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NOTES

A NOTE ON THE RATIONALITY TEST

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The rationality test is based on regressing the actual series of variable in period t on its forecast formed in period $t-1$. The nonrejection of the rationality hypothesis requires expectations' unbiasedness and efficiency. We show that in a model with $t-1$ dating, where expectations strongly and positively affect the economy, the rationality test suffers from low power. It provides a high probability of nonrejection of the rationality hypothesis against numerous alternatives for expectations formation. This result is attained because the realization of the economy is driven by the public's expectations, irrespective of how well they are formed, via the structural relationship between them. The parameters in this test are predetermined by the parameters of the structural model, supporting expectations' unbiasedness under reasonable assumptions. Thus, successfully passing the rationality test could be misleading in terms of interpretation of the quality of the expectations and could lead to questionable accuracy of many applications.

Keywords: Rationality Test, Boundedly Rational Expectations, Naïve Expectations, Ad Hoc Expectations

1. INTRODUCTION

Following the famous papers by Lucas (1972) and Sargent (1973), the rational expectations (RE) approach became the central methodology in macroeconomic modeling, both in central banks and in academia. Although rationality of expectations is a convenient assumption, whether it actually holds in practice is an open empirical question.

The standard rationality test utilized to examine the rationality of expectations (derived from capital markets, surveys, economic reports, etc.) essentially relies on regressing the realization of some variable—for example, inflation—at period t on its expected value from period $t-1$. Under the null hypothesis, expectations are rational. The null is not rejected if the expectations are unbiased (in the rationality test, the slope and the intercept are equal to one and zero, respectively) and their forecast errors are orthogonal to all past information, including past forecast errors.

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The following papers have examined possible problems with the rationality test, such as measurement errors, serial correlation, nonstationarity and small sample bias.

Smyth (1992) showed that when expected inflation is measured with errors, the estimated parameters in the rationality test are biased. He concluded that previous results that did not reject rationality are not valid any more, and after correction for this bias, the rationality hypothesis is rejected.

Rich (1989) proposed to test rationality using general method of moments estimation, which allows more efficient consideration of the serial correlation and heteroskedasticity of the errors. He tested the rationality of survey expectations for inflation from the Michigan survey and found the existence of unbiasedness and efficiency of expectations.

Maddala and Liu (1992) proposed a rationality test where both expectations and realizations are nonstationary, namely $I(1)$. Using the cointegration framework of Engle and Granger (1987), they tested rationality of survey forecasts in foreign exchange markets and found mixed results, depending on the forecast horizons. Applying the same methodology for other variables also yielded mixed results concerning their unbiasedness [see Kim (1997), Grant and Thomas (1999), Osterberg (2000), Mitchell and Pearce (2007)].

Thomas (1999) and Andolfatto et al. (2008) examined the effects of small samples on the rationality test. Thomas (1999) considered inflation expectations from the Livingston, Michigan, and professional forecasters' surveys and found that over the whole sample unbiasedness is not rejected. However, when the sample is split, the results for unbiasedness are mixed. Andolfatto et al. (2008), using data generated by a DSGE model, also pointed out the existence of bias of the rationality test in small samples, even though these expectations had been generated as rational.

In our paper, as in Andolfatto et al. (2008), we also generate a series of the economy's realizations along with the expectations from the model, instead of using empirical data. However, in our setting, the generated expectations are not rational, but boundedly rational, naïve, or formed in an ad hoc manner. Therefore, we examine the properties of the rationality test with nonrational expectations.

Before proceeding, we should define rationality as it is referred to in the paper. We adopt the definition of rationality presented by Evans and Honkapohja (2001)—RE are expectations that are consistent with the model of the economy. This implies that rational agents know (1) the specification of the model (use a correct set of explanatory variables to forecast the endogenous variable) and (2) the parameters of the model. As we will see in Section 3, boundedly rational, naïve, and ad hoc expectations do not satisfy at least one requirement of rationality, according to this definition.

For each type of nonrational expectations, we test the power of the rationality test—that is, the probability of rejecting the null hypothesis against nonrational expectations. We examine the conditions under which the power of the rationality

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test is reduced and thus the probability of accepting the rationality hypothesis when in fact it is not true is increased.

This paper adds a new and important view of the rationality test: we show that in a model with $t-1$ dating—that is, expectations formed in period $t-1$ affect the economy at period t—for any type of nonrational expectations the parameters in the rationality test are predetermined by the parameters of the structural model. The main reason for this phenomenon is that the expectations, irrespective of how well they are formed, drive the economy and hence lead to its self-referential dynamics being dependent on them. Moreover, when there is a strong positive effect of expectations (of any type), so that they have nearly a one-for-one impact on the variable forecasted, then the rationality test has a high probability of erroneous acceptance of the rationality hypothesis. Bullard et al. (2010) made a similar point about the low power of the rationality test for a specific nonrational expectations process in a dynamic model with a strong positive effect of expectations.

Thus, successful passing of the rationality test could be misleading in terms of the interpretation of expectations formation. Therefore, it could be that the unbiasedness and efficiency of the expectations obtained in the empirical literature reflects this self-referential property of the expectations and not their rationality in the sense that the public knows how the economy works.

The paper is organized as follows: Section 2 introduces the standard rationality test; Section 3 introduces the model of the economy; Section 4 explains the source of the low power of the rationality test with nonrational expectations; Section 5 presents Monte Carlo simulations to support the theoretical findings of Section 4; Section 6 introduces practical implications of misinterpretation of the rationality test, and Section 7 concludes.

2. FORMAL PROCEDURE FOR TESTING THE RATIONALITY OF EXPECTATIONS

Let y_t be an endogenous variable and $E_{t-1}[y_t]$ its observable forecast in period t-1. The standard rationality test is based on the following regression estimated by ordinary least squares (henceforth OLS):

$$y_t = k_1 + k_2 E_{t-1}[y_t] + u_t. \quad (1)$$

The nonrejection of rationality requires two conditions:

(a) Unbiasedness of expectations: $k_1 = 0,\ k_2 = 1$, and
(b) Efficiency of expectations: $E(u_t/\Xi_{t-1}) = 0$; u_t should be white noise and orthogonal to all past information Ξ_{t-1} (including past forecast errors).

We examine the power of the rationality test where under the null hypothesis the expectations are rational against the alternative hypothesis (which is true) that they are not rational.

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3. MODEL WITH *t*–1 DATING

Let us assume that the reduced form model of the economy is given by

$$\begin{aligned} y_t &= \mu + \beta E_{t-1}[y_t] + \varphi y_{t-1} + \gamma w_{t-1} + \varepsilon_t, \varepsilon_t \overset{\text{iid}}{\sim} N(0,\ \sigma^2), \\ w_t &= \rho w_{t-1} + \varsigma_t, \text{assuming } |\rho| < 1, \varsigma_t \overset{\text{iid}}{\sim} N\left(0,\ \sigma_\varsigma^2\right), \end{aligned} \tag{2}$$

where y_t is a vector of endogenous variables, and w_t is a vector of exogenous variables which follows AR(1) process. $E_{t-1}[y_t]$ are the public expectations, which are not necessarily rational. The shocks ε_t and ς_t are white noises normally distributed.

For purposes of illustration, we consider only the one-dimensional case. The specification of the model in (2) is general and covers many cases in the literature. We will consider these cases in Section 5.

Throughout the paper we impose the following conditions on the parameters:

$$\beta < 1,\ \varphi \geq 0, \text{ and } |\beta + \varphi| < 1.$$

The condition $\beta < 1$ is needed to ensure E-stability, i.e., stability under least-squares learning, of the RE solution. The condition $\varphi \geq 0$ is natural in most applications, and the condition $|\beta + \varphi| < 1$ then ensures that both the RE solution and the dynamics under naïve expectations (discussed in the following) are nonexplosive.

3.1. Rational Expectations

First, let us assume that expectations are rational—that is, expectations are consistent with the model of the economy in (2), implying a unique RE solution,

$$y_t = Az_{t-1} + \varepsilon_t, \tag{3}$$

where $A \equiv [\alpha^* = \frac{\mu}{1-\beta},\ \varphi^* = \frac{\varphi}{1-\beta}, \gamma^* = \frac{\gamma}{1-\beta}]$ are parameters of the RE equilibrium[1] (henceforth REE), and $z_{t-1} = [1\ \ y_{t-1}\ \ w_{t-1}]'$ is a vector of the explanatory variables. This implies that rational agents know the specification of the model and its parameters, and they therefore use a correct set of explanatory variables to forecast the endogenous variable y_t. This guarantees, as we see immediately, that for any value of the parameter β the forecasts of rational agents are unbiased and efficient (their forecast errors equal ε_t) and cannot be improved further. The actual law of motion of the economy (henceforth ALM) implied by RE is obtained after applying the method of undetermined coefficients on (2).

Thus, the expectations of rational agents ($E^{\text{R}}_{t-1}[y_t]$) from period $t-1$ to period t are

$$E^{\text{R}}_{t-1}[y_t] = Az_{t-1}. \tag{4}$$

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From (3) and (4) we observe that the RE are unbiased and their forecasts are efficient. The forecast errors are $v_t = y_t - E^{\mathrm{R}}_{t-1}[y_t] = \varepsilon_t$, which are white noise shocks.

3.2. Nonrational Expectations

Now we turn to the three types of nonrational expectations and define their formation: In contrast to rational agents, the nonrational agents misspecify the specification of the model, its parameters, or both.

Boundedly rational expectations ($E^{\mathrm{BR}}_{t-1}[y_t]$). Boundedly rational (BR) agents know the specification of the solution of the model in (3) but they do not know its parameters $\alpha^*, \varphi^*, \gamma^*$, and they seek to learn them using econometric tools. The estimated parameters are recursively updated as new information arrives. The perceived law of motion (henceforth PLM) of the model by the BR agents is given by

$$y_t = \tilde{A}_{t-1} z_{t-1} + v_t, \tag{5}$$

where $\tilde{A}_{t-1} = [\tilde{\alpha}\ \tilde{\varphi}\ \tilde{\gamma}]$ is a vector of parameters estimated based on the $t-1$ information set. Thus the forecast of the endogenous variable to period t is

$$E^{\mathrm{BR}}_{t-1}[y_t] = \tilde{A}_{t-1} z_{t-1}. \tag{6}$$

Substituting (6) into the model in (2) (see Appendix A) yields the ALM implied by the PLM of the agents:

$$y_t = T(\tilde{A}_{t-1}) z_{t-1} + \varepsilon_t = (1-\beta) E^{\mathrm{R}}_{t-1}[y_t] + \beta E^{\mathrm{BR}}_{t-1}[y_t] + \varepsilon_t. \tag{7}$$

The T-map $T(\tilde{A}_{t-1})$ is an operator that maps the PLM into an actual model [see Marcet and Sargent (1989), Evans and Honkapohja (2001)]. The evolution of y_t is derived from a weighted average of both rational and boundedly rational expectations (as well as shocks). This does not imply, however, that the share of $1-\beta$ of the public is rational and the complementary share of β is not. All agents in the economy are here assumed to have nonrational expectations, and the use of $E^{R}_{t-1}[y_t]$ in (7) just reflects a tractable representation of the model. $E^{\mathrm{R}}_{t-1}[y_t]$ appears in (7) just because all components in (2), besides $\beta E^{\mathrm{BR}}_{t-1}[y_t]$ (which are $\mu, \varphi y_{t-1}$ and γw_{t-1}), compose the RE solution shown in (3).

The learning algorithm of the BR agents is given by Evans and Honkapohja (2001):

$$\begin{aligned} \tilde{A}_t &= \tilde{A}_{t-1} + \mathrm{gain}_t R_t^{-1} z_{t-1} v_t \\ R_t &= R_{t-1} + \mathrm{gain}_t (z_{t-1} z'_{t-1} - R_{t-1}), \end{aligned} \tag{8}$$

where $\tilde{A}_{t-1} = [\tilde{\alpha}\ \tilde{\varphi}\ \tilde{\gamma}]$, and $\mathrm{gain}_t = g$ is the constant gain (henceforth CG).[2] The forecast error is $v_t = y_t - E^{\mathrm{BR}}_{t-1}[y_t]$. Agents learning with CG allow for structural changes in the economy; therefore, the learning process never stops.[3] As was shown in Evans and Honkapohja (2001), expectations of agents who learn with

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the CG do not converge to the REE but only to a distribution around the REE (for constant gain close to zero).

Naive expectations ($E^{\mathrm{NA}}_{t-1}[y_t]$). Naïve expectations are a particular case of BR expectations under restrictions that $\tilde{\alpha} = 0$, $\tilde{\gamma} = 0$, and $\tilde{\varphi} = 1$, so $E^{\mathrm{NA}}_{t-1}[y_t] = y_{t-1}$, meaning that the forecast is always equal to the endogenous variable in the previous period.

Ad hoc expectations ($E^{\mathrm{AH}}_{t-1}[y_t]$). We also consider ad hoc expectations to demonstrate that even for such an unreasonable type of expectations in practice, the rationality test has low power under very reasonable conditions.

We consider two different types of ad hoc expectations. In the first version, they are orthogonal to the RE, whereas in the second version, they are positively correlated with the RE.

Ad hoc expectations of type 1 ($E^{\mathrm{AH1}}_{t-1}[y_t]$). In each period agents randomly draw their expectations from a normal distribution, such that the following restrictions on the parameters are imposed: $\tilde{\alpha} \overset{\mathrm{iid}}{\sim} N(0, 1)$ and $\tilde{\varphi}, \tilde{\gamma} = 0$.

Ad hoc expectations of type 2($E^{\mathrm{AH2}}_{t-1}[y_t]$). In each period the parameters are chosen from $\tilde{\alpha} \overset{\mathrm{iid}}{\sim} N(0, 1)$ and $\tilde{\varphi} \overset{\mathrm{iid}}{\sim} U(0, 2\varphi^*)$, $\tilde{\gamma} \overset{\mathrm{iid}}{\sim} U(0, 2\gamma^*)$, so that agents randomly choose the intercept from the normal distribution as before, but now the parameters of the explanatory variables $y_{t-1,}$ w_{t-1} are randomly chosen from the uniform distribution around the RE values (in the positive range). This setting enables the ad hoc expectations to be positively correlated with RE.

The ALM of the economy in (7) can be rewritten in a general form for any type of nonrational expectations:

$$y_t = (1 - \beta) E^{\mathrm{R}}_{t-1}[y_t] + \beta E^{\mathrm{NR}}_{t-1}[y_t] + \varepsilon_t, \tag{9}$$

where $E^{\mathrm{NR}}_{t-1}[y_t] \in \{E^{\mathrm{BR}}_{t-1}[y_t], E^{\mathrm{NA}}_{t-1}[y_t], E^{\mathrm{AH}}_{t-1}[y_t]\}$ is a set of nonrational expectations. Equation (9) shows that y_t is derived from a weighted average of the rational and nonrational expectations (and shocks). It should be noted that (9) holds also if $E^{\mathrm{NR}}_{t-1}[y_t] = E^{\mathrm{R}}_{t-1}[y_t]$.

4. EXAMINING THE POWER OF THE RATIONALITY TEST WITH NONRATIONAL EXPECTATIONS

In this section we analyze the source of unbiasedness and efficiency of nonrational expectations as reflected by the rationality test.

4.1. A Consistent Representation of the Rationality Test

First, we express the ALM of the economy in (9) in exactly the same representation as required by the rationality test in (1), where the endogenous variable is explained

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by the expectations formed in the previous period. First, we rearrange (9) by putting de-meaned $(1-\beta)E^{\mathrm{R}}_{t-1}[y_t]$ into the residuals:

$$y_t = \beta E^{\mathrm{NR}}_{t-1}[y_t] + [(1-\beta)E^{\mathrm{R}}_{t-1}[y_t] + \varepsilon_t].$$

After plugging RE from (4) explicitly into the preceding equation we get the rationality test representation:

$$y_t = k_1 + k_2 E^{\mathrm{NR}}_{t-1}[y_t] + u_t, \tag{10}$$

where $k_1 = \frac{(1-\beta)\mu}{1-\beta-\varphi}$, $k_2 = \beta$, and $u_t = \varphi \bar{y}_{t-1} + \gamma w_{t-1} + \varepsilon_t$ are residuals with zero mean. $\bar{y}_{t-1}$ is defined as $\bar{y}_{t-1} = y^{\mathrm{R}}_{t-1} - E(y^{\mathrm{R}}_{t-1})$, where y^{R}_t denotes the RE solution (i.e., the solution if all agents have RE), given by (3) and (4). Here, $E(y^{\mathrm{R}}_{t-1})$ is the unconditional mean of y^{R}_{t-1}, which is given by $E(y^{\mathrm{R}}_{t-1}) = \frac{\mu}{1-\beta-\varphi}$ (see Appendix A).

Equation (10) lays the groundwork for our analysis. We start with an examination of unbiasedness of nonrational expectations.[4]

4.2. Unbiasedness of Nonrational Expectations

Equation (10) shows that the slope k_2 and the intercept k_1 of the rationality test are predetermined by the structural parameters of the model, irrespective of how the expectations (of the public) are formed. The key question is, what will the estimated parameters in (10) be when we apply a linear projection (OLS) of the endogenous variable onto the intercept and the nonrational expectations? We consider two possibilities:

Nonrational expectations orthogonal to RE. Assuming $\mathrm{cov}(E^{\mathrm{NR}}_{t-1}[y_t], u_t) = 0$, estimation of (10) provides unbiased and consistent estimators as $p\lim \tilde{k}_1 = k_1$ and $p\lim \tilde{k}_2 = k_2 = \beta$. To illustrate, let us examine the ad hoc expectations of type 1, $E^{\mathrm{AH1}}_{t-1}[y_t] = \tilde{\alpha}_t$, where $\tilde{\alpha} \overset{\mathrm{iid}}{\sim} N(0,1)$. From (2) we immediately obtain the solution of the model:

$$y_t = \beta\tilde{\alpha}_t + \mu + \varphi y_{t-1} + \gamma w_{t-1} + \varepsilon_t. \tag{11}$$

Now we represent (11) in terms of the rationality test representation shown in (10) such that[5]

$$y_t = \frac{\mu(1-\beta)}{1-\beta-\varphi} + \beta E^{\mathrm{AH1}}_{t-1}[y_t] + u_t, \text{ where } u_t = \varphi\bar{y}_{t-1} + \gamma w_{t-1} + \varepsilon_t. \tag{12}$$

The mapping from the reduced form parameters in (11) into the estimated parameters in the rationality test in (12) is $\tilde{k}_1 = \frac{\mu(1-\beta)}{1-\beta-\varphi} = k_1$ and $\tilde{k}_2 = \beta = k_2$.

We now consider two limiting cases of parameter β: (A) $\beta \to 1$ and (B) $\beta \to -1$.[6] In case A the parameters in the rationality test will converge in the limit to $\tilde{k}_1 \to 0 \ \ \forall\, \mu, \varphi$ and $\tilde{k}_2 \to 1$, implying low power of the rationality test

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(notice that when $\varphi = 0, \tilde{k}_1 = \mu \forall \beta$). In case B, the parameters of the rationality test will converge in the limit to $\tilde{k}_1 \to \frac{2\mu}{2-\varphi}$ and $\tilde{k}_2 \to -1$, implying high power of the rationality test. The second important parameter affecting the estimated intercept in the rationality test is μ. When $\mu \to 0$, $\tilde{k}_1 \to 0 \forall \beta, \varphi$, reducing the power of the rationality test. Therefore, as $\beta \to 1$ and $\mu \to 0$, the power of the rationality test decreases.

Nonrational expectations positively correlated with RE. Assuming $\text{cov}(E^{\text{NR}}_{t-1}[y_t], u_t) > 0$, the estimated slope in (10) is upward biased as $p \lim \tilde{k}_2 > \beta$ and the estimated intercept is downward biased as $p \lim \tilde{k}_1 < k_1$, supporting the nonrejection of the rationality hypothesis even for β strictly below 1. To illustrate, let us consider naïve expectations, $E^{\text{NA}}_{t-1}[y_t] = y_{t-1}$. Solving (2) under naïve expectations yields

$$y_t = \beta y_{t-1} + \mu + \varphi y_{t-1} + \gamma w_{t-1} + \varepsilon_t. \tag{13}$$

Rewriting (13) in terms of the rationality test in (10) yields

$$y_t = \mu + (\beta + \varphi) E^{\text{NA}}_{t-1}[y_t] + u_t, \text{ where } u_t = \gamma w_{t-1} + \varepsilon_t. \tag{14}$$

Equation (14) shows that the estimated parameters in the rationality test are $\tilde{k}_1 = \mu$ and $\tilde{k}_2 = \beta + \varphi$, and they are biased toward nonrejection of the rationality hypothesis: the bias in the slope is given by $\tilde{k}_2 - k_2 = \varphi > 0$ if $\varphi > 0$ and the bias in the intercept is given by $\tilde{k}_1 - k_1 = -\frac{\mu\varphi}{1-\beta-\varphi} < 0$ if $\beta + \varphi < 1$ and $\mu, \varphi > 0$. From here it is evident that naive expectations have a higher probability of passing the rationality test than ad hoc expectations of type 1.

From observing $\tilde{k}_1$ and $\tilde{k}_2$, it follows that high β and low μ act to reduce the power of the rationality test, which is the same conclusion as with ad hoc expectations of type 1.

In the remaining two cases with learning and ad hoc expectations of type 2, the exact analytical mapping from the reduced form parameters into the estimated parameters of the rationality test is very cumbersome. This is because these expectations are not presented by a single variable but are functions of sets of variables. Nevertheless we can identify the direction of the bias in the estimated parameters and examine its determinants.

When we estimate (10) by OLS, $\tilde{k}_2$ is given by

$$\tilde{k}_2 = \beta + \frac{\text{cov}(E^{\text{NR}}_{t-1}[y_t], u_t)}{\text{var}(E^{\text{NR}}_{t-1}[y_t])},$$

where the last term represents the bias. This bias is created because both learning and ad hoc expectations of type 2 are positively correlated with RE via mutual dependence on y_{t-1} and w_{t-1} [namely, $\text{cov}(E^{\text{NR}}_{t-1}[y_t], u_t) > 0$ in (10)]. After

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expressing all terms explicitly we obtain the following expression:

$$\tilde{k}_2 = \beta + \frac{\operatorname{cov}(\tilde{\alpha} + \tilde{\varphi} y_{t-1} + \tilde{\gamma} w_{t-1}, \varphi \bar{y}_{t-1} + \gamma w_{t-1} + \varepsilon_t)}{\operatorname{var}(\tilde{\alpha} + \tilde{\varphi} y_{t-1} + \tilde{\gamma} w_{t-1})}. \tag{15}$$

The bias depends on time-variant parameters $\tilde{\alpha}$, $\tilde{\varphi}$, and $\tilde{\gamma}$ subject to the restrictions of learning and ad hoc expectations, respectively. To get some intuition behind (15), let us assume that $\tilde{\alpha}$, $\tilde{\varphi}$, and $\tilde{\gamma}$ are constant. Under this assumption, (15) can be simplified [exploiting $\varphi^*(1-\beta) = \varphi$ and $\gamma^*(1-\beta) = \gamma$] to

$$\tilde{k}_2 = \beta + (1-\beta)\Psi, \text{ where } \Psi = \frac{\tilde{\varphi}\varphi^* \operatorname{var}(y_{t-1}) + \tilde{\gamma}\gamma^* \operatorname{var}(w_{t-1})}{\tilde{\varphi}^2 \operatorname{var}(y_{t-1}) + \tilde{\gamma}^2 \operatorname{var}(w_{t-1})}. \tag{16}$$

Under RE, $\tilde{\varphi} = \varphi^*$, $\tilde{\gamma} = \gamma^*$, therefore, $\Psi = 1$ and $\tilde{k}_2 = 1$, as expected. Under learning and ad hoc expectations of type 2, $\tilde{\varphi} \neq \varphi^*$, $\tilde{\gamma} \neq \gamma^*$. As long as $\tilde{\varphi}$, $\tilde{\gamma}$ are in the neighborhood of φ^*, γ^*, respectively, the bias of $\tilde{k}_2$ is $(1-\beta)\Psi > 0$, but its precise size depends on $\tilde{\varphi}$ and $\tilde{\gamma}$. We can also observe that as β decreases, the bias increases, as long as $\Psi > 0$. That means that $\tilde{k}_2$ could be high even for low values of β. On the other hand, when $\beta \to 1$, the bias converges to 0 given Ψ, but still $\tilde{k}_2 \to 1$. Regarding $\tilde{k}_1$, from (10) it is evident that the positive bias in $\tilde{k}_2$ leads to a negative bias in $\tilde{k}_1$, supporting the nonrejection of the rationality hypothesis.

4.3. Serial Correlation in the Forecast Errors on the Nonrational Expectations

We now examine the second requirement of the rationality test, the efficiency of the expectations.

First we derive the forecast error using (9):

$$v_t = y_t - E^{\mathrm{NR}}_{t-1}[y_t] = (1-\beta)\{E^{\mathrm{R}}_{t-1}[y_t] - E^{\mathrm{NR}}_{t-1}[y_t]\} + \varepsilon_t, \tag{17}$$

where $E^{\mathrm{NR}}_{t-1}[y_t] \in \{E^{\mathrm{BR}}_{t-1}[y_t], E^{\mathrm{NA}}_{t-1}[y_t], E^{\mathrm{AH}}_{t-1}[y_t]\}$ are the three different types of expectations.

From (17) we see that for any given discrepancy between the rational and nonrational expectations, as β approaches 1, the forecast errors v_t converge to white noise shock ε_t (reducing the serial correlation in the forecast errors to zero), supporting the efficiency of nonrational expectations. As β moves away from 1 and even shifts to be negative, the forecast errors are characterized by serial correlation, increasing the probability of rejecting the efficiency of the expectations.

To summarize, the lower power of the rationality test (for high β and low μ) is obtained because the ALM of the economy from the point of view of the rationality test in (10) is consistent with the PLM of the agents. The ALM in (9), which explicitly contains other variables, is not necessarily consistent with the PLM of the agents, but these variables are not taken into account by the rationality test.

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4.4. The Power of the Rationality Test in Asymptotic Samples

The low power of the rationality test generally does not hold in asymptotic samples unless $\mu = 0$ and $\beta = 1$. This is because for high β and low μ, the estimated parameters $\tilde{k}_1$ and $\tilde{k}_2$ in (10) will asymptotically converge to values that are near zero and one, respectively, but not exactly equal to these values [see (12), (14), and (15)]. In contrast, in finite samples, the estimated parameters $\tilde{k}_1, \tilde{k}_2$ may statistically be indistinguishable from zero and one, respectively, because of the estimator uncertainty captured by its standard deviation and confidence interval.

The next section illustrates these theoretical findings using Monte Carlo simulations in finite samples.

5. ECONOMIC APPLICATIONS AND MONTE CARLO SIMULATIONS

In this section we consider three well-known models that are consistent with the specification of the model in (2). Then we run Monte Carlo simulations (with 1,000 replications and sample size $T = 100$) on these models to examine the power of the rationality test against the hypothesis that agents are nonrational (boundedly rational, naïve, or ad hoc). The three reduced-form models are given as follows:

Model 1 (New Classical Model). Here we present a model from Orphanides and Williams (2005):

$$y_t = \mu + \beta E_{t-1}[y_t] + \varphi y_{t-1} + \gamma w_{t-1} + \varepsilon_t,$$

where $\gamma = 0$, $0 < \beta < 1$, and $\beta + \varphi < 1$.

The calibration of the parameters in Model 1: Orphanides and Williams (2005) assumed $\beta = 0.75$ and $\beta = 0.9$. We extend the range of β to be between 0.1 and 0.9 in order to examine its effect on the power of the rationality test. The other structural parameters, $\mu = \delta\pi^*$ (where $\delta = 0.5 * 0.12$, $\pi^* = 0.02$) and $\varphi = 1 - \beta - \delta$,[7] are taken from Orphanides and Williams (2005). The REE parameters are determined as $\alpha^* = \frac{\mu}{1-\beta}$, $\varphi^* = \frac{\varphi}{1-\beta}$. The value of the constant gain for BR agents is 0.05.[8]

Model 2 (Muth Model). Here we present a model from Chap. 2 in Evans and Honkapohja (2001):

$$y_t = \mu + \beta E_{t-1}[y_t] + \varphi y_{t-1} + \gamma w_{t-1} + \varepsilon_t,$$

where $\varphi = 0$ and $\beta < 0$.

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Model 3 (Lucas Aggregate Supply Model). Here we present a model from Chap. 2 in Evans and Honkapohja (2001):

$$y_t = \mu + \beta E_{t-1}[y_t] + \varphi y_{t-1} + \gamma w_{t-1} + \varepsilon_t,$$

where $\varphi = 0$ and $0 < \beta < 1$.

The calibration of the parameters in Models 2 and 3: We choose different values of β between –0.99 and 0.99 in order to examine its effect on the power of the rationality test. The negative values of β are designed for Model 2, whereas positive values of β are designed for Model 3. The remaining structural parameters are $\mu = 0.01,\ \gamma = 0.3$. The autocorrelation of the exogenous variable is $\rho = 0.5$.

Table B.1 in Appendix B shows results of Model 1, whereas Table B.2 shows results of Models 2 and 3.[9] Both tables present the power of the rationality test introduced in (1), as well as the mean value, over all simulations, of the estimated parameters $\tilde{k}_1, \tilde{k}_2$ and the first-order serial correlation in the forecast errors.[10] Under the null hypothesis, which is false, agents are rational and therefore their expectations are unbiased and efficient. Under the alternative hypothesis, agents are nonrational (boundedly rational, naïve, or ad hoc).

The power in Tables B.1 and B.2 was calculated in the following manner (based on a 5% significance level): For each simulation (out of a total of 1,000 simulations), the series of the model were generated based on the random sample of $T = 100$ periods and then the rationality test was conducted. Once all simulations were made, the power was calculated as #reject/#total, where #reject is the number of simulations where the null hypothesis was rejected and #total is the total number of simulations.

5.1. Boundedly Rational Expectations

In the New Classical model the power of the rationality test is low, in the range of 12–20% for all values of β (Table B.1). This is because the parameters in the rationality test are biased toward nonrejection of the RE hypothesis. As shown in (15) and (16), this bias stems from a positive correlation between the BR expectations and the RE (both of them depend on y_{t-1}). Moreover, (17) implies that the serial correlation in the residuals decreases as β increases. We observe, however, that for $\beta = 0.9$ the power increases slightly to 17.2% because the mean of $\tilde{k}_2$ (over all simulations) decreases to 0.72. This means that there are a non-negligible number of simulations where $\tilde{k}_2$ in the rationality test is low or even negative. This is obtained for simulations where the variation in $E^{\mathrm{BR}}_{t-1}[y_t]$ is much lower than the variation in u_t in the rationality test in (10).[11] In finite samples this could create a negative bias in $\tilde{k}_2$ that has nothing to do with the structural bias of $\tilde{k}_2$ presented in (15).

For negative β (Muth model) and low positive values of β (Lucas model), the power is very high, but decreases to 10% as β approaches unity[12] (Table B.2). The bias in the estimated parameters of the rationality test exists in this case as

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well [see (15)], but it is more moderate than in Table B.1. From (15) the bias in $\tilde{k}_2$ in the New Classical model depends on the dynamics of $\tilde{\alpha}, \tilde{\varphi}$, which are different from $\tilde{\alpha}, \tilde{\gamma}$ in the Muth and Lucas models (implying different dynamics of the endogenous variable as well).

5.2. Naïve Expectations

The power of the rationality test in the New Classical model stands at about 32% for all values of β (Table B.1). In contrast, the power in the Muth and Lucas models is low only for high values of β (Table B.2). The reason for this difference stems from the bias of the parameters in the rationality test in the New Classical model shown in (14), which does not exist in Muth and Lucas models. In the New Classical model the bias exists because the naïve expectations are positively correlated with RE (both of them depend on y_{t-1}). In contrast, in the Muth and Lucas models the naïve expectations are orthogonal to RE (the latter depends on w_{t-1} and constant).

5.3. Ad Hoc Expectations

For low values of β, the power of the rationality test with ad hoc expectations of type 1 is perfect (Tables B.1 and B.2). The Muth model demonstrates complete robustness to different types of expectations, as expected. However, when β approaches unity from below, the power of all models falls to 15–20%. We can also observe that the estimated slope (and intercept) in the rationality test is unbiased [as shown in (12)], because the ad hoc expectations of type 1 are orthogonal to the RE.

The power of the test with ad hoc expectations of type 2 relative to type 1 is lower for high β. This is because the estimated parameters in the rationality test are biased [as shown in (15)] because of the positive correlation between the ad hoc expectations and RE (both of them depend either on y_{t-1} in Table B.1 or on w_{t-1} in Table B.2).

6. IMPLICATIONS OF MISINTERPRETATION OF THE RATIONALITY TEST

The erroneous interpretation of not rejecting the rationality hypothesis may have serious implications on central banks' monetary policy and the welfare of the public. One example is the optimal monetary policy literature, which mostly assumes rationality as we defined it previously. The seminal paper of Clarida et al. (1999) assumed RE when deriving and analyzing optimal monetary policy under discretion and commitment. Dennis (2007) proposed a general algorithm for deriving optimal monetary policy under discretion and commitment (this algorithm recently was installed in DYNARE, a program widely used in central banks). This algorithm assumes RE exactly in the same manner as we referred to here. If the

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central bank mistakenly applies the Dennis (2007) algorithm, justified erroneously by the rationality test results although expectations are nonrational, the derived policy rule may cause welfare losses. This is because the derivation of the policy rule and the analysis accompanying it are based on the rationality assumption, which does not hold. Moreover, Evans and Honkapohja (2003) showed that if the central bank utilized the policy rule implied by the RE assumption, whereas the public is only boundedly rational, such a rule would lead to divergence of the economy accompanied by enormous welfare losses.

Another implication exists in the asset pricing literature. For example, Andreasen (2012) and Rudebusch and Swanson (2012) applied a micro-founded approach to derive a term premium on bonds from a DSGE model. The main point is that the solution for the term premia from the model, as well as for other variables, relied on the RE assumption. Relying on such an assumption, if it does not hold, may lead to a bias in the estimated term premium of the bonds and all other outputs of the model.

7. CONCLUSIONS

The central message of this paper is both straightforward and fundamental. In a model in which expectations are formed in the period previous to the current period, and in which expectations strongly and positively affect the variable being forecasted, the standard rationality test suffers from low power. More specifically, the standard rationality test has difficulty in disentangling rational expectations from several types of nonrational expectations. This finding has serious consequences for applications in economics that rely on the rationality assumption of the public. For example, in a monetary policy framework, an erroneous assumption of RE may cause a bias in monetary policy rules, leading to welfare losses in the economy. Similarly, this could create a bias in the evaluation of the risks of financial assets obtained from microfounded DSGE models, because such evaluation typically relies on the assumption that the public has fully rational expectations.

NOTES

1. To ensure stationarity of the RE solution, $|\varphi^*| < 1$ must hold, implying that $-(1-\beta) < \varphi < 1-\beta$. To ensure E-stability of the RE solution, $\beta < 1$ must hold [see Evans and Honkapohja (2001)].

2. We do not consider the case where agents learn with the decreasing gain $1/t$ because at some stage the expectations converge to the REE and therefore do not contribute to the analysis of the paper.

3. In many papers [for example, Branch and Evans (2006)] it was found that the CG explains the data better than the decreasing gain $(1/t)$.

4. Equation (10) also holds when $E_{t-1}^{NR}[y_t] = E_{t-1}^{R}[y_t]$, that is, when expectations are rational. In such a case, it can be shown that the expectations $E_{t-1}^{NR}[y_t] = E_{t-1}^{R}[y_t]$ are positively correlated with the residual u_t, so that $p\lim(\tilde{k}_1, \tilde{k}_2) = (0, 1)$, consistent with the validity of the unbiasedness test for RE.

5. The residuals u_t exclude the demeaned value of $(1-\beta)E_{t-1}^{R}[y_t]$, which is $\frac{(1-\beta)\mu}{1-\beta-\varphi}$.

6. Negative β appears, for example, in the Muth model [Evans and Honkapohja (2001)], which is examined in Section 5. We consider cases where $-1 < \beta < 1$, because this range covers the majority

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of economic applications, although it should be noted that $\beta < -1$ arises in the Muth cobweb model if the supply elasticity is sufficiently high.

7. In the notation of Orphanides and Williams (2005), $\delta = \theta\alpha$. We set $\theta = 0.5$ and, $\alpha = 0.12$ which are the average values of θ, α assumed by Orphanides and Williams. $\pi^* = 0.02$ is the inflation target.

8. This value of the gain is reasonable in many applications [for example, in Branch and Evans (2006), $g = 0.05$ fitted the inflation data in the United States].

9. Negative values of β in Table B.2 are related to Model 2 and positive values of β are related to Model 3.

10. The unbiasedness hypothesis $k_1 = 0$, $k_2 = 1$ was tested using an F-test. The serial correlation in the forecast errors was tested using the Box–Pierce test.

11. A low variation in $E_{t-1}^{BR}[y_t]$ could stem from the way we calculated φ^* in the calibration of the New Classical model: the higher β, the lower φ^*. For example, if $\beta = 0.7$, then $\varphi^* = 0.8$, and when $\beta = 0.9$, then $\varphi^* = 0.4$. Because the prior for $\tilde{\varphi}$ is randomly chosen from the neighborhood of φ^*, there is a greater likelihood of $\tilde{\varphi}$ as a prior, and in subsequent periods as an estimator, to be close to null when β is high, relative to cases when β is low. This could lead in some random simulations to a very low variation of $E_{t-1}^{BR}[y_t]$ over time, because they barely react to y_{t-1} when $\tilde{\varphi}$ is close to null ($E_{t-1}^{BR}[y_t] = \tilde{\alpha} + \tilde{\varphi} y_{t-1}$). Once the variation in $E_{t-1}^{BR}[y_t]$ is noticeably lower than the variation of u_t in (10), in small samples $\tilde{k}_2$ can be zero or even negative. Thus, for $\beta = 0.9$, the average slope in the rationality test in Table B.1 is slightly lower than for $\beta < 0.9$.

12. For boundedly rational expectations (learning with constant gain), the power of the rationality test is noticeably reduced even for low positive, or negative, values of β, if the agents' initial prior is close to RE and the value of constant gain is small. To demonstrate, for $\beta = -0.5$ and constant gain equal to 0.01, the power of the rationality test declines to 20%, in contrast to 93% in Table B.2.

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APPENDIX A: ANALYTICAL DERIVATIONS AND PROOFS

A.1. A DECOMPOSITION OF y INTO RATIONAL AND NONRATIONAL EXPECTATIONS: THE MODEL WITH $t-1$ DATING

When expectations are rational, it follows from (2) that

$$(1-\beta)E^{\mathrm{R}}_{t-1}[y_t] = \mu + \varphi y_{t-1} + \gamma w_{t-1}. \tag{A.1}$$

Subtracting (A.1) from both sides of (2), we get the result (assuming any type of nonrational expectations $E_{t-1}[y_t] = E^{\mathrm{NR}}_{t-1}[y_t]$)

$$y_t = (1-\beta)E^{\mathrm{R}}_{t-1}[y_t] + \beta E^{\mathrm{NR}}_{t-1}[y_t] + \varepsilon_t. \tag{A.2}$$

■

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A.2. DERIVING THE UNCONDITIONAL MEAN OF THE RE SOLUTION

After taking the RE solution in (3) for period $t-1$, and applying unconditional expectations, we get

$$E\left(y_{t-1}^{\mathrm{R}}\right) = \frac{\mu}{1-\beta} + \frac{\varphi}{1-\beta}E(y_{t-2}) + \frac{\gamma}{1-\beta}E(w_{t-2}) + E(\varepsilon_{t-1}), \tag{A.3}$$

where $E(w_{t-2}) = E(\varepsilon_{t-1}) = 0$. After some algebra on (A.3) we get

$$E\left(y_{t-1}^{\mathrm{R}}\right) = \frac{\mu}{1-\beta-\varphi}. \tag{A.4}$$

■

APPENDIX B: AN EXAMINATION OF THE RATIONALITY TEST USING MONTE CARLO SIMULATIONS

TABLE B.1. The results of the rationality test under different types of expectations and parameter β, $T = 100$: Model 1 (a New Classical model)

Learning algorithm of agents	Impact of expectations (β)					
	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
Learning with constant gain (gain = 0.05)	22.1 0.0 0.94 0.06	16.0 0.0 0.92 0.04	13.4 0.01 0.92 0.04	12.7 0.0 0.88 0.02	12.0 0.01 0.84 0.01	17.2 0.01 0.72 −0.01
Naïve expectations	34.1 −0.01 0.89 −0.03	32.8 0.0 0.90 −0.03	36.3 0.0 0.89 −0.03	34.2 0.0 0.89 −0.03	34.7 0.0 0.90 −0.03	35.3 0.0 0.89 −0.03
Ad hoc expectations (type 1)	100 0.01 0.1 0.63	100 0.01 0.29 0.43	99.9 0.0 0.5 0.29	86 0.0 0.70 0.18	56 0.0 0.79 0.11	20.3 0.0 0.90 0.04
Ad hoc expectations (type 2)	99.9 −0.05 0.61 0.0	99.3 0.02 0.71 0.0	97.1 −0.01 0.78 0.0	72.6 0.0 0.86 0.0	38.4 0.0 0.89 0.0	15.7 0.0 0.92 −0.01

Note: The top number in each cell represents the power of the rationality test; below are presented the estimated intercept $\tilde{k}_1$ and the slope $\tilde{k}_2$ (the mean values over all simulations), and the value at the bottom presents the first-order serial correlation in the forecast errors (the mean over all simulations).

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TABLE B.2. The results of the rationality test under different types of expectations and parameter β, $T = 100$: Model 2 (Muth model, $\beta < 0$) and Model 3 (Lucas aggregate supply model, $\beta > 0$)

	Impact of expectations (β)						
Learning algorithm of agents	$\beta = -0.99$	$\beta = -0.5$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 0.99$
Learning with constant gain (gain = 0.05)	95 0.01 −0.74 0.06	93.1 0.01 −0.23 0.02	70.4 0.01 0.54 0.07	61.3 0.01 0.72 0.07	45.8 0.0 0.92 0.06	22 0.0 1.0 0.02	10.1 0.0 1.0 0.0
Naïve expectations	100 0.01 −0.98 −0.99	100 0.01 −0.46 −0.74	100 0.01 0.34 −0.33	100 0.01 0.54 −0.22	100 0.01 0.72 −0.12	45 0.01 0.90 −0.01	19.3 0.01 0.95 0.05
Ad hoc expectations (type 1)	100 0.01 −0.99 0.01	100 0.01 −0.50 0.02	100 0.01 0.30 0.04	99.3 0.01 0.50 0.04	73.6 0.01 0.70 0.05	22.1 0.01 0.90 0.05	14.8 0.01 0.99 0.05
Ad hoc expectations (type 2)	100 0.01 −0.93 0.0	100 0.01 −0.42 0.0	100 0.01 0.43 0.0	96.6 0.01 0.64 0.0	60.2 0.01 0.84 0.0	21.6 0.01 0.97 0.0	17.6 0.01 1.0 0.0

Note: The top number in each cell represents the power of the rationality test; below are presented the estimated intercept $\tilde{k}_1$ and the slope $\tilde{k}_2$ (the mean values over all simulations), and the value at the bottom presents the first-order serial correlation in the forecast errors (the mean over all simulations).