TWO-DIMENSIONAL POTENTIAL PROBLEMS

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1. Introduction

The solution of problems in two-dimensional potential theory depends, generally speaking, on the discovery of a conformal transformation suitable for the particular region involved, and the degree of success in the applicability of the method seems to be seriously limited by the complexity of the transformation. Only one serious attempt seems to have been made, by W. G. Bickley*, to formulate a general theory coordinating the different types of boundary value problem for general and inclusive classes of curves, but Bickley's analysis is somewhat involved and not very suitable for certain types of problem. In a re-examination of the subject on the basis of certain suggestions made to me by Prof. Livens, I have discovered what appears to be a much simpler and equally general method of formulating directly the solution for a comprehensive class of problem dealing with simple closed cylinders with curved or rectilinear boundaries, and including all those cases for which solutions are known. In its theoretical aspects the method seems to be perfectly general, but to put it into practice it is necessary to determine explicitly the particular form for the conformal transformation of the space outside the cross-section of the cylinder to the space outside a circle with the points at infinity coinciding.

The method in essence depends on expressing the equation of the cross-section curve on the (x, y) plane in the parametric form

$$z = x + iy = e^{-i\xi} f(e^{i\xi}) = e^{-i\xi} \sum_{n=0}^{\infty} a_n e^{ni\xi},$$

where the a's are in general complex.

Bickley, in the paper referred to above, introduces the variable ξ in some of his geometrical discussions but rejects it as unsuitable in the physical discussion. His choice of alternatives and the fact that in a large part of his analysis he still uses the real forms of the general complex expressions, explain why his analysis is more involved than that which follows.

* W. G. Bickley, *Phil. Trans. Roy. Soc.* 228 (1929), 235.

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2. The equation of the cylinder

The equation of the cylinder is taken in the form

$$z = e^{-i\xi}\left(\sum_{n=0}^{\infty} a_n e^{ni\xi}\right).$$

This form, which was actually introduced by D. M. Wrinch in some of her work*, includes most of the cases for which solutions are known, viz.

(*a*) The text-book case of the ellipse

$$z = x+iy = a\cos\xi + ib\sin\xi$$
$$= \tfrac{1}{2}e^{-i\xi}\{(a-b)+(a+b)\,e^{2i\xi}\}$$

including, when $b = 0$, the flat plate.

(*b*) The more general cylinder also treated by Dr Wrinch†

$$z = nae^{-i\xi} + be^{ni\xi}.$$

(*c*) The cylinders whose sections are members of the cardioid family given by

$$z = -c\sec^2\tfrac{1}{2}(\xi+i\alpha)$$

or by

$$z = c\{1+e^{i(\xi+i\alpha)}\}^2.$$

(*d*) The cylinder whose section is one loop of the lemniscate

$$z = c(1+e^{4i\xi})^{\frac{1}{2}}.$$

(*e*) The general cylinder whose section is a simple closed polygonal boundary for which

$$z = A\int^{\xi}\prod_{r=1}^{n}\{\sin\tfrac{1}{2}(\xi-\xi_r)\}^{\frac{\alpha_r}{\pi}-1}d\xi$$
$$= A'\int^{\xi}e^{-i\xi}\prod_{r=1}^{n}\{1-e^{i(\xi-\xi_r)}\}^{\frac{\alpha_r}{\pi}-1}d\xi,$$

wherein the α_r's are the external angles of the polygon, so that $\Sigma\alpha_r = (n+2)\pi$, and the ξ_r's are constants. Particular cases of this transformation are those appropriate to

(i) the rectangle $\quad z = A\int\sqrt{(\cos 2\xi - \cos 2\alpha)}\,d\xi;$

(ii) the regular n-sided polygon

$$z = A\int^{\xi}(\sin\tfrac{1}{2}n\xi)^{2/n}\,d\xi$$
$$= A'\int^{\xi}e^{-i\xi}(1-e^{ni\xi})^{2/n}\,d\xi;$$

* D. M. Wrinch, *Phil. Mag.* 49 (1925), 240.
† D. M. Wrinch, *Phil. Mag.* 48 (1924), 692.

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(iii) the n-bladed propeller recently examined by Westwater*

$$z = c(\cos \tfrac{1}{2}n\xi)^{2/n}$$

or
$$z = 2^{-2/n}ce^{-i\xi}\left\{1 + \sum_{r=1}^{\infty}(-)^{r+1}\frac{2(n-2)\ldots\{(r-1)n-2\}}{n^r r!}e^{rni\xi}\right\};$$

(iv) the "keel and rudder" surface treated by S. D. Daymond and L. Rosenhead†
$$z = Ae^{-i\xi}(e^{i\xi} - e^{-i\alpha})^{1+k}(e^{i\xi} - e^{i\alpha})^{1-k}.$$

Most of these special cases are mentioned by Bickley as being cases which admit of treatment by his method. Finally there is included

(f) The cylinder with the thin airfoil cross-section of which a simple particular case is provided by the Joukowski airfoils,

$$z = z_0 + ae^{-i\xi} + b^2/(z_0 + ae^{-i\xi}), \quad |z_0| < a$$
$$= ae^{-i\xi} + z_0 + \frac{b^2e^{i\xi}}{a}\left\{1 + \sum_{n=1}^{\infty}(-)^n\frac{z_0^n e^{ni\xi}}{a^n}\right\}.$$

3. THE PROBLEMS EXAMINED

The new method of procedure involved is illustrated sufficiently by examining in detail two simple electrical problems; these are the induction problems of the general cylinder:

(i) in a uniform field,

(ii) in the field of a line charge parallel to its length.

The solution of the simpler problem of the single conductor freely charged is, of course, immediately obvious, as is also the solution of the problem involving two of the equipotential surfaces at different potentials. The corresponding hydrodynamical applications of the method are being amply covered in a general discussion of airfoil theory to be published separately, but for completeness in the present discussion and with a view to connecting up with the "internal" problems where the hydrodynamical aspects are the main interest, we propose briefly to review the results which have been obtained in this case. Use is also made of certain results recently obtained, in which it is shown that the components X, Y of the force per unit length on a cylindrical conductor in an electrostatic field are obtained as the constituents of a simple contour integral in the form‡

$$Y + iX = \frac{1}{8\pi}\int_c \left(\frac{\partial\Omega}{\partial z}\right)^2 dz,$$

* F. L. Westwater, *Proc. Camb. Phil. Soc.* 32 (1936), 676.

† S. D. Daymond and L. Rosenhead, *Proc. Camb. Phil. Soc.* 33 (1937), 62. Cf. also Morton, *Phil. Mag.* 22 (1936), 337.

‡ *Phil. Mag.* 23 (1937), 246.

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taken round the contour of the cylinder, where Ω is the complex potential function of the field given by

$$\Omega = \phi + i\psi,$$

ϕ being the potential and the curves $\psi =$ constant being the lines of force.

Also the couple on the cylinder is obtained in the same way, as the real part of the integral

$$\frac{1}{8\pi}\int_c \left(\frac{\partial \Omega}{\partial z}\right)^2 z\,dz,$$

taken round the contour of the cylinder. Throughout the work the complex potential is used consistently, and in fact the essence of the new method of approach consists in using this more general potential in a form which enables its characteristics to be identified more or less by inspection.

4. The general cylinder in a uniform field

The complex potential of a uniform field of intensity E in a direction θ with the x-axis is given by

$$\Omega_0 = -Ee^{-i\theta}z.$$

The transformation for the cylinder is

$$z = e^{-i\zeta}\sum_{n=0}^{\infty} a_n e^{ni\zeta}, \quad \eta = 0,$$

where $\zeta = \xi + i\eta$.

The general undisturbed potential function in terms of ζ is then given by

$$\Omega_0 = -E\sum_{n=0}^{\infty} a_n e^{-i\theta+(n-1)i\zeta}.$$

The new potential when the cylinder is introduced must be such that it is purely imaginary on the cylinder, where the real potential is zero, and the undisturbed form must be preserved at infinity ($\eta = +\infty$), where the effect of the cylinder is zero. Thus we can only add terms which reduce to zero at infinity. Now Ω_0 consists of two types of terms which we shall, for the present, call $F_1(\zeta)$ and $F_2(\zeta)$ and which are distinguished by their behaviour at infinity. $F_1(\zeta)$ is such that it tends to zero at infinity and has a conjugate which tends to infinity there, while $F_2(\zeta)$ becomes infinite at infinity and has a conjugate which tends to zero there. To make Ω, the new potential, purely imaginary on the cylinder where ζ is real, we may subtract the conjugate of the above expression, but the conjugate of $F_1(\zeta)$, being infinite at infinity, introduces terms of the wrong type so that instead of subtracting the conjugate of $F_1(\zeta)$ we subtract this term altogether. Thus

$$\begin{aligned}\Omega &= F_1(\zeta) + F_2(\zeta) - \{F_1(\zeta) + \overline{F}_2(\zeta)\}\\ &= F_2(\zeta) - \overline{F}_2(\zeta).\end{aligned}$$

In the above case

$$F_2(\zeta) = -Ea_0 e^{-i\theta - i\zeta},$$

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so that finally* $$\Omega = -Ea_0 e^{-i\theta - i\zeta} + E\bar{a}_0 e^{i\theta + i\zeta},$$

is the complete expression for the complex potential of the disturbed field.

The components X, Y of the force per unit length on the cylinder are, as mentioned above, now given by

$$Y + iX = \frac{1}{8\pi}\int_c \left(\frac{d\Omega}{dz}\right)^2 dz = \frac{1}{8\pi}\int_0^{2\pi} \left(\frac{d\Omega}{d\zeta}\right)^2 \frac{d\zeta}{dz/d\zeta}.$$

We have, however, $$\frac{d\Omega}{d\zeta} = iEa_0 e^{-i\theta - i\zeta} + iE\bar{a}_0 e^{i\theta + i\zeta},$$

and $$\frac{dz}{d\zeta} = -ia_0 e^{-i\zeta} + ia_2 e^{i\zeta} + 2ia_3 e^{2i\zeta} + \ldots.$$

Therefore $$Y + iX = -\frac{iE^2}{8\pi}\int_0^{2\pi} \frac{(a_0 e^{-i\theta - i\zeta} + \bar{a}_0 e^{i\theta + i\zeta})^2}{a_0 e^{-i\zeta} - a_2 e^{i\zeta} - 2a_3 e^{2i\zeta} \ldots} d\zeta.$$

By expanding the denominator it can be seen that every term in this integrand is of the type $Ke^{ni\zeta}$, where K is a constant, and therefore the value of the integral is obviously zero.

The couple is the real part of the integral

$$\frac{1}{8\pi}\int_c \left(\frac{d\Omega}{dz}\right)^2 z\,dz = \frac{1}{8\pi}\int_0^{2\pi} \left(\frac{d\Omega}{d\zeta}\right)^2 z \frac{d\zeta}{dz/d\zeta},$$

and, as above, this is

$$-\frac{iE^2}{8\pi}\int_0^{2\pi} \frac{(a_0 e^{-i\theta - i\zeta} + \bar{a}_0 e^{i\theta + i\zeta})^2 (a_0 e^{-i\zeta} + a_1 + a_2 e^{i\zeta} + \ldots)}{a_0 e^{-i\zeta} - a_2 e^{i\zeta} - 2a_3 e^{2i\zeta} \ldots} d\zeta,$$

which gives the couple as the real part of

$$-\tfrac{1}{2}iE^2 a_0(\bar{a}_0 + a_2 e^{-2i\theta}).$$

This involves only two of the coefficients in the expansion of z and is therefore easily obtained even in complicated cases. It shows, for example, that the couple always vanishes for any regular polygon with more than two sides, however it is placed in the field. The general result for a rectangle, the angle between the diagonals being 2α, is of the form

$$\tfrac{1}{2}E^2 A^2 \cos 2\alpha \sin 2\theta,$$

A being a constant depending on the dimensions of the rectangle.

5. THE LINE CHARGE PROBLEM

Suppose that the line charge of strength q, parallel to the generators of the cylinder, passes through the point z_0 outside the cylinder. The undisturbed complex potential function is now given by

$$\Omega_0 = \frac{q}{2\pi}\log(z - z_0).$$

* Conforming to the usual practice a bar over a letter denotes the conjugate complex quantity.

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In terms of the complex function ζ, we then have, using ζ_0 for the point on the ζ plane corresponding to z_0,

$$\Omega_0 = \frac{q}{2\pi}\log\{(a_0 e^{-i\zeta} + a_1 + a_2 e^{i\zeta} + \ldots) - (a_0 e^{-i\zeta_0} + a_1 + a_2 e^{i\zeta_0} + \ldots)\}$$

$$= \frac{q}{2\pi}\log(e^{-i\zeta} - e^{-i\zeta_0})\{a_0 - a_2(e^{2i\zeta_0 + i\zeta} - e^{i\zeta_0 + 2i\zeta})\ldots\}$$

$$= \text{constant} + \frac{q}{2\pi}\log\{1 - e^{-i(\zeta-\zeta_0)}\} + \frac{q}{2\pi}\log\{a_0 - \ldots\}.$$

Proceeding along the same lines as sketched above, we construct from this function another function which is imaginary on the cylinder and reduces to the same form as Ω_0 at infinity. This is secured by cancelling the second logarithm term altogether and subtracting the conjugate of the first. We then have finally

$$\Omega = \text{constant} + \frac{q}{2\pi}\log\left[\frac{1 - e^{-i(\zeta-\zeta_0)}}{1 - e^{i(\zeta-\bar{\zeta}_0)}}\right].$$

To find the forces and couple on the cylinder we require the value of $d\Omega/d\zeta$, which is easily seen to be

$$-\frac{iq}{2\pi}\left\{1 - \frac{\sinh\eta_0}{\cosh\eta_0 - \cos(\zeta - \xi_0)}\right\}.$$

The components of the force on the cylinder are again given by

$$Y + iX = \frac{1}{8\pi}\int_0^{2\pi}\left(\frac{d\Omega}{d\zeta}\right)^2\left(\frac{d\zeta}{dz/d\zeta}\right),$$

and therefore we want the value of

$$\int_0^{2\pi}\left\{1 - \frac{\sinh\eta_0}{\cosh\eta_0 - \cos(\zeta - \xi_0)}\right\}^2 \frac{d\zeta}{dz/d\zeta}.$$

This integral is best evaluated by turning it into a simple contour integral in the ζ-plane, taken round the rectangle of sides $\eta = 0$, $\xi = 2\pi$, $\eta = \infty$, $\xi = 0$. The integrand has the same value at two corresponding points $i\eta$ and $2\pi + i\eta$, so that the integrals along $\xi = 0$ and $\xi = 2\pi$ cancel. The only pole inside the contour is $\zeta = \zeta_0$ and the behaviour at infinity is regular, so that the value of the integral itself is determined by the residue at the pole.

Writing $\zeta = \zeta_0 + t$, where t is small, we find approximately

$$\left\{1 - \frac{\sinh\eta_0}{\cosh\eta_0 - \cos(\zeta - \xi_0)}\right\}^2 = -\frac{1}{t^2}\{1 - 2it(1 - \tfrac{1}{2}\coth\eta_0)\},$$

and

$$\frac{dz}{d\zeta} = A + Bt,$$

where

$$A = i(-a_0 e^{-i\zeta_0} + a_2 e^{i\zeta_0} + 2a_3 e^{2i\zeta_0} + \ldots),$$

$$B = -(a_0 e^{-i\zeta_0} + a_2 e^{i\zeta_0} + 2^2 a_3 e^{2i\zeta_0} + \ldots).$$

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With these values it is now easy to see that the residue of the integral is

$$-\frac{1}{A}\left(i\coth\eta_0-2i-\frac{B}{A}\right),$$

so that the components of the force on the cylinder are given by

$$Y+iX=\frac{q^2}{16\pi^2A}\left(\coth\eta_0-2+i\frac{B}{A}\right).$$

To find the couple on the cylinder in this case we can use the contour integral, but it is easier to use the fact that the force on the cylinder is equal and opposite to the force on the line charge. The resultant of the forces on the cylinder must therefore act through the point z_0, so that the moment about the centre, or the couple on the cylinder in the usual sense is

$$Yx_0-Xy_0,$$

which is the real part of

$$(Y+iX)(x_0+iy_0)=(Y+iX)z_0.$$

In this case the couple on the cylinder is therefore the real part of

$$\frac{q^2}{16\pi^2A}\left(\coth\eta_0-2+\frac{iB}{A}\right)(a_0e^{-i\zeta_0}+a_1+a_2e^{i\zeta_0}+\ldots),$$

and is immediately determined once the coefficients in the equation of the cylinder section are known.

6. THE INCLUSION OF INTRINSIC CHARGE ON THE CYLINDER

We can also include in these discussions the cases when the cylinder itself carries an initial charge say Q per unit length. The conditions which the potential functions must satisfy are now modified to the extent that at infinity the potential of the cylinder must behave like $-2Q\log z$. At a great distance from the origin

$$z=a_0e^{-i\zeta},$$

so that

$$\log z=-i\zeta+\text{constant}.$$

Thus all we have to do is to introduce in Ω in both cases a term

$$2Qi\zeta,$$

so that, for example, in the first case when the cylinder is in a uniform field the potential function is given by

$$\Omega=-Ea_0e^{-i\theta-i\zeta}+E\bar{a}_0e^{i\theta+i\zeta}+2Qi\zeta,$$

and in the second case when the initial field is due to a line charge

$$\Omega=\text{constant}+\frac{q}{2\pi}\log\left[\frac{1-e^{-i(\zeta-\zeta_0)}}{1-e^{i(\zeta-\bar{\zeta}_0)}}\right]+2Qi\zeta.$$

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The presence of this extra term considerably influences the force and couple. These results can, however, still be calculated in the same way as before and, taking first the cylinder in the uniform field, the components of the force are given by

$$Y+iX=-\frac{i}{8\pi}\int_0^{2\pi}\frac{\{Ea_0e^{-i\theta-i\zeta}+E\bar{a}_0e^{i\theta+i\zeta}+2Q\}^2}{a_0e^{-i\zeta}-a_2e^{i\zeta}-2a_3e^{2i\zeta}\ldots}d\zeta,$$

which, by expanding the denominator, can easily be found to give

$$Y+iX=-iQEe^{-i\theta},$$

that is,

$$Y=-QE\sin\theta,\quad X=-QE\cos\theta.$$

The couple is, in the same way, the real part of

$$-\frac{i}{8\pi}\int_0^{2\pi}\frac{\{Ea_0e^{-i\theta-i\zeta}+E\bar{a}_0e^{i\theta+i\zeta}+2Q\}^2\{a_0e^{-i\zeta}+a_1+a_2e^{i\zeta}+\ldots\}}{a_0e^{-i\zeta}-a_2e^{i\zeta}-2a_3e^{2i\zeta}\ldots}d\zeta,$$

which is the real part of

$$-\tfrac{1}{2}i(E^2a_0a_2e^{-2i\theta}+E^2a_0\bar{a}_0+2QEa_1e^{-i\theta}),$$

and is therefore still completely and simply determinate. In the case of the cylinder in the field due to a line charge we have similarly

$$Y+iX=-\frac{1}{8\pi}\int_0^{2\pi}\left[\frac{q}{2\pi}\left\{1-\frac{\sinh\eta_0}{\cosh\eta_0-\cos(\zeta-\xi_0)}\right\}-2Q\right]^2\frac{d\zeta}{dz/d\zeta}.$$

We therefore need, in addition to the terms obtained before,

$$\frac{Qq}{4\pi^2}\int_0^{2\pi}\left\{1-\frac{\sinh\eta_0}{\cosh\eta_0-\cos(\zeta-\xi_0)}\right\}\frac{d\zeta}{dz/d\zeta}.$$

Evaluating this integral again as a contour integral in the ζ-plane, we find that it is

$$-\frac{Qq}{2\pi A}.$$

The term

$$\frac{Q^2}{2\pi}\int_0^{2\pi}\frac{d\zeta}{dz/d\zeta}$$

is zero if we assume that there are no singular points of the transformation outside the cylinder. The components of the force on the cylinder are now given by

$$Y+iX=\frac{q^2}{16\pi^2A}\left(\coth\eta_0-2+\frac{iB}{A}\right)-\frac{Qq}{2\pi A}.$$

The couple, as previously, is the real part of $(Y+iX)z_0$, and is again obtained in explicit form.

7. The hydrodynamical cases

In these cases we shall, as previously stated, content ourselves simply with quoting results which are readily obtained by exactly the same procedure as that employed above.

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(*a*) *Cylinder in a uniform stream*

If the stream is given by $\Omega_0 = -\overline{W}z,$

where $\overline{W} = U - iV$ is the conjugate of the complex velocity of the stream at a distance, then for the complete stream when the cylinder is present we have

$$\Omega = -\overline{W}a_0 e^{-i\zeta} - \overline{W}a_1 - W\bar{a}_0 e^{i\zeta},$$

which is real on the cylinder where $\psi = 0$ and reduces to Ω_0 at an infinite distance. Including as well a circulation κ round the cylinder so that

$$\Omega = -\overline{W}a_0 e^{-i\zeta} - \overline{W}a_1 - \overline{W}\bar{a}_0 e^{i\zeta} + \frac{\kappa\zeta}{2\pi},$$

we find the components X, Y of the force on the cylinder, by using the usual Blasius contour integrals, to be given by

$$Y + iX = \rho\kappa\overline{W},$$

and the couple as the real part of

$$\rho\kappa\overline{W}a_1 + 2\pi\rho i\overline{W}^2 a_0 a_2.$$

(*b*) *Cylinder in motion in infinite liquid*

When the cylinder is moving perpendicular to its axis with velocity components u, v along axes Ox, Oy on a normal cross-section of the cylinder, and has an angular velocity ω about any normal to this cross-section, we have to determine a complex potential function Ω such that, on the cylinder, the stream function ψ satisfies the condition

$$\psi = uy - vx - \tfrac{1}{2}\omega(x^2 + y^2) + \text{constant}.$$

It is easy to see that this is satisfied by the imaginary part of

$$\Omega = \bar{w}z(\zeta) - \tfrac{1}{2}i\omega z(\zeta)\,\bar{z}(\zeta) + F(\zeta),$$

where $\bar{w} = u - iv$, and the function $F(\zeta)$ is real when ζ is real. The determination of $F(\zeta)$ to satisfy the condition that Ω is nowhere infinite and vanishes at infinity, follows the lines sketched above. The final result for the given cylinder, including also a circulation κ round it, is that

$$\Omega = \sum_{n=0}^{\infty} \Omega_n e^{ni\zeta} + \frac{\kappa\zeta}{2\pi},$$

where

$$\Omega_0 = ca_1\bar{w} - i\omega c^2 b_0, \quad b_r = \sum_{n=0}^{\infty} a_{n+r}\bar{a}_n,$$

$$\Omega_1 = c(a_2\bar{w} - \bar{a}_0 w) - i\omega c^2 b_1,$$

$$\Omega_n = ca_{n+1}\bar{w} - i\omega c^2 b_n, \quad n > 1.$$

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The force and the couple on the cylinder are more difficult to obtain, but I have found that the components of the force are given by

$$Y+iX=-\tfrac{1}{2}\rho\int_C\left\{\frac{\partial\Omega}{\partial w}\frac{dw}{dt}+\frac{\overline{\partial\Omega}}{\partial w}\frac{d\bar{w}}{dt}+\frac{\partial\Omega}{\partial\bar{w}}\frac{d\bar{w}}{dt}+\frac{\overline{\partial\Omega}}{\partial\bar{w}}\frac{dw}{dt}\right.$$
$$\left.+\frac{\partial\Omega}{\partial\omega}\frac{d\omega}{dt}+\frac{\overline{\partial\Omega}}{\partial\omega}\frac{d\omega}{dt}+\frac{\partial\Omega}{\partial\kappa}\frac{d\kappa}{dt}+\frac{\overline{\partial\Omega}}{\partial\kappa}\frac{d\kappa}{dt}\right\}d\bar{z}$$
$$+\tfrac{1}{2}\rho\int_C(w+i\omega z)(\bar{w}-i\omega\bar{z})\,d\bar{z}-\tfrac{1}{2}\rho\int_C\left(\bar{w}-i\omega\bar{z}-\frac{\partial\Omega}{\partial z}\right)^2dz,$$

and the couple is the real part of

$$-\tfrac{1}{2}\rho\int_C\left\{\frac{\partial\Omega}{\partial w}\frac{dw}{dt}+\frac{\partial\Omega}{\partial\bar{w}}\frac{d\bar{w}}{dt}+\frac{\overline{\partial\Omega}}{\partial w}\frac{d\bar{w}}{dt}+\frac{\overline{\partial\Omega}}{\partial\bar{w}}\frac{dw}{dt}+\frac{\partial\Omega}{\partial\omega}\frac{d\omega}{dt}\right.$$
$$\left.+\frac{\overline{\partial\Omega}}{\partial\omega}\frac{d\omega}{dt}+\frac{\partial\Omega}{\partial\kappa}\frac{d\kappa}{dt}+\frac{\overline{\partial\Omega}}{\partial\kappa}\frac{d\kappa}{dt}\right\}z\,d\bar{z}$$
$$+\tfrac{1}{2}\rho\int_C(w+i\omega z)(\bar{w}-i\omega\bar{z})\,z\,d\bar{z}-\tfrac{1}{2}\rho\int_C\left(\bar{w}-i\omega\bar{z}-\frac{d\Omega}{dz}\right)^2z\,dz.$$

These are generalizations of the contour integral expressions obtained by Blasius and used extensively in the static cases. Using the above form for Ω in these integrals we get immediately

$$Y+iX=\pi\rho i\left(A\frac{d\bar{w}}{dt}-2\bar{B}\frac{dw}{dt}-i\frac{d\omega}{dt}\bar{D}\right)$$
$$+\pi\rho\omega(A\bar{w}-2\bar{B}w-i\omega\bar{D})-i\rho\omega\kappa a_1+\rho\bar{w}\kappa-\rho\frac{d\kappa}{dt}\left(\bar{a}_0+\sum_{n=2}^{\infty}\bar{a}_n\right),$$

and the couple is the real part of

$$\pi\rho C\frac{d\omega}{dt}+i\pi\rho D\frac{d\bar{w}}{dt}+i\pi\rho w(A\bar{w}-2\bar{B}w-i\omega\bar{D})$$
$$-i\rho\omega\kappa b_0+\rho\bar{w}\kappa a_1-\tfrac{1}{2}\frac{d\kappa}{dt}\rho\left\{\sum_{r=1}^{\infty}(b_r+\bar{b}_r)\right\},$$

where
$$A=a_0\bar{a}_0+\sum_{n=1}^{\infty}na_{n+1}\bar{a}_{n+1};$$
$$B=a_0a_2;\qquad\bar{B}=\bar{a}_0\bar{a}_2;\qquad C=\sum_{n=1}^{\infty}nb_n\bar{b}_n;$$
$$D=b_1a_0+\sum_{n=1}^{\infty}n\bar{b}_na_{n+1};\qquad\bar{D}=\bar{b}_1\bar{a}_0+\sum_{n=1}^{\infty}nb_n\bar{a}_{n+1}.$$

(c) *The cylinder in infinite liquid in motion due to a line vortex*

The results for this case are almost identical with those obtained for the electrical charge.

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8. CONCLUSION

It appears to be possible to give a simple formulation of all the standard potential problems for the very general and inclusive type of cylinder with which we started, and, once the equation of the cylinder is known in the form in which we have taken it, the results of all these potential problems can be deduced by mere substitution of the coefficients in the expansion.

Although the general discussion is given in terms of the series expansion of z in powers of $\exp i\zeta$, much of the discussion is independent of the necessity of formulating the actual series. For example, in obtaining the force and couple in the case of the line charge outside the cylinder, it is only the values of $dz/d\zeta$ and $d^2z/d\zeta^2$ at $\zeta = \zeta_0$ that are required. In some cases, however, although it is possible to formulate the discussion in finite terms I have found that the general results can be obtained more readily by using the series expansions.

In cases such as those presented by the problems with rectilinear polygonal boundaries, the method does not apply directly, because the singular points of the transformation occur on the boundary. This difficulty, however, can always be overcome by obtaining first the results for the more general equipotential of the series defined by the transformation obtained by writing $\xi + i\alpha$ for ξ in the equation $z = f(\xi)$ and then letting $\alpha \to 0$ in the final result. This is, for example, equivalent to deducing the case of the flat plate from that of the elliptic cylinder. The application of this process of rounding off the corners of the cylinders involved, which can be applied partially and to different extents at different corners, seems to open up possibilities for obtaining, from a given polygonal boundary, several series of generalized curves for which the solutions of these potential problems are now immediate.

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