References

1. John Derbyshire, Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics *AMC* **10** (2004) p. 12.
2. Marek Capinski and Peter E. Kopp. *Measure, Integral and Probability*, Springer Science & Business Media (2004).

10.1017/mag.2018.126

HAZAR AYDIN

High School Student, Istanbul Köy Hizmetleri Anadolu Lisesi, Anadolu Cd. No:4, 34870 Kartal/Istanbul, Turkey

e-mail: *hazar.aydin@yandex.com*

102.47 An equilateral triangle associated with a hexagon

In [1], Dao Thanh Oai discusses two equilateral triangles associated with an arbitrary hexagon. Our aim in this paper is to display two more equilateral triangles based on the same configuration.

Theorem

Let $A_1A_2A_3A_4A_5A_6$ be a hexagon, with equilateral triangles constructed externally on the sides with apexes B_i (for $1 \leqslant i \leqslant 6$) as shown in Figure 1. Let C_i be the midpoint of B_iB_{i+1} (working throughout with indices modulo 6) and let P_1, P_2 and P_3 be the midpoints of C_1C_4, C_2C_5 and C_3C_6 respectively. Then

(a) the segments C_1C_4, C_2C_5 and C_3C_6 are equal in length;

(b) the triangle $P_1P_2P_3$ is equilateral.

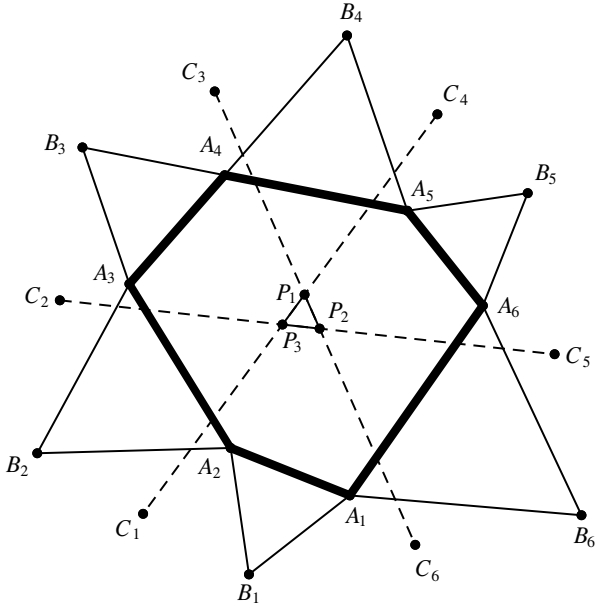


FIGURE 1

https://doi.org/10.1017/mag.2018.128 Published online by Cambridge University Press

Proof:

We use complex numbers, where $\bar{z}$ is the complex conjugate of z. The complex number which represents a point X is called the *affix* of X. In Figure 1, let α_i be the affix of A_i.

The complex number $\omega = e^{\frac{i\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a sixth root of unity, with $\omega^3 = -1$ and $\omega + \bar{\omega} - 1 = 0$. The effect of multiplying z by ω is to rotate it anticlockwise by $\frac{\pi}{3}$.

The affix of the apex B_i is $\beta_i = \alpha_i + \omega(\alpha_{i+1} - \alpha_i) = \bar{\omega}\alpha_i + \omega\alpha_{i+1}$.

The affix of C_i is the mean of those of B_i and B_{i+1}, so this is $\gamma_i = \frac{1}{2}(\bar{\omega}\alpha_i + \alpha_{i+1} + \omega\alpha_{i+2})$.

Now we have distances as follows:

$$C_1C_4 = |\gamma_1 - \gamma_4| = \tfrac{1}{2}|\bar{\omega}(\alpha_1 - \alpha_4) + (\alpha_2 - \alpha_5) + \omega(\alpha_3 - \alpha_6)|$$

$$C_2C_5 = |\gamma_2 - \gamma_5| = \frac{1}{2}|\bar{\omega}(\alpha_2 - \alpha_5) + (\alpha_3 - \alpha_6) - \omega(\alpha_1 - \alpha_4)| = |-\omega||\gamma_1 - \gamma_4|$$

$$C_3C_6 = |\gamma_3 - \gamma_6| = \frac{1}{2}|\bar{\omega}(\alpha_3 - \alpha_6) - (\alpha_1 - \alpha_4) - \omega(\alpha_2 - \alpha_5)| = |-\omega||\gamma_2 - \gamma_5|$$

which are equal as $|-\omega| = 1$. Since $|\gamma_2 - \gamma_5| = |-\omega||\gamma_1 - \gamma_4|$, the angle at P_1 is $\frac{\pi}{3}$, and similarly the angle at P_2 is $\frac{\pi}{3}$, so the triangle $P_1P_2P_3$ is equilateral.

Corollary: The same is true when the equilateral triangles are drawn inwardly from the sides.

Proof: This is just a matter of replacing ω by $\bar{\omega}$ in the previous argument.

The reader can explore similar results based on polygons with an even number of sides with equilateral triangles or squares constructed on the sides.

The author would like to acknowledge the help of an anonymous referee and the Editor in rewriting this paper.

Reference

1. Dao Thanh Oai, Equilateral triangles and Kiepert perspectors in complex numbers, *Forum Geometricorum* **15** (2015) pp. 105-114.

10.1017/mag.2018.128

DASARI NAGA VIJAY KRISHNA
Department of Mathematics, Narayana Educational Institute, Machilipatnam, Bengalore, India
e-mail: *vijay9290009015@gmail. com*

https://doi.org/10.1017/mag.2018.128 Published online by Cambridge University Press