}(y_i), \psi_{j_2}(y_i')) + \operatorname{dist}(\psi_{j_2}(y_i'), \psi_{j_2}(y_i)) \\ &\le \operatorname{dist}(\psi_{j_1}(y_i), x) + \operatorname{dist}(x, \psi_{j_2}(y_i')) + \operatorname{dist}(\psi_{j_2}(y_i'), \psi_{j_2}(y_i)) \\ &\le \|\psi\|_U + \tfrac{1}{2}\Delta(\psi, U) < \Delta(\psi, U),\end{aligned}$$

which is a contradiction. □

Remark 4.4. If an m-section ψ satisfies equation (6), and we modify ψ by applying a permutation to the indices $j = 1, \dots, m$ of the components $\psi_j(y)$ for all y in some component Y_i of Y, then equation (6) remains true.

4.3. *Monotone sections.* Suppose that we have a linear ordering $\prec$ on T as in §3. We begin with some remarks on comparability.

Remark 4.5. (Comparability)

(1) Suppose $x \prec x'$ and x is not a vertex. Then there exist neighborhoods U, U' of x and x' which are comparable: that is, $z \prec z'$ for all $z \in U$ and $z' \in U'$, and we write $U \prec U'$.

(2) An interval disjoint from $\mathsf{B}(T)$ is comparable with any point not contained in it.

This is an immediate consequence of Remark 3.2.

Remark 4.6. Suppose J is a non-trivial open interval contained in $f(\langle z, z'\rangle)$ and disjoint from $\mathsf{B}(T)$. Then there exists an open interval $J' \subset \langle z, z'\rangle$ with $f(J') = J$.

To see this, consider the preimage sets for the two endpoints of J; these are disjoint non-empty closed subsets of the interval $\langle z, z'\rangle$, and it is easy to see that in such a situation some component of the complement of their union is an interval with one endpoint in each, and it must map onto J.

We say that ψ is a *monotone section* if $\psi_i(y) \prec \psi_{i+1}(y)$ for each $i = 1, \dots, m-1$ and $y \in Y$.

Remark 4.7. Suppose f has an m-section ψ defined on a cocountable open set $Y \subset T$ satisfying equation (6). Then we can renumber the indices of the components by a permutation on each component of Y so that the m-section is also monotone. By throwing out from Y any vertices or images of vertices (a finite set), we obtain an m-section satisfying equation (6) and:

(1) if y and y' belong to the same component of Y, and $1 \le j < j'$, then

$$\psi_j(y) \prec \psi_{j'}(y');$$

(2) if $j \ne j'$ and for some sequences $y_i, y_i' \in Y$ we have

$$\lim \psi_j(y_i) = \lim \psi_{j'}(y_i') = u$$

then $y := f(u) \notin Y$.

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4.4. *Extreme preimages.* Given the linear ordering $\prec$ on T as in §3, for any point $y \in T$ we define

$$m_y := \min f^{-1}[y], \tag{7}$$

$$M_y := \max f^{-1}[y]. \tag{8}$$

Given the set Y, we can, without reference to any m-section, define

$$H_- := \operatorname{clos}\{m_y \mid y \in Y\}, \tag{9}$$

$$H_+ := \operatorname{clos}\{M_y \mid y \in Y\}, \tag{10}$$

$$H_\# = H_- \cap H_+. \tag{11}$$

Remark 4.8. Suppose $x \in H_\#$.

(1) If J is a set which is comparable to some neighborhood U' of x, then no neighborhood U of x has $f(U) \subset f(J)$.

(2) In particular, f does not collapse any neighborhood U of x.

To see the first statement, suppose $J \prec U$ and $f(U) \subset f(J)$. Then for every $x' \in U$ there exists $x'' \in J$ with $f(x') = f(x'') := y$, but since $x'' \prec x'$ we have that $x' \neq m_y$; hence $x \notin H_-$. Similarly, if $U \prec J$ then $x \notin H_+$.

LEMMA 4.9. *Suppose $x \in H_\#$, together with $f(x)$, is not a branchpoint of T. Then each of the two closed intervals into which x divides the edge containing it is mapped into a single closed branch of T relative to $f(x)$.*

Proof. If not, then there exists a closed subinterval J of the edge, disjoint from x, whose image contains $f(x)$ in its interior. Then picking a neighborhood U of x so small that $f(U) \subset f(J)$, we obtain a contradiction to Remark 4.8(1). □

Given the linear ordering on T, we call an m-section on $Y \subset T$ *spanning* if, for every $y \in Y$,

$$\psi_1(y) = m_y,$$
$$\psi_m(y) = M_y.$$

PROPOSITION 4.10. *If $f : T \to T$ is cocountably m-fold, and $\prec$ is a linear ordering on T as in §3, then there exists a monotone spanning m-section ψ defined on a cocountable open set $Y \subset T$ for which the corresponding m-shift system $\mathcal{H}$ defined by*

$$H_j := \operatorname{clos}\{\psi_j(y) \mid y \in Y\}, \quad j = 1, \ldots, m$$

satisfies

$$f(\mathfrak{K}(\mathcal{H})) \cap Y = \emptyset,$$

and in particular $\mathfrak{K}(\mathcal{H})$ is eventually countable.

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Proof. From Proposition 4.2 and Remark 4.4, we can find a monotone m-section on a cocountable open set $Y \subset T$ satisfying equation (6); using Remark 4.7, we can also insure that Y has no vertices or images of vertices, and that any point in $\mathfrak{K}(\mathcal{H})$ has image outside Y. Now, if we replace $\psi_1(y)$ (respectively $\psi_m(y)$) with m_y (respectively M_y), we do not change monotonicity, and we gain spanning. We need to show that

$$\begin{aligned} H_- \cap H_j \cap f^{-1}[Y] &= \emptyset \quad \text{for } j > 1, \\ H_+ \cap H_j \cap f^{-1}[Y] &= \emptyset \quad \text{for } j < m, \\ H_\# \cap f^{-1}[Y] &= \emptyset. \end{aligned}$$

Suppose u belongs to one of the sets above, and let U (respectively $\tilde{U}$) be a neighborhood of u (respectively $y = f(u)$) with $f(U) \subset \tilde{U}$, with $\tilde{U}$ contained in a single component $\bar{Y}$ of Y, and the length of U sufficiently small that any sequence $\psi_j(y_i)$ contained in U has j constant. This means that there is an index j' such that $\psi_{j'}(y')$ separates $\psi_j(y')$ from $m_{y'}$ (respectively $M_{y'}$) for every $y' \in U$, and in particular u cannot be a limit of points of the form m_{y_i} (respectively M_{y_i}). □

5. *The center*

In this section we concentrate on the second hypothesis of Theorem 2.8: the existence of a set containing all minimal sets in the core that locally divides the shift system $\mathcal{H}$. Given $f : T \to T$ a cocountably m-fold map on the tree T, Proposition 4.10 has given us a monotone spanning m-section ψ for which the kernel is eventually countable. Note that this implies in particular that every minimal set in the core is a periodic orbit, since infinite minimal sets are uncountable, while the core is contained in every image of the kernel. Thus we are interested in the behavior of orbit segments near a periodic orbit in the core.

We begin by noting some simplifying assumptions concerning f that we can make without loss of generality. Note that any closed connected subset $T' \subset T$ is itself a tree. A subtree is *natural* if it is a union of edges of T; it is *proper* if it is neither all of T nor a single point.

We concentrate on maps $f : T \to T$ satisfying the following assumption.

Assumption 5.1.

(1) If a branchpoint y of T is f-preperiodic, then $f(y)$ is a fixed point of f:

$$\bigcup_{n=0}^{\infty} f^{-n}[\mathrm{Per}(f)] \cap \mathsf{B}(T) \subset f^{-1}[\mathrm{Fix}(f)] \cap \mathsf{B}(T).$$

(2) There is no proper f-invariant natural subtree of T.

Remark 5.2. It suffices to prove Theorem 1.1 for the class of maps satisfying Assumption 5.1.

To see that we can assume (1), note that since $\mathsf{B}(T)$ is a finite set, the required property holds for some iterate f^n of f. However, if f is cocountably m-fold then f^n is cocountably (m^n)-fold, and $h_{\mathrm{top}}(f^n) = nh_{\mathrm{top}}(f)$, so the estimates $h_{\mathrm{top}}(f^n) \geq \log m^n$ and $h_{\mathrm{top}}(f) \geq \log m$ are equivalent.

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To see that we can assume (2), suppose that T' is an f-invariant proper natural subtree. We distinguish the following two subcases.

- If the restriction $f|T'$ is cocountably m-fold, then we replace $f : T \to T$ with $f : T' \to T'$ using the fact that $h_{\text{top}}(f|T') \leq h_{\text{top}}(f)$ to complete the argument.
- If $f|T'$ is *not* cocountably m-fold, we collapse T' to a point; it is easy to see that the quotient space is a tree and the induced action of f on this tree is again cocountably m-fold. Since entropy is non-increasing under factors, we are done.

We will assume from now on that f has both properties above.

Suppose $y \in \mathsf{B}(T) \cap \text{Fix}(f)$. By a *branch germ* at y we mean the intersection of a branch at y with some (sufficiently small) neighborhood of y in T. We call a branch (or edge) at y *monotone* at y if some branch germ maps into a single closed branch at y. A *non-monotone* branch (or edge) at y is one for which there are points arbitrarily near y whose images belong to distinct branches at y.

We call the numbering of left-open edges of T (and hence the induced ordering $\prec$ on the points of T) *f-adjusted* if for every $y \in \mathsf{B}(T) \cap \text{Fix}(f)$ every *monotone* outgoing edge at y is numbered lower than every *non-monotone* outgoing edge at y. (We cannot *a priori* rule out the possibility that the incoming branch is non-monotone at y; this must be handled separately.)

We can always pick our numbering to be f-adjusted, and we assume from now on that this property holds.

5.1. *Periodic branchpoints in the core.* In this section we show that any periodic branchpoint in the core locally divides $\mathcal{H}$.

Given $y \in \mathsf{B}(T) \cap \text{Fix}(f)$ of outdegree ν, and assuming the ordering $\prec$ is f-adjusted, we have numbered the branches at y as

$$T \setminus \{y\} = \bigcup_{i=0}^{\nu} B_i(y)$$

consistent with $\prec$: that is, $B_{i_1}(y) \prec B_{i_2}(y)$ for $0 < i_1 < i_2$. For a neighborhood U of y in T, we use the notation

$$U_i := B_i(y) \cap U.$$

(Remember that $B_i(y)$ does not include y.) Set $\mathfrak{G} := \{H_1, H_m\}$ and $H_* := H_1 \cup H_m$. For $i = 0, \ldots, \nu$ and $j = 1, m$, or $*$, let

$$G_i^j := U_i \cap H_j.$$

Remark 5.3. If $f(U_i) = \{y\}$ then $U_i \cap H_* = \emptyset$.

We will successively shrink the neighborhood U to insure a number of conditions as the section progresses; thus we are concerned with the germ of the behavior at y. In what follows, the ith branch at y will be denoted simply as B_i.

Remark 5.4. Suppose V is a closed subinterval of a branch at y such that $f(V) \supsetneq \{y\}$. Then for a sufficiently small neighborhood U of y (in particular, one disjoint from V) the following conditions apply.

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(1) There exists at least one branch germ U_i such that $f(V) \supset U_i$.
(2) If $V \subset B_{i_0}$, where $i_0 > 0$ (i.e. V is contained in an outgoing branch) and $z \in U_i \subset f(V)$, then for any i', the branch germ $U_{i'}$ does not contain m_z (respectively M_z) if $i' > i_0$ (respectively $i' < i_0$).
(3) If $V \subset \{x \mid x \prec y\}$ and $z \in U_i \subset f(V)$, then for any $i' \geq 0$, the branch germ $U_{i'}$ does not contain m_z.

This is because if U is disjoint from V, and then z has a preimage in V, which is $\prec$ (respectively $\succ$) any point of $U_{i'}$ for $i' \neq i_0$. If $i' = i_0$, V is farther from y than $U_{i'}$, which implies that $V \prec U_{i'}$ (respectively $V \succ U_{i'}$) when $i_0 = 0$ (respectively $i_0 > 0$).

LEMMA 5.5. *If B_i is not monotone at y, then for U sufficiently small:*
(1) *if $i = 0$, $G_i^1 = \emptyset$;*
(2) *if $1 \leq i \leq \nu$, $G_i^m = \emptyset$.*

Proof. By Remark 5.3, we can assume that U_i is not collapsed to y by f. By assumption, there exist points $x_k \in B_i$ converging to y with $f(x_k) = y$. For each branch $B_{i'}$ intersecting $f(B_i)$, we can find a closed interval $V_{i'} \subset B_i$ such that $f(V_{i'})$ contains a neighborhood of y in $\{y\} \cup B_{i'}$. A finite number of these suffice to fill a neighborhood of y in $f(B_i)$. Now the result follows from Remark 5.4. □

Set

$$\Phi := \{i \mid U_i \cap H_* \neq \emptyset \text{ for all neighborhoods } U \text{ of } y\};$$

we call branches B_i with $i \in \Phi$ *active branches* (and the corresponding neighborhoods U_i as *active branch germs*). We will often refer to an active branch germ U_i solely via its index i. We say $j \in \{1, m\}$ is a *color* for the active branch germ U_i (or, by abuse of language, for i) if $G_i^j \neq \emptyset$. Note that a given branch may have up to two colors. A branch germ is *monochrome* if it has precisely one color.

Remark 5.6. Active non-monotone branch germs are monochrome, with

$$\begin{cases} j = m & \text{if } i = 0, \\ j = 1 & \text{otherwise.} \end{cases}$$

This is an immediate corollary of Lemma 5.5.

One can find a neighborhood U of y so that whenever $\tilde{U} \subset U$ is a subneighborhood of y, the colors of U_i and $\tilde{U}_i$ agree for $i = 0, \ldots, \nu$; we call any such U a *determining* neighborhood of y. We will write

$$(i, j) \to (i', j') \tag{12}$$

if for some (hence any) determining neighborhood,

$$f(G_i^j) \cap G_{i'}^{j'} \neq \emptyset.$$

We form a *branch graph* $\mathcal{B}$ whose vertices are the active branch germs, and with a directed edge from U_i to $U_{i'}$ (denoted $i \xrightarrow{\mathcal{B}} i'$) if there exist $j, j' \in \{1, m\}$ such that $(i, j) \to (i', j')$ (in the sense of equation (12)).

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LEMMA 5.7.

(1) *Every vertex in $\mathcal{B}$ has indegree at most 2.*

(2) *Every monotone vertex in $\mathcal{B}$ has outdegree at most 1.*

(3) *If $i_1 \xrightarrow{\mathcal{B}} i'$ and $i_2 \xrightarrow{\mathcal{B}} i'$ with i_1 and i_2 distinct, then i_1 and i_2 are both monotone and monochrome, with different colors.*

Proof. To see (1), suppose that $i_k \xrightarrow{\mathcal{B}} i'$ for three distinct i_k, $k = 1, 2, 3$. We can assume that in an f-adjusted ordering $B_{i_1} \prec B_{i_2} \prec B_{i_3}$. Then by Remark 5.4, $G^m_{i_1} = G^m_{i_2} = \emptyset$ and $G^1_{i_2} = G^1_{i_3} = \emptyset$; in particular, B_{i_2} is not active, a contradiction.

(2) is clear, by the definition of monotonicity.

To see (3), suppose $i_1 < i_2$, so that by Remark 5.4 $M_z \notin U_{i_1}$ and $m_z \notin U_{i_2}$ for any $z \in U_{i'}$. Thus we must have $G^1_{i_1}$ and $G^m_{i_2}$ both non-empty; by Lemma 5.5 this means that to be active B_{i_2} must be monotone, and hence (since the ordering is f-adjusted) either B_{i_1} is monotone or $B_{i_1=0}$ is non-monotone. In this last case, $G^1_{i_1} = \emptyset$ by Lemma 5.5 again, contradicting the assumption that B_{i_1} is active. □

We call a path or loop in $\mathcal{B}$ *monotone* if every vertex occurring along the path is monotone at y.

LEMMA 5.8. *Any monotone loop in $\mathcal{B}$ contains at least one monochrome vertex.*

Proof. If the loop contains all the active branch germs at y, then since f is cocountably m-fold, some branch contains a non-trivial preimage of y, and by Remark 5.4, this implies some branch B_i has G^1_i or G^m_i empty.

If some branch is not an element of our loop, then since the union of the (closed) branches in the loop is a proper natural subtree, it is not invariant (by our basic assumption). So at least one of these branches must map to a union of two or more branches (one in the loop, the other out of the loop) and again by Remark 5.4 the loop contains a monochrome branch germ. □

LEMMA 5.9. *Every monotone path γ in $\mathcal{B}$ can be written as a concatenation*

$$\gamma = \alpha\lambda^k$$

where λ is a loop and α is a path with no repetitions (in particular, $|\alpha| < \operatorname{card}\Phi$).

Proof. This is an almost immediate consequence of the fact that every active monotone branch germ has outdegree 1 in $\mathcal{B}$ (Lemma 5.7). □

Let U be a determining neighborhood of y. We refer to a point x such that $f^k(x) \in [U \cap H_*] \setminus \{y\}$ for $k = 0, \ldots, n-1$ as a *satellite* of y with *time of flight* n. Such a point has two kinds of itinerary: a *branch itinerary* $\{i_k\}_{k=0}^{n-1}$ defined by $f^k(x) \in U_{i_k}$, and a *color itinerary* $\{j_k\}_{k=0}^{n-1}$ satisfying $f^k(x) \in H_{j_k}$, $k = 0, \ldots, n-1$. The branch itinerary is unique, but *a priori* a satellite may have more than one color itinerary. However, Lemma 5.5, Remark 5.6, and Lemma 5.7 give limitations on the color itineraries that can occur in conjunction with a given branch itinerary. We call a choice of color itinerary *legitimate* for a given branch itinerary if it is consistent with these limitations.

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5.1.1. *Forbidden color words.* We wish to find a finite color word that does not appear in any color itinerary of any satellite of y. This means that $\{y\}$ locally divides $\mathcal{H}$ (Definition 2.2). We will do this in Corollary 5.14, but first we need a substantial digression. A *colored path* of length n in $\mathcal{B}$ is a path of length n together with a legitimate choice of color for each branch germ U_{i_k}:

$$\gamma = (i_0, j_0), \ldots, (i_{n-1}, j_{n-1}), \quad i_k \in \Phi,\ j_k \in \{1, m\} \quad \text{for } k = 0, \ldots, n-1.$$

The branch itinerary, together with a choice of color itinerary, for any satellite of y with time of flight (at least) n, determines a colored path in $\mathcal{B}$ of length n, so the number of color itineraries of length n that occur among the satellites of y is bounded above by the number $\mathcal{N}_n$ of color itineraries occurring among the colored paths of length n in $\mathcal{B}$. Since the number of (abstract) color words of length n is 2^n, it will suffice to prove the following proposition.

PROPOSITION 5.10. *For n sufficiently large,*

$$\mathcal{N}_n < 2^n.$$

Given a colored path $\gamma = (i_0, j_0) \ldots (i_{n-1}, j_{n-1})$, we have a color word

$$\mathfrak{c}(\gamma) := j_0 \ldots j_{n-1}$$

consisting of the sequence of colors appearing along γ. We can think of $\mathfrak{c}$ as a projection map from colored paths to color words. To estimate $\mathcal{N}_n$, we construct, for each (legitimate) colored vertex $(b, c) \in \Phi \times \{1, m\}$, a rooted tree $\Gamma(b, c)$ whose vertices $\gamma \in (b, c)_n$ at level $n = 0, 1, \ldots$ are the colored paths

$$\gamma = (i_0, j_0) \ldots [(i_n, j_n) = (b, c)]$$

of length $n+1$ that end at (b, c), and an edge connecting each vertex $\gamma = (i_0, j_0) \ldots (i_n, j_n)$ at level $n > 0$ with the colored path $\gamma' = (i_1, j_1) \ldots (i_n, j_n) \in (b, c)_{n-1}$ at level $n-1$ obtained by truncating the first colored vertex. The standard orientation of this edge is from γ' to γ (that is, in the direction of increasing level), which may at first appear counter-intuitive. However, we make use of this orientation only briefly (cf. the paragraph preceding Lemma 5.12). Denote the number of color words (of length n) occurring for vertices at level $n-1$ of $\Gamma(b, c)$ by

$$\mathcal{C}_n(b, c) := \operatorname{card} \mathfrak{c}((b, c)_{n-1}).$$

Clearly, $\mathcal{N}_n \leq \sum_{(b,c)\in\Phi\times\{1,m\}} \mathcal{C}_n(b, c)$. If $\ell = \operatorname{card} \Phi$ denotes the number of active branches at y, we will establish the inequality

$$\mathcal{C}_{p\ell+1}(b, c) \leq 2^{p(\ell-1)}(1 + 6p) \tag{13}$$

for $p \geq 0$ from which the proposition will follow easily.

Our previous results yield the following information about the graph $\Gamma(b, c)$.

LEMMA 5.11. *For each vertex $\gamma = (i_0, j_0) \ldots (b, c)$ in $(b, c)_n$:*

(1) *if $n > 0$ there is precisely one vertex $\gamma' = (i_1, j_1) \ldots (b, c) \in (b, c)_{n-1}$ at level $n-1$ joined to γ;*

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(2) *in general, there are at most two vertices* $\gamma_- = (i_{-1}, j_{-1})(i_0, j_0)\dots(b, c)$ *at level* $n+1$ *joined to* γ*;*

(3) *if* U_{i_0} *is non-monotone, then no vertex in* $(b, c)_n$ *other than* γ *is joined to* γ'.

Proof. (1) is trivial.

To see (2) and (3), note that if $\gamma_- = (i_{-1}, j_{-1})(i_0, j_0)\dots(b, c)$ then $i_{-1} \xrightarrow{\mathcal{B}} i_0$. From Lemma 5.7(1), there are at most two possibilities for i_{-1}, given i_0. If two distinct possibilities $i_{-1}^{(1)}$, $i_{-1}^{(2)}$ exist, then both are monotone, with $j_{-1}^{(1)} \neq j_{-1}^{(2)}$; otherwise, i_{-1} is unique, and can be colored in at most two ways. By Remark 5.6, a non-monotone germ can be colored in at most one way. □

To establish equation (13), we will distinguish colored paths according to the branch paths they represent. For $p = 0, \dots, ,$ let $\mathcal{M}_p(b, c)$ denote the number of color words coming from colored paths of length $p\ell + 1$ in which the branch path is monotone, and $\mathcal{S}_p(b, c)$ the number coming from colored paths of length $p\ell+1$ going through at least one non-monotone branch germ. (The reason for this peculiar numbering will become clearer in what follows.)

Let us first estimate $\mathcal{M}_p(b, c)$. Note that $\mathcal{M}_0(b, c) = 1$, and if $p > 0$, then for $\mathcal{M}_p(b, c)$ to be non-zero we need b to belong to a monotone loop λ in $\mathcal{B}$. We denote the length of λ by $\mathrm{len}(\lambda)$. To obtain an estimate on $\mathcal{M}_p(b, c)$ independent of b, we let q denote the maximum length of all monotone loops in $\mathcal{B}$ (note that these are disjoint, by Lemma 5.7(2), and hence there are finitely many); note that

$$q \leq \ell.$$

Given a monotone loop λ containing b, Lemma 5.8 implies that λ must contain at least one monochrome vertex, and there are at most $2^{\mathrm{len}(\lambda)-1}$ legitimate colorings of λ, and at most

$$(2^{\mathrm{len}(\lambda)-1})^k = 2^{\mathrm{len}(\lambda^k)(1-(k/\mathrm{len}(\lambda^k)))} \leq 2^{\mathrm{len}(\lambda^k)(1-(1/q))}$$

legitimate colorings of λ^k, the concatenation of λ with itself k times. By Lemma 5.9, every monotone path ending at b is a concatenation of the form $\gamma = \alpha\lambda^k$ for some k, where α is a non-repetitive monotone path, whose length is therefore bounded by the number of monotone vertices, and hence by ℓ. There is a unique path of length $p\ell + 1$ which is a subpath of some power of λ, and the number of colorings of it is bounded by $2^{p\ell-\lfloor p\ell/q\rfloor} \leq 2^{pl(1-(1/q))+1}$. There is also the possibility of replacing the initial subword of this with a non-repetitive monotone path (i.e. α); the number of legitimate colorings of α is bounded above by 2^ℓ. Thus we have the estimate

$$\mathcal{M}_p(b, c) \leq 2^{pl(1-(1/q))+1} + 2^\ell 2^{(p-1)l(1-(1/q))+1};$$

factoring out $2^{p\ell}$ and using the fact that $q \leq \ell$, we obtain the estimate

$$\mathcal{M}_p(b, c) \leq 2^{p\ell}[2^{-p} + 2^{-(p-1)}] \cdot 2 = 2^{p(\ell-1)} \cdot 6. \tag{14}$$

Now consider $\mathcal{S}_p(b, c)$. To estimate this we need an excursion into abstract graph theory. By construction, the graph $\Gamma(b, c)$ is an (infinite) tree; Lemma 5.11((1) and (2)) tells us that (if we adopt the convention that edges are oriented in the direction of increasing level) every vertex except the root has indegree 1 and every vertex has outdegree at most 2.

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We refer to such a graph as a *stump* and to a vertex with outdegree 0 (respectively 1) as an *end* (respectively *cutpoint*) of the graph. Each end has a unique path to the root; we call it a *cut end* if it is not the root, and this path contains at least one cutpoint.

LEMMA 5.12. *In any stump, the number of cut ends at level ℓ is bounded above by $2^{\ell-1}$.*

Proof. First, assign to each edge in the tree a 0 or 1; if the edge is leaving a cutpoint, make sure it is assigned a 0 (the assignment to edges leaving a vertex with outdegree 2 can be chosen in an arbitrary way). Then each vertex at level ℓ is assigned a sequence of 0's and 1's corresponding to the unique path from it to the root. If v is a cut end at level ℓ, consider the sequence obtained from its path by replacing the first 0 associated to a cut point with a 1; this leads to a sequence that does not occur in the graph, and is a one-to-one map from cut ends into the set of 'missing' ends. Since there are 2^ℓ sequences all together and the set of 'missing' sequences is disjoint from the set of extant ones, we have that the number of cut ends at level ℓ plus the number of 'missing' image sequences adds to at most 2^ℓ; but the number of image sequences equals the number of cut ends, and we are done. □

COROLLARY 5.13. *For any colored vertex $(b, c) \in \Phi \times \{1, m\}$,*

$$\mathcal{S}_1(b, c) \leq 2^{\ell-1}.$$

Proof. Since a colored path contains at least one non-monotone vertex, we see that some positive level must contain a non-monotone vertex, and it follows from Lemma 5.11(3) that a non-monotone vertex at level k in $\Gamma(b, c)$ means that the corresponding vertex at level $k-1$ is a cut point. In particular, the number of non-monotone colored paths of length $\ell+1$ ending at a given vertex is at most $2^{\ell-1}$, and this is a bound on $\mathcal{S}_1(b, c)$. □

Proof of equation (13). If b is not part of a monotone loop (in particular, if b itself is not monotone), then $\mathcal{M}_p(b, c) = 0$ and every path ending at (b, c) hits a non-monotone vertex at least once in every ℓ steps. The number of legitimate colorings for all such paths is bounded above by a product of terms of the form $\mathcal{S}_1(b_i, c_i)$, where the subscripted vertices are those occurring at precise multiples of ℓ steps; each of these is bounded by $2^{\ell-1}$, by Corollary 5.13. In this case,

$$\mathcal{C}_{p\ell+1}(b, c) \leq \mathcal{S}_p(b, c) \leq 2^{p(\ell-1)} \leq 2^{p(\ell-1)}(1 + 6p).$$

When b *is* part of a monotone loop, we can still imagine paths ending at (b, c) of the type analyzed above, but of course there are others. For any given path we let k be the maximum integer for which the last $k\ell + 1$ vertices are monotone; then this path consists of a path of length $(p-k)\ell + 1$ of the type above fused with a monotone path of length $k\ell + 1$. (The case above is $k = 0$.) The first part(s) can be colored in at most $2^{(p-k)(\ell-1)}$ ways, as above, while the last part(s) can be colored in at most $\mathcal{M}_k(b, c)$ different ways.

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Using equation (14) with k in place of p, we obtain equation (13)

$$\begin{aligned}\mathcal{C}_{p\ell+1}(b,c) &\le \sum_{k=0}^{p} 2^{(p-k)(\ell-1)}\mathcal{M}_k(b,c)\\ &\le 2^{p(\ell-1)} + \sum_{k=1}^{p} 2^{(p-k)(\ell-1)}\cdot 6\cdot 2^{k(\ell-1)}\\ &= 2^{p(\ell-1)}\left[1+\sum_{k=1}^{p} 6\right]\\ &= 2^{p(\ell-1)}[1+6p]\end{aligned}$$

as required. □

Proof of Proposition 5.10. The number of colored vertices $(b,c)\in\Phi\times\{1,m\}$ is bounded by 2ℓ, so substituting in equation (13) we have

$$\mathcal{N}_{p\ell+1} \le \sum_{(b,c)\in\Phi\times\{1,m\}} \mathcal{C}_{p\ell+1}(b,c) \le 2\ell\cdot 2^{p(\ell-1)}[1+6p].$$

Since $1+6p$ grows more slowly than 2^p, we can find a sufficiently large value of p so that $\ell\cdot[1+6p]<2^p$; then for n equal to this value of $p\ell+1$ we have

$$\mathcal{N}_n \le 2\ell\cdot 2^{p(\ell-1)}[1+6p] < 2\cdot 2^{p(\ell-1)}2^p = 2^{p\ell+1} = 2^n$$

as required. □

We are now in a position to produce a forbidden color word.

COROLLARY 5.14. *There exists a word in the letters $\{1,m\}$ that does not appear in any color itinerary for any satellite of y.*

Proof. Proposition 5.10 shows that the number of words of sufficient length that appear in legitimate colorings of paths in $\Gamma(b,c)$ (which bounds the number of words appearing in color itineraries of satellites of y) is strictly less than the number of abstract words in $\{1,m\}$. □

5.2. *Periodic non-branchpoints in the core.* In the previous section, we showed that any periodic branchpoint in the core locally divides $\mathcal{H}$. We now proceed to the more difficult task of finding a set containing all periodic points in the core, but away from $\mathsf{B}(T)$, which locally divides $\mathcal{H}$.

Definition 5.15. We denote the set of periodic non-branchpoints in the core by

$$\mathcal{P} := [\mathfrak{Z}_0(\mathcal{H}) \cap \mathrm{Per}(f)] \setminus \mathsf{B}(T).$$

PROPOSITION 5.16. *$\mathcal{P}$ is closed.*

Proof. We begin with the following observations.

CLAIM 5.17. *A periodic point which is an accumulation point of $\mathcal{P}$ has an orbit disjoint from $\mathsf{B}(T)$.*

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Proof. This is an immediate corollary of Corollary 5.14: since $\mathfrak{Z}_0(\mathcal{H})$ is closed, the orbit belongs to $\mathfrak{Z}_0(\mathcal{H})$. However, if it intersects $\mathsf{B}(T)$, by our assumptions on f it consists of a fixed point in $\mathsf{B}(T)$, and since it is an accumulation point of $\mathcal{P}$, there exist periodic points in $\mathfrak{Z}_0(\mathcal{H})$ arbitrarily near to (but distinct from) the point. These points are satellites of the fixed point with arbitrarily long time of flight, and since they belong to $\mathfrak{Z}(\mathcal{H})$, they have all possible words in their itinerary, contrary to Corollary 5.14. □

Now, suppose $\{q_n\}$ is a sequence of points of $\mathcal{P}$ converging to $y \notin \mathcal{P}$; denote by Q_n the orbit of q_n. Since $y \in \operatorname{clos}\mathcal{P} \subset \mathfrak{Z}_0(\mathcal{H})$, its ω-limit set must also belong to the invariant closed set $\mathfrak{Z}_0(\mathcal{H})$, and since the latter is contained in the countable set $f^i(\mathfrak{K}(\mathcal{H}))$, any minimal subset is a periodic orbit. Thus we can pick a cycle $P = \{p_0, \ldots, p_{N-1}\}$ in $\omega(y)$, which *a fortiori* also belongs to $\mathcal{P}$. We can pick a neighborhood U of P that is disjoint from $\mathsf{B}(T)$ (by the claim) and from some neighborhood of y; going to a subsequence if necessary, we can also pick points $q_n' \in Q_n$ converging to p_0 from one side.

Now pick a closed one-sided neighborhood $J_0 \subset U$ of p_0 containing q_n' for all sufficiently large n. We know from Lemma 4.9 and the fact that the q_n' are periodic points whose (forward) orbit leaves U that $J_1 := f(J_0)$ is again a closed one-sided neighborhood of p_1; iterating this procedure (reducing J_0 if necessary) we obtain closed one-sided neighborhoods $J_i := f^i(J_0) \subset U$ of p_i, $i = 0, \ldots, N-1$. Note that $f(J_{N-1})$ is a closed one-sided neighborhood of p_0, on either the same or the opposite side as J_0; in the first case, it must properly contain J_0 (otherwise the union $\bigcup_{i=0}^{N-1} J_i$ is an invariant neighborhood of P in U, contradicting the fact that $q_n \to y$) while in the second case we can iterate the procedure up to J_{2N}, which properly contains either J_0 or J_N; in any case (again reducing J_0 if necessary), we obtain a family of N disjoint closed intervals $\tilde{J}_i = J_i$ or $J_i \cup f^N(J_i)$, $i = 0, \ldots, N-1$, contained in U with $p_i \in \tilde{J}_i$ (possibly as an endpoint), such that $f(\tilde{J}_i) = \tilde{J}_{i+1}$ for $i = 0, \ldots, N-2$, and $\tilde{J}_0 \subsetneq f(\tilde{J}_{N-1}) \subset U$.

Since the periodic orbits Q_n intersect both a neighborhood of y and $V := \bigcup_{i=0}^{N-1} \tilde{J}_i$, for each sufficiently large n we can find $q = q_n'' \in Q_n$ such that $q \notin V$ but $f(q) \in \operatorname{int} V$, say $f(q) \in \operatorname{int} \tilde{J}_{i+1}$. We know that some neighborhood U' of q is comparable to $\tilde{J}_i$. However, since $f(\tilde{J}_i)$ contains $\tilde{J}_{i+1}$, it also contains the image of a neighborhood of q, and this contradicts the assumption that $q = q_n'' \in H_\#$, by Remak 4.8(1), since $\tilde{J}_i \subset U$ contains no branchpoints and $q \in \mathcal{P}$. □

LEMMA 5.18. *Suppose p_i, $i = 1, 2, 3$, are distinct elements of $\mathcal{P}$ contained in a single edge, with p_2 between p_1 and p_3. If $f(p_1)$ and $f(p_3)$ lie in the same edge, then $f(p_2)$ lies between them; in particular, it belongs to the same edge.*

Proof. Since they are distinct periodic points, their images under f are also distinct; Lemma 4.9 applied to p_1 (respectively p_3) implies $f(p_1)$ (respectively $f(p_3)$) cannot separate $f(p_2)$ from $f(p_3)$ (respectively $f(p_1)$), and if $f(p_1)$ and $f(p_3)$ lie in the same edge, this implies $f(p_2)$ lies between them on this edge. □

Definition 5.19. The *edge itinerary* of a point $p \in \mathcal{P}$ of least period N is the sequence $\mathsf{ei}(p) = \{E_0, \ldots, E_{N-1}\}$ of edges of T visited by p during one period: $f^i(p) \in E_i$, $i = 0, \ldots, N-1$.

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A *t-fold repetition* of $\{E_0, \dots, E_{N-1}\}$ ($t \in \mathbb{N}$) is a sequence of edges $\{E'_0, \dots, E'_{tN-1}\}$ with $E'_i = E_j$ whenever $i \equiv j \mod N$; in particular, it is a *doubling* if $t = 2$. An edge itinerary is *repetitive* if it is a t-fold repetition of some shorter sequence of edges.

We call two points $p, q \in \mathcal{P}$ *edge equivalent* (and write $p \sim q$) if one of their edge itineraries is a t-fold repetition of the other for some $t \in \mathbb{N}$. Denote the edge equivalence class of $p \in \mathcal{P}$ by $[p]$, and its convex hull in T by $\langle p \rangle$.

PROPOSITION 5.20.

(1) *If $p \in \mathcal{P}$ and $t > 2$ then $\mathsf{ei}(p)$ cannot be a t-fold repetition of any sequence of edges. Thus at most two itineraries can occur among the elements of one edge equivalence class, and in this case one is a doubling of the other.*

(2) *If $p \sim q$ with $\mathsf{ei}(q)$ doubling $\mathsf{ei}(p)$, then p lies between q and $f^N(q) \sim q$, where N is the period of p.*

(3) *For any $p \in \mathcal{P}$:*
   (a) *$\langle p \rangle$ is a closed (possibly degenerate) interval interior to a single edge;*
   (b) *$\langle p \rangle$ and $\langle q \rangle$ are disjoint unless $p \sim q$;*
   (c) *$f(\langle p \rangle) = \langle f(p) \rangle$.*

Proof. (1) If for some $p \in \mathcal{P}$ the itinerary $\mathsf{ei}(p)$ is a t-fold repetition of an itinerary of length k, then its period $N = kt$ and the orbit of p under f^k consists of t points, all belonging to $\mathcal{P}$ and having the same edge itinerary, which are permuted by f^k. Inductive application of Lemma 5.18 shows that the extreme points of this set are preserved by f^k, which means that if $t \geq 3$, no internal point of this set can map to an extreme point of its image under any iterate of f, contradicting the assumption that they form a periodic orbit under f.

(2) Suppose $p \sim q$ and the period of q is greater than that of p; denote the period of p by N. By the preceding item, $\mathsf{ei}(q)$ is a doubling of $\mathsf{ei}(p)$, so $S := \{p, q, f^N(q)\}$ is contained in one edge, for each i, $f^i(S)$ is contained in a single edge, and $f^N(S) = S$. By Lemma 5.18, the middle point of S cannot map to either extreme point of S, which forces p to be the middle point of S.

(3)(a) This is a trivial consequence of condition (1) and Proposition 5.16.

(3)(b) We need to show that if $p \in \mathcal{P}$ belongs to $\langle q \rangle$ then $p \sim q$. Suppose $p \in \langle q \rangle$ but $p \not\sim q$. Then there exists $q'' \sim q$ so that p lies between q and q'', and by Lemma 5.18 the same holds for the images of these three points by any iterate f^i. However, Lemma 5.18 forces q to have the same edge itinerary as p, or else to be a doubling of it.

(3)(c) We need to show that a point $x \in \langle p \rangle$ (not assuming $x \in \mathcal{P}$) has $f(x) \in \langle f(p) \rangle$. This follows from application of Lemma 4.9 to the endpoints of $\langle p \rangle$, together with the observation that $\langle p \rangle$ contains no branchpoints, by the earlier arguments in this proof. □

LEMMA 5.21. *$\mathcal{P}$ has finitely many edge-equivalence classes.*

Proof. Suppose $p_n \in \mathcal{P}$, $n = 1, 2, \dots$, belong to pairwise non-edge-equivalent orbits, with respective periods N_n. Passing to a subsequence, we can assume the p_n converge inside a single edge E_0 to a point p which by Proposition 5.16 belongs to $\mathcal{P}$ and hence is periodic, say with period N. For each n, let

$$t_n := \min\{t > 0 \mid f^{N_n - t}(p_n),\ f^{N-t}(p) \text{ belong to different edges}\}.$$

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We can assume by passing to a subsequence that all the t_n are congruent modulo N, and all the $f^{N_n - t_n}(p_n)$ belong to the same edge. By applying an iterate of f to the whole picture, we can assume that $t_n = 1$ for all n: thus we have a sequence $q_n := f^{N_n - 1}(p_n)$ all contained in an edge different from that containing $p' := f^{N-1}(p)$.

Let q be an accumulation point of the q_n; by Proposition 5.16 $q \in \mathcal{P}$ is periodic, and distinct from p'; but both are periodic and both map to p, which is a contradiction. □

For any $p \in \mathcal{P}$, the union

$$Z(p) := \bigcup f^i(\langle p \rangle)$$

consists of N disjoint closed (possibly degenerate) intervals

$$Z(p) = \bigcup_{j=0}^{N-1} Z_j(p)$$

where N is the least period among the points in the edge equivalence class $[p]$ and the numbering is via the action of f:

$$f(Z_j(p)) = Z_{j+1}(p)$$

(indices taken mod N).

Remark 5.22. If for some $p \in \mathcal{P}$ the restriction $f|Z(p)$ is cocountably m-fold, then

$$h_{\text{top}}(f) \geq \log m.$$

This is Theorem 4.3 in [**Bob05**], which gives our desired inequality for m-fold interval maps, applied to $f^N|Z_0(p)$, which is non-degenerate and cocountably (m^N)-fold.

In view of Remark 5.22, we can *assume for the rest of this section* that the following assumption holds.

Assumption 5.23. The restriction of f to each set $Z(p)$, $p \in \mathcal{P}$, fails to be cocountably m-fold.

If $Z(p)$ is not connected ($N \geq 2$), at least one component of the complement $T \setminus Z(p)$ has common boundary points with at least two of the intervals $Z_j(p)$. We call the closure of such a component a *central component* relative to $Z(p)$, and denote the union of all the central components by $C(p)$. A *peripheral component* of $T \setminus Z(p)$ is one attached to a unique interval $Z_j(p)$; we label its closure $P_j(p)$ and denote the union of these by $P(p)$. This yields a partition of the tree corresponding to any edge equivalence class $[p]$

$$T = C(p) \cup Z(p) \cup P(p).$$

Each interval $Z_j(p)$ of $Z(p)$ touches $C(p)$ in at least one endpoint; we will write

$$Z_j(p) = [z_j^-, z_j^+]$$

where z_j^- is a common boundary point of $Z_j(p)$ and $C(p)$; if $Z_j(p)$ is non-degenerate, then the other endpoint z_j^+ may touch another central component, or the peripheral component $P_j(p)$, or be an end of T, in which case we refer to a 'trivial peripheral component' $P_j(p) = \emptyset$. When $Z_j(p)$ is a single point, we will nonetheless refer separately to its endpoints z_j^- and z_j^+.

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If $N = 1$, i.e. $Z(p) = Z_0(p) = [z_0^-, z_0^+]$, then $T \setminus Z(p)$ has at most two components, both peripheral; we associate to each endpoint a (possibly trivial) peripheral component $P_0^{\pm}(p)$. In this case, our Assumption 5.23 says that at least one of the peripheral components has interior points mapping to $Z(p)$, and we make sure $P_0^+(p)$ is of this type.

Under Assumption 5.23, we will construct a set W containing $\mathcal{P}$ that locally divides $\mathcal{H}$. This will be made up of sets of the form $W(p) \supset Z(p)$ for various edge equivalence classes in $\mathcal{P}$. Each such set $W(p)$ will be either $Z(p)$ itself or its union with $C(p)$ or $P(p)$ if either contains no interior points mapping to $Z(p)$. We begin by showing that if this occurs then the partition $T = C(p) \cup Z(p) \cup P(p)$ has a particular structure.

Suppose first that $f^{-1}[Z(p)] \cap P(p) = \emptyset$.

Remark 5.24. If $N \geq 2$ and no point interior to $P(p)$ maps into $Z(p)$, then either $P(p) = \emptyset$ (so z_i^+ is an endpoint of T for $i = 0, \ldots, N-1$) or else each $Z_i(p)$ is attached to a non-trivial peripheral component $P_i(p)$, and for $i = 0, \ldots, N-1$, $f(P_i(p)) \subset P_{i+1}(p)$ (indices taken mod N). In either case, $T \setminus Z(p)$ has exactly one central component.

This follows immediately from Lemma 4.9. □

We would like to establish an analogous picture when no interior point of $C(p)$ maps to $Z(p)$.

PROPOSITION 5.25. *If $C(p) \neq \emptyset$ but $f^{-1}[Z(p)] \cap C(p) = \emptyset$, then:*

(1) *$T \setminus Z(p)$ has a unique central component;*
(2) *each component $Z_i(p)$ is attached to a non-trivial peripheral component $P_i(p)$;*
(3) *for $i = 0, \ldots, N-1$ (taking indices mod N),*

$$P_{i+1}(p) \subset f(P_i(p)) \subset C(p) \cup Z_{i+1}(p) \cup P_{i+1}(p).$$

Proof. To establish the first statement, we will show that if $T \setminus Z(p)$ has at least two central components, then some interior point of $C(p)$ maps into $Z(p)$. If there are at least two central components, $Z(p)$ includes three intervals $Z_{i_j}(p)$, $j = 1, 2, 3$, such that $Z_{i_2}(p) \subset \langle Z_{i_1}(p), Z_{i_3}(p) \rangle$, and no other intervals $Z_i(p)$, $i \neq i_1, i_2, i_3$, intersect $\langle Z_{i_1}(p), Z_{i_3}(p) \rangle$. Since $f(Z_i(p)) = Z_{i+1}(p)$ for all i (indices taken mod N) and the $Z_i(p)$ are permuted, we can assume that $Z_{i_2-1}(p)$ is not contained in $\langle Z_{i_1-1}(p), Z_{i_3-1}(p) \rangle$. However, the interior of this hull is disjoint from all peripheral components and contains a point mapping into $Z_{i_2}(p)$, which must then be interior to $C(p)$.

This establishes the uniqueness of the central component. If $Z_i(p)$ touches a peripheral component, we have already called it $P_i(p)$; if not, then z_i^+ is an endpoint of the tree T, and we say $P_i(p)$ is trivial. Since $f|Z(p)$ is not cocountably m-fold and $Z(p)$ has no preimages in $C(p)$, at least one peripheral component contains an interior point mapping into $Z(p)$, and in particular at least one $P_i(p)$ is non-trivial.

Now, suppose that for some j, $P_j(p)$ is non-trivial. Since the map f is surjective on T and $C(p) \cup Z(p)$ is invariant, $P_j(p) \subset f(P(p))$. We claim that only $P_{j-1}(p)$ (mod N) can contain preimages of interior points of $P_j(p)$. Note that since $C(p) \neq \emptyset$ requires $N \geq 2$, the points z_i^+ are not fixed points, and in particular (by Assumption 5.1) have no preimages that are branchpoints. Now suppose some $z' \in P_i(p)$, $i \neq j-1$, has $f(z') \in \operatorname{int} P_j(p)$. Then the interval $[z_i^+, z']$ maps across a neighborhood of z_j^+, which

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we can assume to contain no branchpoints. By Remark 4.6, we can find a subinterval J of $[z_i^+, z']$ that is disjoint from $\mathsf{B}(T)$ mapping exactly onto this neighborhood. However, some neighborhood U of z_{j-1}^+ (also disjoint from $\mathsf{B}(T)$) has $f(U) \subset f(J)$, so since both U and J are comparable and disjoint from $\mathsf{B}(T)$, Remark 4.8(1) tells us that $z_j^+ \notin H_\#$, contradicting $z_j^+ \in \mathfrak{Z}_0(\mathcal{H})$.

Note, however, that this shows that if $P_j(p)$ is non-trivial, then so is $P_{j-1}(p)$, and no other peripheral branch can have points mapping to its interior. Inductively, this proves the proposition. □

To continue our analysis, we need to track the dynamics of neighborhoods of the points $z_j^\pm$. To this end, we assign to each endpoint $z_j^\pm$ of $Z_j(p)$ a set $U_j^\pm(p)$ as follows:

(1) if z_j^+ is an endpoint of T, then $U_j^+(p) := Z_j(p)$;
(2) otherwise, $U_j^\pm(p)$ is a one-sided neighborhood of $z_j^\pm$, contained in the closed component of $T \setminus Z(p)$ attached to $z_j^\pm$.

Reducing the sets $U_j^\pm(p)$ in case (2) if necessary, we can assume that they are pairwise disjoint, and that the interior of each one is disjoint from the finite set $\mathsf{B}(T) \cup f(\mathsf{B}(T))$.

Given $j \in \{0, \dots, N-1\}$ and $\sigma \in \{+, -\}$, we can track the action of f on $U_j^\sigma(p)$ by specifying a set $\varphi(U_j^\sigma(p)) \in \{U_{j+1}^-(p), U_{j+1}^+(p), Z_{j+1}(p)\}$, well defined in view of Lemma 4.9 applied to $z_j^\pm \in \mathcal{P} \subset T \setminus \mathsf{B}(T)$, according to the following rules:

(1) if $f(U_j^\sigma(p)) \subset Z_{j+1}(p)$, then $\varphi(U_j^\sigma(p)) := Z_{j+1}(p)$;
(2) otherwise, $\varphi(U_j^\sigma(p)) \neq Z_{j+1}(p)$ has non-trivial intersection with $f(U_j^\sigma(p))$.

Extending φ to $\varphi(Z_j(p)) = Z_{j+1}(p)$, we have a self-map of the finite set $\{U_j^-(p), U_j^+(p), Z_j(p) \mid j = 0, \dots, N-1\}$ into itself, each orbit of which is eventually periodic. Automatically, φ has the *trivial* cycle $Z_0(p) \mapsto Z_1(p) \mapsto \cdots \mapsto Z_{N-1}(p) \mapsto Z_0(p)$. As for non-trivial cycles, the two possibilities are:

(1) a single non-trivial cycle of length $2N$, with $\varphi^N(U_j^\pm(p)) = U_j^\mp(p)$ for each j (and all $U_j^\pm(p) \neq Z_j(p)$);
(2) one or two non-trivial cycles (disjoint if there are two) of length N ($\varphi^N(U_j^\pm(p)) = U_j^\pm(p)$).

Let $U := \bigcup_{j=0}^{N-1} U_j^-(p) \cup U_j^+(p)$; abusing the terminology of §5.1, we refer to a point $x \in U$ as a *satellite* of $Z(p)$ with *time of flight* t if $f^k x \in U \setminus Z(p)$ for $k = 0, \dots, t-1$. If $t \geq 2N$, it is a *persistent satellite*. The *local itinerary* of a satellite x is the sequence of non-trivial sets $U_j^\sigma(p)$ containing successive iterates of x: note that if $f^k(x) \in U_j^\sigma(p) \setminus Z(p)$ and $k < t-1$ then $f^{k+1}(x) \in \varphi(U_j^\sigma(p)) \setminus Z(p)$. We are interested in satellites whose orbit segments are contained in $H_* := H_1 \cup H_m$; for any such satellite, a color itinerary is a word $w_0, \dots, w_{t-1}$ with $w_i \in \{1, m\}$ such that $f^i(x) \in H_{w_i}$ for $i = 0, \dots, t-1$.

To build a set $W(p)$ that locally divides $\mathcal{H}$ we will try to find words in the letters $\{1, m\}$ that do not occur in any color itinerary of any persistent satellite of $Z(p)$.

Remark 5.26.

(1) If $\varphi^N(U_j^\sigma(p)) = Z_j(p)$, then $U_j^\sigma(p)$ contains no persistent satellites of $Z(p)$.
(2) Any persistent satellite x has $f^N(x)$ contained in an element of a non-trivial φ-cycle.

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The following observations lead to a lemma that will be useful in proving the existence of 'forbidden words' for satellites of $Z(p)$. We will say that a point z'' *separates* the points z and z' if it belongs to the interior of their convex hull (this means that z and z' belong to different branches at z'').

LEMMA 5.27. *Suppose* $y \in \langle z', z\rangle$ *such that* $\operatorname{int} f(\langle y, z\rangle)$ *is disjoint from* $\langle f(z'), f(z)\rangle$, $B(T)$, *and* $f(B(T))$. *Then* $\operatorname{int}\langle y, z\rangle$ *is disjoint from at least one of the sets* H_1, H_m.

Proof. First apply Remark 4.6 to the interval $\operatorname{int} f(\langle y, z\rangle)$ to find a subinterval of $\langle y, z\rangle$ mapping exactly onto it; we can assume without loss of generality that y is an endpoint mapping to an endpoint of $f(\langle y, z\rangle)$, so that $f(\langle y, z\rangle) = \langle f(y), f(z)\rangle$. Now apply Remark 4.6 again, this time to find a subinterval J' of $\langle z', y\rangle$ mapping onto $\operatorname{int} f(\langle y, z\rangle)$. By assumption, J' is disjoint from $B(T)$ and hence comparable to $\langle y, z\rangle$, and the lemma follows from Remark 4.8(1). □

The next three lemmas provide the basis for constructing 'forbidden words' for satellites of $Z(p)$. We fix $p \in \mathcal{P}$ for these three results.

LEMMA 5.28. *Suppose* $\operatorname{int} U_j^-(p)$ *is contained in a central component of* $T \setminus Z(p)$ *and* $\varphi(U_j^-(p)) \subset P_{j+1}$. *Then at least one of* $H_1 \cap \operatorname{int} U_j^-(p)$ *and* $H_m \cap \operatorname{int} U_j^-(p)$ *is empty.*

Proof. Let $z = z_j^-$, y an endpoint of $U_j^-(p)$, and pick $z' \in Z(p) \setminus Z_j(p)$ another point in the central component of $T \setminus Z(p)$ containing $U_j^-(p)$. Since $f(z') \in Z(p) \setminus Z_{j+1}(p)$ we have the situation of Lemma 5.27 and our conclusion follows. □

LEMMA 5.29. *Suppose* $\{U_0^+(p), \ldots, U_{N-1}^+(p)\}$ *is a non-trivial* φ*-cycle with* $U_i^+(p) \subset P_i$ *for* $i = 0, \ldots, N-1$. *If there exist points of* $P(p) \setminus Z(p)$ *mapping into* $Z(p)$, *then for some* $i \in \{0, \ldots, N-1\}$ *and* $j \in \{1, m\}$, $H_j \cap U_i^+(p) = \emptyset$.

Proof. Suppose $z' \in P_i \setminus Z(p)$ with $f(z') \in Z_{i+1}(p)$, and let $U_i^+(p) = [z_i^+, z_i^+ + \varepsilon)$. Then Lemma 5.27 applied to $z = z_i^+$, $y = z_i^+ + \varepsilon$ and z' gives the desired conclusion. □

LEMMA 5.30. *Suppose a non-trivial* φ*-cycle has all elements in central components of* $T \setminus Z(p)$. *If some point* $z' \notin Z(p)$ *contained in a central component maps into* $Z(p)$, *then some element of the cycle has interior disjoint from either* H_1 *or* H_m.

Proof. Suppose $f(z') \in Z_{j+1}(p)$, and $U_j^-(p) = \langle z_j^- - \varepsilon, z_j^-\rangle$; apply Lemma 5.27 to $z = z_j^-$, $y = z_j^- - \varepsilon$ and z'. □

PROPOSITION 5.31. *Suppose* $p \in \mathcal{P}$ *with* $f|Z(p)$ *not cocountably* m*-fold. Define a set* $W(p) \supset Z(p)$ *as follows.*

(1) *If* $T \setminus Z(p)$ *has exactly one central component, and no interior point of* $C(p)$ *(respectively of* $P(p)$*) maps to* $Z(p)$, *set*

$$W(p) := C(p) \cup Z(p) \quad \text{(respectively } P(p) \cup Z(p)\text{)}.$$

(Note that in these cases $N \geq 2$.*)*

(2) *If* $N = 1$ *and one peripheral component* $P_0^-(p)$ *has no interior points mapping to* $Z(p)$ *or to the other peripheral component, then*

$$W(p) := Z(p) \cup P_0^-(p).$$

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(3) *In all other cases,*

$$W(p) := Z(p).$$

Then:

(1) $f(W(p)) \subset W(p)$*;*

(2) *for any open set* U *containing* $Z(p)$, $U \setminus W(p) \neq \emptyset$*;*

(3) $W(p)$ *locally divides* $\mathcal{H}$.

Proof. The first property is trivial, and the second is nearly so: in any case, since some points outside $Z(p)$ map to it, $T \setminus Z(p)$ is non-empty; furthermore, when we do adjoin a set to $Z(p)$ to form $W(p)$, there is always another endpoint of each $Z_j(p)$ to which a non-trivial component of $T \setminus Z(p)$ is attached.

We are left with the third property. We need to show that every persistent satellite of $Z(p)$ has its $\mathfrak{G}$-itinerary in some proper subset $\Lambda(p) \subset \Omega_m$ that depends only on $Z(p)$.

Since every persistent satellite lands in a non-trivial φ-cycle after at most N applications of f, we limit our attention to these instances.

If a non-trivial φ-cycle includes subsets of both central and peripheral components of $T \setminus Z(p)$, then Lemma 5.28 insures that a persistent satellite with local itinerary in this cycle cannot have a color itinerary containing N successive occurrences of some $j \in \{1, m\}$ (where j depends only on the cycle). Note that this situation *must* occur if either the cycle is of length $2N$ or the cycle contains at least one peripheral element and the total number of central components exceeds 1.

If the cycle is entirely peripheral (respectively entirely central) and the peripheral (respectively central) components of $T \setminus Z(p)$ contain at least one preimage of $Z(p)$, then Lemma 5.29 (respectively Lemma 5.30) again insures that any satellite with local itinerary in this cycle cannot display N successive occurrences of some $j \in \{1, m\}$.

The exception that remains is when the union of the peripheral (respectively central) components of $T \setminus Z(p)$ is f-invariant and contains our φ-cycle. In these cases, we know that the union $W(p)$ of $Z(p)$ with all the peripheral (respectively central) components is f-invariant, and that there must be preimages of $Z(p)$ outside $W(p)$. In particular, no open set containing $Z(p)$ is contained in $W(p)$. If there is no non-trivial φ-cycle outside $W(p)$, then $W(p)$ has no persistent satellites, while if there is one then Lemma 5.29 (respectively Lemma 5.30) applies to it. □

PROPOSITION 5.32. *Under Assumption 5.23, we can find a subset* $\mathcal{Q} \subset \mathcal{P}$ *such that*

$$W(\mathcal{Q}) := \bigcup_{q \in \mathcal{Q}} W(q)$$

is a proper subset of T *that contains* $Z(\mathcal{P}) := \bigcup_{p \in \mathcal{P}} Z(p)$.

Proof. Note that if, for some $p \in \mathcal{P}$, $W(p)$ is a connected proper superset of $Z(p)$, then either:

(1) $W(p) = C(p) \cup Z(p)$; or

(2) $Z(p)$ has a single component $Z_0(p)$, and $W(p) = Z_0(p) \cup P_0^-(p)$.

If any such points $p \in \mathcal{P}$ exist, we pick p_0 if possible of the first type and in any case so that $W(p_0)$ is maximal (in the sense that it is not a proper subset of $W(p)$ for any $p \in \mathcal{P}$

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of the same type). Now set

$$\mathcal{Q} := \{p_0\} \cup \{q \in \mathcal{P} \mid Z(q) \not\subset W(p_0)\}.$$

In this case, we make the following claim.

CLAIM 5.33. *For any $q \in \mathcal{Q} \setminus \{p_0\}$, $W(q) \cap W(p_0) = \emptyset$.*

Proof. To see this in case (1), note that every such $Z(q)$ must be contained in peripheral components of $T \setminus Z(p_0)$, of which there is more than one and by Proposition 5.25, these are permuted transitively by f. In particular, q cannot also be of type (1), since then its central component $C(q)$ (and hence $W(q)$) would contain $W(p_0)$, contradicting the maximality of the latter. It follows that $W(q)$ can consist only of $Z(q)$ possibly together with peripheral components of $T \setminus Z(q)$, which are separated from $W(p_0)$ by $Z(q)$.

In case condition (1) fails but condition (2) holds, each $Z(q)$ is contained in the single peripheral component $P_0^+(p_0)$, and $W(q)$ consists of $Z(q)$ with the possible addition of peripheral components of $T \setminus Z(q)$, which again are separated from $W(p_0)$ by $Z(q)$. □

Now, if no $p \in \mathcal{P}$ satisfies condition (1) or (2), then for each $p \in \mathcal{P}$ either $W(p)$ consists of $Z(p)$ together with peripheral components, or else $W(p) = Z(p)$. If $W(p) = Z(p)$ for all $p \in \mathcal{P}$, then clearly $\mathcal{Q} = \mathcal{P}$ has the required property, while if some $p \in \mathcal{P}$ has $W(p) = Z(p) \cup P(p)$ (with $P(p) \neq \emptyset$), then $C(p)$ is connected and we can pick p_0 of this type for which $W(p_0)$ is maximal. Then

$$\mathcal{Q} = \{p_0\} \cup \{q \in \mathcal{P} \mid Z(q) \subset C(p_0)\}$$

works, since any $q \in \mathcal{Q} \setminus \{p_0\}$ with $W(q) \neq Z(q)$ has $P(q)$ separated from $W(p_0)$ by $Z(q)$. □

Remark 5.34. If $\mathcal{Q}$ is a set of the type described in Proposition 5.32, then

$$W := W(\mathcal{Q})$$

contains $\mathcal{P}$ and locally divides $\mathcal{H}$.

To see this, note that since $f(H_i) = T$ for all i and $W \neq T$, it follows that $H_i \setminus W \neq \emptyset$ for each i, and Proposition 5.31 and Lemma 2.4 ensure that every orbit segment in a neighborhood of W either terminates in W or has its itineraries in a set $\Lambda \subset \Omega_m$ with entropy strictly less than $\log m$.

6. *Proof of our main theorem*

In this section we indicate how the various threads of this paper can be pulled together to prove Theorem 1.1, by fulfilling the hypotheses of Theorem 2.8. The key point is to combine Proposition 4.10, giving a spanning m-section with eventually countable kernel, and the results of §5, characterizing the center of a spanning section. The following is the analogue of Theorem 6.7 in [**BN05**].

LEMMA 6.1. *Suppose $f : T \to T$ is a cocountably m-fold self-map of a tree.*

Under Assumptions 5.1 and 5.23, there exist a cocountable open set $Y \subset T$ and an m-section on Y such that the associated m-shift system $\mathcal{H}$ satisfies:

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(1) *$f(\mathfrak{R}(\mathcal{H}))$ is (at most) countable;*
(2) *every minimal set in $\mathfrak{Z}_0(\mathcal{H})$ is a periodic orbit;*
(3) *there is a set W that contains every periodic orbit in $\mathfrak{Z}_0(\mathcal{H})$ and divides $\mathcal{H}$.*

Proof. By Proposition 4.10 there exist a cocountable open set $Y \subset T \setminus [\mathsf{V}(T) \cup f(\mathsf{V}(T))]$ and a monotone, spanning m-section ψ on Y for which the corresponding m-shift system $\mathcal{H}$ satisfies

$$f(\mathfrak{R}(\mathcal{H})) \cap Y = \emptyset, \tag{15}$$

which immediately implies the first condition above.

In particular, this also implies that any minimal set of f that is contained in the core $\mathfrak{Z}_0(\mathcal{H})$ (and hence in the kernel) is a periodic orbit, since otherwise it would have to be uncountable.

Then, a combination of Corollary 5.14 and Proposition 5.32 (together with Remark 5.34 and Lemma 2.4) gives the existence of a set W that contains all periodic orbits in the core and that locally divides $\mathcal{H}$. □

By Remark 5.2, we can assume Assumption 5.1 holds, while failure of Assumption 5.23 clearly gives the conclusion of Theorem 1.1. However, this means that Lemma 6.1 gives us the hypotheses of Theorem 2.8, and Theorem 1.1 follows immediately.

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