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A characterization of generalized Hughes complexes

By R. Y. SHARP
Pure Mathematics Section, The University, Sheffield S3 7RH
and M. TOUSI
Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395–1795, Tehran, Iran

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0. *Introduction*

While trying to place the grade-theoretic analogue of the Cousin complex (for a commutative Noetherian ring A) of K. R. Hughes [7], and the complexes of modules of generalized fractions studied by the first author and H. Zakeri [18], and by L. O'Carroll [10, p. 420], on a similar footing, Sharp and M. Yassi [15] introduced the concept of generalized Hughes complex. Such complexes are described as follows.

A *system of ideals of* A is a non-empty set Φ of ideals of A such that, whenever $\mathfrak{a}$, $\mathfrak{b} \in \Phi$, there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$.

Such a system of ideals Φ gives rise to an additive, left exact functor

$$D_\Phi := \varinjlim_{\mathfrak{b} \in \Phi} \mathrm{Hom}_A(\mathfrak{b}, \bullet)$$

from $\mathscr{C}(A)$, the category of all A-modules and A-homomorphisms, to itself; D_Φ is called the *generalized ideal transform determined by* Φ, or, more briefly, the Φ-*transform*. There is a morphism of functors $\eta_\Phi : \mathrm{Id} \to D_\Phi$ which is such that, for each A-module G and each $g \in G$ and each $\mathfrak{a} \in \Phi$, the image $\eta_\Phi(G)(g)$ is the natural image in $D_\Phi(G)$ of the A-homomorphism $f_{\mathfrak{a},g} : \mathfrak{a} \to G$ defined by $f_{\mathfrak{a},g}(r) = rg$ for all $r \in \mathfrak{a}$.

Let M be an A-module. To define a generalized Hughes complex for M, we need a family $\mathscr{S} = (\Phi_i)_{i \in \mathbb{N}}$ of systems of ideals of A. (We use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{N}_0$ to denote the set of non-negative integers.) The *generalized Hughes complex for* M *with respect to* $\mathscr{S}$ has the form

$$0 \longrightarrow M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \ldots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \ldots$$

and is described as follows. Write $K^{-2} = 0$, $K^{-1} = M$, and use $h^{-2} : K^{-2} \to K^{-1}$ to denote the zero homomorphism. Then, for all $n \in \mathbb{N}_0$, $K^n := D_{\Phi_{n+1}}(\mathrm{Coker}\, h^{n-2})$, while $h^{n-1} : K^{n-1} \to K^n$ is the composition of the natural epimorphism from K^{n-1} to $\mathrm{Coker}\, h^{n-2}$ and the homomorphism $\eta_{\Phi_{n+1}}(\mathrm{Coker}\, h^{n-2}) : \mathrm{Coker}\, h^{n-2} \to D_{\Phi_{n+1}}(\mathrm{Coker}\, h^{n-2}) = K^n$. This complex is denoted by $\mathscr{H}(\mathscr{S}, M)$ and is a generalization of Hughes' grade-theoretic analogue of the Cousin complex constructed in [7].

These generalized Hughes complexes provide an 'umbrella concept' which covers all the algebraic Cousin complexes previously studied by the first author [11], [12], and all the complexes of modules of generalized fractions of the type described in [10,

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p. 420]: it was proved by Sharp and P. Schenzel[**13**, theorem 2·3] that every Cousin complex for M (of the type described in [**12**, §1]) is (isomorphic to) a generalized Hughes complex for M, and it was proved by Sharp and M. Yassi[**15**, theorem 3·5] that every complex of A-modules of generalized fractions of the above-mentioned type is (isomorphic to) a generalized Hughes complex.

The purposes of this paper are to provide a characterization of generalized Hughes complexes, and to illustrate the use of this by (i) providing rather short proofs of some known results, and (ii) verifying that the Cousin complexes of the type described in [**12**, §1] can be constructed using (an algebraic analogue of) the technique used by C. Bănică and O. Stănăşilă in [**1**, pp. 230–231] (see also R. Hartshorne[**6**, chapter IV]). We shall also show that the algebraic analogue of the above-mentioned technique of Bănică and Stănăşilă, when used in generality comparable to theirs, can produce complexes which are not generalized Hughes complexes.

Throughout this paper, A will denote a commutative Noetherian ring (with non-zero identity), and M will denote an A-module. Also, $\mathscr{C}(A)$ will denote the category of all A-modules and A-homomorphisms, and $V(\mathfrak{b})$, for an ideal $\mathfrak{b}$ of A, will denote the *variety* $\operatorname{Supp}(A/\mathfrak{b})$ of $\mathfrak{b}$.

1. *A characterization of generalized Hughes complexes*

In this section, we answer the following question: what conditions on the terms and homomorphisms in a complex

$$0 \longrightarrow M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \ldots \longrightarrow C^i \xrightarrow{e^i} C^{i+1} \longrightarrow \ldots$$

are necessary and sufficient for the complex to be isomorphic (over Id_M) to the generalized Hughes complex $\mathscr{H}(\mathscr{S}, M)$ for M with respect to the family of systems of ideals $\mathscr{S}$ of A? (To say that two complexes $C^\bullet_\alpha = (C^i_\alpha)_{i \geqslant -2} (\alpha = 1, 2)$ of A-modules and A-homomorphisms of the form

$$0 \xrightarrow{d^{-2}_{C^\bullet_\alpha}} M \xrightarrow{d^{-1}_{C^\bullet_\alpha}} C^0_\alpha \xrightarrow{d^0_{C^\bullet_\alpha}} C^1_\alpha \longrightarrow \ldots \longrightarrow C^n_\alpha \xrightarrow{d^n_{C^\bullet_\alpha}} C^{n+1}_\alpha \longrightarrow \ldots \quad (\alpha = 1, 2)$$

are *isomorphic over* Id_M is to say that there is an isomorphism of complexes $\Psi = (\psi^i)_{i \geqslant -2} : C^\bullet_1 \to C^\bullet_2$ which is such that $\psi^{-1} : M \to M$ is the identity mapping Id_M. Similarly, to say that a morphism of complexes $\Theta = (\theta^i)_{i \geqslant -2} : C^\bullet_1 \to C^\bullet_2$ is *over* Id_M is simply to say that $\theta^{-1} = \mathrm{Id}_M$.)

Reminders **1·1.** (See [**13**, 1·3(i), 1·4(i)].) Let Φ be a system of ideals of A.

(i) The system Φ determines the Φ-*torsion functor* $\Gamma_\Phi : \mathscr{C}(A) \to \mathscr{C}(A)$: see [**15**, 1·1]. This is the subfunctor of the identity functor on $\mathscr{C}(A)$ for which

$$\Gamma_\Phi(G) = \{g \in G : \mathfrak{a}g = 0 \text{ for some } \mathfrak{a} \in \Phi\}$$

for each A-module G. Also, for each $i \in \mathbb{N}_0$, we let $H^i_\Phi := \varinjlim_{\mathfrak{b} \in \Phi} \operatorname{Ext}^i_A(A/\mathfrak{b}, \bullet)$, a functor which is naturally equivalent to the ith right derived functor of the left exact functor Γ_Φ.

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(ii) For each A-module G, there is an exact sequence

$$0 \longrightarrow \Gamma_\Phi(G) \longrightarrow G \xrightarrow{\eta_\Phi(G)} D_\Phi(G) \longrightarrow H^1_\Phi(G) \longrightarrow 0,$$

where $\eta_\Phi(G)$ is as described in the Introduction. Note that

$$\operatorname{Ker} \eta_\Phi(G) = \Gamma_\Phi(G) \quad \text{and} \quad \operatorname{Coker} \eta_\Phi(G) \cong H^1_\Phi(G).$$

(iii) We say that an A-module G is a Φ-*torsion* module if $\Gamma_\Phi(G) = G$. If G is a Φ-torsion A-module, then all submodules and all A-homomorphic images of G are also Φ-torsion. Consequently, for each A-module L, the module $H^i_\Phi(L)$ is Φ-torsion for every $i \in \mathbb{N}_0$.

As preparation for our characterization of generalized Hughes complexes, we begin by noting some properties of such complexes.

Remark 1·2. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A. Write the generalized Hughes complex $\mathcal{H}(\mathcal{S},M)$ (as described in the Introduction) as

$$0 \longrightarrow M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \ldots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \ldots.$$

Also, write $K^{-2} = 0$, $K^{-1} = M$, and use $h^{-2}: K^{-2} \to K^{-1}$ to denote the zero homomorphism.

(i) For each $n \in \mathbb{N}_0$,

$$\operatorname{Coker} h^{n-1} \cong \operatorname{Coker} (\eta_{\Phi_{n+1}}(\operatorname{Coker} h^{n-2})) \cong H^1_{\Phi_{n+1}}(\operatorname{Coker} h^{n-2})$$

(by 1·1(ii)), and so $\operatorname{Coker} h^{n-1}$ is Φ_{n+1}-torsion.

(ii) For each $n \in \mathbb{N}_0$,

$$H^{n-1}(\mathcal{H}(\mathcal{S},M)) = \operatorname{Ker} (\eta_{\Phi_{n+1}}(\operatorname{Coker} h^{n-2})) = \Gamma_{\Phi_{n+1}}(\operatorname{Coker} h^{n-2})$$

(by 1·1(ii)), and so $H^{n-1}(\mathcal{H}(\mathcal{S},M))$ is Φ_{n+1}-torsion.

(iii) For each $n \in \mathbb{N}_0$,

$$\eta_{\Phi_{n+1}}(K^n): K^n \to D_{\Phi_{n+1}}(K^n)$$

is an isomorphism (by [**13**, lemma 1·5(iii)] or [**20**, corollary (3·1·11)]).

The next lemma, which uses ideas similar to ones used by K. Suominen in [**19**, §1], provides the key for the main result of this paper. The influence of the ideas of P. Gabriel [**3**] should also be acknowledged.

LEMMA 1·3. *Let Φ be a system of ideals of A. Let $e: C \to C'$ and $\psi: C \to K$ be homomorphisms of A-modules. Set $K' := D_\Phi(K)$ and $h := \eta_\Phi(K): K \to D_\Phi(K) = K'$.*

Assume that $\operatorname{Ker} e$ *and* $\operatorname{Coker} e$ *are both Φ-torsion.*

(i) *The map $D_\Phi(e): D_\Phi(C) \to D_\Phi(C')$ is an isomorphism.*

(ii) *There is a unique A-homomorphism $\psi': C' \to K'$ such that the diagram*

$$\begin{array}{ccc} C & \xrightarrow{e} & C' \\ {\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\omega'} \\ K & \xrightarrow{h} & K' \end{array}$$

commutes. In fact, $\psi' = D_\Phi(\psi) \circ D_\Phi(e)^{-1} \circ \eta_\Phi(C')$.

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(iii) *If ψ and $\eta_\Phi(C')\colon C' \to D_\Phi(C')$ are both isomorphisms, then the homomorphism ψ' of part* (ii) *is also an isomorphism.*

Proof. (i) We shall use the exact sequences

$$0 \to \operatorname{Ker} e \xrightarrow{\tau} C \xrightarrow{\lambda} \operatorname{Im} e \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} e \xrightarrow{\rho} C' \xrightarrow{\sigma} \operatorname{Coker} e \longrightarrow 0,$$

in which the maps are the obvious homomorphisms. Now $e = \rho \circ \lambda$; it is therefore enough for us to show that $D_\Phi(\rho)$ and $D_\Phi(\lambda)$ are both isomorphisms.

The first of the above exact sequences induces an exact sequence

$$0 \longrightarrow D_\Phi(\operatorname{Ker} e) \xrightarrow{D_\Phi(\tau)} D_\Phi(C) \xrightarrow{D_\Phi(\lambda)} D_\Phi(\operatorname{Im} e) \longrightarrow \mathscr{R}^1 D_\Phi(\operatorname{Ker} e).$$

However, $\mathscr{R}^1 D_\Phi(\operatorname{Ker} e) \cong H^2_\Phi(\operatorname{Ker} e)$ by [**13**, 1·4(iii)]. By hypothesis, $\operatorname{Ker} e$ is Φ-torsion. Hence $D_\Phi(\operatorname{Ker} e) = H^2_\Phi(\operatorname{Ker} e) = 0$, by [**13**, 1·4(i) and 1·5(i)]. Therefore $D_\Phi(\lambda)$ is an isomorphism.

Next, from the exact sequence $0 \to \operatorname{Im} e \xrightarrow{\rho} C' \xrightarrow{\sigma} \operatorname{Coker} e \to 0$ we obtain an induced exact sequence

$$0 \longrightarrow D_\Phi(\operatorname{Im} e) \xrightarrow{D_\Phi(\rho)} D_\Phi(C') \xrightarrow{D_\Phi(\sigma)} D_\Phi(\operatorname{Coker} e).$$

However, by hypothesis, $\operatorname{Coker} e$ is Φ-torsion, and so $D_\Phi(\operatorname{Coker} e) = 0$, by [**13**, 1·5(i)]. Hence $D_\Phi(\rho)$ is an isomorphism.

(ii) Application of the morphism of functors $\eta_\Phi\colon \mathrm{Id} \to D_\Phi$ to the modules and homomorphisms in the diagram

$$\begin{array}{ccc} C & \xrightarrow{e} & C' \\ {\scriptstyle\psi}\downarrow & & \\ K & \xrightarrow{h} & K' \end{array}$$

yields a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\eta_\Phi(C)} & D_\Phi(C) & & \\ {\scriptstyle\psi}\downarrow & \searrow^{e} & \downarrow{\scriptstyle D_\Phi(\psi)} & \cong & \searrow^{D_\Phi(e)} \\ K & & C' & \xrightarrow{\eta_\Phi(C')} & D_\Phi(C') \\ & \searrow_{h} & \downarrow{\scriptstyle \eta_\Phi(K)} & & \\ & & D_\Phi(K) & \cong & \searrow^{D_\Phi(h)} \\ & & K' & \xrightarrow[\cong]{\eta_\Phi(K')} & D_\Phi(K'). \end{array}$$

It should be noted that $D_\Phi(e)$ is an isomorphism, by part (i), that $D_\Phi(K) = K'$, and that, by [**13**, 1·5(iii)],

$$D_\Phi(h) = D_\Phi(\eta_\Phi(K)) = \eta_\Phi(D_\Phi(K)) = \eta_\Phi(K')$$

is also an isomorphism.

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Thus, if there were an A-homomorphism $\psi' : C' \to K'$ such that $\psi' \circ e = h \circ \psi$, then it would satisfy $D_\Phi(\psi') \circ D_\Phi(e) = D_\Phi(h) \circ D_\Phi(\psi)$ and we would have to have

$$\begin{aligned}\psi' &= D_\Phi(h)^{-1} \circ \eta_\Phi(K') \circ \psi' = D_\Phi(h)^{-1} \circ D_\Phi(\psi') \circ \eta_\Phi(C') \\ &= D_\Phi(\psi) \circ D_\Phi(e)^{-1} \circ \eta_\Phi(C').\end{aligned}$$

On the other hand, one can easily verify by means of an elementary diagram chase that $D_\Phi(\psi) \circ D_\Phi(e)^{-1} \circ \eta_\Phi(C') \circ e = h \circ \psi$.

(iii) This is immediate from part (ii), since if ψ is an isomorphism, so too is $D_\Phi(\psi)$.

We can now prove the main result of this section.

THEOREM 1·4. *Let $\mathscr{S} = (\Phi_i)_{i \in \mathbb{N}}$ be a family of systems of ideals of A. Let*

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \dots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \dots$$

be a complex $C^\bullet$ of A-modules and A-homomorphisms such that, for each $n \in \mathbb{N}_0$, both $H^{n-1}(C^\bullet)$ and $\operatorname{Coker} e^{n-1}$ are Φ_{n+1}-torsion.

Then there is a unique morphism of complexes

$$\Psi = (\psi^n)_{n \geqslant -2} : C^\bullet \to \mathscr{H}(\mathscr{S}, M)$$

over Id_M.

In fact, we can give the following information about the constituent homomorphisms of Ψ. Of course, $\psi^{-1} = \mathrm{Id}_M$. For $n \in \mathbb{N}_0$, let $\overline{\psi^{n-1}} : \operatorname{Coker} e^{n-2} \to \operatorname{Coker} h^{n-2}$ be the homomorphism induced by ψ^{n-1}, and let $\overline{e^{n-1}} : \operatorname{Coker} e^{n-2} \to C^n$ and $\overline{h^{n-1}} : \operatorname{Coker} h^{n-2} \to K^n$ be the homomorphisms induced by e^{n-1} and h^{n-1} respectively. Then ($D_{\Phi_{n+1}}(\overline{e^{n-1}})$ is an isomorphism and)

$$\psi^n = D_{\Phi_{n+1}}(\overline{\psi^{n-1}}) \circ D_{\Phi_{n+1}}(\overline{e^{n-1}})^{-1} \circ \eta_{\Phi_{n+1}}(C^n).$$

Moreover, this morphism Ψ is an isomorphism of complexes if and only if $\eta_{\Phi_{n+1}}(C^n) : C^n \to D_{\Phi_{n+1}}(C^n)$ is an isomorphism for every $n \in \mathbb{N}_0$.

Proof. Use the notation

$$0 \xrightarrow{h^{-2}} M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \dots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \dots$$

for the generalized Hughes complex $\mathscr{H}(\mathscr{S}, M)$. It will be convenient to write $C^{-2} = K^{-2} = 0$, $C^{-1} = K^{-1} = M$. Let $\psi^{-2} : C^{-2} \to K^{-2}$ be the zero map.

Let $n \in \mathbb{N}_0$ and suppose that it has been shown that there is a unique choice of A-homomorphisms $\psi^i : C^i \to K^i$ for $i = -1, 0, \dots, n-1$ such that the diagram

$$\begin{array}{ccccccccc} C^{-2} & \xrightarrow{e^{-2}} & C^{-1} & \longrightarrow & \dots & \longrightarrow & C^{n-2} & \xrightarrow{e^{n-2}} & C^{n-1} \\ \downarrow{\scriptstyle \psi^{-2}} & & \downarrow{\scriptstyle \psi^{-1}} & & & & \downarrow{\scriptstyle \psi^{n-2}} & & \downarrow{\scriptstyle \psi^{n-1}} \\ K^{-2} & \xrightarrow{h^{-2}} & K^{-1} & \longrightarrow & \dots & \longrightarrow & K^{n-2} & \xrightarrow{h^{n-2}} & K^{n-1} \end{array}$$

commutes and $\psi^{-1} : C^{-1} \to K^{-1}$ is the identity mapping on M, and suppose that it has also been shown that $\psi^0, \dots, \psi^{n-1}$ must all be isomorphisms if $\eta_{\Phi_{j+1}}(C^j) : C^j \to D_{\Phi_{j+1}}(C^j)$ is an isomorphism for all $j = 0, \dots, n-1$. This is certainly the case when $n = 0$.

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There is a unique A-homomorphism $\overline{\psi^{n-1}}: \operatorname{Coker} e^{n-2} \to \operatorname{Coker} h^{n-2}$ such that the diagram

$$\begin{array}{ccc} C^{n-1} & \xrightarrow{\sigma_{n-1}} & \operatorname{Coker} e^{n-2} \\ \downarrow{\scriptstyle \psi^{n-1}} & & \downarrow{\scriptstyle \overline{\psi^{n-1}}} \\ K^{n-1} & \xrightarrow{\pi_{n-1}} & \operatorname{Coker} h^{n-2} \end{array}$$

(in which σ_{n-1} and π_{n-1} are the canonical epimorphisms) commutes, and $\overline{\psi^{n-1}}$ is an isomorphism if $\psi^{-1}, \dots, \psi^{n-1}$ are. The inductive step can now be completed by application of Lemma 1·3 to the diagram

$$\begin{array}{ccc} \operatorname{Coker} e^{n-2} & \xrightarrow{\overline{e^{n-1}}} & C^n \\ \downarrow{\scriptstyle \overline{\psi^{n-1}}} & & \\ \operatorname{Coker} h^{n-2} & \xrightarrow{\overline{h^{n-1}}} & K^n. \end{array}$$

We conclude that there is a unique homomorphism $\psi^n: C^n \to K^n$ (given by the formula in the statement of the theorem) for which the diagram

$$\begin{array}{ccc} C^{n-1} & \xrightarrow{e^{n-1}} & C^n \\ \downarrow{\scriptstyle \psi^{n-1}} & & \downarrow{\scriptstyle \psi^n} \\ K^{n-1} & \xrightarrow{h^{n-1}} & K^n \end{array}$$

commutes, and, also, ψ^n must be an isomorphism if $\eta_{\Phi_{j+1}}(C^j): C^j \to D_{\Phi_{j+1}}(C^j)$ is an isomorphism for all $j = 0, \dots, n$.

To complete the proof, simply note that, if there is an isomorphism of complexes $\Psi: C^\bullet \to \mathscr{H}(\mathscr{S}, M)$ over Id_M, then $C^n \cong K^n$ for all $n \in \mathbb{N}_0$, and so, by 1·2(iii), we must have that $\eta_{\Phi_{n+1}}(C^n): C^n \to D_{\Phi_{n+1}}(C^n)$ is an isomorphism for every $n \in \mathbb{N}_0$.

We can use 1·1(ii) and 1·2 to obtain the following reformulation of part of Theorem 1·4.

COROLLARY 1·5. *Let $\mathscr{S} = (\Phi_i)_{i \in \mathbb{N}}$ be a family of systems of ideals of A. Let*

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \dots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \dots$$

be a complex $C^\bullet$ of A-modules and A-homomorphisms.

Then $C^\bullet$ is isomorphic (over Id_M) to the generalized Hughes complex $\mathscr{H}(\mathscr{S}, M)$ if and only if

(a) $\operatorname{Coker} e^{n-1}$ *is Φ_{n+1}-torsion for all $n \in \mathbb{N}_0$,*

(b) $H^{n-1}(C^\bullet)$ *is Φ_{n+1}-torsion for all $n \in \mathbb{N}_0$, and*

(c) $\Gamma_{\Phi_{n+1}}(C^n) = H^1_{\Phi_{n+1}}(C^n) = 0$ *for all $n \in \mathbb{N}_0$.*

Moreover, when these conditions are satisfied, there is exactly one isomorphism of complexes $C^\bullet \xrightarrow{\cong} \mathscr{H}(\mathscr{S}, M)$ over Id_M.

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2. *Some quick proofs of known results*

In this section, we show briefly how the results of Section 1 can be used to provide some quick proofs of known results, and an extension of one of those results.

We begin with complexes of modules of generalized fractions. Whenever we have a chain $\mathscr{U}$ of triangular subsets on A (see [**10**, p. 420]), we can construct a complex of modules of generalized fractions $C(\mathscr{U},M)$. One of the main results of [**15**] is theorem 3·5, which shows that, given a chain $\mathscr{U}$ of triangular subsets on A, there is a family $\mathscr{S}(\mathscr{U})$ of systems of ideals of A such that the complex of modules of generalized fractions $C(\mathscr{U},M)$ is isomorphic (over Id_M) to the generalized Hughes complex $\mathscr{H}(\mathscr{S}(\mathscr{U}),M)$. Our first aim in this section is to use the methods of Section 1 to prove a strengthening of this result which shows that, under certain conditions, a generalized Hughes complex is isomorphic to a complex of modules of generalized fractions, so that the algebra of generalized fractions (see [**16**]) can be used when one wishes to work with elements in the terms of the generalized Hughes complex.

THEOREM 2·1. *Let $\mathscr{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A. Suppose that there exists a chain $\mathscr{U} = (U_i)_{i\in\mathbb{N}}$ of triangular subsets on A such that, for each $i\in\mathbb{N}$, we have $\sum_{j=1}^{i} Au_j \in \Phi_i$ for all $(u_1,\dots,u_i)\in U_i$, and, for each ideal $\mathfrak{b}\in\Phi_i$, there exists $(v_1,\dots,v_i)\in U_i$ such that $\sum_{j=1}^{i} Av_j \subseteq \mathfrak{b}$.*

Then there is a unique isomorphism of complexes $C(\mathscr{U},M) \xrightarrow{\cong} \mathscr{H}(\mathscr{S},M)$ over Id_M.

Proof. The complex $C(\mathscr{U},M)$ has the form

$$0 \longrightarrow M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} \dots \longrightarrow U_n^{-n}M \xrightarrow{e^n} U_{n+1}^{-n-1}M \longrightarrow \dots,$$

where $e^0(m) = m/(1)$ for all $m\in M$ and

$$e^n\left(\frac{m}{(u_1,\dots,u_n)}\right) = \frac{m}{(u_1,\dots,u_n,1)}$$

for all $n\in\mathbb{N}$, $m\in M$ and $(u_1,\dots,u_n)\in U_n$.

By 1·5, it is enough for us to show that

(*a*) $\operatorname{Coker} e^n$ is Φ_{n+1}-torsion for all $n\in\mathbb{N}_0$,

(*b*) $H^{n-1}(C(\mathscr{U},M))$ is Φ_{n+1}-torsion for all $n\in\mathbb{N}_0$ (where we are regarding the term $U_1^{-1}M$ as being in the 0th position, so that $H^{n-1}(C(\mathscr{U},M)) = \operatorname{Ker} e^n/\operatorname{Im} e^{n-1}$ for $n\in\mathbb{N}$ and $H^{-1}(C(\mathscr{U},M)) = \operatorname{Ker} e^0$), and

(*c*) $\Gamma_{\Phi_{n+1}}(U_{n+1}^{-n-1}M) = H^1_{\Phi_{n+1}}(U_{n+1}^{-n-1}M) = 0$ for all $n\in\mathbb{N}_0$.

It follows from the proof of [**4**, theorem 3·6] (the part of that proof that we need uses Proposition 3·3 and Lemma 3·4, and the definition of the *sequence of sets of primes induced by $\mathscr{U}$ and M* on page 255, of the same paper) that, for each $n\in\mathbb{N}_0$,

$$\operatorname{Supp}(\operatorname{Coker} e^n) \subseteq \bigcup_{(u_1,\dots,u_{n+1})\in U_{n+1}} V\left(\sum_{i=1}^{n+1} Au_i\right)$$

and

$$\operatorname{Supp}(H^{n-1}(C(\mathscr{U},M))) \subseteq \bigcup_{(u_1,\dots,u_{n+1})\in U_{n+1}} V\left(\sum_{i=1}^{n+1} Au_i\right).$$

Thus, if x is an element of $\operatorname{Coker} e^n$ or $H^{n-1}(C(\mathscr{U},M))$, and $\mathfrak{p}$ is a minimal prime of

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$(0:x)$, then there exists $(u_1,\dots,u_{n+1})\in U_{n+1}$ such that $\sum_{i=1}^{n+1}Au_i\subseteq\mathfrak{p}$. It is now easy to use properties of the triangular set U_{n+1} to complete the verification of points (a) and (b).

For each $u=(u_1,\dots,u_n,u_{n+1})\in U_{n+1}$, let W_u denote the triangular subset

$$\{(u_1^{\alpha_1},\dots,u_{n+1}^{\alpha_{n+1}}):\alpha_1,\dots,\alpha_{n+1}\in\mathbb{N}\}$$

of A^{n+1}. By [**16**, 3·5], we have

$$\varinjlim_{u\in U_{n+1}} W_u^{-n-1}M\cong U_{n+1}^{-n-1}M.$$

Let $i\in\mathbb{N}_0$. For an ideal $\mathfrak{a}$ of A, since the A-module $A/\mathfrak{a}$ is finitely generated, the functor $\operatorname{Ext}_A^i(A/\mathfrak{a},\bullet)$ commutes with direct limits. In order to check point (c), it is therefore enough for us to show that

$$\varinjlim_{u\in U_{n+1}}\operatorname{Ext}_A^i(A/\mathfrak{b},W_u^{-n-1}M)=0\quad\text{for each}\quad\mathfrak{b}\in\Phi_{n+1}.$$

This, however, is easy: given $u\in U_{n+1}$ and $\mathfrak{b}\in\Phi_{n+1}$, there exist $v=(v_1,\dots,v_n,v_{n+1})\in U_{n+1}$ and a lower triangular $(n+1)\times(n+1)$ matrix H with entries in A such that $Hu^T=v^T$ and $\sum_{j=1}^{n+1}Av_j\subseteq\mathfrak{b}$; the fact (see [**17**, 2·1]) that multiplication by v_{n+1} provides an automorphism of $W_v^{-n-1}M$ ensures that $\operatorname{Ext}_A^i(A/\mathfrak{b},W_v^{-n-1}M)=0$.

Remark 2.2. It should be noted that a special case of Theorem 2·1 above provides a comparatively short proof of the main part of theorem 3·5 of [**15**].

Next, we consider the denominator systems of M. Kersken [**8**].

Reminder 2·3. In [**8**], M. Kersken defined a *denominator system* over A to be a set $\mathscr{L}\subseteq\bigcup_{i\in\mathbb{N}_0}A^i$ (where A^0 is interpreted as $\{\varnothing\}$) for which the following conditions are satisfied:

(a) $\mathscr{L}\neq\varnothing$;

(b) whenever $i\in\mathbb{N}$ and $(u_1,\dots,u_i)\in\mathscr{L}^i:=\mathscr{L}\cap A^i$, then $(u_1,\dots,u_j)\in\mathscr{L}$ for all $j=0,\dots,i$;

(c) whenever $i\in\mathbb{N}$ and $u=(u_1,\dots,u_i)\in\mathscr{L}^i$, then

$$S(u):=\{u_{i+1}\in A:(u_1,\dots,u_i,u_{i+1})\in\mathscr{L}\}$$

is a multiplicatively closed subset of A; and

(d) whenever $i\in\mathbb{N}$ and $u=(u_1,\dots,u_i),v=(v_1,\dots,v_i)\in\mathscr{L}^i$ such that $\sum_{j=1}^i v_jA\subseteq\sum_{j=1}^i u_jA$, then $S(v)\subseteq S(u)\subseteq\overline{(S(v)+\sum_{j=1}^i u_jA)}$, where $\overline{T}$, for a multiplicatively closed subset T of A, denotes the saturation of T.

Given such a denominator system $\mathscr{L}$, Kersken constructed (see [**8**, (2·4)]) a complex $\tilde{C}^\bullet(\mathscr{L};M)$ which he called the *Cousin complex of M with respect to $\mathscr{L}$*. In [**5**, 2·3], M. A. Hamieh and H. Zakeri showed that every one of Kersken's Cousin complexes is isomorphic to a complex of modules of generalized fractions. It follows from this and [**15**, theorem 3·5] that every one of Kersken's Cousin complexes is isomorphic to a generalized Hughes complex. Our next aim in this section is to show that the methods of Section 1 can be used to provide a quick direct proof of the latter fact.

THEOREM 2·4. *Let $\mathscr{L}$ be a denominator system over A, as described in* 2·3. *For each*

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$i \in \mathbb{N}$, *set* $\Phi_i := \{\sum_{j=1}^{i} u_j A : (u_1, \ldots, u_i) \in \mathscr{L}^i = \mathscr{L} \cap A^i\}$. *Then* $\mathscr{S} := (\Phi_i)_{i \in \mathbb{N}}$ *is a family of systems of ideals of* A, *and there is a unique isomorphism of complexes (over* Id_M*) between Kersken's Cousin complex* $\tilde{C}^\bullet(\mathscr{L}; M)$ *and the generalized Hughes complex* $\mathscr{H}(\mathscr{S}, M)$.

Proof. It is immediate from [**8**, (1·1)] that Φ_i is a system of ideals of A (for all $i \in \mathbb{N}$).

Write $\tilde{C}^\bullet(\mathscr{L}; M)$ as

$$0 \xrightarrow{\delta^{-2}} M \xrightarrow{\delta^{-1}} \tilde{C}^0(\mathscr{L}; M) \xrightarrow{\delta^0} \ldots \longrightarrow \tilde{C}^n(\mathscr{L}; M) \xrightarrow{\delta^n} \tilde{C}^{n+1}(\mathscr{L}; M) \longrightarrow \ldots .$$

By 1·5, it is enough for us to show that

(*a*) $\mathrm{Coker}\, \delta^{n-1}$ is Φ_{n+1}-torsion for all $n \in \mathbb{N}_0$,

(*b*) $H^{n-1}(\tilde{C}^\bullet(\mathscr{L}; M))$ is Φ_{n+1}-torsion for all $n \in \mathbb{N}_0$, and

(*c*) $\Gamma_{\Phi_{n+1}}(\tilde{C}^n(\mathscr{L}; M)) = H^1_{\Phi_{n+1}}(\tilde{C}^n(\mathscr{L}; M)) = 0$ for all $n \in \mathbb{N}_0$.

Let $n \in \mathbb{N}_0$. By the definition of the δ^i in Satz (2·4), the definition of '$\mathscr{L}$-htm' on page 390, and lemma (2·2) of Kersken's paper [**8**], each element of $\mathrm{Coker}\, \delta^{n-1}$ is annihilated by an ideal in Φ_{n+1}, and the same is true for each element of $H^{n-1}(\tilde{C}^\bullet(\mathscr{L}; M))$. Hence points (*a*) and (*b*) have been verified.

Let $i \in \mathbb{N}_0$. Since, by [**2**, (2·4)], the functor $H^i_{\Phi_{n+1}}$ commutes with direct limits, it follows from [**8**, (2·2)] that it is enough for us to show that, for an arbitrary finitely generated submodule D of $\mathrm{Coker}\, \delta^{n-2}$ and a sequence $u := (u_1, \ldots, u_n) \in \mathscr{L}^n$ for which $(\sum_{j=1}^{n} u_j A) D = 0$, we have

$$H^i_{\Phi_{n+1}}((S(u))^{-1} D) = 0.$$

To see this, let $x \in H^i_{\Phi_{n+1}}((S(u))^{-1} D)$. There exists $v := (v_1, \ldots, v_{n+1}) \in \mathscr{L}^{n+1}$ such that $(\sum_{j=1}^{n+1} v_j A) x = 0$. By [**8**, (1·1)], there exists $(w_1, \ldots, w_{n+1}) \in \mathscr{L}^{n+1}$ such that

$$\sum_{j=1}^{n} w_j A \subseteq \left(\sum_{j=1}^{n} u_j A\right) \cap \left(\sum_{j=1}^{n} v_j A\right)$$

and $w_{n+1} \in \sum_{j=1}^{n+1} v_j A$. Thus $(\sum_{j=1}^{n+1} w_j A) x = 0$ and $w_{n+1} \in S(u)$, by 2·3(*d*). Thus multiplication by w_{n+1} yields an automorphism of $H^i_{\Phi_{n+1}}((S(u))^{-1} D)$, so that $x = 0$.

Thus point (*c*) has been verified, and the proof is complete.

Remark 2·5. Let the situation and notation be as in 2·4.

Note that, if the denominator system $\mathscr{L}$ is saturated [**8**, p. 391], then it follows from [**8**, (1·1) and (1·3)] that $(\mathscr{L}^i)_{i \in \mathbb{N}}$ is a chain $\mathscr{U}$ of triangular subsets on A; it thus follows from Theorem 2·1 that there is a unique isomorphism of complexes

$$C(\mathscr{U}, M) \xrightarrow{\cong} \mathscr{H}(\mathscr{S}, M)$$

over Id_M; therefore, use of Theorem 2·4 enables us to deduce that, in these circumstances, there is a unique isomorphism of complexes

$$C(\mathscr{U}, M) \xrightarrow{\cong} \tilde{C}^\bullet(\mathscr{L}; M)$$

over Id_M.

Since one can replace $\mathscr{L}$ by its saturation (without affecting $\tilde{C}^\bullet(\mathscr{L}; M)$ up to isomorphism over Id_M), Theorem 2·4 therefore provides another approach to the

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theorem of Hamieh and Zakeri [5, 2·3] that $\tilde{C}^\bullet(\mathscr{L};M)$ is isomorphic (over Id_M) to a complex of modules of generalized fractions.

3. *Another approach to Cousin complexes*

Notation 3·1. In this section, we shall employ the following notation. We shall let $\mathscr{F} = (F_i)_{i\in\mathbb{N}_0}$ denote a filtration of $\mathrm{Spec}\,(A)$ which admits M (in the sense of [12, 1·1]). Thus

$$\mathrm{Spec}\,(A) \supseteq F_0 \supseteq F_1 \supseteq \ldots \supseteq F_i \supseteq F_{i+1} \supseteq \ldots,$$

$\mathrm{Supp}\,(M) \subseteq F_0$ and, for each $i\in\mathbb{N}_0$, each member of $\partial F_i := F_i \backslash F_{i+1}$ is a minimal member of F_i with respect to inclusion.

The Cousin complex $C(\mathscr{F},M)$ for M with respect to $\mathscr{F}$ of [12, 1·3] will be denoted by

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \ldots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \ldots.$$

Thus $M^n = \bigoplus_{\mathfrak{p}\in\partial F_n} (\mathrm{Coker}\, d^{n-2})_{\mathfrak{p}}$ for each $n\in\mathbb{N}_0$.

We quote now one of the main results of [13].

THEOREM 3·2. [13, theorem 2·3]. *Let $W \subseteq \mathrm{Spec}\,(A)$ be such that $W \supseteq \mathrm{Supp}\,(M)$, and let $\mathscr{S}(\mathscr{F},W) = (\Phi_n(\mathscr{F},W))_{n\in\mathbb{N}}$ be the family of systems of ideals of A for which*

$$\Phi_n(\mathscr{F},W) = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of } A \text{ such that } V(\mathfrak{b}) \cap W \subseteq F_n\}$$

for all $n\in\mathbb{N}$. Then the Cousin complex $C(\mathscr{F},M)$ is isomorphic (over Id_M) to the generalized Hughes complex $\mathscr{H}(\mathscr{S}(\mathscr{F},W),M)$.

In fact, Theorem 1·4 can be used, in conjunction with [2, (2·4) and (2·5)], to provide an alternative proof of Theorem 3·2; we omit the details. Instead, we shall use the above theorem and our characterization Theorem 1·4 to verify that $C(\mathscr{F},M)$ can be constructed using (an algebraic analogue of) an approach of Bănică and Stănăşilă in [1, pp. 230–231]. For this, some further notation is necessary.

Further notation 3·3. Let Φ and Φ' be systems of ideals of A such that $\Phi' \subseteq \Phi$.

(i) For each A-module L, we have $\Gamma_{\Phi'}(L) \subseteq \Gamma_{\Phi}(L)$, and we define $\Gamma_{\Phi/\Phi'}(L) := \Gamma_{\Phi}(L)/\Gamma_{\Phi'}(L)$. Moreover, for a homomorphism $f: L\to N$ of A-modules, the commutative diagram

$$\begin{array}{ccc} \Gamma_{\Phi'}(L) & \longrightarrow & \Gamma_{\Phi}(L) \\ \downarrow{\scriptstyle \Gamma_{\Phi'}(f)} & & \downarrow{\scriptstyle \Gamma_{\Phi}(f)} \\ \Gamma_{\Phi'}(N) & \longrightarrow & \Gamma_{\Phi}(N) \end{array}$$

(in which the horizontal maps are the inclusion homomorphisms) induces a homomorphism

$$\Gamma_{\Phi/\Phi'}(f): \Gamma_{\Phi/\Phi'}(L) \to \Gamma_{\Phi/\Phi'}(N).$$

These assignments result in an additive functor $\Gamma_{\Phi/\Phi'}: \mathscr{C}(A) \to \mathscr{C}(A)$, and, for each $i\in\mathbb{N}_0$, we denote the ith right derived functor of $\Gamma_{\Phi/\Phi'}$ by $H^i_{\Phi/\Phi'}$.

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(ii) Take an injective resolution

$$I^\bullet: 0 \to I^0 \to I^1 \to \ldots \to I^i \to I^{i+1} \to \ldots$$

of M. Then there is a sequence of complexes and morphisms of complexes

$$0 \to \Gamma_{\Phi'}(I^\bullet) \to \Gamma_{\Phi}(I^\bullet) \to \Gamma_{\Phi/\Phi'}(I^\bullet) \to 0$$

which is such that, for each $i \in \mathbb{N}_0$, the sequence

$$0 \to \Gamma_{\Phi'}(I^i) \to \Gamma_{\Phi}(I^i) \to \Gamma_{\Phi/\Phi'}(I^i) \to 0$$

is exact. Therefore, there results a long exact sequence

$$\begin{aligned}
0 &\longrightarrow H^0_{\Phi'}(M) \longrightarrow H^0_{\Phi}(M) \longrightarrow H^0_{\Phi/\Phi'}(M) \\
&\longrightarrow H^1_{\Phi'}(M) \longrightarrow H^1_{\Phi}(M) \longrightarrow H^1_{\Phi/\Phi'}(M) \\
&\longrightarrow \ldots \qquad\qquad\qquad \ldots \\
&\longrightarrow H^i_{\Phi'}(M) \longrightarrow H^i_{\Phi}(M) \longrightarrow H^i_{\Phi/\Phi'}(M) \\
&\longrightarrow H^{i+1}_{\Phi'}(M) \longrightarrow \ldots
\end{aligned}$$

and, up to isomorphism of complexes, this sequence does not depend on the choice of injective resolution of M.

Remark 3·4. Let $F \subseteq \operatorname{Spec}(A)$, and let

$$\Phi := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } A \text{ such that } V(\mathfrak{a}) \subseteq F\}.$$

Then Φ is a system of ideals of A, and, for an A-module G, we have that G is Φ-torsion if and only if $\operatorname{Supp}(G) \subseteq F$.

We shall need the following well-known facts about indecomposable injective A-modules.

Reminders 3·5. (See [**9**, section 18], for example.) Let $\mathfrak{p} \in \operatorname{Spec}(A)$.

(i) For $a \in A \setminus \mathfrak{p}$, multiplication by a provides an automorphism of $E_A(A/\mathfrak{p})$; moreover, each element of $E_A(A/\mathfrak{p})$ is annihilated by some power of $\mathfrak{p}$.

(ii) Let S be a multiplicatively closed subset of A such that $\mathfrak{p} \cap S = \varnothing$. By (i), the indecomposable injective A-module $E_A(A/\mathfrak{p})$ has a natural structure as $S^{-1}A$-module; as such, it is isomorphic to $E_{S^{-1}A}(S^{-1}A/S^{-1}\mathfrak{p})$.

Proposition 3·6. (Compare Bănică and Stănăşilă [**1**, chapter VII, proposition 1·1] and Hartshorne [**6**, chapter IV, §1, motif F].) *Let $F' \subseteq F \subseteq \operatorname{Spec}(A)$, and assume that each element of $F \setminus F'$ is a minimal member of F with respect to inclusion. Let*

$$\Phi := \{\mathfrak{a} : \mathfrak{a} \textit{ is an ideal of } A \textit{ such that } V(\mathfrak{a}) \subseteq F\}$$

and $\Phi' := \{\mathfrak{a} : \mathfrak{a}$ is an ideal of A such that $V(\mathfrak{a}) \subseteq F'\}$. Then

(i) $(\Phi \setminus \Phi') \cap \operatorname{Spec}(A) = (F \setminus F') \cap \Phi$, *and this is equal to $F \setminus F'$ if F is closed under specialization (that is, whenever $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in F$, then $\mathfrak{q} \in F$ too); and*

(ii) *for all $i \in \mathbb{N}_0$,*

$$H^i_{\Phi/\Phi'}(M) \cong \bigoplus_{\mathfrak{p} \in (F \setminus F') \cap \Phi} H^i_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. (i) Let $\mathfrak{p} \in (\Phi \setminus \Phi') \cap \operatorname{Spec}(A)$. Then $V(\mathfrak{p}) \subseteq F$, and so $\mathfrak{p} \in F$. Also, there exists $\mathfrak{q} \in V(\mathfrak{p}) \setminus F'$; we cannot have $\mathfrak{p} \subset \mathfrak{q}$ (we reserve the symbol '$\subset$' to denote strict

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inclusion), for then $\mathfrak{q}$ would be a member of $F\backslash F'$ but not a minimal member of F. Hence $\mathfrak{p} \in (F\backslash F') \cap \Phi$. All the remaining claims of this part are easy to prove.

(ii) Let

$$I^\bullet : 0 \to I^0 \to I^1 \to \ldots \to I^i \to I^{i+1} \to \ldots$$

be an injective resolution of M. For each $i \in \mathbb{N}_0$, there is a family $(E_\lambda)_{\lambda \in \Lambda_i}$ of indecomposable injective A-modules such that $I^i = \bigoplus_{\lambda \in \Lambda_i} E_\lambda$, and, for each $\lambda \in \Lambda_i$, there exists $\mathfrak{p}_\lambda \in \operatorname{Spec}(A)$ such that $E_\lambda \cong E_A(A/\mathfrak{p}_\lambda)$.

It follows from 3·5(i) that, for $\mathfrak{p} \in \operatorname{Spec}(A)$,

$$\Gamma_\Phi(E_A(A/\mathfrak{p})) = \begin{cases} 0 & \text{if } \mathfrak{p} \notin \Phi, \\ E_A(A/\mathfrak{p}) & \text{if } \mathfrak{p} \in \Phi, \end{cases}$$

and a similar comment applies to Φ'. It therefore follows from part (i) that, for all $i \in \mathbb{N}_0$,

$$\Gamma_{\Phi/\Phi'}(I^i) \cong \bigoplus_{\substack{\lambda \in \Lambda_i \\ \mathfrak{p}_\lambda \in (F\backslash F') \cap \Phi}} E_\lambda.$$

Next, the results quoted in 3·5(i) mean that, if $\mathfrak{p}, \mathfrak{q}$ are two distinct members of $F\backslash F'$, so that $\mathfrak{p} \not\subseteq \mathfrak{q}$, then $\operatorname{Hom}_A(E_A(A/\mathfrak{p}), E_A(A/\mathfrak{q})) = 0$. We can now use both parts of 3·5 to see that there is an isomorphism of complexes

$$\Gamma_{\Phi/\Phi'}(I^\bullet) \cong \bigoplus_{\mathfrak{p} \in (F\backslash F') \cap \Phi} \Gamma_{\mathfrak{p}A_\mathfrak{p}}((I^\bullet)_\mathfrak{p}),$$

that is, the direct sum, taken over $\mathfrak{p} \in (F\backslash F') \cap \Phi$, of the $\Gamma_{\mathfrak{p}A_\mathfrak{p}}((I^\bullet)_\mathfrak{p})$ (each viewed as a complex of A-modules and A-homomorphisms in the natural way). Since, for each $\mathfrak{p} \in \operatorname{Spec}(A)$, the complex $(I^\bullet)_\mathfrak{p}$ is an injective resolution of the $A_\mathfrak{p}$-module $M_\mathfrak{p}$, the result follows.

COROLLARY 3·7. *Let the situation be as in* 3·6. *For* $\mathfrak{p} \in \operatorname{Supp}(M)$, *we refer to* $\dim_{A_\mathfrak{p}} M_\mathfrak{p}$ as the M-height of $\mathfrak{p}$, *and denote this by* $\operatorname{ht}_M \mathfrak{p}$.

Let $i \in \mathbb{N}_0$ *be such that* $i > \operatorname{ht}_M \mathfrak{p}$ *for all* $\mathfrak{p} \in (F\backslash F') \cap \Phi \cap \operatorname{Supp}(M)$. *Then*

(i) $H^{i+j}_{\Phi/\Phi'}(M) = 0$ *for all* $j \in \mathbb{N}_0$, *and*

(ii) *in the long exact sequence of* 3·3(ii), *the map* $H^i_{\Phi'}(M) \to H^i_\Phi(M)$ *is surjective, while* $H^{i+j}_{\Phi'}(M) \to H^{i+j}_\Phi(M)$ *is an isomorphism for all* $j \in \mathbb{N}$.

Proof. Part (i) is immediate from 3·6(ii) and Grothendieck's Vanishing Theorem for local cohomology, while part (ii) is then an immediate consequence of the long exact sequence of 3·3(ii).

We now propose to consider a commutative algebra analogue of the construction described by Bănică and Stănăşilă in [**1**, pp. 230–231].

Definition 3·8. Let $\mathscr{F}' = (F'_i)_{i \in \mathbb{N}_0}$ be a family of subsets of $\operatorname{Spec}(A)$ such that

$$\operatorname{Spec}(A) \supseteq F'_0 \supseteq F'_1 \supseteq \ldots \supseteq F'_i \supseteq F'_{i+1} \supseteq \ldots$$

and $\operatorname{Supp}(M) \subseteq F'_0$. (It should be noted that we are *not* here assuming that, for each $i \in \mathbb{N}_0$, each member of $F'_i\backslash F'_{i+1}$ is a minimal member of F'_i with respect to inclusion.)

For each $i \in \mathbb{N}_0$, let $\Phi'_i := \{\mathfrak{a} : \mathfrak{a}$ is an ideal of A such that $V(\mathfrak{a}) \subseteq F'_i\}$, a system of ideals of A. By 3·4, we have $M = \Gamma_{\Phi'_0}(M)$, and if we compose the resulting natural isomorphism $M \xrightarrow{\cong} H^0_{\Phi'_0}(M)$ with the homomorphism $H^0_{\Phi'_0}(M) \to H^0_{\Phi'_0/\Phi'_1}(M)$ of the long

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exact sequence of 3·3(ii) for Φ'_0 and Φ'_1, we obtain a homomorphism $M \to H^0_{\Phi'_0/\Phi'_1}(M)$. Also, for each $n \in \mathbb{N}_0$, we can compose the homomorphism $H^n_{\Phi'_n/\Phi'_{n+1}}(M) \to H^{n+1}_{\Phi'_{n+1}}(M)$ of the long exact sequence of 3·3(ii) for Φ'_n and Φ'_{n+1} with the homomorphism $H^{n+1}_{\Phi'_{n+1}}(M) \to H^{n+1}_{\Phi'_{n+1}/\Phi'_{n+2}}(M)$ from the corresponding long exact sequence for Φ'_{n+1} and Φ'_{n+2} to obtain a homomorphism

$$H^n_{\Phi'_n/\Phi'_{n+1}}(M) \to H^{n+1}_{\Phi'_{n+1}/\Phi'_{n+2}}(M).$$

It is then clear that if we put these various homomorphisms together, we obtain a complex

$$0 \to M \to H^0_{\Phi'_0/\Phi'_1}(M) \to \ldots \to H^n_{\Phi'_n/\Phi'_{n+1}}(M) \to H^{n+1}_{\Phi'_{n+1}/\Phi'_{n+2}}(M) \to \ldots,$$

which we shall refer to as the *generalized Cousin complex for M with respect to $\mathscr{F}'$*. This complex is a commutative algebra analogue of a complex constructed by Bănică and Stănăşilă in [**1**, pp. 230–231]; it should be noted that the earlier work of Hartshorne in [**6**, chapter IV, 2·3–2·6] was effected under more restrictive hypotheses.

For us, an important question is whether the generalized Cousin complex for M with respect to $\mathscr{F}'$ constructed in 3·8 is isomorphic, over Id_M, to the generalized Hughes complex $\mathscr{H}(\mathscr{S}', M)$, where $\mathscr{S}' = (\Phi'_i)_{i\in\mathbb{N}}$. The following example shows that this is not always the case.

Example 3·9. Let K be a field, let

$$A := K[[X_1, X_2, X_3, X_4, X_5]]/(X_1X_2X_3X_4X_5),$$

(where $X_1, \ldots, X_5$ are indeterminates) and denote the natural image of X_i in A by x_i (for $1 \leqslant i \leqslant 5$). Let $\mathfrak{m}$ denote the maximal ideal of A.

Let $F'_0 = \mathrm{Spec}(A)$, let $F'_1 = \mathrm{Spec}(A)\backslash\{(x_1)\}$, let

$$F'_i = \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathrm{ht}\,\mathfrak{p} \geqslant i-1\} \quad \text{for} \quad i = 2, 3,$$

let $F'_4 = \{\mathfrak{m}\}$, and let $F'_i = \varnothing$ for all $i \geqslant 5$.

We use the notation and terminology of 3·8, and consider the generalized Cousin complex for A itself with respect to $\mathscr{F}' = (F'_i)_{i\in\mathbb{N}_0}$. It follows from 3·6 and Grothendieck's Vanishing Theorem for local cohomology that this generalized Cousin complex has the form

$$0 \to A \to H^0_{(x_1)A_{(x_1)}}(A_{(x_1)}) \to 0 \to 0 \to H^3_{\Phi'_3/\Phi'_4}(A) \to H^4_{\mathfrak{m}}(A) \to 0 \to \ldots.$$

Since $\dim A = 4$, we must have $H^4_{\mathfrak{m}}(A) \neq 0$. This complex cannot be isomorphic, over Id_A, to a generalized Hughes complex for A, since it follows from the construction of such complexes that, if the ith term, for some $i \in \mathbb{N}_0$, of a generalized Hughes complex for M is zero, then the $(i+j)$th term is also zero for all $j \in \mathbb{N}$.

Thus, speaking loosely, not every generalized Cousin complex is a generalized Hughes complex.

However, if we restrict attention to a filtration $\mathscr{F} = (F_i)_{i\in\mathbb{N}_0}$ of $\mathrm{Spec}(A)$ which admits M, as considered in 3·1, so that each member of $\partial F_i := F_i\backslash F_{i+1}$ is a minimal member of F_i with respect to inclusion, then the situation is better. This restricted situation is an analogue of the situation considered by Hartshorne in [**6**, chapter IV, 2·3–2·6]. Our final theorem in this paper ties together some of the ends in this restricted situation: we show that the generalized Cousin complex for M with respect to $\mathscr{F}$ is indeed isomorphic (over Id_M) to the 'obvious' generalized Hughes complex

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connected with this situation, and so, by 3·2, is isomorphic (over Id_M) to the ordinary Cousin complex $C(\mathscr{F}, M)$. We are thus able to conclude that the Cousin complexes of the type described in [**12**, §1] can be constructed using (an algebraic analogue of) the technique used by Bănică and Stănăşilă in [**1**, pp. 230–231].

THEOREM 3·10. *Let the notation be as in* 3·1. *For each* $i \in \mathbb{N}_0$, *let*

$$\Phi := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } A \text{ such that } V(\mathfrak{a}) \subseteq F_i\},$$

and let $\mathscr{S} := (\Phi_i)_{i \in \mathbb{N}}$.

Then the generalized Cousin complex (*see* 3·8)

$$0 \to M \to H^0_{\Phi_0/\Phi_1}(M) \to \ldots \to H^n_{\Phi_n/\Phi_{n+1}}(M) \to H^{n+1}_{\Phi_{n+1}/\Phi_{n+2}}(M) \to \ldots$$

for M *with respect to* $\mathscr{F}$ *is isomorphic* (*over* Id_M) *to the generalized Hughes complex* $\mathscr{H}(\mathscr{S}, M)$, *and so, by* 3·2, *is isomorphic* (*over* Id_M) *to the ordinary Cousin complex* $C(\mathscr{F}, M)$.

Proof. Abbreviate the generalized Cousin complex $C^\bullet$ for M with respect to $\mathscr{F}$ by

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \ldots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \ldots.$$

We shall use Corollary 1·5, and so we check conditions (*a*), (*b*) and (*c*) of that result. Let $n \in \mathbb{N}_0$.

First of all, $\operatorname{Coker} e^{-1} \cong H^1_{\Phi_1}(M)$, and so it is Φ_1-torsion. (Note that $H^1_{\Phi_0}(M) = 0$.) Now suppose that $n \geqslant 1$. By 3·7 (ii) and [**14**, 2·3 (iv)], we have

$$H^n_{\Phi_{n-1}}(M) \cong H^n_{\Phi_{n-2}}(M) \cong H^n_{\Phi_{n-3}}(M) \cong \ldots \cong H^n_{\Phi_0}(M)$$

if $n \geqslant 2$, so that, in any event (that is, if $n \geqslant 2$ or if $n = 1$) we must have $H^n_{\Phi_{n-1}}(M) = 0$ since M is Φ_0-torsion by 3·4. Hence the homomorphism $H^{n-1}_{\Phi_{n-1}/\Phi_n}(M) \to H^n_{\Phi_n}(M)$ of the long exact sequence of 3·3 (ii) for Φ_{n-1} and Φ_n is surjective, and $\operatorname{Coker} e^{n-1}$ is just the cokernel of the homomorphism $H^n_{\Phi_n}(M) \to H^n_{\Phi_n/\Phi_{n+1}}(M)$ from the corresponding long exact sequence for Φ_n and Φ_{n+1}; since we now know that this latter cokernel is isomorphic to $H^{n+1}_{\Phi_{n+1}}(M)$ (because $H^{n+1}_{\Phi_n}(M) = 0$), we see that $\operatorname{Coker} e^{n-1}$ is Φ_{n+1}-torsion. Thus condition 1·5 (*a*) is satisfied.

Next, $H^{-1}(C^\bullet) \cong H^0_{\Phi_1}(M)$, and so is Φ_1-torsion. Further, for $n \geqslant 1$, the facts described in the immediately preceding paragraph enable us to see that $H^{n-1}(C^\bullet)$ is a homomorphic image of $H^n_{\Phi_{n+1}}(M)$, and so is Φ_{n+1}-torsion. Thus condition 1·5 (*b*) is satisfied.

Finally, by 3·6 (ii),

$$C^n = H^n_{\Phi_n/\Phi_{n+1}}(M) \cong \bigoplus_{\mathfrak{p} \in \partial F_n \cap \Phi_n} H^n_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}).$$

Let $i \in \mathbb{N}_0$. By [**2**, (2·4) and (2·5)],

$$H^i_{\Phi_{n+1}}(C^n) \cong \bigoplus_{\mathfrak{p} \in \partial F_n \cap \Phi_n} H^i_{\Phi_{n+1}}(H^n_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}))$$

$$\cong \bigoplus_{\mathfrak{p} \in \partial F_n \cap \Phi_n} H^i_{\Phi_{n+1}A_\mathfrak{p}}(H^n_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})),$$

where $\Phi_{n+1}A_\mathfrak{p}$ denotes the system $\{\mathfrak{b}A_\mathfrak{p} : \mathfrak{b} \in \Phi_{n+1}\}$ of ideals of $A_\mathfrak{p}$. But, for each

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$\mathfrak{p} \in \partial F_n \cap \Phi_n$, the system $\Phi_{n+1} A_{\mathfrak{p}}$ is just $\{A_{\mathfrak{p}}\}$. Therefore $H^i_{\Phi_{n+1}}(C^n) = 0$. Thus condition 1·5(*c*) is satisfied.

We can now apply 1·5 to complete the proof.

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