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Ergodic properties of random iterations of analytic functions

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Abstract. It is a known fact that an iterated function system (IFS) of entire functions is not necessarily ergodic. In this paper we show that if an IFS of analytic functions is defined in a domain whose boundary contains more than two points (in the extended complex plane) then the system possesses an ergodic property.

1. *Introduction and statement of results*

The main result of this paper (formulated for the unit disk for the sake of convenience) is the following theorem (for the definitions see §3 below).

THEOREM 1. *Let* $\mathcal{F} = \{f_j, p_j\}_{j=1}^N$, $\sum_1^N p_j = 1$, $p_j > 0$, *be an iterated function system with probabilities of analytic (holomorphic) functions mapping* $D = \{|z| < 1\}$ *into itself. Assume that* $\{f_j\}$ *have no common fixed point in* D *in the case when all these functions are Möbius transformations of* D *onto itself. Let* $\{F_n\}_{n\geq 1}$ *be independent identically distributed (i.i.d.) random variables with values in* $\{f_j\}$ *with* $P(F_n = f_j) = p_j$. *Then the distributions* μ_n *of the random variables* $Z_n(z_0) = F_n \circ F_{n-1} \circ \cdots \circ F_1(z_0)$ *have a weak star limit* μ, *where* μ *is independent of the initial point* $z_0 \in D$ *and* $\mu(D)$ *is zero or one. If* $\mu(D) = 1$ *then* μ *is the unique invariant probability distribution of* $\mathcal{F}$.

Remark 1. The assertion of Theorem 1 can be extended to any domain $D \subset \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with more then two boundary points in $\overline{\mathbb{C}}$, and this last condition is necessary for the assertion (see the corresponding examples below).

If all f_j are Möbius transformations mapping D onto itself with a common fixed point in D, then the system is equivalent (conjugate) to rotations of D about $z = 0$, and the assertion of Theorem 1 is trivially not true.

The proof of Theorem 1 uses the following two fundamental concepts, one from analysis (about the hyperbolic metric) and the other from probability theory (about the

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coupling method). The main fact about the hyperbolic metric frequently used below is that the hyperbolic distances are non-increasing functions under analytic transformations. In particular, if $h(\cdot,\cdot)$ denotes the hyperbolic metric in $D = \{|z| < 1\}$ (see §3 below for the definition), then (see [**Gol**, §8.2])

$$h(f_j(x), f_j(y)) \le h(x, y), \quad x, y \in D, \ j = 1, \dots, N. \tag{1}$$

In [**AW1**] the case when all f_j are Möbius transformations of D onto itself (without a common fixed point in D) is considered, it is shown that the limit distribution μ in this case is zero (see also [**BWK**] for a different type of convergence result and [**KW**] for general discussions and results about random iterations of Möbius transformations and their applications to continued fractions). Here we prove Theorem 1 for the case when at least one of the functions, say f_1, is not a Möbius transformation of D (onto itself), which means that

$$h(f_1(x), f_1(y)) < h(x, y), \quad x, y \in D. \tag{2}$$

The proof of Theorem 1 can be extended to any system of functions $\{f_j\}$ (not necessarily analytic) satisfying (1) and (2). We do not consider general metric spaces in this paper; we just say here that the open unit disk in Theorem 1 can be replaced by any domain in the complex plane whose boundary contains more than two points. This is exactly the condition which allows the introduction of the hyperbolic metric in a domain (see [**Gol**, §8.2]). The following simple examples show that Theorem 1 is not true for arbitrary domains. Let $f_1(z) = z^2$ and $f_2(z) = z^2 + 0.1$. These functions are analytic in the whole complex plane $\mathbb{C}$, whose boundary (with respect to the extended complex plane) consists of the single point at infinity. Then the orbit $Z_n(z_0)$ is bounded or unbounded depending whether the initial point z_0 is taken sufficiently close or far from the origin. The same argument shows that the system $f_1(z) = z^2$ and $f_2(z) = -z^3$ mapping $\mathbb{C} \setminus \{0\}$ into itself (without common fixed points there) is not ergodic. Note that in this case the boundary consists of two points.

Condition (2) (together with (1)) does not necessarily guarantee for the limit distribution μ in Theorem 1 that $\mu(D) = 1$. The question about necessary and sufficient conditions for this equality remains open; a sufficient condition is that at least one of the functions $\{f_j\}$ (say f_1) maps D strictly inside itself.

THEOREM 2. *If under the conditions of Theorem 1 we have $\overline{f_1(D)} \subset D$, then the limit distribution μ satisfies $\mu(D) = 1$.*

Here $\overline{f_1(D)}$ is the closure of $f_1(D)$.

Remark 2. Theorem 2 can be generalized to the case when there is a finite sequence of indices $\alpha_1, \alpha_1, \dots, \alpha_m$ for which the closure of the set $f_{\alpha_m} \circ f_{\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1}(D)$ belongs to D.

The condition $\overline{f_1(D)} \subset D$ also guarantees a 'nice' behaviour for the backward iterations.

THEOREM 3. *Let $\overline{f_1(D)} \subset D$. Then under the conditions of Theorem 1 the limit*

$$W = \lim_{n\to\infty} (F_1 \circ F_2 \circ \cdots \circ F_n(z_0))$$

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exists almost surely, is independent of the initial point $z_0 \in D$, and has the distribution μ, which is the limit distribution in Theorem 2 (and consequently $\mu(D) = 1$).

Theorem 3 is also valid under the more general condition of Remark 2.

2. *Discussions*

For IFSs of type $\{f_j, p_j\}_{j=1}^N$ with compact phase spaces, satisfying conditions (1) and (2), the assertion of Theorem 1 is known (the limit distribution μ in this case is of course a unit measure). See, for example, [**DF**]. See also [**Kai**] for general Markov chains in compact metric spaces. In the case of IFSs with compact phase spaces the assertion of Theorem 1 is a simple consequence of the fact that if K denotes the phase space of such a system, then the set $\cap_{n=1}^{\infty} f_1^n(K)$ is a single point. In our case with non-compact phase space we cannot guarantee of course that $f_1^n(D)$ shrinks to a point as n grows; moreover, if $x, y \in D$, we cannot guarantee that the sequence $h(f_1^n(x), f_1^n(y))$, which decreases monotonically as n grows (due to (2)), decreases to zero. This in particular means that the norm of f_1, defined by

$$\|f_1\| = \sup_{x \neq y} \frac{h(f_1(x), f_1(y))}{h(x, y)}$$

may be one. This prevents us from using the techniques developed in [**BEH**] for IFSs with 'average contractivity'.

The assertion of Theorem 2 can be derived from Theorem 1 by using a tightness argument for the set of iterated distributions μ_n (see [**HM**, p. 87]). The result in Theorem 2 can be used to create a large class of fractals as the attractors of the IFSs; see [**BH**]. Also see [**Bar**] and [**PR**] for IFSs of analytic functions whose attractors are Julia sets. For this purpose both the forward and the backward iterations can be used. The fact that the distribution μ for the limits of the reversed iterations in Theorem 3 is the same as the limit distribution of the direct iterations in Theorem 2 follows from the results in [**Let**] (see the proof of Theorem 3 below). See also [**CL**].

In [**Fur**] an interesting result (see Theorem 1.3) is obtained about random iterations of a system of Möbius transformations $\{f_j\}$ (possibly infinite) of D onto itself, which is not contained in a proper subgroup of all such Möbius transformations; the assertion there is that the reverse iterations have a limit almost surely and the limit point lies on the boundary of D. In [**AW2**] we provide a very explicit way to determine whether or not the required conditions in Furstenberg's theorem are satisfied.

3. *Notation and terminology*

Throughout this paper we use the following notation and definitions:

$D = \{\|z\| < 1\}$	the open unit disk in $\mathbb{C}$.
$\mathcal{F} = \{f_j, p_j\}_{j=1}^N$	IFS with probabilities of analytic functions from D to D, $\sum_1^N p_j = 1$, $p_j > 0$
f^n	$f^n = f \circ \cdots \circ f$, where f is taken n times in this composition.

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$\{F_n\}_{n=1}^{\infty}$	independent identically distributed (i.i.d.) random variables with values in $\{f_j\}_{j=1}^{N}$ with distribution $P(F_n = f_j) = p_j$.
Z_0	a random variable with values in D.
$\{Z_n(Z_0)\}_{n=0}^{\infty}$	random variables with values in D: $Z_n(Z_0) = F_n \circ F_{n-1} \circ \cdots \circ F_1(Z_0)$, $n \geq 1$.
μ_n	the probability distribution of Z_n, $n \geq 0$, $\mu_n(D) = 1$.
$\mu_n \to \mu$	the weak star convergence of measures (possibly signed) in D: $\int \phi \, d\mu_n \to \int \phi \, d\mu$, as $n \to \infty$, for any continuous function ϕ on D with compact support. (Sometimes we call this weak convergence in this paper.)
$\mathcal{F}^*$	a mapping from the set of distribution measures on D into itself: $\mathcal{F}^*(\mu)(B) = \sum_{j=1}^{N} p_j \mu(f_j^{-1}(B))$ for all Borel sets $B \subset D$. Informally, if μ is a distribution of $Z \in D$, then $\mathcal{F}^*(\mu)$ is the distribution of $F_1(Z)$.
Invariant distribution	we say that a distribution μ is an invariant distribution for an IFS $\mathcal{F}$ if $\mathcal{F}^*(\mu) = \mu$.
$\mathcal{F}^{*n}$	$\mathcal{F}^{*n} = \mathcal{F}^* \circ \cdots \circ \mathcal{F}^*$, where $\mathcal{F}^*$ is taken n times in this composition.
$h(x, y)$	the hyperbolic distance between x and y, $x, y \in D$, defined by the differential element $h(x, x + dx) = \lvert dx \rvert / (1 - \lvert x \rvert)$.
H_R	the hyperbolic disk of radius R, centered at the origin: $H_R = \{z : h(z, 0) < R\}$. We recall that H_R is also a disk in the Euclidean metric.
$\lvert x - y \rvert$	the Euclidean distance in $\mathbb{C}$.
Möbius transformation	a rational function of the form $f(z) = (az + b)/(cz + d)$, $ad - bc \neq 0$.

4. *Proofs*

Proof of Theorem 1. The steps of the proof of Theorem 1 (in the case when not all f_j are Möbius transformations mapping D onto itself) are given in the following three lemmas.

LEMMA 1. *For any two probability distributions μ_0 and μ_0' of initial points Z_0 and Z_0' we have $\mu_n - \mu_n' \to 0$, as $n \to \infty$, weakly in D, where $\mu_n = \mathcal{F}^{*n}(\mu_0)$ and $\mu_n' = \mathcal{F}^{*n}(\mu_0')$.*

LEMMA 2. *If the given iterated function system $\mathcal{F}$ has no invariant probability distribution, then*

$$\frac{\mu_0 + \mu_1 + \cdots + \mu_n}{n + 1} \to 0,$$

as $n \to \infty$, weakly in D.

LEMMA 3. *If the given iterated function system $\mathcal{F}$ has no invariant probability distribution, then*

$$\mu_n \to 0,$$

as $n \to \infty$, weakly in D.

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Before we prove these lemmas we check that they give Theorem 1. Actually, from Lemma 3 we have that either $\mu_n \to 0$ (and the assertion of Theorem 1 is trivially true) or there is an invariant probability distribution μ for the given IFS. If now we take in Lemma 1 μ_0' equal to μ, we get that $\mathcal{F}^*(\mu) = \mu$ and $\mu_n - \mu \to 0$, as $n \to \infty$. This proves Theorem 1.

Proof of Lemma 1. Let Z_0 and Z_0' be two independent random points in D with the initial distributions μ_0 and μ_0' respectively. Let $\{F_n\}_{n\geq 1}$ be a sequence of independent identically distributed (i.i.d.) random variables with values in $\{f_j\}_{j=1}^{j=N}$ with distribution $P(F_n = f_j) = p_j$. Consider the two random orbits $Z_n = Z_n(Z_0) = F_n \circ F_{n-1} \circ \cdots \circ F_1(Z_0)$ and $Z_n' = Z_n'(Z_0') = F_n \circ F_{n-1} \circ \cdots \circ F_1(Z_0')$, $n \geq 1$, generated by the same random sequence $\{F_n\}_{n\geq 1}$. In other words, we couple the random variables Z_n and Z_n' at the moment $n = 1$. (For the coupling method see, for example, [**Lin**].) So Z_n and Z_n' are no longer independent (as $n \geq 1$), but their distributions μ_n and μ_n' are the same.

Without loss of generality we may suppose that the initial points Z_0 and Z_0' are not random, but some fixed points $Z_0 = z_0$, $Z_0' = z_0'$ in D (i.e. the initial distributions μ_0 and μ_0' are Dirac measures concentrated at z_0 and z_0' respectively).

Assume that $\mu_n \to 0$, as $n \to \infty$. Then, due to (1) we have $\mu_n' \to 0$, as $n \to \infty$, and evidently $\mu_n - \mu_n' \to 0$ (as $n \to \infty$).

Suppose now that we do not have the limit relation $\mu_n \to 0$. Then there exists a compact set $K \subset D$ and a number $\delta > 0$, such that $\mu_n(K) > \delta$ for infinitely many n. From this it follows that

$$P(Z_n \in K \text{ infinitely often}) > \delta > 0. \tag{3}$$

This means that

$$P\left(\lim_{n\to\infty} |Z_n| = 1\right) < 1. \tag{4}$$

It is not difficult to see (due to (1)) that Kolmogorov's zero-one law applies to the event in (4) (which then says that the left-hand side in (4) is either zero or one) and we get

$$P\left(\lim_{n\to\infty} |Z_n| = 1\right) = 0. \tag{5}$$

From (5) in turn it follows that the number δ in (3) can be assumed to be arbitrarily close to 1 (by choosing K sufficiently large).

Let $d = h(z_0, z_0')$ be the hyperbolic distance between the two initial points and denote by K_d the closed neighbourhood of K of radius d. Then, due to (1), $Z_n' \in K_d$ whenever $Z_n \in K$, and from (3)

$$P(Z_n, Z_n' \in K_d \text{ infinitely often}) > \delta > 0. \tag{6}$$

If $Z_n, Z_n' \in K_d$ and if $F_{n+1} = f_1$, then due to (2) we have

$$h(z_{n+1}, z_{n+1}') \leq c \cdot h(z_n, z_n'), \tag{7}$$

where $c = c(K_d) < 1$.

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Since the random variables $\{F_n\}_{n\geq 1}$ are independent of each other and $P(F_n = f_1) = p_1 > 0$, we conclude from (6) that

$$P(Z_n, Z'_n \in K_d,\ F_{n+1} = f_1,\ \text{infinitely often}) > \delta > 0.$$

This together with (7) gives that

$$P\left(\lim_{n\to\infty} h(Z_n, Z'_n) = 0\right) > \delta.$$

But since δ can be chosen arbitrarily close to 1, we get

$$P\left(\lim_{n\to\infty} h(Z_n, Z'_n) = 0\right) = 1.$$

The last limit relation is stronger then the convergence in probability:

$$\lim_{n\to\infty} P(h(Z_n, Z'_n) > \epsilon) = 0,$$

for arbitrary $\epsilon > 0$. From the arbitrariness of ϵ in this limit relation we get for any continuous function φ, with a compact support in D, that

$$\lim_{n\to\infty}\left(\int \varphi\, d\mu_n - \int \varphi\, d\mu'_n\right) = 0$$

(any such function φ is bounded and uniformly continuous in D).

Lemma 1 is proved. □

Proof of Lemma 2. Let $\bar{\mu}_n = (\mu_0 + \mu_1 + \cdots + \mu_n)/(n+1)$. Then there exists an infinite subset Λ of the set of natural numbers such that the distributions $\bar{\mu}_n$ tend weakly in D to a distribution μ as $n \to \infty$, $n \in \Lambda$ (see [**HM**, p. 71]). It is enough to prove that μ is an invariant distribution for $\mathcal{F}$. Then if the limit relation in Lemma 2 is not fulfilled, we may assume that $\mu(D) \neq 0$ and the normalized distribution $\mu/\mu(D)$ will be an invariant probability distribution on D. This is a standard scheme for proving existence of an invariant probability distribution for systems with compact phase spaces (see, for example, [**Sin**]). We do not know of any reference for non-compact cases and we provide a short proof of this fact.

Consider

$$\begin{aligned}\mathcal{F}^*(\bar{\mu}_n) &= \mathcal{F}^*((\mu_0 + \cdots + \mu_n)/(n+1)) = (\mathcal{F}^*(\mu_0) + \cdots + \mathcal{F}^*(\mu_n))/(n+1)\\ &= (\mu_1 + \cdots + \mu_{n+1})/(n+1).\end{aligned}$$

Then

$$\mathcal{F}^*(\bar{\mu}_n) - \bar{\mu}_n = (\mu_{n+1} - \mu_0)/(n+1) \to 0,$$

weakly as $n \to \infty$. Now using the continuity of the operator $\mathcal{F}^*$ we get

$$0 = \lim_{n\to\infty, n\in\Lambda} (\mathcal{F}^*(\bar{\mu}_n) - \bar{\mu}_n) = \mathcal{F}^*(\mu) - \mu.$$

So, by the definition, μ is an invariant distribution.

Lemma 2 is proved. □

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Proof of Lemma 3. Suppose on the contrary that the given iterated function system $\mathcal{F}$ has no invariant probability distribution, but that the distributions μ_n do not tend to zero. Then there exists a compact set $K \in D$, such that

$$\limsup_{n\to\infty} \mu_n(K) > 0. \tag{8}$$

On the other hand, from Lemma 2 we have

$$\liminf_{n\to\infty} \mu_n(K) = 0 \tag{9}$$

(and this is true for all compact subsets of D). We show below that the last two limit relations lead to a contradiction.

Let

$$\alpha := \sup_K \limsup_{n\to\infty} \mu_n(K) > 0, \tag{10}$$

where the supremum is taken over all compact subsets of D. (Positivity of α follows from (8).)

Let H_R denote the closed disk in D, centered at $z = 0$, with hyperbolic radius R. We assume that

$$\limsup_{n\to\infty} \mu_n(H_R) > \alpha - \epsilon, \tag{11}$$

for some positive ϵ, with $\epsilon < \alpha$.

We also consider the disk H_{5R}, centered at $z = 0$, with the hyperbolic radius $5R$. Due to (11) and (9) we may choose two natural numbers m and n, $m < n$, such that

$$\mu_m(H_R) > \alpha - \epsilon$$

and

$$\mu_n(H_{5R}) < \epsilon.$$

From the last two inequalities we get

$$\begin{aligned} \epsilon > \mu_n(H_{5R}) &= P(Z_n \in H_{5R}) \geq P(Z_n \in H_{5R}, Z_m \in H_R) \\ &= P(Z_n \in H_{5R} \mid Z_m \in H_R) \cdot P(Z_m \in H_R). \end{aligned}$$

This in turn yields

$$P(Z_n \in H_{5R} \mid Z_m \in H_R) < \frac{\epsilon}{P(Z_m \in H_R)} < \frac{\epsilon}{\alpha - \epsilon}. \tag{12}$$

Any realization $\{f_{\alpha_i}\}_{i\geq 1}$ of $\{F_i\}_{i\geq 1}$ which takes a point $z_m \in H_R$ out of H_{5R} after $n - m$ steps will take any other point $z_m^* \in H_R$ out of $H_{3R} = \{h(z, 0) \leq 3R\}$ after $n - m$ steps; this follows from (1) and the triangle inequality.

Fix $l := n - m$. Using the independence of $\{F_j\}$ we get from (12) for any index $k \geq 0$ that

$$P(Z_{k+l} \in H_{3R} \mid Z_k \in H_R) < \frac{\epsilon}{\alpha - \epsilon}. \tag{13}$$

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Denote by H a big closed disk (centered at $z = 0$) such that for any $k \geq 0$

$$P(Z_{k+l} \in H \mid Z_k \in H_R) = 1. \tag{14}$$

From (13) and (14) we have

$$P(Z_{k+l} \in H \backslash H_{3R} \mid Z_k \in H_R) > 1 - \frac{\epsilon}{\alpha - \epsilon} = \frac{\alpha - 2\epsilon}{\alpha - \epsilon}, \tag{15}$$

$k \geq 0$, l fixed.

From (11) it follows that there is an infinite subset $\Lambda \subset \mathbb{N}$ of indices k for which

$$P(Z_k \in H_R) > \alpha - \epsilon. \tag{16}$$

For any such index k we have by (15) that

$$\begin{aligned} P(Z_{k+l} \in H \backslash H_{3R}) &\geq P(Z_{k+l} \in H \backslash H_{3R},\ Z_k \in H_R) \\ &= P(Z_{k+l} \in H \backslash H_{3R} \mid Z_k \in H_R) \cdot P(Z_k \in H_R) \\ &> \frac{\alpha - 2\epsilon}{\alpha - \epsilon} \cdot (\alpha - \epsilon) = \alpha - 2\epsilon. \end{aligned} \tag{17}$$

Now from Lemma 1 (taking $\mu_0' = \mu_l$ in this lemma) we get

$$\mu_{k+l} - \mu_k \to 0,$$

weakly, as $k \to \infty$.

Finally, the last limit relation together with (16) and (17) gives

$$\limsup_{k \to \infty, k \in \Lambda} \mu_k(H) \geq (\alpha - \epsilon) + (\alpha - 2\epsilon) \geq 2\alpha - 3\epsilon,$$

which contradicts (10) if $\epsilon < \frac{1}{3}\alpha$.

Lemma 3 is proved. □

Proof of Theorem 2. A proof of Theorem 2 can be derived from the proof of Theorem 3 (see below), but here we give a more direct proof using Theorem 1. We have

$$\mu_n(f_1(D)) = P(Z_n \in f_1(D)) \geq P(F_n = f_1) = p_1 > 0.$$

From this inequality it follows that the distributions μ_n cannot tend weakly to zero in D (as $n \to \infty$), since $f_1(D)$ is separated from the boundary of D. Then by Theorem 1 the limit distribution is a unit measure. This proves Theorem 2. □

It is interesting to note here that the distributions $\{\mu_n\}$ in fact possess a stronger property (than that in the last inequality) called tightness, which means that for any $\epsilon > 0$ there exists a compact subset $K \subset D$ such that $\mu_n(K) > 1 - \epsilon$, for all $n \geq 0$. Any tight sequence of unit distributions is weakly compact in the weak star topology (see [**HM**, p. 87]). From this fact it immediately follows that if a tight sequence of unit distributions has a weak star limit then the limit is also a unit distribution. To show that the sequence μ_n in Theorem 2 is tight we denote by B_l the set of all points which can be reached starting at some point of $f_1(D)$ and in less then l steps. Then $\overline{B_l} \subset D$ and for all sufficiently large n we have

$$\mu_n(B_l) = P(Z_n \in B_l) \geq P(F_j = f_1 \text{ for some } j = n, n-1, \ldots, n-l+1) = 1-(1-p_1)^l.$$

From this inequality the tightness of $\{\mu_n\}$ follows, which in turn implies that the limit distribution μ in Theorem 2 is a unit measure.

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Proof of Theorem 3. We start by recalling a more precise version of inequality (2) (see [**Gol**]). If B is a subset of D, such that $\overline{B} \subset D$, then

$$h(f_1(x), f_1(y)) < c \cdot h(x, y)$$

for all $x, y \in B$, where $c = c(B) < 1$. In particular, we get that if U is any subset of $f_1(D)$, then

$$\operatorname{diam}(f_1(U)) < c \cdot \operatorname{diam} U, \tag{18}$$

where $c = c(f_1(D)) < 1$ and diam is the diameter in the hyperbolic metric.

Consider the sequence of i.i.d. random variables

$$F_1, F_2, \ldots, F_n, \ldots, \quad \text{where } P(F_n = f_1) = p_j \tag{19}$$

and the sequence of nested sets $\{F_1 \circ F_2 \circ \cdots \circ F_n(D)\}_{n\geq 1}$. Let $h_n = \operatorname{diam}(F_1 \circ F_2 \circ \cdots \circ F_n(D))$, $h_1 \geq h_2 \geq \ldots$. It is enough to show that

$$\lim_{n\to\infty} h_n = 0 \tag{20}$$

almost surely.

First note that the probability that the sequence in (19) contains the function f_1 is one. From this together with (1) it follows that with probability one the limit in (20) is finite and no larger than $\operatorname{diam}(f_1(D))$. Furthermore, if in the sequence (19) we have for some $n \geq 1$ that $F_n = F_{n+1} = f_1$, then (18) gives that $h_{n+1} < c \cdot h_n$, where $c < 1$ is independent of n. But evidently

$$P(F_n = F_{n+1} = f_1 \text{ for infinitely many } n) = 1.$$

This proves the limit relation (20) and the first part of Theorem 3 about the existence of the limit W. To prove that the distribution of W coincides with the limit distribution of the direct iterations in Theorem 2, we use the following result (see Proposition 1 in [**CL**], see also [**Let**]), which we formulate in the weaker form we need.

Let $\{f_j, p_j\}_{j=1}^N$, $\sum_1^N p_j = 1$, $p_j > 0$, be an iterated function system with probabilities of continuous functions $\{f_j\}$ mapping a locally compact space E with countable basis into itself. Let $\{F_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with values in $\{f_j\}$, and let $P(F_n = f_j) = p_j$, $j = 1, 2, \ldots, N$. Consider for $n \geq 0$ and $x \in E$ the following:

$$Z_n(x) = F_n \circ F_{n-1} \circ \cdots \circ F_1(x) \quad \text{and} \quad W_n(x) = F_1 \circ F_2 \circ \cdots \circ F_n(x).$$

Assume that $W = \lim_{n\to\infty} W_n(x)$ exists and does not depend on x almost surely. Then the distribution of W is an invariant probability distribution of the Markov chain $(Z_n(x))_{n=0}^\infty$, and this chain has only one invariant probability distribution. □

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