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On normal numbers for continued fractions

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Abstract. In this paper we will show that any number normal with respect to the *regular continued fraction* (RCF) is also normal with respect to the *continued fraction from below* (BCF) and the *nearest integer continued fraction* (NICF). Furthermore, we will show that in case of the NICF a number is normal with respect to the RCF-expansion if and only if it is NICF-normal. This can be generalized to a larger class of semi-regular continued fraction expansions.

1. *Introduction*

A well-known result concerning the expansion of numbers x in the unit interval in any integer base b (with $b \geq 2$) is that almost all numbers x are normal to base b. That is, any finite sequence $a_1, \ldots, a_k$ with $a_i \in \{0, 1, \ldots, b-1\}$ occurs in the b-adic expansion of x in the right frequency, viz. b^{-k}. A proof of this follows easily from applying the ergodic theorem to the transformation $Sx = bx \bmod 1$ on $[0, 1)$.

Given an integer base b, it is already very difficult—if not impossible by the present methods—to determine whether a number is normal in that base or not. It is not known whether π, e or any irrational algebraic number (like $\sqrt{2}$) is normal to any base. Furthermore, no numbers are known to be normal to two bases which are relatively prime. For example, no number is known which is normal to both base 2 and 3. Of course, from the above we know that almost every number (in the sense of Lebesgue) is normal to both base 2 and 3.

Another, and in a way a more 'geometric' and 'base-free', expansion of numbers x in the unit interval is the continued fraction expansion

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \quad a_i \in \mathbb{N},\ n \geq 1. \tag{1}$$

We denote (1) by $x = [0; 1/a_1, 1/a_2, \ldots]$.

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Again it follows from the ergodic theorem that almost all numbers are normal (now for the continued fraction expansion), and in [**AKS**] a Champernowne-type construction is given to obtain a normal number for the continued fraction expansion, see also [**P**] for a different construction.

Every irrational real number x has a unique continued fraction, which is of the form (1) in the case $x \in [0, 1)$. This expansion is called the *regular continued fraction expansion* (RCF) of x. Apart from the RCF there are very many other continued fraction expansions of x. Here we mention:

(i) the *continued fraction from below* (BCF)

$$x = \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ddots}}, \quad b_i \geq 2,\ n \geq 1;$$

(ii) the *nearest integer continued fraction* (NICF)

$$x = \cfrac{\varepsilon_1}{c_1 + \cfrac{\varepsilon_2}{c_2 + \ddots}}, \quad c_i \geq 2, \varepsilon_i \in \{-1, 1\},\ \varepsilon_{i+1} + c_i \geq 2,\ i \geq 1.$$

The BCF has an underlying dynamical system which is ergodic, and which has a σ-finite measure, see also [**I**]. The NICF is an example of a so-called α-expansion [**N**], which itself is contained in the much larger class of S-expansions [**K**]. It has an underlying dynamical system which is ergodic (even weakly Bernoulli), and which has a finite invariant measure. Metrical properties have been studied by Rieger [**Rie**], Rockett [**Roc**] and by the second author [**N**].

As in the case of the RCF one can show that for almost all x the BCF/NICF/S-expansion of x is normal with respect to the continued fraction under consideration (clearly the definition of normality should be different in the case of the BCF from the other mentioned cases, see also §2). In particular, one has that the Adler–Keane–Smorodinsky number

$$x = [0; 1/2, 1/3, 1/1, 1/2, 1/4, 1/2, 1/1, 1/3, 1/5, \ldots] = 0.440\,3388\ldots,$$

obtained by concatenating the (finite) RCF expansions of the rationals

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{7}, \tfrac{2}{7}, \ldots,$$

is normal with respect to the RCF, BCF and NICF expansions.

In this paper we will show that any number normal with respect to the RCF is also normal with respect to the BCF and the NICF. In fact, we will show that in the case of the NICF a number is normal with respect to the RCF expansion (R-normal) if and only if it is NICF-normal. The latter can be generalized to all S-expansions, but we leave this to the reader.

Our method is based on a construction of the natural extension of a continued fraction transformation. For example, the NICF transformation is obtained from the RCF transformation as an induced transformation of the natural extension of the RCF transformation with bounded return time. This fact, essentially, implies the equivalence of the normalities of the RCF and the NICF.

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2. *Normal numbers for the continued fraction from below*

In this section we consider a relation between the regular continued fraction (RCF) and the *continued fraction from below* (BCF), see [**I**] and [**Z**] (in particular, *Aufgabe 3* in [**Z**, p. 131]). We start with some definitions.

For $x \in [0, 1) \setminus \mathbb{Q}$, we define maps F_+ and F_- as follows:

$$\begin{cases} F_+(x) = \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor, \\ F_-(x) = \left\lceil \dfrac{1}{x} \right\rceil - \dfrac{1}{x}. \end{cases}$$

Here $\lfloor \xi \rfloor$ denotes the *floor* (or *entier*) value of ξ and $\lceil \xi \rceil$ the *ceiling* of ξ, i.e. $\lfloor \xi \rfloor$ is the greatest integer not exceeding ξ and $\lceil \xi \rceil$ is $\lfloor \xi \rfloor + 1$ in case $\xi \notin \mathbb{Z}$ (and equals ξ otherwise). In the following we will always assume that x is irrational.

If $x \in (1/a, 1/(a-1))$, $a \geq 2$, then

$$F_+(x) = \frac{-(a-1)x + 1}{x} \quad \text{and} \quad F_-(x) = \frac{ax - 1}{x},$$

and these maps correspond to the matrices

$$\begin{pmatrix} -(a-1) & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix},$$

respectively, as linear fractional transformations. It is well known that both F_+ and F_- have absolutely continuous invariant measures, given by

$$d\mu_+ = \frac{1}{\log 2} \frac{dx}{1+x} \quad \text{and} \quad d\mu_- = \frac{dx}{1-x},$$

respectively. Notice that the latter measure is σ-finite, but infinite.

Setting

$$\begin{cases} a_n(x) = \left\lfloor \dfrac{1}{F_+^{n-1}(x)} \right\rfloor, \\ b_n(x) = \left\lceil \dfrac{1}{F_-^{n-1}(x)} \right\rceil, \end{cases}$$

for $n \geq 1$, then the RCF and BCF expansions of x are given by

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} = [0; 1/a_1, 1/a_2, 1/a_3, \dots]$$

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and

$$x = \cfrac{1}{b_1(x) - \cfrac{1}{b_2(x) - \cfrac{1}{b_3(x) - \ddots}}} = [0; 1/b_1, -1/b_2, -1/b_3, \dots],$$

respectively.

For any given sequence $(a_1, a_2, \dots)$ of positive integers there exists a unique irrational number x in the unit interval, such that $a_1(x) = a_1, a_2(x) = a_2, \dots$. In the case of the BCF, for any given sequence $(b_1, b_2, \dots)$ of positive integers greater than 1 such that $b_n \geq 3$ for infinitely many n, there exists a unique irrational number x in the unit interval, such that $b_1(x) = b_1, b_2(x) = b_2, \dots$.

We denote *fundamental intervals* (or *cylinder sets*) by

$$\langle a_1, a_2, \dots, a_n\rangle_+ = \{x \in [0, 1) : a_1(x) = a_1, \dots, a_n(x) = a_n\},$$
$$\langle b_1, b_2, \dots, b_n\rangle_- = \{x \in [0, 1) : b_1(x) = b_1, \dots, b_n(x) = b_n\},$$

for any sequences $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$. We note that

$$\mu_-(\langle \underbrace{2, 2, \dots, 2}_{k}\rangle_-) = \infty,$$

for any $k \geq 1$.

Now we give the definition of a normal number associated with the RCF and BCF expansions.

Definition 1. An irrational number $x \in [0, 1)$ is said to be a normal number associated with the RCF expansion ('an R-normal number') if

$$\lim_{N\to\infty} \frac{1}{N}\#\{n : 0 \leq n \leq N - 1, a_{n+1}(x) = a_1, \dots, a_{n+k}(x) = a_k\}$$

exists and equals $\mu_+(\langle a_1, \dots, a_k\rangle_+)$, for any $(a_1, \dots, a_k)$.

Definition 2. An irrational number $x \in [0, 1)$ is said to be a normal number associated with the continued fraction expansion from below ('a BCF-normal number') if

$$\lim_{N\to\infty} \frac{\#\{n : 0 \leq n \leq N - 1, b_{n+1}(x) = b_1, \dots, b_{n+k}(x) = b_k\}}{\#\{n : 0 \leq n \leq N - 1, b_{n+1}(x) = c_1, \dots, c_{n+\ell}(x) = c_\ell\}} = \frac{\mu_-(\langle b_1, \dots, b_k\rangle_-)}{\mu_-(\langle c_1, \dots, c_\ell\rangle_-)},$$

for any $\langle b_1, \dots, b_k\rangle_-$ and $\langle c_1, \dots, c_\ell\rangle_-$ for which $(b_1, \dots, b_k) \neq (2, \dots, 2)$ and $(c_1, \dots, c_\ell) \neq (2, \dots, 2)$.

To discuss a relation between these notions of normality, we consider a relation between the maps F_+ and F_-. For this purpose, we define a jump transformation F_j of F_- as follows (see Schweiger [**S2**], for the general definition of jump transformations):

$$F_j(x) = F_-^{n(x)+1}(x), \quad x \in [0, 1) \setminus \mathbb{Q},$$

with

$$n(x) = \min\{n \geq 0 : F_-^n(x) \in [0, \tfrac{1}{2})\}.$$

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F_j has an absolutely continuous invariant measure, which is finite. In order to find this measure we consider a natural extension map $\hat{F}_-$ of F_-, defined on $[0, 1) \times [1, \infty)$:

$$\hat{F}_-(x, y) = \left(-\frac{1}{x} + b, -\frac{1}{y} + b\right) \quad \text{if } x \in \left[\frac{1}{b}, \frac{1}{b-1}\right).$$

The map $\hat{F}_-$ is one-to-one and onto (a.e.), and an absolutely continuous invariant measure $\hat{\mu}_-$ for $\hat{F}_-$ is given by

$$d\hat{\mu}_- = \frac{dx\,dy}{(x-y)^2}.$$

Then $[\frac{1}{2}, 1) \times [1, \infty)$ is mapped to $[0, 1) \times [1, 2)$ by $\hat{F}_-$. Thus it turns out that the induced transformation of $\hat{F}_-$ to $A = [0, 1) \times [2, \infty)$ is a natural extension of F_j. Clearly $d\hat{\mu}_-|_A$ is an invariant measure for $\hat{F}_-|_A$ and we, moreover, have that its marginal distribution to the first coordinate is an absolutely continuous invariant measure, which is finite. Indeed, the marginal density is given by

$$\int_2^\infty \frac{dy}{(x-y)^2} = \frac{1}{2-x}.$$

We denote it by $d\mu_j$:

$$d\mu_j = \frac{1}{\log 2}\frac{dx}{2-x},$$

where $1/\log 2$ is a normalizing constant.

On the level of the continued fractions it is easy to understand the relation between the RCF and the BCF. To this end, we define two operations on the partial quotients (i.e. digits) of a (semi-regular) continued fraction expansion, see also [**K**, §§2 and 4].

(i) *'Plug-in' of* $\varepsilon = -1$, $d_2 = 1$. Let d_1, d_3 be two positive integers and let $\xi \in [0, 1)$. Suppose, moreover, that $d_3 \geq 2$, then

$$d_1 + \frac{1}{d_3 + \xi} = d_1 + 1 + \frac{-1}{1 + \dfrac{1}{d_3 - 1 + \xi}}.$$

(ii) *'Singularization' of* $\varepsilon = 1$, $d_2 = +1$. Let d_1, d_3 be two positive integers and let $\xi \in [0, 1)$, then

$$d_1 + \frac{1}{1 + \dfrac{1}{d_3 + \xi}} = d_1 + 1 + \frac{-1}{d_3 + 1 + \xi}.$$

Given the RCF expansion of any irrational number x, say

$$x = [0; 1/a_1, \ldots, 1/a_n, \ldots],$$

the BCF expansion of $x = [0; 1/d_1, -1/d_2, \ldots, -1/d_n, \ldots]$ can be obtained from the RCF expansion of x in the following way.

(a) Let $k = \min\{n \geq 2 : \varepsilon_n(x) = +1\}$, where $[d_0; \varepsilon_1/d_1, \ldots, \varepsilon_n/d_n, \ldots]$ is some (semi-regular) continued fraction expansion of x. Then apply the *plug-in* operation exactly before the kth digit in the case when this digit is larger than 1. In the case when the kth digit equals 1, apply the *singularization* operation to the kth digit.

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In either way we will find a new (semi-regular) continued fraction expansion of x.

(b) Repeat (a) to this new (semi-regular) continued fraction expansion of x.

Example. Let $x = [0; 1/2, 1/3, 1/7, \dots]$. To apply the above algorithm, we see that $k = 2$ and that we have to *plug-in* just before the second partial quotient $a_2 = 3$. Doing so yields the following continued fraction expansion of x:

$$x = [0; 1/3, -1/1, 1/2, 1/7, \dots],$$

and we see that for this new continued fraction expansion of x one has $k = 3$. Again we must use the *plug-in* operation, which yields

$$x = [0; 1/3, -1/2, -1/1, 1/1, 1/7, \dots].$$

For the latter continued fraction of x one sees that $k = 4$; however, since the fourth digit of this continued fraction expansion of x equals 1 we must singularize this fourth digit, and we find

$$x = [0; 1/3, -1/2, -1/2, -1/8, \dots],$$

etc. Since the BCF expansion of any irrational number x is unique, we thus find the BCF expansion of x from the RCF expansion of x. Clearly a 'converse' algorithm also exists.

The above described algorithm can also be understood in terms of matrix multiplications, see also [**K**, §2]. We have the following proposition.

PROPOSITION 1. *The dynamical systems (F_+^2, μ_+) and (F_j, μ_j) are conjugate to each other by the conjugacy $\phi : x \mapsto 1 - x$ for $x \in (0, 1)$.*

Proof. It is easy to see that μ_+ is mapped to μ_j by $x \mapsto 1 - x$. We will show that $F_+^2 \circ \phi = \phi \circ F_j$. The following is easy.

LEMMA 1. *For $a \geq 3$ and $k \geq 0$ we have*

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}}_{k \text{ times}} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -(a-2) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -(k+1) & 1 \\ 1 & 0 \end{pmatrix}.$$

Suppose $x = [0; 1/b_1, -1/b_2, -1/b_3, \dots]$, then x can be represented by an infinite matrix product as follows:

$$x = \begin{pmatrix} 0 & 1 \\ -1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & b_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & b_3 \end{pmatrix} \cdots.$$

Choose $(n_1, n_2, \dots)$ so that

$$b_1 = 2, b_2 = 2, \dots, b_{n_1-1} = 2, b_{n_1} \neq 2 \ (n_1 = 1 \text{ is possible}),$$
$$b_{n_1+1} = 2, b_{n_1+2} = 2, \dots, b_{n_1+n_2-1} = 2, b_{n_1+n_2} \neq 2.$$

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Then we have from the above lemma

$$\begin{aligned} x &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ -1 & b_{n_1} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\cdots \\ &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & n_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_{n_1}-2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & n_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_{n_1+n_2}-2 \end{pmatrix}\cdots \\ &= 1-\hat{x}, \end{aligned}$$

where

$$\hat{x} = [0; 1/n_1, 1/(b_{n_1}-2), 1/n_2, 1/(b_{n_1+n_2}-2), \dots].$$

From the definition of F_j we see

$$F_j(x) = F_-^{n_1}(x) = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}}_{n_2-1 \text{ times}}\begin{pmatrix} 0 & 1 \\ -1 & b_{n_1+n_2} \end{pmatrix}\cdots.$$

On the other hand,

$$F_+^2(\hat{x}) = [0; 1/n_2, 1/b_{n_1+n_2}-2, \dots],$$

thus we have

$$F_+^2 \circ \phi = \phi \circ F_j,$$

which completes the proof of the proposition. □

Definition 3. An irrational number $x \in [0, 1)$ is said to be a jump BCF-normal number if

$$\lim_{N\to\infty} \frac{\#\{n : 0 \le n \le N-1, F_j^n(x) \in I\}}{N} = \mu_j(I),$$

for any cylinder set $I = \langle b_1, \dots, b_k\rangle_-$.

Remark. Since μ_j is a finite measure, it follows that the above equation holds for all cylinder sets if it holds for all cylinders sets I of the form $I = \langle b_1, \dots, b_k\rangle_- \neq \langle 2, 2, \dots, 2\rangle_-$.

PROPOSITION 2. *An irrational number* $\in [0, 1)$ *is jump BCF-normal if and only if* $1-x$ *is* F_+^2*-normal, that is, if*

$$\begin{aligned} &\lim_{N\to\infty} \frac{\#\{n : 0 \le n \le N-1, a_{2n+1}(1-x) = a_1, \dots, a_{2n+k}(1-x) = a_k\}}{N} \\ &\quad = \mu_+(\langle a_1, a_2, \dots, a_k\rangle_+) \end{aligned}$$

for all cylinder sets $\langle a_1, a_2, \dots, a_k\rangle_+$.

Proof. When $1-x$ is F_+^2-normal, it follows from the approximation method that

$$\lim_{N\to\infty} \frac{\#\{n : 0 \le n \le N-1, F_+^{2n}(1-x) \in I\}}{N} \le \mu_+(I),$$

for any interval I in $[0, 1)$, which implies that

$$\lim_{N\to\infty} \frac{\#\{n : 0 \le n \le N-1, F_+^{2n}(1-x) \in I\}}{N} = \mu_+(I),$$

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for any interval I in $[0,1)$. Also we see

$$\lim_{N\to\infty}\frac{\#\{n : 0\le n\le N-1, F_+^{2n}(1-x)\in I\}}{N}=\mu_j(I),$$

for any interval I in $[0,1)$ when x is jump BCF-normal. Then the assertion of Proposition 2 is an easy consequence of Proposition 1. □

Next we claim the equivalence between R-normality and F_+^2-normality.

THEOREM 1. *An irrational number x is R-normal if and only if x is F_+^2-normal.*

The proof of this theorem is essentially the same with the case of 2-adic and 4-adic normality. However, the number of digits are infinite in the case of continued fractions and the assertion of the theorem is a special case of a result by F. Schweiger, see also [**S1**].

COROLLARY 1. *An irrational number $x\in[0,1)$ is R-normal if and only if it is jump BCF-normal.*

Proof. Suppose $0<x<\frac{1}{2}$, then it is easy to see that $a_1(x)\neq 1$, and if $x=[0;1/a_1,1/a_2,\dots]$, one has

$$1-x=[0;1/1,1/(a_1-1),1/a_2,\dots].$$

Thus, the R-normality of x is equivalent to that of $1-x$ for any x. The assertion of this corollary now follows from Theorem 1 and Proposition 2. □

THEOREM 2. *If an irrational number x is jump normal, then it is BCF-normal.*

Proof. Let x be a jump normal number, and consider the following two cylinder sets $\langle b_1,\dots,b_k\rangle_-$ and $\langle c_1,\dots,c_\ell\rangle_-$ (both $\neq\langle 2,2,\dots,2\rangle_-$). Since there exist s and t such that

$$\begin{cases}1\le s\le k \quad\text{and}\quad 1\le t\le \ell,\\ b_1=\cdots=b_{s-1}=2, b_s\ge 3 \quad\text{and}\quad c_1=\cdots=c_{t-1}=2, c_s\ge 3,\end{cases}$$

we see

$$\begin{aligned}&\lim_{N\to\infty}\frac{\#\{n : 0\le n\le N-1, b_{n+1}(x)=b_1,\dots,b_{n+k}(x)=b_k\}}{\#\{n : 0\le n\le N-1, b_{n+1}(x)=c_1,\dots,b_{n+\ell}(x)=c_\ell\}}\\ &\quad=\lim_{N\to\infty}\frac{\#\{n : 1\le n\le N, b_{n-s}(x)=b_1,\dots,b_{n-s+k-1}(x)=b_k\}}{\#\{n : 0\le n\le N-1, b_{n-t}(x)=c_1,\dots,b_{n-t+\ell-1}(x)=c_\ell\}}\\ &\quad=\lim_{N\to\infty}\frac{\#\{n : 1\le n\le N, b_{n-s}(x)=b_1,\dots,b_{n-s+k-1}(x)=b_k\}/J_n}{\#\{n : 0\le n\le N-1, b_{n-t}(x)=c_1,\dots,b_{n-t+\ell-1}(x)=c_\ell\}/J_n},\end{aligned}\tag{$*$}$$

where $J_n=\#\{n : 1\le n\le N, b_n\ge 3\}$, if the limit exists. From the assumption that x is BCF-normal it follows that $J_n\to\infty$ as $n\to\infty$. Now let $m\ge 1$ be the smallest positive integer for which $b_m(x)\ge 3$, and put

$$y=b_m(x)-\cfrac{1}{b_{m-1}-\cfrac{1}{b_{m-2}-\ddots-\cfrac{1}{b_1}}}.$$

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Then it turns out that

$$\hat{F}_-^m(x,\infty) = (F_-^m(x), y) =: (z, y) \quad \text{and} \quad y \geq 2.$$

So $y \in A$, which is the domain of the natural extension of F_j. If $b_n(x) \geq 3$, then $\hat{F}_-^n(x,\infty) = \hat{F}_j^p(z, y)$ for some p. It is easy to see that

$$b_{n-s}(x) = b_1, \ldots, b_{n-s+k-1}(x) = b_k$$

occurs if and only if

$$\hat{F}_j^p(z, y) \in \langle b_1, \ldots, b_k\rangle_- \times I$$

with

$$I = \begin{cases} ([-b_1; \underbrace{-1/2, -1/2, \ldots, -1/2}_{m-1}], [b_1; \underbrace{-1/2, -1/2, \ldots, -1/2}_{m-2}, -1/3]), & \text{if } m \geq 2, \\ (-(b_1 - 1), -b_1], & \text{if } m = 1. \end{cases}$$

Thus, the numerator of $(*)$ is

$$\#\{p : 0 \leq p \leq P, \hat{F}_j^p(z, y) \in \langle b_1, \ldots, b_k\rangle_- \times I\}$$

for some P, where P depends on N and x. It is easy to see that the set $\hat{F}_j^{-1}(\langle b_1, \ldots, b_k\rangle_- \times I)$ is a countable disjoint union of the form

$$\langle \underbrace{2, 2, \ldots, 2}_{\geq 0}, b_1, \ldots, b_k\rangle_- \times [2, \infty).$$

It follows from the approximation method with the jump-normality of x, that

$$\frac{\#\{p : 0 \leq p \leq P, \hat{F}_j^p(z, y) \in \langle b_1, \ldots, b_k\rangle_- \times I\}}{P} \to \mu_j(\cup\langle 2, 2, \ldots, 2, b_1, \ldots, b_k\rangle_-) = \frac{1}{\log 2}\hat{\mu}_-(\langle b_1, \ldots, b_k\rangle_- \times I),$$

as $P \to \infty$. Since F_j preserves μ_j and $\hat{F}_j$ preserves $d\hat{\mu}_-|_A$, the right-hand expression equals

$$\frac{1}{\log 2}\hat{\mu}_-(\hat{F}_-^{-m}\langle b_1, \ldots, b_k\rangle_- \times I) = \frac{1}{\log 2}\mu_-(\langle b_1, \ldots, b_k\rangle_-).$$

In the same manner one finds that the denominator of $(*)$ divided by P tends to $(\log 2)^{-1}\mu_-(\langle c_1, \ldots, c_\ell\rangle_-)$ as $P \to \infty$. Consequently, we have

$$\lim_{N\to\infty} \frac{\#\{n : 0 \leq n \leq N-1, b_{n+1}(x) = b_1, \ldots, b_{n+k}(c) = b_k\}}{\#\{n : 0 \leq n \leq N-1, b_{n+1}(x) = c_1, \ldots, b_{n+\ell}(c) = c_\ell\}} = \frac{\mu_-(\langle b_1, \ldots, b_k\rangle_-)}{\mu_-(\langle c_1, \ldots, c_\ell\rangle_-)}.$$

This proves the assertion of this theorem. □

The following example shows that the converse of Theorem 2 does not hold in general.

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Example. Let

$$x = [0; 1/a_1, 1/a_2, \dots]$$

be R-normal. Suppose that $(\varepsilon_n)_{n\geq 1}$ monotonically decreases to 0, i.e.

$$\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n \to 0.$$

For any sequence $(d_1, \dots, d_k)$ we have

$$\frac{\#\{n : 0 \leq n \leq N-1, a_{n+1} = d_1, \dots, a_{n+k} = d_k\}}{N} \to \mu_+(\langle d_1, \dots, d_k\rangle_+).$$

We put $k = n_1 = 2$ and choose $d_1, \dots, d_k \leq 2$, $1 \leq i \leq 2$. Then there exists a positive integer n_2^* such that

$$\left|\frac{\#\{n : 0 \leq n \leq N-1, a_{n+1} = d_1, a_{n+2} = d_2\}}{N} - \mu_+(\langle d_1, d_2\rangle_+)\right| < \varepsilon_1,$$

for any $N \geq n_2^*$ and any d_1, d_2 with $d_i \in \{1, 2\}$, $i = 1, 2$.

Now define $n_2 = (n_1 + n_2^*) \vee (n_1 + 10^{n_1})$. Next we find n_3^* for which

$$\left|\frac{\#\{n : 0 \leq n \leq N-1, a_{n+1} = d_1, \dots, a_{n+n_2} = d_{n_2}\}}{N} - \mu_+(\langle d_1, \dots, d_{n_2}\rangle_+)\right| < \varepsilon_2,$$

for any $N \geq n_3^*$, and any $d_1, \dots, d_{n_2}$, with $1 \leq d_i \leq n_2$, $1 \leq i \leq n_2$. Now define $n_3 = (n_2 + n_3^*) \vee (n_2 + 10^{n_2})$. In this way we can define a subsequence of positive integers $n_1, n_2, \dots$ inductively (we can choose them all as even integers). Finally, we put

$$\tilde{x} = [0; 1/a_1, 1/a_2, \underbrace{1/n_1, \dots, 1/n_1}_{n_1 \text{ times}}, 1/a_3, \dots, 1/a_{n_2}, \underbrace{1/n_2, \dots, 1/n_2}_{n_2 \text{ times}}, 1/a_{n_2+1}, \dots].$$

Then it is possible to show that $\tilde{x}$ is BCF-normal, but not R-normal. Indeed, it is clear that $\tilde{x}$ is not normal. The proof of BCF-normality is also easy. For fixed $\langle b_1, \dots, b_k\rangle_-$ and $\langle c_1, \dots, c_\ell\rangle_-$ (both $\neq \langle 2, 2, \dots, 2\rangle_-$), the inserted sequences $n_i, \dots, n_i$ do not have any effect on

$$\lim_{N\to\infty} \frac{\#\{n : 0 \leq n \leq N-1, b_{n+1}(x) = b_1, \dots, b_{n+k}(x) = b_k\}}{\#\{n : 0 \leq n \leq N-1, b_{n+1}(x) = c_1, \dots, b_{n+\ell}(x) = c_\ell\}}$$

if i is sufficiently large. So the R-normality of x shows the BCF-normality of $\tilde{x}$.

Remark. The fact that jump-normality implies BCF-normality but not conversely has a counterpart for g-adic expansions, see also [**J**]. Let g be a positive integer, $g \geq 2$, and suppose $x \in [0, 1)$ has a g-adic expansion

$$x = .a_1 a_2 \dots a_n \dots, \quad a_i \in \{0, 1, \dots, g-1\},\ i \geq 1,$$

where we demand that this representation is not of the form $.a_1 a_2 \dots a_n \overline{g-1}$. Furthermore, let $b \in \{0, 1, \dots, g-1\}$, then we define the jump-transformation $T_b : [0, 1) \to [0, 1)$ by

$$T_b(x) := \begin{cases} .a_{m+1}a_{m+2}\dots, & \text{if } a_m = b, a_i \neq b \text{ for } 1 \leq i \leq m-1, \\ 0, & \text{if } a_i \neq b \text{ for } i \geq 1. \end{cases}$$

In [**J**] the following result by Schweiger (for the special case $g = 10$) is proved.

THEOREM. (F. Schweiger) *A necessary and sufficient condition for the number x to be normal to base g is the uniform distribution of the g sequences*

$$\{T_b(x)\}_{n\geq 1}, \quad b = 0, 1, \dots, g-1.$$

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3. *NICF normal numbers*

The *nearest integer continued fraction* (NICF) was introduced by Hurwitz and Minnegerode over a century ago. For $x \in [-\frac{1}{2}, \frac{1}{2}) \setminus \mathbb{Q}$ we define the map $F_{1/2}$ as follows:

$$F_{1/2}(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + \frac{1}{2} \right\rfloor,$$

and these maps correspond to the matrices

$$\begin{pmatrix} -(a-1) & \pm 1 \\ 1 & 0 \end{pmatrix}.$$

We define

$$\varepsilon_i(x) = \operatorname{sgn}(x), \quad c_i = \left\lfloor \left| \frac{1}{F_{1/2}^{i-1}} \right| + \frac{1}{2} \right\rfloor,$$

for $i \geq 1$ and denote by $\langle \varepsilon_1 c_1, \ldots, \varepsilon_k c_k \rangle$ the cylinder set given by $(\varepsilon_1, \ldots, \varepsilon_k)$ and $(c_1, \ldots, c_k)$. Moreover, if this set has positive Lebesgue measure, then we call the sequence $(\varepsilon_1 c_1, \ldots, \varepsilon_k c_k)$ an admissible sequence and $\langle \varepsilon_1 c_1, \ldots, \varepsilon_k c_k \rangle$ an admissible cylinder.

In [**K**] it was shown that the NICF can be obtained from the RCF using the following algorithm.

Algorithm. Let $x = [a_0; a_1, a_2, \ldots, a_n, \ldots]$ be the RCF expansion of x. Then singularize in each block of m partial quotients $a_{n+1} = 1, a_{n+2}, \ldots, a_{n+m} = 1$, with $a_n \neq 1$ (or $n = 0$ and $a_{n+m+1} \neq 1$ (or $m = \infty$) the first, third, ... partial quotient.

This algorithm can be reformulated in terms of ergodic theory, using induced transformations and the natural extension of the RCF transformation. The key concept in this reformulation is that of a 'singularization area' S; one induces the natural transformation map $\hat{F}$ of the RCF transformation on S^c.

In this paper we will use a natural extension RCF transformation which is an isomorphic copy of the natural extension of the RCF transformation as used in [**N**] and [**K**]. For our present purposes this natural extension is easier to work with. Of course, all the results obtained in [**N**] and [**K**] can easily be reformulated in terms of this isomorphic copy; we leave this to the reader. Put $X_1 = [0, 1) \times [-\infty, -1)$ and define a map $\hat{F}_+(x, y)$ on X_1 by

$$\hat{F}_+(x, y) = \left(\frac{1}{x} - a, \frac{1}{y} - a \right), \quad \text{for } x \in \left[\frac{1}{a+1}, \frac{1}{a} \right).$$

As in the previous section, $\hat{F}_+$ is one-to-one, onto (a.s.) and has an absolutely continuous invariant measure $\hat{\mu}_+$, given by

$$d\hat{\mu}_+ = \frac{1}{\log 2} \frac{dx\, dy}{(x-y)^2}.$$

On the other hand, we put

$$X_{1/2} = [-\tfrac{1}{2}, 0) \times [-\infty, -(G+1)) \cup [0, \tfrac{1}{2}) \times [-\infty, -G),$$

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where $G = (\sqrt{5}+1)/2$ and we define $\hat{F}_{1/2}(x, y)$ on $X_{1/2}$ by

$$\hat{F}_{1/2}(x, y) = \left(\frac{\varepsilon(x)}{x} - c, \frac{\varepsilon(x)}{y} - c \right),$$

where $\varepsilon(x) = \operatorname{sgn}(x)$ and $c = \lfloor |\frac{1}{x}| + \frac{1}{2} \rfloor$. One easily sees that $\hat{F}_{1/2}$ is one-to-one and onto (a.s.) on $X_{1/2}$, and that $\hat{F}_{1/2}$ has an absolutely continuous invariant measure $\hat{\mu}_{1/2}$ on $X_{1/2}$, given by

$$d\hat{\mu}_{1/2} = \frac{1}{\log G} \frac{dx\, dy}{(x-y)^2}.$$

The systems $(X_1, \hat{\mu}_+)$, respectively $(X_{1/2}, \hat{\mu}_{1/2})$, are natural extensions of the RCF, respectively the NICF.

Let $A = [0, 1) \times [-\infty, -G)$, and consider the induced transformation of $\hat{F}_+$ on A. Then we see that $\hat{F}_+|_A$ is isomorphic to $\hat{F}_{1/2}$ by the translation $(x, y) \mapsto (x-1, y-1)$ for $\frac{1}{2} \le x < 1$ and identity otherwise. Moreover, it is easy to see that $\hat{\mu}_+|_A$ is mapped to $\hat{\mu}_{1/2}$ by this isomorphism. We also note that $[\frac{1}{2}, 1) \times [-\infty, -G)$ is the singularization area in the sense of [**K**]. Now we define normality with respect to the NICF.

Definition 4. An irrational number $x \in [-\frac{1}{2}, \frac{1}{2})$ is said to be NICF-normal if

$$\lim_{N\to\infty} \frac{1}{N} \#\{0 \le n \le N-1 : \varepsilon_{n+1}(x) = \varepsilon_1, c_{n+1}(x) = c_1, \ldots, \varepsilon_{n+k}(x) = \varepsilon_k, c_{n+k}(x) = c_k\}$$

equals

$$\hat{\mu}_{1/2}\langle \epsilon_1 c_1, \ldots \epsilon_k c_k \rangle,$$

for every admissible sequence $(\varepsilon_1 c_1, \ldots, \varepsilon_k c_k)$, where $\mu_{1/2}$ is the projection of $\hat{\mu}_{1/2}$ on the first coordinate.

The main theorem of this section is the following theorem.

THEOREM 3. *Every irrational number $x \in [-\frac{1}{2}, \frac{1}{2})$ is NICF-normal if and only if x (or $x+1$) is R-normal.*

For the proof of Theorem 3 we need a definition and some elementary lemmas.

Definition 5. We say that $(x, y) \in X_1$ (or $\in A$, respectively) is $\hat{F}_+$-normal (or A-normal, respectively), if

$$\lim_{N\to\infty} \frac{1}{N} \#\{0 \le n \le N-1 : \hat{F}_+^n(x, y) \in \langle a_{-k}, \ldots, a_\ell \rangle|_{-k}^{\ell}\} = \hat{\mu}_+(\langle a_{-k}, \ldots, a_\ell \rangle|_{-k}^{\ell})$$

for any cylinder set with $k, \ell \ge 0$, or

$$\lim_{N\to\infty} \frac{1}{N} \#\{0 \le n \le N-1 : \hat{F}_+^n(x, y) \in \langle a_{-k}, \ldots, a_\ell \rangle|_{-k}^{\ell} \cap A\} = \hat{\mu}_+(\langle a_{-k}, \ldots, a_\ell \rangle|_{-k}^{\ell} \cap A)/\hat{\mu}_+(A),$$

again for any cylinder set with $k, \ell \ge 0$, respectively. Here

$$\langle a_{-k}, \ldots, a_\ell \rangle|_{-k}^{\ell} = \hat{F}_+^{k+1}\langle a_{-k}, \ldots, a_\ell \rangle.$$

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LEMMA 2. *If $y < -2$, then $\hat{F}_+^n(x, y) \in A$ if and only if $\hat{F}_+^n(x, -\infty) \in A$.*

LEMMA 3. *Let x be an irrational number. Then x is R-normal if and only if $(x, -\infty)$ is $\hat{F}_+$-normal.*

In the same way as for the RCF, we can define $\hat{F}_{1/2}$-normality. We have the following lemma.

LEMMA 4. *Let x be an irrational number. Then x is NICF-normal if and only if $(x, -\infty)$ is $\hat{F}_{1/2}$-normal.*

Due to the isomorphism $(x, y) \mapsto (x - 1, y - 1)$ for $\frac{1}{2} \le x < 1$ and identity otherwise, we have the following lemma.

LEMMA 5. *(x, y) is A-normal if and only if (x, y) (or $(x - 1, y - 1)$ when $\frac{1}{2} < x < 1$) is $\hat{F}_{1/2}$-normal.*

Because of the above lemmas, our main theorem, Theorem 3, at once follows as soon as the following proposition is proved.

PROPOSITION 3. *(x, y) is $\hat{F}_+$-normal if and only if (x, y) is A-normal.*

Proof of Proposition 3. Suppose that (x, y) is $\hat{F}_+$-normal. First we will prove the following lemma.

LEMMA 6. *If (x, y) is $\hat{F}_+$-normal, then*

$$\lim_{N\to\infty} \frac{1}{N} \#\{n : 0 \le n \le N - 1, \hat{F}_+^n(x, y) \in A\} = \hat{\mu}_+(A).$$

Proof of Lemma 6. We decompose A and A^c as follows:

$$A = \bigcup_{p=0}^{\infty} \bigcup_{b=2}^{\infty} \langle b, 1, \ldots, 1\rangle|_{-2p}^0 \quad \text{and} \quad A^c = \bigcup_{p=0}^{\infty} \bigcup_{b=2}^{\infty} \langle b, 1, \ldots, 1\rangle|_{-2p-1}^0.$$

To approximate A by a finite union of cylinder sets, we define

$$A_M = \bigcup_{p=0}^{M} \bigcup_{b=2}^{M} \langle b, 1, \ldots, 1\rangle|_{-2p}^0 \quad \text{and} \quad B_M = \bigcup_{p=0}^{M} \bigcup_{b=2}^{M} \langle b, 1, \ldots, 1\rangle|_{-2p-1}^0.$$

Then we have

$$A_M \subset A, \bigcup A_M = A \quad \text{and} \quad A \subset B_M^c, \bigcap B_M^c = A.$$

Now fix some $\varepsilon > 0$, then

$$\frac{\#\{n : 0 \le n \le N - 1, \hat{F}_+^n(x, y) \in A\}}{N} \ge \frac{\#\{n : 0 \le n \le N - 1, \hat{F}_+^n(x, y) \in A_M\}}{N}. \quad (2)$$

Notice that the right-hand side of (2) tends to $\hat{\mu}_+(A_M)$ as N tends to infinity, and that $\hat{\mu}_+(A_M) > \hat{\mu}_+(A) - \varepsilon$ for sufficiently large M. Furthermore, we also have that

$$\frac{\#\{n : 0 \le n \le N - 1, \hat{F}_+^n(x, y) \in A\}}{N} \le \frac{\#\{n : 0 \le n \le N - 1, \hat{F}_+^n(x, y) \in B_M^c\}}{N}. \quad (3)$$

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The right-hand side of (3) tends to $\hat{\mu}_+(A_M)$ as N tends to infinity and, moreover, $\hat{\mu}_+(B_M^c) < \hat{\mu}_+(A) + \varepsilon$ for sufficiently large M. Thus, we find that

$$\lim_{N\to\infty} \frac{1}{N} \#\{n : 0 \le n \le N-1, \hat{F}_+^n(x,y) \in A\} = \hat{\mu}_+(A),$$

which proves Lemma 6. □

In general, if $a_0 \neq 1$, then

$$\langle a_{-k}, \dots, a_\ell \rangle|_{-k}^{\ell} \subset A \quad \text{for } k, \ell \ge 0.$$

So we see

$$\begin{aligned} &\frac{\#\{n : 0 \le n \le N-1, \hat{F}_+|_A^n(x,y) \in \langle a_{-k}, \dots, a_\ell \rangle|_{-k}^{\ell} \cap A\}}{N} \\ &\quad = \frac{\#\{n : 0 \le n \le N-1, \hat{F}_+|_A^n(x,y) \in \langle a_{-k}, \dots, a_\ell \rangle|_{-k}^{\ell}\}}{N} \\ &\quad = \frac{N^*}{N} \frac{\#\{n : 0 \le n \le N-1, \hat{F}_+|_A^n(x,y) \in \langle a_{-k}, \dots, a_\ell \rangle|_{-k}^{\ell}\}}{N^*}, \end{aligned}$$

where we choose N^* so that

$$\#\{n : 0 \le n \le N^*-1, \hat{F}_+^n(x,y) \in A\} = N.$$

Due to the previous lemma, Lemma 6, one has that

$$\frac{\#\{n : 0 \le n \le N-1, \hat{F}_+|_A^n(x,y) \in \langle a_{-k}, \dots, a_\ell \rangle|_{-k}^{\ell}\}}{N^*}$$

converges to

$$\frac{1}{\hat{\mu}_+(A)} \hat{\mu}_+(\langle a_{-k}, \dots, a_\ell \rangle|_{-k}^{\ell}),$$

as $N^* \to \infty$, which also implies that $N \to \infty$.

If $a_{-2p} \neq 1$ (or $a_{-2p-1} \neq 1$), then

$$\langle a_{-2p}, 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-2p}^{\ell} \subset A$$

(or $\langle a_{-2p-1}, 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-2p-1}^{\ell} = \emptyset$, respectively), and the same result follows.

Thus we only need to check the case $\langle 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-k}^{\ell}$.

If $k = 2p$ ($p \ge 0$), then

$$\langle 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-k}^{\ell} = \bigcup_{b=1}^{\infty} \langle b, 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-2p-1}^{\ell}$$

and

$$\bigcup_{b=2}^{\infty} \langle b, 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-2p-1}^{\ell} \subset A.$$

On the other hand, if $k = 2p+1$ ($p \ge 0$), then

$$\langle 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-k}^{\ell} = \bigcup_{b=1}^{\infty} \langle b, 1, \dots, 1, a_1, a_2, \dots, a_\ell \rangle|_{-2p-2}^{\ell}$$

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and

$$\bigcup_{b=2}^{\infty}\langle b, 1, \ldots, 1, a_1, a_2, \ldots, a_\ell\rangle|_{-2p-2}^{\ell} \subset A^c.$$

Now using the same argument as in the proof of Lemma 6, it follows that (x, y) is A-normal whenever it is $\hat{F}_+$-normal.

Now suppose that (x, y) is A-normal. Since $\hat{F}_+^{-1}A^c = A \cap \{(x, y) \in X_1, \frac{1}{2} \le x < 1\}$, the following lemma, similar to Lemma 6, will finish the proof of Proposition 3.

LEMMA 7. *If (x, y) is A-normal, then*

$$\lim_{N\to\infty} \frac{\#\{n : 0 \le n \le N-1, \hat{F}_+^n(x, y) \in A\}}{N} = \hat{\mu}_+(A).$$

Proof of Lemma 7. Put $N^* = \#\{n : 0 \le n \le N-1, \hat{F}_+^n(x, y) \in A\}$, then

$$N = N^* + \#\{n : 0 \le n \le N^* - 1, \hat{F}_+|_A^n(x, y) \in \hat{F}_+^{-1}(A^c)\},$$

and we see that $\#\{n : 0 \le n \le N-1, \hat{F}_+^n(x, y) \in A\}/N$ equals

$$\frac{N^*}{N^* + \#\{n : 0 \le n \le N^* - 1, \hat{F}_+|_A^n(x, y) \in \hat{F}_+^{-1}(A^c)\}}. \tag{4}$$

Since the fraction in (4) tends to

$$\frac{1}{1 + \hat{\mu}_+|_A(\hat{F}_+^{-1}A^c)} = \frac{1}{1 + [\hat{\mu}_+(A^c)/\hat{\mu}_+(A)]} = \hat{\mu}_+(A),$$

as $N^* \to \infty$, the lemma follows. □

Due to Lemma 7, the rest of the proof of Proposition 3 is essentially similar to the first part of the proof. We will only show that the limit

$$\lim_{N\to\infty} \frac{\#\{n : 0 \le n \le N-1, \hat{F}_+^n(x, y) \in \langle 1, \ldots, 1, a_1, \ldots, a_\ell\rangle|_{-k}^{\ell}\}}{N}$$

equals $\hat{\mu}_+(\langle 1, \ldots, 1, a_1, \ldots, a_\ell\rangle|_{-k}^{\ell})$. In order to show this, notice that

$$\langle 1, \ldots, 1, a_1, \ldots, a_\ell\rangle|_{-k}^{\ell} = B_1 \cup B_2,$$

where

$$B_1 = \langle 1, \ldots, 1, a_1, \ldots, a_\ell\rangle|_{-k}^{\ell} \cap A \quad \text{and} \quad B_2 = \langle 1, \ldots, 1, a_1, \ldots, a_\ell\rangle|_{-k}^{\ell} \cap A^c.$$

Furthermore,

$$\hat{F}_+^{-1}(B_2) = B_1,$$

and one finds that

$$\frac{\#\{n : 0 \le n \le N-1, \hat{F}_+^n(x, y) \in \langle 1, \ldots, 1, a_1, \ldots, a_\ell\rangle|_{-k}^{\ell}\}}{N}$$

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equals

$$\frac{\#\{n : 0 \le n \le N-1, \hat{F}_+^n(x,y) \in B_1\} + \#\{n : 0 \le n \le N-1, \hat{F}_+^n(x,y) \in B_2\}}{N},$$

which is in its turn equals

$$\frac{\#\{n : 0 \le n \le N-1, \hat{F}_+^n(x,y) \in B_1\} + \#\{n : 0 \le n \le N-1, \hat{F}_+^{n-1}(x,y) \in \hat{F}_+^{-1}B_2\}}{N}.$$

An easy calculation shows that this last fraction converges to

$$\hat{\mu}_+(\langle 1, \ldots, 1, a_1, \ldots, a_\ell \rangle|_{-k}^{\ell}),$$

by A-normality of (x, y). This proves the proposition. □

4. *α-expansions*

The NICF is an example of an α-expansion (see also [**N**]) or, more generally, an S-expansion (see [**K**]). Let $\alpha \in [\frac{1}{2}, 1]$, then for $x \in [\alpha - 1, \alpha)$ we define the map $F_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$ by

$$F_\alpha(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor.$$

Clearly, for $\alpha = 1$ we find the RCF and for $\alpha = \frac{1}{2}$ we find the NICF. For each $\alpha \in [\frac{1}{2}, 1)$ one can find a suitable 'induced area', which links α-expansions to S-expansions; for more details, see [**K**, §§4 and 5]. In fact one has to discern between $\alpha \in [\frac{1}{2}, g]$ and $\alpha \in (g, 1)$, with $g = G^{-1}$, and one finds

$$A_\alpha = \begin{cases} \left[0, \dfrac{2\alpha - 1}{1 - \alpha}\right) \times [-\infty, -2) \\ \quad \cup \left[\dfrac{2\alpha - 1}{1 - \alpha}, \dfrac{1 - \alpha}{\alpha}\right) \times [-\infty, -G) \\ \quad \cup \left[\dfrac{1 - \alpha}{\alpha}, 1\right) \times [-\infty, -1), & \frac{1}{2} \le \alpha \le g, \end{cases}$$

$$A_\alpha = \begin{cases} \left[0, \dfrac{1 - \alpha}{\alpha}\right) \times [-\infty, -2) \\ \quad \cup \left[\dfrac{1 - \alpha}{\alpha}, 1\right) \times [-\infty, -1), & g < \alpha < 1, \end{cases}$$

and the corresponding singularization areas are

$$S_\alpha = \begin{cases} \left[\alpha, \dfrac{1 - \alpha}{\alpha}\right) \times [-\infty, -G) \cup \left[\dfrac{1 - \alpha}{\alpha}, 1\right) \times [-\infty, -1), & \frac{1}{2} \le \alpha \le g, \\ [\alpha, 1) \times [-\infty, -1), & g < \alpha < 1. \end{cases}$$

For both cases (i.e. $\alpha \in [\frac{1}{2}, g]$ and $\alpha \in (g, 1)$), the isomorphism between the natural extension $\hat{F}_\alpha$ of F_α and $\hat{F}_+|_{A_\alpha}$ is given by $(x, y) \mapsto (x - 1, y - 1)$ for $\alpha \le x < 1$ and

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identity, otherwise. Since the boundaries of the sets A_α are rather 'easy', the method of the previous section for the NICF can be carried over to any α-expansion. Obviously, for a general S-expansion the situation is more difficult, due to the fact that the boundary ∂S of the singularization area S is, in general, more complicated than ∂S_α (for $\alpha \in [\frac{1}{2}, 1)$). For example, if

$$A_D = \{(x, y) \in X_1, y < x - 2\},$$

with singularization area S_D given by

$$S_D = \left\{(x, y) \in X_1, \frac{1}{2} < x < 1, y < \frac{x}{1 - 2x}\right\}$$

and with the translation $(x, y) \mapsto (x - 1, y - 1)$ on S_D, we find Minkowski's *diagonal continued fraction*. As before, we can define normality for this continued fraction and get the equivalence to the R-normality. Clearly the method to do this is similar to that of §2, but more complicated.

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