Check for updates

J. Appl. Probab. **63,** 788–797 (2026)
*doi:*10.1017/jpr.2025.10058

ANY RANDOM VARIABLE WITH RIGHT-UNBOUNDED DISTRIBUTIONAL SUPPORT IS THE MINIMUM OF INDEPENDENT AND VERY HEAVY-TAILED RANDOM VARIABLES

ANTON SERGEEVICH TARASENKO,* *Sobolev Institute of Mathematics*
SERGEY FOSS,** *Heriot-Watt University and Sobolev Institute of Mathematics*
GEORGIY KRIVTSOV,*** *Novosibirsk State University and New Economic School*

Abstract

A random variable ξ has a *light-tailed* distribution (for short, is light-tailed) if it possesses a finite exponential moment, $\mathbb{E}\exp(\lambda\xi) < \infty$ for some $\lambda > 0$, and has a *heavy-tailed* distribution (is heavy-tailed) if $\mathbb{E}\exp(\lambda\xi) = \infty$ for all $\lambda > 0$. Leipus *et al.* (2023 *AIMS Math.* **8**, 13066–13072) presented a particular example of a light-tailed random variable that is the minimum of two independent heavy-tailed random variables. We show that this phenomenon is universal: *any* light-tailed random variable with right-unbounded support may be represented as the minimum of two independent heavy-tailed random variables. Moreover, a more general fact holds: these two independent random variables may have as heavy-tailed distributions as we wish. Further, we extend the latter result to the minimum of any finite number of independent random variables. We also comment on possible generalizations of our result to the case of dependent random variables.

Keywords: Light tail; heavy tail; long tail; subexponentiality; minimum of random variables

2020 Mathematics Subject Classification: Primary 60E05
Secondary 60E15

1. Introduction and Results

Questions about the closure of certain distribution classes under various operations (summation, the taking maximum or the minimum, etc.) arise naturally in tail asymptotics analysis and in a variety of applications (see, e.g., [2–7]).

We restrict our attention mostly to independent random variables, and there are several known simple closure properties. For example, the sum of two variables is heavy-tailed if at least one of the summands is heavy-tailed, and is light-tailed otherwise; the sum is long-tailed if at least one of the summands is long-tailed; the minimum and the maximum of light-tailed random variables are light-tailed; the minimum and the maximum of long-tailed random variables are long-tailed; and the maximum of heavy-tailed random variables is heavy-tailed.

Received 13 May 2025; accepted 2 September 2025.
* Postal address: Prospekt Akademika Koptyuga 4, Novosibirsk, Novosibirskaya Oblast, 630090, Russia. Email: sunakdb@gmail.com
** Postal address: School of Mathematical and Computer Sciences, Riccarton Campus, Edinburgh, EH14 4AS, UK. Email: s.foss@hw.ac.uk
*** Postal address: New Economic School, ul. Nobel 3, Moscow, 121205, Russia. Email: gkrivtsov@nes.ru

© The Author(s), 2026. Published by Cambridge University Press on behalf of Applied Probability Trust.

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

However, it was shown by example in [1] that the minimum of two independent heavy-tailed random variables may be light-tailed. The example, however, is somewhat involved and does not fully elucidate the mechanism behind this phenomenon. This was our motivation for the present work: we provide a systematic construction for generating such distributions. We produce a simple scheme for constructing random variables with arbitrarily heavy tails whose minimum has an arbitrarily light tail.

To formulate our results, we need to introduce some notation and notions. Throughout this paper, we use the same notation F for a probability distribution on the real line and for its distribution function $F(x)$, and we denote by

$$\overline{F}(x) = 1 - F(x)$$

its right tail. The following definitions are standard and can be found, for example, in [2, 3].

Definition 1.1. A distribution F (or a random variable ξ following distribution F) has a heavy tail if for all $\lambda > 0$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x} F(dx) \equiv \mathbb{E}\mathrm{e}^{\lambda \xi} = \infty,$$

and has a light tail if $\mathbb{E}\mathrm{e}^{\lambda \xi} < \infty$ for some $\lambda > 0$.

A distribution F is *long-tailed* if it has a right-unbounded support (i.e. $F(x) < 1$ for all x) and

$$\lim_{x \to \infty} \frac{\overline{F}(x+1)}{\overline{F}(x)} = 1.$$

It is known (see, e.g., [3]) that *any long-tailed distribution is heavy-tailed*.

Here is our first result.

Theorem 1.1. *Let F be any distribution with right-unbounded support, and let g be any non-negative increasing function satisfying $\lim_{x\to\infty} g(x) = \infty$.*

Then there exist independent random variables ξ_1 and ξ_2 such that the distribution of $\min(\xi_1, \xi_2)$ is F, and

$$\mathbb{E}g(\xi_1) = \mathbb{E}g(\xi_2) = \infty.$$

As a particular corollary, we show the universality of the fact that *any light-tailed random variable may be represented as the minimum of two independent heavy-tailed random variables*.

Corollary 1.1. *Given any light-tailed distribution F, we can choose two heavy-tailed distribution H_1 and H_2 such that for independent random variables ξ_1 with distribution H_1 and ξ_2 with distribution H_2, the random variable $\eta = \min(\xi_1, \xi_2)$ has distribution F.*

Clearly, distributions H_1 and H_2 in Corollary 1.1 cannot be long-tailed (otherwise the minimum would be also long-tailed and therefore heavy-tailed).

Consider the following practical scenario, which illustrates the idea behind the construction of the two distributions in the proof of Theorem 1.1. Imagine two developer teams, each already overwhelmed with tasks. A crucial client reports a critical bug that must be fixed without delay. Initially, the bug is assigned to both teams, but because each is already overloaded, it lands at

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

the bottom of their respective queues. Recognizing the urgency, management instructs one team to drop everything and address the bug immediately, granting the team a fixed amount of time. If that team fails to resolve the issue within the allotted time, the bug is reassigned to the other team, this time with a more generous time limit, while the first team returns to its previous work. If both teams fail in their first attempts, the cycle continues with successively longer time limits until the bug is finally fixed.

This set-up reflects the core idea examined in this paper. While the expected time for either team to solve the bug might be very long, suggesting a heavy-tailed distribution, the overall expected resolution time for the bug remains short, indicating a light-tailed distribution.

We extend this idea to the minimum of several random variables. To do so, we need a way to compare distributions. One common way to compare distributions is via the classical stochastic dominance (see, e.g., [5]).

Definition 1.2. We say that a distribution F stochastically dominates a distribution G if

$$F(x) \leqslant G(x) \quad \text{for all } x \in \mathbb{R}.$$

This allows us to establish a more general result concerning the minimum of many random variables.

Theorem 1.2. *Let F be any distribution with right-unbounded support, and let g be a positive, strictly increasing function satisfying* $\lim_{x\to\infty} g(x) = \infty$.

Then for any $n \in \mathbb{N}$ *and any* $1 < k \leqslant n$, *there exist independent random variables* $\xi_1, \ldots, \xi_n$ *such that the following properties hold:*

(i) *For any set of indices*

$$1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n,$$

the distribution F stochastically dominates the distribution of $\min(\xi_{i_1}, \ldots, \xi_{i_k})$.

(ii) *For any set of indices*

$$1 \leqslant i_1 < i_2 < \cdots < i_{k-1} \leqslant n,$$

we have

$$\mathbb{E}g\big(\min(\xi_{i_1}, \ldots, \xi_{i_{k-1}})\big) = \infty.$$

2. Proof of Theorem 1.1 and of Corollary 1.1, and Examples

Remark 2.1. Without loss of generality, we may reduce our consideration to non-negative random variables. Indeed, for any function g satisfying the conditions of the Theorem 1.1, $\mathbb{E}g(\xi)$ is finite if and only if $\mathbb{E}g(\max(\xi, 0))$ is finite, and, clearly, $\min(\xi_1, \xi_2)^+ = \min(\xi_1^+, \xi_2^+)$.

We need the following notion. The *cumulative hazard function* (for brevity, *cumulative hazard*) of a distribution F is defined as

$$R_F(x) = -\log \overline{F}(x).$$

The cumulative hazard offers a useful way to quantify how rapidly the tail $\overline{F}(x)$ decays as x increases. The remarks below explain the relationship between the cumulative hazard, a general non-decreasing function, and the minimum of independent random variables.

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

Remark 2.2. Any non-decreasing and right-continuous function R that satisfies

$$\lim_{x\to-\infty} R(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} R(x) = \infty$$

is the cumulative hazard of some distribution F, where

$$\overline{F}(x) = \mathrm{e}^{-R(x)}.$$

Remark 2.3. The cumulative hazard of the minimum of two independent random variables ξ_1 and ξ_2 equals the sum of their individual cumulative hazards. That is, if $\eta = \min(\xi_1, \xi_2)$ then

$$R_\eta(x) = R_{\xi_1}(x) + R_{\xi_2}(x).$$

Proof of Theorem 1.1. Let H_1 and H_2 be the distribution functions of ξ_1 and ξ_2. The idea of the proof is to let them essentially 'take turns' on successive intervals. We divide the positive real axis into consecutive intervals $(a_i, a_{i+1}]$. On even-indexed intervals, we 'freeze' the cumulative hazard R_1 while letting R_2 increase together with R_F. On odd-indexed intervals, we do the opposite: keep R_2 fixed and let R_1 follow R_F. By carefully stitching these pieces together, we ensure that the sum $R_1 + R_2$ (i.e. the cumulative hazard of the minimum) matches R_F at all points. Hence, $\min(\xi_1, \xi_2)$ has exactly the distribution F. At each step, we also choose the lengths of these intervals in such a way that each distribution's tail is made 'heavy enough'. This "alternating" approach guarantees that while each random variable is heavy-tailed (in the sense prescribed by g), their minimum still reproduces the desired lighter-tailed distribution F.

We construct the sequence $\{a_i\}_{i=0}^{\infty}$ and the cumulative hazards R_1 and R_2 inductively. First, set $a_0 = 0$ and define $R_1(x) = R_2(x) = 0$ for all $x \leqslant 0$. Assume a_l is defined. We then choose a_{l+1} and specify R_1 and R_2 on $(a_l, a_{l+1}]$ as follows:

- For even values of l, we let
$$R_1(x) = R_1(a_l),\ R_2(x) = R_2(a_l) + \big(R_F(x) - R_F(a_l)\big)$$
for $x \in (a_l, a_{l+1}]$.
- If l is odd, we let
$$R_1(x) = R_1(a_l) + \big(R_F(x) - R_F(a_l)\big),\ R_2(x) = R_2(a_l)$$
for $x \in (a_l, a_{l+1}]$.

This ensures R_1 and R_2 remain right-continuous at each a_l. We can choose a_{l+1} arbitrarily provided $a_{l+1} - a_l \geqslant 1$, so the intervals $(a_i, a_{i+1}]$ cover all of $\mathbb{R}^+$, and, moreover, the inequalities

$$(a_{l+1} - a_l)\exp\{-R_1(g^{-1}(a_l))\} \geqslant 1 \quad (\text{if } l \text{ is even}),$$

$$(a_{l+1} - a_l)\exp\{-R_2(g^{-1}(a_l))\} \geqslant 1 \quad (\text{if } l \text{ is odd})$$

hold, ensuring the expectation of each constructed random variable is infinite.

Let H_1 and H_2 be the distributions corresponding to R_1 and R_2. We verify that these distributions satisfy the stated properties.

First, we show that *the minimum of the random variables has the correct distribution.* Let $R_{\min}$ be the cumulative hazard of $\min(\xi_1, \xi_2)$. Clearly, $R_{\min}(a_0) = 0 = R_F(a_0)$.

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

Assume there is some l such that $R_{\min}(a_l) = R_F(a_l)$. If l is odd then for any $x \in (a_l, a_{l+1}]$, we have

$$R_{\min}(x) - R_{\min}(a_l) = \big(R_1(x) - R_1(a_l)\big) + \big(R_2(x) - R_2(a_l)\big)$$
$$= R_F(x) - R_F(a_l),$$

so $R_{\min}(x) = R_F(x)$ in the interval.

A similar argument shows that $R_{\min}(x) = R_F(x)$ for $x \in (a_l, a_{l+1}]$. Hence, $R_{\min} \equiv R_F$ and $\min(\xi_1, \xi_2)$ has distribution F.

Second, we show that the distributions of ξ_1 and ξ_2 are sufficiently heavy. We show this for ξ_1. An identical argument applies to ξ_2. Observe that

$$\mathbb{E}g(\xi_1) = \int_0^\infty \mathbb{P}\big(g(\xi_1) > t\big)\,\mathrm{d}t = \sum_{j=0}^{\infty} \int_{a_j}^{a_{j+1}} \mathbb{P}\big(g(\xi_1) > t\big)\,\mathrm{d}t$$
$$\geqslant \sum_{j=0}^{\infty} \int_{a_{2j}}^{a_{2j+1}} \mathbb{P}\big(g(\xi_1) > t\big)\,\mathrm{d}t.$$

Since R_1 is constant on each interval $(a_{2j}, a_{2j+1}]$, we have

$$\int_{a_{2j}}^{a_{2j+1}} \mathbb{P}\big(g(\xi_1) > t\big)\,\mathrm{d}t = (a_{2j+1} - a_{2j}) \exp\{-R_1\big(g^{-1}(a_{2j})\big)\} \geqslant 1.$$

Thus,

$$\mathbb{E}g(\xi_1) \geqslant \sum_{j=0}^{\infty} 1 = \infty.$$

□

Proof of Corollary 1.1. We can apply the Theorem 1.1 by taking a light-tailed distribution F and any function g such that $\mathrm{e}^{-cx} = o(g(x))$ as $x \to \infty$ for any $c > 0$. For example, we may take $g(x) = x^+$. Then both ξ_1 and ξ_2 have infinite first moments and therefore all their exponential moments are infinite too.

For Corollary 1.1, there is an alternative argument. Theorem 2.6 in [3] says that a distribution F is heavy-tailed if and only if its cumulative hazard satisfies $\liminf_{x\to\infty} R(x)/x = 0$. Then we can simplify the above proof of Corollary 1.1 by defining a sequence $\{a_l\}$ in such a way that $R_1(a_{2i})/a_{2i+1} \leq 1/i$ and $R_2(a_{2i+1})/a_{2i+2} \leq 1/i$ for all i. □

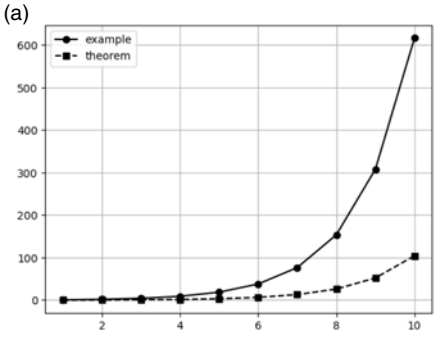
Let us provide a few examples illustrating how quickly the sequence $\{a_i\}_{i=0}^{\infty}$ constructed in the Theorem 1.1 can grow.

To make precise how close our explicit choices of the segment endpoints a_k are to the smallest values allowed by the recursive inequalities in the proof of Theorem 1.1, we computed a 'minimal' sequence $\{a_k^*\}$ by enforcing

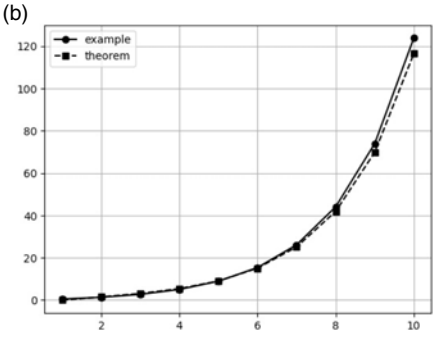
$$a_{k+1}^* - a_k^* = \max\big\{1, \exp\big(R_F\big(g^{-1}(a_k^*)\big)\big)\big\}, \quad a_0^* = 0.$$

In Figure 1 we plot both $\log a_k^*$ (dashed lines) and $\log a_k$ (solid curves) on the same logarithmic (or, where noted, double-logarithmic) scale. This highlights that even when our explicit a_k are noticeably larger in some cases, they nevertheless lie on the same asymptotic scale as the theoretical minima.

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press



Example 2.1. Log-scale plot of the "minimal" recursive endpoints $\{a_k^*\}$ (dashed lines) versus our explicit $\{a_k\}$ from the example (solid curve) for the exponential distribution $\overline{F}(x) = e^{-\alpha x}$ with $g(x) = e^{\beta x}$ ($\beta < \alpha$).



Example 2.2. Log-scale plot of the "minimal" recursive endpoints $\{a_k^*\}$ (dashed lines) versus our explicit $\{a_k\}$ from the example (solid curve) for the power-tailed distribution $\overline{F}(x) = (1+x)^{-\alpha}$ with $g(x) = x^{\beta}$ ($\beta + 1 < \alpha$).

FIGURE 1. Comparison of explicit $\{a_k\}$ and recursive $\{a_k^*\}$ for two cases: (a) exponential distribution and (b) power-tailed distribution.

Example 2.1. Let F be an exponential distribution with $\overline{F}(x) = \mathrm{e}^{-\alpha x}$, $x \geq 0$, and let $g(x) = \mathrm{e}^{\beta x}$, where $\beta < \alpha$. To construct ξ_1 and ξ_2 according to the Theorem 1.1, we may take

$$a_k = 2^{\sum_{j=1}^{k} \left(\frac{\alpha}{\beta}\right)^j}.$$

Example 2.2. Let F be a distribution with tail $\overline{F}(x) = \frac{1}{(1+x)^{\alpha}}$, $x \geq 0$, and let $g(x) = x^{\beta}$, where $\beta + 1 < \alpha$. Then we may also take

$$a_k = 2^{\sum_{j=1}^{k} \left(\frac{\alpha}{\beta}\right)^j}.$$

Note, however, that this construction is not always necessary. Let F be any distribution and define a new distribution $F^{\uparrow}$ by

$$\overline{F}^{\uparrow}(x) = \sqrt{\overline{F}(x)} = \mathrm{e}^{-R_F(x)/2}.$$

If ξ_1 and ξ_2 are independent random variables with distribution $F^{\uparrow}$, then $\min(\xi_1, \xi_2)$ will have distribution F. Thus, for the construction in Theorem 1.1 to be needed, ξ_1 and ξ_2 must be heavier-tailed than $F^{\uparrow}$.

In particular, in Example 2.1, if $\beta > \alpha/2$, we can simply take ξ_1 and ξ_2 to be exponentially distributed with $\overline{F}(x) = \mathrm{e}^{-\alpha x/2}$, and the minimum of those variables will still have the desired distribution F.

3. Archimedean Survival Copulas and Dependent Minima

In the preceding sections we focused on independent random variables. However, this assumption may be considered restrictive in applications. For example, in actuarial models, risks or lifetimes are often dependent, and copulas provide the standard way to model such dependence. A particularly convenient class is given by Archimedean survival copulas, which offer tractable families of dependence models widely used not only in survival analysis but

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

also in finance and in many other areas of applied probability (see, e.g., [8] for a systematic introduction, including Archimedean survival copulas in particular).

These copulas also fit naturally with our cumulative-hazard-based construction. If $(X_1, \ldots, X_n)$ has an Archimedean survival copula with generator ϕ (non-increasing, $\phi(1) = 0$), then for all x,

$$\phi\Big(\mathbb{P}\Big(\min_{i=1,\ldots,n} X_i > x\Big)\Big) = \sum_{i=1}^{n} \phi\big(\mathbb{P}(X_i > x)\big).$$

Thus, the same additive decomposition that underpins the independent case appears here as well, only expressed through the transformed function ϕ. This suggests that we can mimic our earlier scheme by requiring heavy-tailed marginals, yet still obtain a prescribed light-tailed distribution for the minimum under dependence. A full exploration of these possibilities lies beyond the scope of this paper, but the structural similarity indicates that our approach is not limited to independence and may extend naturally to certain dependent settings. In particular, the following result holds.

Theorem 3.1. *Let F be any distribution with right-unbounded support, and let g be any non-negative increasing function satisfying* $\lim_{x\to\infty} g(x) = \infty$.

Suppose h is a strictly increasing function that is continuous in a neighbourhood of 0, with $h(0) = 0$ and $h(1) = 1$. Then there exist random variables ξ_1 and ξ_2 such that

$$h\big(\overline{F}(x)\big) = h\big(\mathbb{P}(\xi_1 > x)\big)\, h\big(\mathbb{P}(\xi_2 > x)\big)$$

and

$$\mathbb{E}g(\xi_1) = \mathbb{E}g(\xi_2) = \infty.$$

Remark 3.1. We do not impose the usual generator conditions that ensure that $h^{-1}(h(u)h(v))$ is a genuine survival copula. If we also want existence of copula-based joint law, additional conditions on h (e.g. convexity of $-\log h$) are required and lie outside our scope.

The proof can be obtained by adapting the argument of Theorem 1.1. But instead of decomposing cumulative hazards, we decompose survival functions so that the copula identity is preserved. More precisely, on each interval $(a_l, a_{l+1}]$ we proceed as follows:

- For even l, hold $\overline{F}_{\xi_1}$ constant at the value $\overline{F}_{\xi_1}(a_l)$, and let $\overline{F}_{\xi_2}(x)$ vary so that
$$h\big(\overline{F}(x)\big) = h\big(\overline{F}_{\xi_1}(a_l)\big)\, h\big(\overline{F}_{\xi_2}(x)\big), \qquad x \in (a_l, a_{l+1}].$$
- For odd l, hold $\overline{F}_{\xi_2}$ constant at the value $\overline{F}_{\xi_2}(a_l)$, and let $\overline{F}_{\xi_1}(x)$ vary so that
$$h\big(\overline{F}(x)\big) = h\big(\overline{F}_{\xi_1}(x)\big)\, h\big(\overline{F}_{\xi_2}(a_l)\big), \qquad x \in (a_l, a_{l+1}].$$

Since h is continuous near 0, the procedure can always be carried out initially by selecting a_1 sufficiently large and putting $\overline{F}_{\xi_1}(x) = 1$ and $\overline{F}_{\xi_2}(x) = \overline{F}(x)$ for $x \in [0, a_1)$. The subsequent points $a_2, a_3, \ldots$ of the partition can then be chosen large enough to ensure that both ξ_1 and ξ_2 are heavy-tailed in the sense required by g.

4. Proof of Theorem 1.2

We again restrict our consideration to non-negative random variables.

Proof. Let $H_1, H_2, \ldots, H_n$ be the distribution functions of $\xi_1, \ldots, \xi_n$. We present them via their cumulative hazards, building on the cumulative hazard R_F of F. We partition $\mathbb{R}^+$ into

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

intervals $\{(a_i, a_{i+1}]\}_{i=0}^{\infty}$, and on each interval we keep the cumulative hazards corresponding to a selected set of $k-1$ indices constant. This ensures that the minimum of the associated random variables exhibits a heavy tail. By cyclically considering all possible subsets of $k-1$ indices, we guarantee that every such subset yields a heavy-tailed minimum. Simultaneously, on every interval, each of the remaining $n-(k-1)$ cumulative hazards is allowed to grow at the same rate as R_F, ensuring that in every interval there is at least one cumulative hazard that grows quickly enough.

Construction of intervals and distributions. Let $I_0, \dots, I_{M-1}$ be the collection of all possible subsets of $k-1$ indices from $\{1, \dots, n\}$, where $M = \binom{n}{k-1}$. Extend these subsets into an infinite sequence $\{I_j\}_{j=0}^{\infty}$ by repeating them cyclically; that is, identify I_i with I_j whenever $i \equiv j \pmod{M}$.

Next, we inductively define a sequence $\{a_i\}_{i=0}^{\infty}$ and cumulative hazards $R_1, \dots, R_n$. Set $a_0 = 0$ and define $R_i(x) = 0$ for all i and for all $x \leqslant 0$. Assuming the functions are defined up to a_l, choose a_{l+1} and define the functions on the interval $(a_l, a_{l+1}]$ as follows:

- For any $i \in I_l$, set $R_i(x) = R_i(a_l)$ for all $x \in (a_l, a_{l+1}]$.
- For any $i \notin I_l$, define

$$R_i(x) = R_i(a_l) + \big(R_F(x) - R_F(a_l)\big)$$

  for all $x \in (a_l, a_{l+1}]$.

This construction ensures that the functions $R_1, \dots, R_n$ remain right-continuous at a_l. The point a_{l+1} may be chosen arbitrarily provided that:

- $a_{l+1} - a_l \geqslant 1$, so that the intervals $(a_i, a_{i+1}]$ are guaranteed to cover $\mathbb{R}^+$.
- The inequality

$$(a_{l+1} - a_l)\ \exp\left\{-\sum_{i \in I_l} R_i(g^{-1}(a_l))\right\} \geqslant 1$$

  holds, which will guarantee that the expectation of the minimum for any set of $k-1$ indices is infinite.

Let $H_1, \dots, H_n$ be the distributions corresponding to the cumulative hazards $R_1, \dots, R_n$. We now verify that these distributions satisfy the stated properties.

We show now that *the minimum of k random variables is sufficiently light.* Consider any subset of indices $1 \leqslant i_1 < \cdots < i_k \leqslant n$ with corresponding independent random variables $\xi_{i_1}, \dots, \xi_{i_k}$. Let $R_{\min}$ denote the cumulative hazard of their minimum, $\min(\xi_{i_1}, \dots, \xi_{i_k})$. By the construction of $\{I_j\}_{j=0}^{\infty}$, on any interval $(a_l, a_{l+1}]$ there exists at least one index $j_l \in \{i_1, \dots, i_k\}$ with property $j_l \notin I_l$. Consequently, for any $x \in (a_l, a_{l+1}]$ we have

$$R_{\min}(x) - R_{\min}(a_l) = \sum_{j \in \{i_1, \dots, i_k\}} (R_j(x) - R_j(a_l)) \geqslant R_{j_l}(x) - R_{j_l}(a_l) = R_F(x) - R_F(a_l).$$

It follows that $R_{\min}(x) \geqslant R_F(x)$ for all $x \geqslant 0$, so F stochastically dominates the distribution of $\min(\xi_{i_1}, \dots, \xi_{i_k})$.

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

It is left to show that the minimum of $k-1$ random variables is sufficiently heavy. Consider any subset of indices $1 \leqslant i_1 < \cdots < i_{k-1} \leqslant n$, and let $\eta = \min(\xi_{i_1}, \ldots, \xi_{i_{k-1}})$. Then

$$\mathbb{E}g(\eta) = \int_0^\infty \mathbb{P}(g(\eta) > t)\,\mathrm{d}t = \sum_{j=0}^{\infty} \int_{a_j}^{a_{j+1}} \mathbb{P}(g(\eta) > t)\,\mathrm{d}t.$$

The chosen set of indices corresponds to I_m for some $0 \leqslant m < M$, and we can use the following lower bound:

$$\mathbb{E}g(\eta) \geqslant \sum_{\substack{j=0 \\ j \equiv m \,(\mathrm{mod}\, M)}}^{\infty} \int_{a_j}^{a_{j+1}} \mathbb{P}(g(\eta) > t)\,\mathrm{d}t.$$

Moreover, for each such interval $(a_j, a_{j+1}]$,

$$\int_{a_j}^{a_{j+1}} \mathbb{P}(g(\eta) > t)\,\mathrm{d}t = (a_{j+1} - a_j)\ \exp\left\{-\sum_{i \in I_j} R_i\big(g^{-1}(a_j)\big)\right\}.$$

By the choice of a_{j+1}, we ensure that

$$(a_{j+1} - a_j)\ \exp\left\{-\sum_{i \in I_j} R_i\big(g^{-1}(a_j)\big)\right\} \geqslant 1.$$

Therefore,

$$\mathbb{E}g(\eta) \geqslant \sum_{\substack{j=0 \\ j \equiv m \,(\mathrm{mod}\, M)}}^{\infty} 1 = \infty.$$

□

Acknowledgements

The authors are grateful to the anonymous referees for their constructive comments and suggestions, which helped to improve the clarity and scope of this paper. In particular, we thank them for encouraging us to extend the discussion to dependent settings via Archimedean survival copulas and for their helpful remarks on presentation, references, and notation.

Funding information

The study of AT was supported by the Program for Fundamental Scientific Research of the Siberian Branch of the Russian Academy of Sciences, project FWNF-2026-0030.

Competing interests

There are no competing interests to declare that arose during the preparation of or the publication process for this article.

References

[1] LEIPUS, R., SIAULYS, J. AND KONSTANTINIDES, D. (2023). Minimum of heavy-tailed random variables is not heavy tailed. *AIMS Mathematics* **8** (6), 13066–13072.

[2] EMBRECHTS, P., KLÜPPELBERG, C. AND MIKOSCH, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press

[3] FOSS, S., KORSHUNOV, D. AND ZACHARY, S. (2013). *An Introduction to Heavy-Tailed and Subexponential Distributions*, 2nd edn. Springer, New York.
[4] LEIPUS, R., SIAULYS, J. AND KONSTANTINIDES, D. (2023). *Closure Properties for Heavy-Tailed and Related Distributions*, Springer.
[5] STOYAN, D. (1983). *Comparison Methods for Queues and Other Stochastic Models*. Wiley, Chichester.
[6] MATSUI, M. AND WATANABE, T. (2025). Convolution closure properties of subexponential densities. *Journal of Mathematical Analysis and Applications*, **545** (1), 129–158.
[7] XU, H., FOSS, S. AND WANG, Y. (2015) Convolution and convolution-root properties of long-tailed distributions. *Extremes*, 18(4), 605–628.
[8] NELSEN, R. B. (2006). *An Introduction to Copulas*, 2nd edn. Springer, New York.

https://doi.org/10.1017/jpr.2025.10058 Published online by Cambridge University Press