Ergod. Th. & Dynam. Sys., (2026), **46**, 2313–2360 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10302

Existence of large deviations rate function for any *S*-unimodal map

HIROKI TAKAHASI† and MASATO TSUJII‡

† *Department of Mathematics, Keio University, Yokohama, Japan*
(e-mail: hiroki@math.keio.ac.jp)
‡ *Department of Mathematics, Kyushu University, Fukuoka, Japan*
(e-mail: tsujii@math.kyushu-u.ac.jp)

(*Received* 16 *November* 2021 *and accepted in revised form* 11 *March* 2026)

Dedicated to Professor Emeritus Hiroshi Kokubu
on the occasion of his retirement at age 65

Abstract. For an arbitrary negative Schwarzian unimodal map with a non-flat critical point, we establish the level-2 large deviation principle for empirical distributions. We also give an example of a bimodal map for which the level-2 large deviation principle does not hold.

Key words: large deviation principle, *S*-unimodal map, renormalization
2020 Mathematics Subject Classification: 37E05, 60F10 (Primary); 37D25, 37A50 (Secondary)

1. *Introduction*

A main objective of the ergodic theory of smooth dynamical systems is to understand long-term behavior of typical orbits for a majority of systems. Much effort has been dedicated to constructing physically relevant invariant measures which statistically predict typical asymptotic behaviors. A refined description of the dynamics requires the analysis of atypical or transient behaviors before orbits settle down to equilibrium. The theory of large deviations is concerned with such rare events. The large deviation principle (LDP) asserts the existence of the rate function which controls probabilities of rare events on an exponential scale.

It is now classical that a transitive uniformly hyperbolic (Axiom A) attractor supports a unique Sinaĭ–Ruelle–Bowen measure [2, 3, 31], and Lebesgue almost every orbit in the basin of attraction is asymptotically distributed with respect to this measure. The LDP for Axiom A attractors was established by Kifer [19], Orey and Pelikan [24] and Takahashi [32]. For one-dimensional non-hyperbolic systems, after much progress [6, 26, 27], a major advance was made in [7] which establishes the LDP for an arbitrary $C^{1+\alpha}$ multimodal map with non-flat critical points that is topologically exact. The aim of this paper is to treat what is left out in [7]: the LDP for renormalizable unimodal maps,

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

including infinitely renormalizable ones. The conclusion is that the LDP holds for an arbitrary S-unimodal map with a non-flat critical point.

We introduce our setting and results in more precise terms. Let $X \subset \mathbb{R}$ be a compact non-degenerate interval. A C^1 map $f : X \to X$ is called *unimodal* if it has a unique critical point c, which is contained in $\mathrm{int}(X)$ and is an extremum, and satisfies $f(\partial X) \subset \partial X$. An *$S$-unimodal map* f is a unimodal map of class C^3 on $X \setminus \{c\}$ with negative Schwarzian derivative $f'''/f' - \frac{3}{2}(f''/f')^2 < 0$ such that if $x \in \partial X$ is a fixed point of f then $|f'(x)| > 1$. We say that the critical point c is *non-flat* if there exist $l > 1$ and C^3 diffeomorphisms φ and ψ defined on a neighborhood of c and $f(c)$ respectively such that $\varphi(c) = 0 = \psi(f(c))$ and $|\varphi(x)|^l = |\psi(f(x))|$ for any x near c.

Let $\mathcal{M}$ denote the space of Borel probability measures on X endowed with the weak* topology. The empirical measure at time n with initial point $x \in X$ is the uniform probability distribution on the orbit $\{x, f(x), \ldots, f^{n-1}(x)\}$, denoted by

$$\delta_x^n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \in \mathcal{M},$$

where $\delta_{f^k(x)}$ denotes the unit point mass at $f^k(x)$. For a Borel set $A \subset X$, we write $|A|$ for its Lebesgue measure. Our main result is stated as follows.

THEOREM A. (Level-2 LDP) *For an arbitrary S-unimodal map $f : X \to X$ with a non-flat critical point, the level-2 LDP holds, namely, there exists a convex lower semicontinuous function $I : \mathcal{M} \to [0, \infty]$ such that*

$$\text{(\textit{lower bound})} \quad \liminf_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta_x^n \in \mathcal{G}\}| \geq -\inf_{\mathcal{G}} I$$

for any open set $\mathcal{G} \subset \mathcal{M}$, and

$$\text{(\textit{upper bound})} \quad \limsup_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta_x^n \in \mathcal{C}\}| \leq -\inf_{\mathcal{C}} I$$

for any closed set $\mathcal{C} \subset \mathcal{M}$.

Hereafter we follow the convention that $\sup \emptyset = -\infty$, $\inf \emptyset = \infty$, $\log 0 = -\infty$. The function I is called *the rate function*. Since $\mathcal{M}$ is a metrizable space, the LDP determines the rate function uniquely [28, Theorem 2.13].

In order to explain the meaning of Theorem A, let us recall that an S-unimodal map f with a non-flat critical point is classified into the following mutually exclusive cases [10, 14]:

(I) f has an attracting periodic orbit;
(II) f is infinitely renormalizable;
(III) f is at most finitely renormalizable and has no attracting periodic orbit.

Guckenheimer [14] proved this classification for negative Schwarzian C^3 unimodal maps with non-degenerate critical points. The same holds in our slightly more general setting. For details, see [10, Ch. III, §4]. The dynamics is relatively simple in cases (I) and (II): the empirical measure along the orbit of Lebesgue almost every initial point converges in the weak* topology to the measure supported on the attracting periodic orbit or the attracting

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Cantor set. The dynamics in case (III) is much more complicated and displays a rich array of different statistical behaviors (see, for example, [5, 15, 17, 18]). As a prototypical example, consider the quadratic map $x \in [0, 1] \mapsto ax(1 - x)$ with $1 < a \leq 4$. It is well known that:

- the set of a-values corresponding to case (I) is open and dense in the parameter space [13];
- the set of a-values corresponding to case (II) is non-empty [11, 12], and has zero Lebesgue measure [22];
- the set of a-values corresponding to case (III) has positive Lebesgue measure [1, 17].

Irrespective of rich bifurcations, Theorem A states that the LDP continues to hold for an arbitrary parameter a.

The rate function I in Theorem A is given as follows. Let $\mathcal{M}(f)$ denote the set of elements of $\mathcal{M}$ which are f-invariant. For each $\mu \in \mathcal{M}(f)$, the limit

$$\chi(x) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|$$

exists for μ-almost every $x \in X$ and belongs to $\mathbb{R} \cup \{-\infty\}$, because $\log |f'|$ is uniformly bounded from above. We set

$$\chi^+(x) = \max\{\chi(x), 0\} \in \mathbb{R},$$

and define a (non-negative) *Lyapunov exponent* of μ by

$$\chi^+(\mu) = \int \chi^+(x)\, d\mu(x). \tag{1.1}$$

Let $h(\mu)$ denote the measure-theoretic entropy of μ with respect to f. Define a *free energy* $F : \mathcal{M} \to [-\infty, 0]$ by

$$F(\mu) = \begin{cases} h(\mu) - \chi^+(\mu) & \text{if } \mu \in \mathcal{M}(f), \\ -\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

The rate function I in Theorem A is defined to be the negative of the upper semicontinuous regularization of F:

$$I(\mu) = -\inf_{\mathcal{G}\ni\mu} \sup_{\mathcal{G}} F(\nu).$$

Here, the infimum is taken over all open sets $\mathcal{G}$ in $\mathcal{M}$ containing μ. Note that the entropy and the Lyapunov exponent are upper semicontinuous as functions of measures. In cases (I) and (II), the Lyapunov exponent is in fact continuous (see Lemma 2.10) and thus $I = -F$. In case (III), the Lyapunov exponent may fail to be lower semicontinuous [4, §2], which implies $I \neq -F$.

The next theorem asserts that Theorem A cannot be extended to maps with multiple critical points.

THEOREM B. (The breakdown of the LDP) *There exists a C^3 interval map with exactly two non-degenerate critical points for which the level-2 LDP does not hold.*

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

The rest of this paper consists of four sections. In §2 we introduce several basic definitions and results on S-unimodal maps. In §3 we prove the lower bound for open sets in Theorem A. In §4 we prove the upper bound for closed sets in Theorem A. In §5 we prove Theorem B.

For only finitely renormalizable maps, the dynamics of typical orbits consist of two stages: transition to the deepest renormalization cycle, and circulation within that cycle. The LDP restricted to the deepest renormalization cycle mostly follows from the known result [7]. For infinitely renormalizable maps, orbits contained in sufficiently deep cycles are approximated by the post-critical measure supported on the attracting Cantor set. Therefore, all we have to do is to analyze the transitions between renormalization cycles. The main point of §3 is to prove that any f-invariant measure supported on a hyperbolic set is approximated with orbit segments from finitely many subintervals of X (see Proposition 3.1). This claim is standard if the measure is ergodic, and if non-ergodic then we glue orbit segments together which approximate its ergodic components.

The main technique used in §4 to derive the upper bound is a 'coarse graining approach'. Here the set of orbits with prescribed time averages of continuous functions are coarse grained (see §4.3), and estimates on the resultant 'clusters' are transferred to the large deviations upper bound on empirical measures (see Propositions 4.1 and 4.6). The estimate on each cluster consists of contributions from the uniformly hyperbolic dynamics on each renormalization cycle and the transitions between them.

A counterexample of a bimodal map we construct in the proof of Theorem B is non-transitive and has two non-degenerate critical points, one of which is non-recurrent. With a slight modification of our construction, one can find a counterexample in polynomial maps of degree 3 (see Remark 5.5). It is plausible that there is an S-unimodal map with a non-recurrent flat critical point for which the level-2 LDP fails. For a transitive S-unimodal map with a non-recurrent flat critical point and all periodic orbits hyperbolic repelling, the level-2 LDP was shown in [8] under an assumption that the criticality increases at some specific rates.

2. *Preliminaries*

This section introduces several basic definitions and results on the dynamics of S-unimodal maps and its renormalization. In §2.1 we classify attracting periodic orbits. In §2.2 we prove a lemma on the Lyapunov exponent of invariant measures. In §2.3 we define renormalization of S-unimodal maps. In §2.4 we introduce basic structures associated with the renormalization. In §2.5–§2.7 we are concerned with the dynamics on each renormalization cycle. In §2.8 we introduce some notation which will be frequently used later. In §2.9 we summarize a few results on infinitely renormalizable maps.

2.1. *Classification of attracting periodic orbits.*

Let $f : X \to X$ be a unimodal map, and let $\{f^k(x)\}_{k=0}^{p-1}$ be a periodic orbit of f with prime period p. We say that $\{f^k(x)\}_{k=0}^{p-1}$ is:

- *hyperbolic attracting* if $|(f^p)'(x)| < 1$;
- *neutral* if $|(f^p)'(x)| = 1$;
- *hyperbolic repelling* if $|(f^p)'(x)| > 1$.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

The *basin* of the periodic orbit $\{f^k(x)\}_{k=0}^{p-1}$ is the set of points in X whose omega-limit set is contained in the set $\{f^k(x)\}_{k=0}^{p-1}$. We say that $\{f^k(x)\}_{k=0}^{p-1}$ is *attracting* if its basin contains an open set. In this case, the union of the connected components of the basin containing a point from $\{f^k(x)\}_{k=0}^{p-1}$ is called the *immediate basin* of $\{f^k(x)\}_{k=0}^{p-1}$. Each connected component of the immediate basin contains exactly one point from $\{f^k(x)\}_{k=0}^{p-1}$.

If $f: X \to X$ is S-unimodal, a neutral periodic point x with prime period p satisfies either

(a) $(f^p)''(x) \neq 0$, or

(b) $(f^p)''(x) = 0$ and $(f^p)'''(x)/(f^p)'(x) < 0$.

In case (a), the periodic point x is locally attracting from only one side and in case (b) it is attracting from both sides. Hence, a periodic point is attracting if and only if it is hyperbolic attracting or neutral. The immediate basin of an attracting periodic orbit contains the critical point [**10**, Ch. II, Lemma 6.1, Theorem 6.1]. Therefore, there is at most one attracting periodic orbit, denoted by $O(f)$. Let $B(f)$ denote the immediate basin of $O(f)$. If f has no attracting periodic orbit, we put $B(f) = \emptyset$ for convenience. Let $\delta_{O(f)}$ denote the element of $\mathcal{M}(f)$ that is supported on $O(f)$. The attracting periodic orbit $O(f)$ is called *two-sided attracting* if it is hyperbolic attracting or neutral in case (b). Otherwise (i.e., if it is neutral in case (a)), an attracting periodic orbit $\{f^k(x)\}_{k=0}^{p-1}$ is called *one-sided attracting*.

2.2. *Lyapunov exponents.* Since our definition of the Lyapunov exponent in (1.1) is a little non-standard, we will use the next lemma in §3.

LEMMA 2.1. *Let $f: X \to X$ be an S-unimodal map with a non-flat critical point. Then the map $\mu \in \mathcal{M}(f) \mapsto \chi^+(\mu)$ is affine. Moreover, the following assertions hold.*

(a) *If f has no hyperbolic attracting periodic orbit, then $\chi^+(\mu) = \int \log|f'|\, d\mu$ for any $\mu \in \mathcal{M}(f)$.*

(b) *If f has a hyperbolic attracting periodic orbit, then $\chi^+(\mu) = \int \log|f'|\, d\mu$ for any $\mu \in \mathcal{M}(f)$ satisfying $\mu(O(f)) = 0$.*

Proof. Assertion (a) follows from [**29**, Proposition A.1]. Hence, $\chi^+(\cdot)$ is affine if f has no hyperbolic attracting periodic orbit. Assertion (b) follows from Mañé's theorem [**23**, Theorem A] which asserts that the complement of $O(f)$ is a hyperbolic set. Then $\chi^+(\cdot)$ is affine on the subspace of measures in $\mathcal{M}(f)$ not charging $O(f)$. Since every element of $\mathcal{M}(f)$ can be written uniquely as a convex combination of $\delta_{O(f)}$ and a measure not charging $O(f)$, $\chi^+(\cdot)$ is also affine in this case. □

2.3. *Renormalization of S-unimodal maps.* Let $f: X \to X$ be a unimodal map with a non-flat critical point c. A proper closed subinterval J of X is *restrictive* of *period* $p \geq 2$ if the following conditions hold (cf. [**10**, pp. 139]):

- the interiors of $J, \ldots, f^{p-1}(J)$ are pairwise disjoint;
- $f^p(J) \subset J$ and $f^p(\partial J) \subset \partial J$;

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

- one of the intervals $J, \dots, f^{p-1}(J)$ contains c in its interior;
- J is maximal with respect to these properties—if $J' \supset J$ is a closed interval which is strictly contained in X and satisfies the previous three properties with the same integer p, then $J' = J$.

We define a strictly decreasing sequence of closed intervals

$$X = J_0 \supsetneq J_1 \supsetneq J_2 \supsetneq \cdots$$

which contain the critical point c, and a strictly increasing sequence of integers

$$1 = p_0 < p_1 < p_2 < \cdots$$

so that J_m is restrictive of period p_m for $m \geq 1$, inductively as follows. Given J_m and p_m for some $m \geq 0$, note that $f^{p_m}|_{J_m} \colon J_m \to J_m$ is a unimodal map. If $f^{p_m}|_{J_m}$ has a restrictive interval, then define J_{m+1} to be the restrictive interval of $f^{p_m}|_{J_m}$ containing c and with the smallest period r_m. We define $p_{m+1} = p_m r_m$. If $f^{p_m}|_{J_m}$ has no restrictive interval, then we do not define J_{m+1} and stop the definition setting $\bar{m}(f) = m$. If this inductive definition continues for arbitrarily large m, we set $\bar{m}(f) = \infty$. We say that f is:

- *non-renormalizable* if $\bar{m}(f) = 0$;
- *renormalizable* if $\bar{m}(f) \geq 1$;
- *only finitely renormalizable* if $\bar{m}(f) < \infty$;
- *infinitely renormalizable* if $\bar{m}(f) = \infty$.

We assume that f has negative Schwarzian derivative, and review the dynamics of the unimodal map $f^{p_m}|_{J_m}$, $m \geq 0$. We classify such dynamics into the following cases:

(A) $f^{p_m}|_{J_m}$ is non-renormalizable and has no attracting fixed point;
(B) $f^{p_m}|_{J_m}$ is renormalizable;
(C) $f^{p_m}|_{J_m}$ is non-renormalizable and has an attracting fixed point.

For each $m \geq 0$, let L_m denote the closed interval in J_m bordered by $f^{p_m}(c)$ and $f^{2p_m}(c)$. Note that L_m is possibly a singleton, that is, a degenerate closed interval.

In case (A), we have $f^{p_m}(L_m) = L_m$ and $f^{p_m}|_{L_m}$ is topologically exact: for any relatively open subset U of L_m there is an integer multiple $n \geq 1$ of p_m such that $f^n(U) = L_m$. One can check this by combining [10, Theorem V.1.3] and [30, Theorem 2.19 and Proposition 2.34], for example.

In case (B), we have $c \in \mathrm{int}(L_m)$ and $f^{p_m}(L_m) \subset L_m$, and the orbit of any point in $\mathrm{int}(J_m)$ eventually falls into L_m. There exists a restrictive interval J_{m+1} that contains the critical point c and the set $\bigcup_{k=0}^{p_{m+1}/p_m} f^{kp_m}(J_{m+1})$ is forward invariant with respect to f^{p_m}. We will consider two subcases:

(B1) $p_{m+1}/p_m \neq 2$;
(B2) $p_{m+1}/p_m = 2$.

Compare these in Figure 1. In case (B2) we have $L_m = f^{p_m}(J_{m+1}) \cup f^{2p_m}(J_{m+1})$ and $L_m \subset \bigcup_{k=0}^{p_{m+1}-1} f^k(J_{m+1})$.

In case (C), $m = \bar{m}(f)$ and J_m contains a unique point z_f from $O(f)$. Indeed, if J_m contained two points from $O(f)$, then one would find a restrictive interval for $f^{p_m}|_{J_m}$ as

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

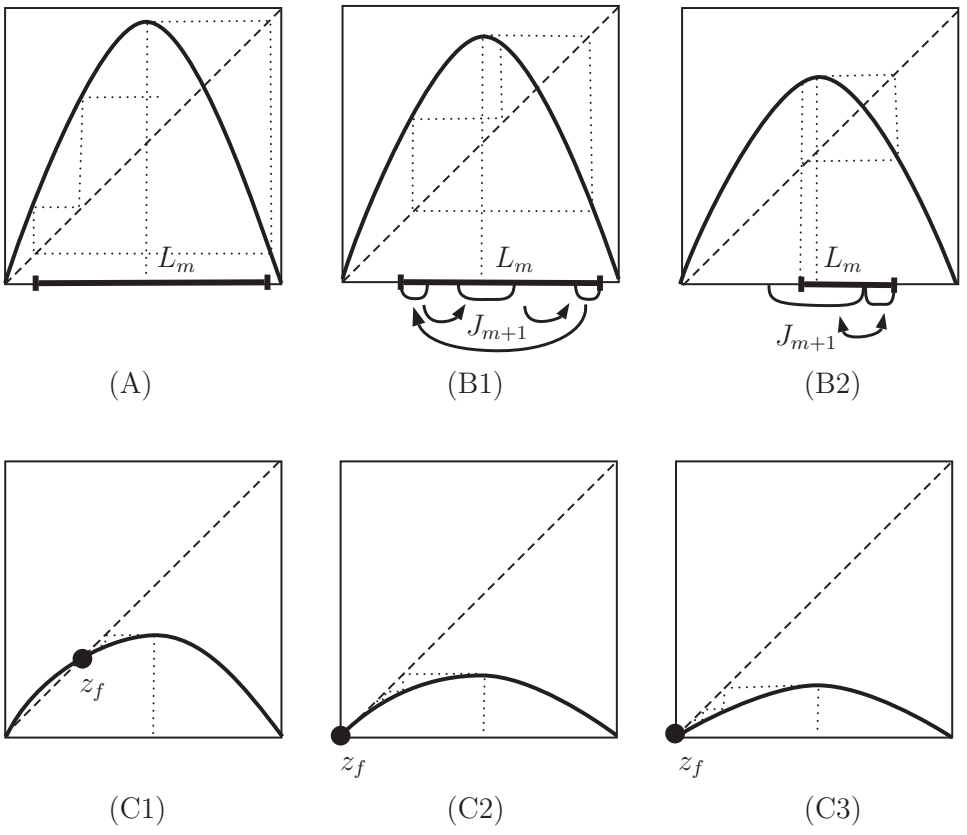


FIGURE 1. The graphs of the renormalized unimodal maps $f^{p_m}|_{J_m} : J_m \to J_m$.

an immediate basin of $O(f)$. In particular, the point $z_f \in O(f) \cap J_m$ is a fixed point of the unimodal map $f^{p_m}|_{J_m}$. We will consider three subcases:

(C1) $z_f \in \text{int}(J_m)$;

(C2) $z_f \in \partial J_m$ and $O(f)$ is neutral;

(C3) $z_f \in \partial J_m$ and $O(f)$ is hyperbolic attracting.

Compare these in Figure 1. In case (C1), $O(f)$ is hyperbolic attracting and $\text{int}(J_m)$ is the connected component of $B(f)$ containing z_f. For any closed interval J contained in $\text{int}(J_m)$, the following uniform convergence holds:

$$\lim_{n\to\infty} \sup_{x\in J} |f^{p_m n}(x) - z_f| = 0. \tag{2.1}$$

In cases (C2) and (C3), since f has negative Schwarzian derivative, $f^{p_m}(c)$ lies in between z_f and c and the following uniform convergence on J_m holds:

$$\lim_{n\to\infty} \sup_{x\in J_m} |f^{p_m n}(x) - z_f| = 0. \tag{2.2}$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

If, moreover, z_f is two-sided attracting and $m \geq 1$, the dynamics of the previous renormalization $f^{p_{m-1}}|_{J_{m-1}}$ is similar to that in case (C1) from Lemma 2.2 below. For this reason, if $1 \leq \bar{m}(f) < \infty$ and $\partial J_{\bar{m}(f)}$ contains a two-sided attracting fixed point of $f^{p_{\bar{m}(f)}}|_{J_{\bar{m}(f)}}$, then we zoom out to the previous renormalization by setting

$$m(f) = \bar{m}(f) - 1.$$

In all other cases, we set

$$m(f) = \bar{m}(f).$$

LEMMA 2.2. *Let $f: X \to X$ be a renormalizable S-unimodal map, and let J be a restrictive interval with period p containing c and not contained in any other restrictive interval with period smaller than p. If $z_f \in \partial J$ is periodic and two-sided attracting, then z_f is a fixed point of f, $p = 2$ and its immediate basin coincides with* $\mathrm{int}(X)$. *Further, for any closed interval J contained in* $\mathrm{int}(X)$, *we have the uniform convergence (2.1) on J with $p_m = 1$.*

Proof. Let B denote the connected component of the immediate basin of the periodic orbit of z_f that contains z_f. Then we have $f^p(B) \subset B$. Since the renormalized map $f^p|_J$ belongs to case (C2) or (C3) and since z_f is two-sided attracting, the interval J is strictly contained in B.

We claim that $f(B) \subset B$. Suppose that this is not the case. Then, by the definition of immediate basin, we have that $f(B) \cap B = \emptyset$. This implies that $\mathrm{cl}(B)$ is a restrictive interval and contradicts the assumption of the lemma.

Since the immediate basin B contains only one point in the orbit of z_f, the point z_f is a fixed point of f. Then it follows that $p = 2$. Since $f(\partial B) \subset \partial B$, it is immediate to see that B coincides with $\mathrm{int}(X)$. □

Remark 2.3. To summarize, in the case $m(f) < \infty$ the dynamics of the renormalization $f^{p_{m(f)}}|_{J_{m(f)}}$ is either in case (A), or one of the following:

(C-I) there is a two-sided attracting fixed point in $\mathrm{int}(J_{m(f)})$ whose immediate basin contains $\mathrm{int}(J_{m(f)})$;

(C-II) there is a one-sided attracting fixed point in $\partial J_{m(f)}$ whose immediate basin contains $J_{m(f)}$.

2.4. *Structures associated with renormalization.* Let $f: X \to X$ be an S-unimodal map with a non-flat critical point c such that $m(f) \geq 1$. For each integer m with $0 \leq m < m(f)$, we define the mth *renormalization cycle* K_m by

$$K_m = \bigcup_{k=0}^{p_m - 1} f^k(J_m).$$

For each integer m with $0 \leq m < m(f) - 1$, we denote by $\tilde{\mathscr{P}}_m$ the collection of the connected components of $K_m \setminus \bigcup_{k=0}^{p_{m+1}-1} \mathrm{int}(f^k(J_{m+1}))$.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

In the case $m(f) < \infty$, we set

$$K_{m(f)} = \begin{cases} \bigcup_{k=0}^{p_{m(f)}-1} f^k(J_{m(f)}) & \text{if } f \text{ has an attracting periodic orbit,} \\ \bigcup_{k=0}^{p_{m(f)}-1} f^k(L_{m(f)}) & \text{if } f \text{ has no attracting periodic orbit.} \end{cases} \tag{2.3}$$

Moreover, we define $\tilde{\mathscr{P}}_{m(f)-1}$ to be the collection of the connected components of the following sets: $K_{m(f)-1} \setminus \bigcup_{k=0}^{p_{m(f)}-1} \operatorname{int}(f^k(J_{m(f)}))$ if f has an attracting periodic orbit; $K_{m(f)-1} \setminus \bigcup_{k=0}^{p_{m(f)}-1} \operatorname{int}(f^k(L_{m(f)}))$ if f has no attracting periodic orbit.

The elements of $\tilde{\mathscr{P}}_m$ with $0 \le m < m(f)$ are closed intervals in K_m, possibly singletons, and $\tilde{\mathscr{P}}_m$ has the Markov property: if $P, Q \in \tilde{\mathscr{P}}_m$ and $f(P) \cap Q \neq \emptyset$ then $f(P) \supset Q$. Let $\mathscr{P}_m$ denote the set of elements of $\tilde{\mathscr{P}}_m$ that are contained in $\bigcup_{k=0}^{p_m-1} f^k(L_m)$. Note that $\mathscr{P}_m$ is non-empty.

Remark 2.4. Case (B2) is rather exceptional. The set $\mathscr{P}_m$ consists of the singletons $f^k(J_{m+1}) \cap f^{p_m+k}(J_{m+1})$, $0 \le k \le p_m - 1$, which form a hyperbolic repelling periodic orbit of prime period p_m.

For each integer m with $0 \le m < m(f)$, we define

$$K_{m,m+1} = \operatorname{cl}(K_m \setminus K_{m+1})$$

and

$$\Gamma_m = \bigcap_{n=0}^{\infty} f^{-n}\Big(\bigcup_{P \in \mathscr{P}_m} P \Big).$$

Note that Γ_m for $0 \le m < m(f)$ contains an attracting periodic orbit only if $m = m(f) - 1$, and this periodic orbit is one-sided attracting. In this case, $K_{m,m+1}$ is disjoint from the interior of the immediate basin of the attracting periodic orbit. For convenience we set $\Gamma_{-1} = \partial X$. In the case $m(f) < \infty$ we further define

$$\Gamma_{m(f)} = \begin{cases} O(f) & \text{if } f \text{ has a two-sided attracting periodic orbit,} \\ \bigcup_{k=0}^{p_{m(f)}-1} f^k(L_{m(f)}) & \text{otherwise.} \end{cases}$$

The sets Γ_m, $-1 \le m \le m(f)$ are non-empty closed sets, and $\Gamma_m \cap \Gamma_{m'} = \emptyset$ holds for all distinct integers m, m' in $\{-1, \dots, m(f) - 1\}$. Note that $\Gamma_{m(f)-1}$ intersects $\Gamma_{m(f)}$ if and only if $L_{m(f)} = J_{m(f)}$.

For each integer m with $-1 \le m \le m(f)$ we set

$$\mathcal{M}_m(f) = \{\mu \in \mathcal{M}(f) \colon \operatorname{supp}(\mu) \subset \Gamma_m\},$$

where $\operatorname{supp}(\mu)$ denotes the smallest closed subset of X with full μ-measure. The sets $\mathcal{M}_m(f)$ are pairwise disjoint, and if $m(f) < \infty$ then $\mathcal{M}(f)$ is the convex hull of $\bigcup_{m=-1}^{m(f)} \mathcal{M}_m(f)$. If $m(f) < \infty$ and f has a one-sided attracting periodic orbit, then $L_{m(f)}$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

is contained in the basin of this attracting periodic orbit, and so $\mathcal{M}_{m(f)}(f) = \emptyset$. The set $\mathcal{M}_{-1}(f)$ is a singleton that is supported on the fixed point in ∂X.

2.5. *Symbolic dynamics on each cycle.* Let $f\colon X \to X$ be an S-unimodal map such that $m(f) \geq 1$. For each integer m with $0 \leq m < m(f)$, there is a topological Markov chain over the finite alphabet $\tilde{\mathscr{P}}_m$ determined by the transition matrix

$$(M_{PQ})_{P,Q\in\tilde{\mathscr{P}}_m}, \quad M_{PQ} = \begin{cases} 1 & \text{if } f(P) \supset Q, \\ 0 & \text{otherwise.} \end{cases}$$

Let $n \geq 2$ be an integer and let $P_0, P_1, \ldots, P_{n-1} \in \tilde{\mathscr{P}}_m$. The word $P_0P_1 \cdots P_{n-1}$ of length n is *admissible* if $M_{P_kP_{k+1}} = 1$ holds for $0 \leq k \leq n-2$. Let E_m^n denote the set of admissible words of elements of $\tilde{\mathscr{P}}_m$ of length n. Let Σ_m denote the set of one-sided infinite sequences $\{P_k\}_{k=0}^{\infty}$ of elements of $\tilde{\mathscr{P}}_m$ such that $P_0 \cdots P_{n-1} \in E_m^n$ holds for every $n \geq 2$. We endow Σ_m with the restriction of the product topology of the discrete topology of $\tilde{\mathscr{P}}_m$. Let $\sigma_m\colon \Sigma_m \to \Sigma_m$ denote the left shift: $\sigma_m(\{P_k\}_{k=0}^{\infty}) = \{P_k\}_{k=1}^{\infty}$. For each $P_0P_1 \cdots P_{n-1} \in E_m^n$, we set $I_{P_0P_1\cdots P_{n-1}} = \bigcap_{k=0}^{n-1} f^{-k}(P_k)$, and define

$$\pi_m\colon \Sigma_m \to \bigcap_{n=0}^{\infty} f^{-n}\Bigg(\bigcup_{P\in\tilde{\mathscr{P}}_m} P\Bigg)$$

by

$$\pi_m(\{P_k\}_{k=0}^{\infty}) \in \bigcap_{n=1}^{\infty} I_{P_0\cdots P_{n-1}}. \tag{2.4}$$

By [21, Main Theorem], f has no wandering interval, and so any homterval is contained in the basin of an attracting periodic orbit. Since $\bigcap_{n=1}^{\infty} I_{P_0\cdots P_{n-1}}$ is not contained in the basin of an attracting periodic orbit, it is not a homterval, namely, a singleton. Hence, π_m is well defined.

PROPOSITION 2.5. *Let $f\colon X \to X$ be an S-unimodal map with a non-flat critical point such that $m(f) \geq 1$. For each integer m with $0 \leq m < m(f)$, the restriction of f to $\bigcap_{n=0}^{\infty} f^{-n}(\bigcup_{P\in\tilde{\mathscr{P}}_m} P)$ is topologically conjugate by the conjugacy map π_m to the topological Markov chain $\sigma_m\colon \Sigma_m \to \Sigma_m$.*

Proof. By definition, π_m is continuous and surjective. Since the elements of $\tilde{\mathscr{P}}_m$ are pairwise disjoint, π_m is injective and has a continuous inverse. For each $P_0P_1 \cdots P_{n-1} \in E_m^n$, the Markov property of $\tilde{\mathscr{P}}_m$ implies $f^{k-1}(I_{P_0\cdots P_{n-1}}) = I_{P_{k-1}\cdots P_{n-1}}$ for every $1 \leq k \leq n$. Hence, $f \circ \pi_m = \pi_m \circ \sigma_m$ holds. We have verified that the restriction of f to $\bigcap_{n=0}^{\infty} f^{-n}(\bigcup_{P\in\tilde{\mathscr{P}}_m} P)$ is topologically conjugate to σ_m by π_m. □

We will use the following lemma a few times.

LEMMA 2.6. *Let $f\colon X \to X$ be an S-unimodal map with a non-flat critical point such that $m(f) \geq 1$. For each integer m with $0 \leq m < m(f)$, we have*

$$\lim_{n\to\infty} \sup\{|I_\omega| : \omega \in E_m^n\} = 0.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Proof. Suppose there exist $\varepsilon > 0$ and an infinite subset F of $\bigcup_{n=1}^{\infty} E_m^n$ such that $|I_\omega| > \varepsilon$ for all $\omega \in F$. Since X has a finite diameter, we can choose a sequence $\{\omega^{(n)}\}_{n=1}^{\infty}$ in F such that $\{I_{\omega^{(n)}}\}_{n=1}^{\infty}$ is a nested sequence. Since $\bigcap_{n=1}^{\infty} I_{\omega^{(n)}}$ is not a singleton, it is a homterval. However, it is not contained in the basin of an attracting periodic orbit. We obtain a contradiction to [21, Main Theorem]. □

2.6. *Distortion estimate on each cycle.* Let Y be a non-degenerate compact interval in X and let $g\colon Y \to X$ be a C^1 map. Let J be a subinterval of Y such that the restriction of g to J is a diffeomorphism onto its image. By a *distortion of g on J* we mean the quantity

$$\sup_{x,y\in J} \frac{|g'(x)|}{|g'(y)|}.$$

We will frequently use the following distortion estimates on each cycle.

PROPOSITION 2.7. *Let $f\colon X \to X$ be an S-unimodal map with a non-flat critical point such that $m(f) \geq 1$. For each integer m with $0 \leq m < m(f)$ the following assertions hold.*

(a) *If Γ_m does not contain a neutral periodic orbit, then there exists a constant $\gamma_m \geq 1$ such that if $n \geq 1$ then the distortion of f^n on any connected component of $\bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1})$ is bounded by γ_m.*

(b) *If Γ_m contains a neutral periodic orbit, then for any $\varepsilon > 0$ there exists $N \geq 1$ such that if $n \geq N$ then the distortion of f^n on every connected component of $\bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1})$ is bounded by $e^{\varepsilon n}$.*

Proof. Since f is C^2, the restriction of $\log|f'|$ to the union of the elements of $\tilde{\mathscr{P}}_m$ not containing a point from an attracting periodic orbit is Lipschitz continuous. Moreover, by Mañé's theorem [23, Theorem A], the maximal f-invariant set in this union is a hyperbolic set. Hence, (a) holds.

As for (b), if Γ_m contains a neutral periodic orbit, then $m = m(f) - 1$ and $\partial J_{m(f)}$ contains a point from the neutral periodic orbit. Let $\varepsilon > 0$, let $n \geq 2$ and let W be a connected component of $\bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1})$. There exists $\omega \in E_m^n$ such that $W = I_\omega$. Since $K_{m,m+1}$ does not contain the critical point of f, the infimum of $|f'|$ over this set is positive. For all $x, y \in I_\omega$ we have

$$\log \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \leq \left(\sup_{K_{m,m+1}} \frac{|f''|}{|f'|}\right) \sum_{k=0}^{n-1} |f^k(I_\omega)| \leq \left(\sup_{K_{m,m+1}} \frac{|f''|}{|f'|}\right) \sum_{k=0}^{n-1} \sup_{\omega\in E_m^{n-k}} |I_\omega|.$$

By Lemma 2.6, the last number is bounded by εn for a sufficiently large n. The proof of Proposition 2.7 is complete. □

2.7. *Topological exactness on each cycle.* Let Y be a non-degenerate interval in X and $g\colon Y \to Y$ be a continuous map. Let U be a non-degenerate subinterval of Y and let $n \in \mathbb{N}$. Each connected component of $g^{-n}(U)$ is called *a pullback of U by g^n*. If V is a pullback of U by g^n and $g^n|_V$ is a diffeomorphism, then V is called *a diffeomorphic pullback of U by g^n*.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

The next lemma will be used later in §3.2 for the proof of the lower bound in Theorem A, in order to glue different orbits in different cycles together to form one orbit with required properties. It will also be used later in §4.2 for the proof of the upper bound in Theorem A.

LEMMA 2.8. *Let $f: X \to X$ be an S-unimodal map with a non-flat critical point such that $m(f) \geq 1$. Let $0 \leq m < m(f)$ and suppose $p_{m+1}/p_m \neq 2$. For any open interval $J \subset L_m$ that intersects Γ_m, there exists an integer $n \geq 1$ such that $f^{p_m n}(J) = L_m$, and for any $n' \geq n$ there exists a diffeomorphic pullback of* $\mathrm{int}(L_m)$ *by $f^{p_m n'}$ that is contained in J.*

Proof. We assume $m = 0$ in the proof below. In the case $m > 0$, we can apply the same argument to the renormalized map $f^{p_m}: J_m \to J_m$ to obtain the conclusion of the lemma.

Without loss of generality we may assume $X = [0, 1]$ and the critical point c is the maxima of f. We claim that the topological entropy $h_{\mathrm{top}}(f)$ of $f: X \to X$ is greater than or equal to $\log\sqrt{2}$. Indeed, the assumption $p_1 = p_1/p_0 > 2$ implies that if we take an interval $[x', x]$ such that $x' < c < x$ and $f(x) = x = f(x')$, then $f^2(x) = x$, $f^2(c) < x'$ and $[x', x] \subset f^2([x', x])$. Since $[x', x]$ contains an f^2-invariant set on which f^2 is topologically conjugate to the full shift on two symbols, we obtain $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f^2)/2 \geq \log\sqrt{2}$.

By the Milnor–Thurston kneading theory (see [10, Ch. 2, Theorem 8.1]), there exists a semiconjugacy from $f: X \to X$ to a piecewise linear map with slopes $\pm s$ where $s = e^{h_{\mathrm{top}}(f)} \geq \sqrt{2}$. More precisely, there exists a continuous and surjective monotone map $\lambda: X \to X$ satisfying $T \circ \lambda = \lambda \circ f$ for the tent map

$$T: X \to X, \quad T(x) = \min\{sx, s(1 - x)\}.$$

The map λ sends a non-trivial interval in X to a point only if it is eventually mapped by f into the restrictive interval J_1. (For this, we use the non-existence of wandering intervals; see [10, III.4, Proposition 4.3].) Hence, λ maps an open interval $J \subset L_0$ that intersects the Cantor set Γ_0 onto a non-trivial interval J'.

If $s = \sqrt{2}$, then T is renormalizable with period 2 and so is f, contradicting the assumption $p_1 > 2$. So we may suppose $s > \sqrt{2}$. Then there is $n \geq 0$ such that both $T^n(J')$ and $T^{n+1}(J')$ contain the turning point $1/2$ of T in their interiors (see [10, III.4, Proposition 4.4 and Exercise 4.1]). Consequently, $f^n(J)$ and $f^{n+1}(J)$ contain c in their interiors and therefore $f^{n+2}(J) \supset L_0 = [f^2(c), f(c)]$. Then we have $f^{n'}(J) \supset f^{n'-n-2}(L_0) = L_0$ for $n' \geq n + 2$.

Since f is unimodal, for any non-degenerate subinterval I of X we can take a subinterval $I' \subset I$ such that $f: I' \to f(I)$ is a homeomorphism. Applying this procedure to the sequence of maps

$$J \xrightarrow{f} f(J) \xrightarrow{f} \cdots \xrightarrow{f} f^{n'-1}(J) \xrightarrow{f} f^{n'}(J) \supset L_0,$$

we find an interval $J' \subset J$ such that $f^{n'}: J' \to L_0$ is a homeomorphism. Then $\mathrm{int}(J')$ is a diffeomorphic pullback of $\mathrm{int}(L_0)$ by $f^{n'}$, as required in the lemma. □

Let $f: X \to X$ be an S-unimodal map with a non-flat critical point such that $m(f) \geq 1$. Suppose $p_{m+1}/p_m \neq 2$ for some $0 \leq m < m(f)$. In view of Lemma 2.8, for each $P \in \mathscr{P}_m$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

that is contained in L_m we fix $q(P) \in \mathbb{N}$ such that $f^{p_m q(P)}(P) = L_m$, and for any $n \geq q(P)$ there exists a diffeomorphic pullback of $\operatorname{int}(L_m)$ by $f^{p_m n}$ that is contained in P. For later uses and convenience we set

$$M_m = \max\{p_m q(P) \colon P \in \mathscr{P}_m \text{ and } P \subset L_m\} \quad \text{and} \quad M_{-1} = 2. \tag{2.5}$$

2.8. *Notation.* Let $C(X)$ denote the set of real-valued continuous functions on X. For $\phi \in C(X)$ and $n \in \mathbb{N}$, write $S_n\phi$ for the sum $\sum_{k=0}^{n-1} \phi \circ f^k$. For $\ell \in \mathbb{N}$ define

$$C(X)^\ell = \{\vec{\phi} = (\phi_1, \ldots, \phi_\ell) \colon \phi_j \in C(X) \text{ for } 1 \leq j \leq \ell\}.$$

For $\vec{\phi} = (\phi_1, \ldots, \phi_\ell) \in C(X)^\ell$, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{R}^\ell$ and a measure $\mu \in \mathcal{M}$, the expression $\int \vec{\phi}\, d\mu > \vec{\alpha}$ indicates that $\int \phi_j\, d\mu > \alpha_j$ holds for $1 \leq j \leq \ell$. We set

$$A_n(\vec{\phi}, \vec{\alpha}) = \left\{x \in X : \int \vec{\phi}\, d\delta_x^n > \vec{\alpha}\right\}$$

and

$$S_n\vec{\phi} = (S_n\phi_1, \ldots, S_n\phi_\ell).$$

For $a \in \mathbb{R}$ and $\vec{v} = (v_1, \ldots, v_\ell) \in \mathbb{R}^\ell$, we write

$$\vec{a} = (a, a, \ldots, a) \in \mathbb{R}^\ell \quad \text{and} \quad \|\vec{v}\| = \max_{1 \leq j \leq \ell} |v_j|.$$

For $\vec{v}, \vec{w} \in \mathbb{R}^\ell$, the expression $\vec{v} \geq \vec{w}$ indicates that $v_j \geq w_j$ holds for $1 \leq j \leq \ell$.

2.9. *Infinitely renormalizable maps.* Let $f \colon X \to X$ be an infinitely renormalizable S-unimodal map with a non-flat critical point. The omega-limit set of Lebesgue almost every initial point $x \in X$ coincides with the closed invariant set

$$\Lambda = \bigcap_{m=0}^{\infty} K_m,$$

and the empirical measure δ_x^n converges to *the post-critical measure* μ_∞ supported on Λ. The restriction of f to Λ is uniquely ergodic and the post-critical measure μ_∞ is the unique invariant measure on Λ. Moreover, $h(\mu_\infty) = \chi^+(\mu_\infty) = 0$ holds (see [4, Theorem 3.4(b)]).

In order to 'approximate' μ_∞, we will use the following lemma, which in particular asserts that the post-critical measure is approximated by empirical measures along sufficiently long orbit segments in sufficiently deep cycles.

LEMMA 2.9. [33, Lemma 2.3] *Let $f \colon X \to X$ be an S-unimodal map with a non-flat critical point such that $m(f) = \infty$. Let $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$. For any $\varepsilon > 0$ there exist m_*, $N_* \in \mathbb{N}$ such that the following assertions hold.*

(a) *If $n \geq N_*$ and $A_n(\vec{\phi}, \vec{\alpha}) \cap K_{m_*} \neq \emptyset$ then*

$$\int \vec{\phi}\, d\mu_\infty > \vec{\alpha} - \vec{\varepsilon}.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

(b) *If $n \geq N_*$ then*

$$\sup_{K_{m_*}} \left\| \frac{1}{n} S_n \vec{\phi} - \int \vec{\phi}\, d\mu_\infty \right\| < \varepsilon.$$

For infinitely renormalizable maps, the Lyapunov exponent depends continuously on invariant measures.

LEMMA 2.10. [33, Lemma 2.6] *Let $f : X \to X$ be an S-unimodal map with a non-flat critical point such that $m(f) = \infty$. Then $\mu \in \mathcal{M}(f) \mapsto \chi^+(\mu)$ is continuous.*

3. *Large deviations lower bound*

In this section we complete the proof of the lower bound in Theorem A. In §3.1 we show that this follows from a key lower bound stated in Proposition 3.1. The rest of this section is dedicated to the proof of Proposition 3.1.

3.1. *Key lower bound.*

The next proposition allows us to approximate a given invariant measure with finitely many intervals in a particular sense.

PROPOSITION 3.1. *Let $f : X \to X$ be an S-unimodal map with a non-flat critical point. Let $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$ and let $\mu \in \mathcal{M}(f)$ satisfy $\int \vec{\phi}\, d\mu > \vec{\alpha}$. For any $\varepsilon > 0$ there exists $N \geq 1$ such that for each integer $n \geq N$, there exists a finite collection $\mathscr{A}_n$ of closed intervals in $A_n(\vec{\phi}, \vec{\alpha})$ with pairwise disjoint interiors such that*

$$\#\mathscr{A}_n > \exp((h(\mu) - \varepsilon)n) \quad \text{and} \tag{a}$$

$$|A| > \exp(-(\chi^+(\mu) + \varepsilon)n) \quad \text{for every } A \in \mathscr{A}_n. \tag{b}$$

In order to prove Proposition 3.1, we approximate the given measure μ with a finite convex combination of measures each supported on some Γ_m or Λ, and then for each of these measures construct a collection of finitely many intervals. We then glue orbits from these intervals together to construct a collection intervals with the desired properties.

The proof of Proposition 3.1 is given in §3.4. Below we deduce a corollary to Proposition 3.1 and complete the proof of the lower bound in Theorem A.

COROLLARY 3.2. *Let $f : X \to X$ be an S-unimodal map with a non-flat critical point. Let $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$ and let $\mu \in \mathcal{M}(f)$ satisfy $\int \vec{\phi}\, d\mu > \vec{\alpha}$. Then*

$$\liminf_{n\to\infty} \frac{1}{n} \log |A_n(\vec{\phi}, \vec{\alpha})| \geq F(\mu).$$

Proof. From Proposition 3.1 and (1.2), for any $\varepsilon > 0$ there exists $N \geq 1$ such that $|A_n(\vec{\phi}, \vec{\alpha})| > e^{(F(\mu)-2\varepsilon)n}$ holds for every $n \geq N$. Taking logarithms of both sides, dividing by n and letting $n \to \infty$ and then $\varepsilon \to 0$, we obtain the desired inequality. □

Proof of the lower bound in Theorem A. Note that subsets of $\mathcal{M}$ of the form $\{\mu \in \mathcal{M} : \int \vec{\phi}\, d\mu > \vec{\alpha}\}$ with $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$ constitute a base of the weak*

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

topology of $\mathcal{M}$. So, any non-empty open subset $\mathcal{G}$ of $\mathcal{M}$ is written as the union $\mathcal{G} = \bigcup_\lambda \mathcal{G}_\lambda$ of sets $\mathcal{G}_\lambda$ of this form. For each $\mathcal{G}_\lambda$, Corollary 3.2 gives

$$\liminf_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta_x^n \in \mathcal{G}_\lambda\}| \geq \sup_{\mathcal{G}_\lambda} F.$$

Hence, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta_x^n \in \mathcal{G}\}| \geq \sup_\lambda \sup_{\mathcal{G}_\lambda} F = \sup_{\mathcal{G}} F = -\inf_{\mathcal{G}} I,$$

as required in Theorem A. □

3.2. *Approximation of a measure on each cycle.* The rest of this section is entirely dedicated to a proof of Proposition 3.1. We fix $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$ in the statement of Proposition 3.1 until the end of §3.4, and assume $f\colon X \to X$ is an S-unimodal map with a non-flat critical point c.

The next lemma allows us to approximate an invariant measure on each single cycle with a finite collection of intervals.

LEMMA 3.3. *Let $f\colon X \to X$ satisfy $m(f) \geq 1$. Let $0 \leq m < m(f)$ and suppose $p_{m+1}/p_m \neq 2$. Let $\mu \in \mathcal{M}_m(f)$, $P \in \mathscr{P}_m$ satisfy $P \subset L_m$ and $\mu(P) > 0$, and let P' be a non-degenerate closed interval in L_m. For any $\varepsilon > 0$, there exists $N \geq 1$ such that for each integer $q \geq N$, there exists a finite collection $\mathscr{B}_{p_m q}(P, P')$ of pullbacks of P' by $f^{p_m q}$ that are contained in P such that*

$$\left| \frac{1}{p_m q} \log \#\mathscr{B}_{p_m q}(P, P') - h(\mu) \right| < \varepsilon, \tag{a}$$

$$\sup_{B \in \mathscr{B}_{p_m q}(P,P')} \sup_B \left\| \frac{1}{p_m q} S_{p_m q}\vec{\phi} - \int \vec{\phi}\, d\mu \right\| < \varepsilon. \tag{b}$$

If, moreover, $P' \cap \{f(c), \ldots, f^{M_m - 1}(c)\} = \emptyset$ for the integer $M_m \geq 1$ in (2.5), then every pullback $B \in \mathscr{B}_{p_m q}(P, P')$ is diffeomorphic and satisfies

$$\sup_B \left| \frac{1}{p_m q} \log |(f^{p_m q})'| - \chi^+(\mu) \right| < \varepsilon. \tag{c}$$

Proof. From the definition of π_m in (2.4) and Proposition 2.5, $f|_{\Gamma_m}$ is topologically conjugate to a topological Markov chain via the finite Markov partition $\mathscr{P}_m$. For $q \in \mathbb{N}$ let $\tilde{\mathscr{B}}_{p_m q}(P)$ denote the collection of the pullbacks of elements of $\mathscr{P}_m$ by $f^{p_m q}$ that are contained in P. Lemma 2.6 gives $\lim_{q\to\infty} \sup_{B \in \tilde{\mathscr{B}}_{p_m q}(P)} |B| = 0$, and so for any continuous function $\varphi\colon K_{m,m+1} \to \mathbb{R}$ we have

$$\lim_{q\to\infty} \sup_{B \in \tilde{\mathscr{B}}_{p_m q}(P)} \sup_{x,y \in B} \frac{1}{p_m q} |S_{p_m q}\varphi(x) - S_{p_m q}\varphi(y)| = 0. \tag{3.1}$$

The proof of Lemma 3.3 divides into two cases. We first treat the case where μ is ergodic. Since $m < m(f)$, μ is not supported on the orbit of a hyperbolic attracting periodic orbit (see §2.4). Let $\varepsilon > 0$. Since $\mu(L_m) = 1/p_m$, the normalized restriction of μ

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

to L_m is an $f^{p_m}|_{L_m}$-invariant Borel probability measure with entropy $p_m h(\mu)$. For $q \in \mathbb{N}$, we set $r = q - M_m/p_m$. Since $P \in \mathscr{P}_m$, $\tilde{\mathscr{B}}_{p_m r}(P)$ coincides with the collection of the pullbacks of elements of $\mathscr{P}_m$ by $f^{p_m r}$ that are contained in P. We claim that for all sufficiently large q there exists a subset $\mathscr{B}_{p_m r}(P)$ of $\tilde{\mathscr{B}}_{p_m r}(P)$ for which the following inequalities hold:

$$\left|\frac{1}{p_m r}\log \#\mathscr{B}_{p_m r}(P) - h(\mu)\right| < \frac{\varepsilon}{3};$$
$$\sup_{B\in\mathscr{B}_{p_m r}(P)} \sup_{B} \left\|\frac{1}{p_m r} S_{p_m r}\vec{\phi} - \int \vec{\phi}\, d\mu\right\| < \frac{\varepsilon}{3};$$
$$\sup_{B\in\mathscr{B}_{p_m r}(P)} \sup_{B} \left|\frac{1}{p_m r}\log |(f^{p_m r})'| - \chi^+(\mu)\right| < \frac{\varepsilon}{3}.$$

Indeed, the first inequality follows from Shannon, McMillan and Breiman's theorem [9] for the normalized restriction. The second one follows from (3.1) and Birkhoff's ergodic theorem. The third one follows from Proposition 2.7, Birkhoff's ergodic theorem and $\chi^+(\mu) = \int \log |f'|\, d\mu$ by Lemma 2.1.

By the definition of M_m in (2.5), if $Q \in \mathscr{P}_m$ and $Q \subset L_m$ then $f^{M_m}(Q) = L_m$. For each $B \in \mathscr{B}_{p_m r}(P)$, the interval $f^{p_m r}(B)$ is an element of $\mathscr{P}_m$ that is contained in L_m. We fix a connected component W_B of $f^{p_m r}(B) \cap f^{-M_m}(P')$. Let B' denote the diffeomorphic pullback of W_B by $f^{p_m r}$ that is contained in B. We define

$$\mathscr{B}_{p_m q}(P, P') = \{B' \colon B \in \mathscr{B}_{p_m r}(P)\}.$$

Clearly we have $\#\mathscr{B}_{p_m q}(P, P') = \#\mathscr{B}_{p_m r}(P)$, and the elements of $\mathscr{B}_{p_m q}(P, P')$ are pullbacks of P' by $f^{p_m q}$. If $P' \cap \{f(c), \dots, f^{M_m - 1}(c)\} = \emptyset$ then for every $B \in \mathscr{B}_{p_m r}(P)$, W_B is a diffeomorphic pullback of P' by f^{M_m}. So, every element of $\mathscr{B}_{p_m q}(P, P')$ is a diffeomorphic pullback of P' by $f^{p_m q}$. Therefore, for all sufficiently large $q \geq 1$ we obtain inequalities (a), (b), (c) of Lemma 3.3 from the corresponding three estimates for $\mathscr{B}_{p_m r}(P)$ as above. To deduce (c) we have noted that the orbits of points in elements of $\mathscr{B}_{p_m q}(P, P')$ up to time $p_m q$ are uniformly bounded away from the critical point c by a distance independent of q.

It remains to treat the case where μ is non-ergodic. By Lemma 2.1, the map $\nu \in \mathcal{M}_m(f) \mapsto \chi^+(\nu) \in \mathbb{R}$ is affine. By virtue of the ergodic decomposition theorem and Jacobs's theorem on the decomposition of entropy [16], for any $\varepsilon > 0$ there exist a finite number of ergodic measures $\mu_1, \dots, \mu_s$ in $\mathcal{M}_m(f)$ and constants $\rho_1, \dots, \rho_s$ in $(0, 1)$ adding up to 1 such that the measure $\mu' = \sum_{i=1}^{s} \rho_i \cdot \mu_i$ in $\mathcal{M}_m(f)$ satisfies

$$|h(\mu) - h(\mu')| < \varepsilon, \quad \left\|\int \vec{\phi}\, d\mu - \int \vec{\phi}\, d\mu'\right\| < \varepsilon, \quad |\chi^+(\mu) - \chi^+(\mu')| < \varepsilon,$$

and $\mu'(P)$ is sufficiently close to $\mu(P)$ so that $\mu'(P) > 0$. With no loss of generality we may assume that μ is written as a convex combination of ergodic measures $\mu_i \in \mathcal{M}_m(f)$:

$$\mu = \rho_1 \cdot \mu_1 + \rho_2 \cdot \mu_2 + \cdots + \rho_s \cdot \mu_s,$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

with $\mu_1(P) > 0$ and $\rho_1, \dots, \rho_s$ adding up to 1. We fix $P_1, \dots, P_s \in \mathscr{P}_m$ so that $P_1 = P$, $P_i \subset L_m$ and $\mu_i(P_i) > 0$ for $1 \le i \le s$. We set $P_{s+1} = P'$.

Let $\varepsilon > 0$. Let $q \ge s$ be a large integer, and write it as a sum of positive integers

$$q = q_1 + q_2 + \cdots + q_s,$$

with $|p_m q_i - p_m \rho_i q| \le p_m$ for $1 \le i \le s$. From the previous argument in the ergodic case, if q is sufficiently large then for each i there is a finite collection $\mathscr{B}_{p_m q_i}(P_i, P_{i+1})$ of diffeomorphic pullbacks of P_{i+1} by $f^{p_m q_i}$ that is contained in P_i for which the following three inequalities hold:

$$\left| \frac{1}{p_m q_i} \log \#\mathscr{B}_{p_m q_i}(P_i, P_{i+1}) - h(\mu_i) \right| < \frac{\varepsilon}{2}; \tag{3.2}$$

$$\sup_{B \in \mathscr{B}_{p_m q_i}(P_i, P_{i+1})} \sup_B \left\| \frac{1}{p_m q_i} S_{p_m q_i} \vec{\phi} - \int \vec{\phi}\, d\mu_i \right\| < \frac{\varepsilon}{2}; \tag{3.3}$$

$$\sup_{B \in \mathscr{B}_{p_m q_i}(P_i, P_{i+1})} \sup_B \left| \frac{1}{p_m q_i} \log |(f^{p_m q_i})'| - \chi^+(\mu_i) \right| < \frac{\varepsilon}{2}. \tag{3.4}$$

We now define $\mathscr{B}_{p_m q}(P, P')$ to be the collection of the diffeomorphic pullbacks of P' by the composition $g_s \circ g_{s-1} \circ \cdots \circ g_1$ of diffeomorphisms g_i of the form $f^{p_m q_i}|_B$ with $B \in \mathscr{B}_{p_m q_i}(P_i, P_{i+1})$. If q is sufficiently large, then by (3.2) we have

$$\begin{aligned}
\left| \frac{1}{p_m q} \log \#\mathscr{B}_{p_m q}(P, P') - h(\mu) \right| &\le \frac{1}{p_m q} \left| \sum_{i=1}^{s} \log \#\mathscr{B}_{p_m q_i}(P_i, P_{i+1}) - p_m q \sum_{i=1}^{s} \rho_i h(\mu_i) \right| \\
&\le \frac{1}{p_m q} \sum_{i=1}^{s} |\log \#\mathscr{B}_{p_m q_i}(P_i, P_{i+1}) - p_m q_i h(\mu_i)| \\
&\quad + \sum_{i=1}^{s} \left| \frac{q_i}{q} - \rho_i \right| h(\mu_i) < \frac{\varepsilon}{2} + \frac{s}{q} h_{\mathrm{top}}(f) < \varepsilon,
\end{aligned}$$

where $h_{\mathrm{top}}(f)$ denotes the topological entropy of f. Moreover, for each $B \in \mathscr{B}_{p_m q}(P, P')$, using (3.3) for $1 \le i \le s$, we have

$$\begin{aligned}
\sup_B \left\| \frac{1}{p_m q} S_{p_m q} \vec{\phi} - \int \vec{\phi}\, d\mu \right\| &\le \frac{1}{p_m q} \sum_{i=1}^{s} \sup_{B' \in \mathscr{B}_{p_m q_i}(P_i, P_{i+1})} \sup_{B'} \left\| S_{p_m q_i} \vec{\phi} - p_m q_i \int \vec{\phi}\, d\mu_i \right\| \\
&\quad + \sum_{i=1}^{s} \left| \frac{q_i}{q} - \rho_i \right| \|\vec{\phi}\| < \frac{\varepsilon}{2} + \frac{s}{q} \|\vec{\phi}\| < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the above two estimates respectively imply (a) and (b) in Lemma 3.3. We can prove (c) in the same way as (b), using (3.4) and noting that the orbits of points in elements of $\mathscr{B}_{p_m q}(P, P')$ up to time $p_m q$ are uniformly bounded away from the critical point c by a distance independent of q. This completes the proof of Lemma 3.3. □

3.3. *The initial transition.* Let μ_{-1} denote the element of the singleton $\mathcal{M}_{-1}(f)$. Recall that μ_{-1} is the unit point mass at the fixed point in ∂X (see §2.4). Put $p_{-1} = 1$. Recall

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

that L_0 is the closed interval in X bordered by $f(c)$ and $f^2(c)$ (see §2.3). Let x_f denote the fixed point of f in ∂X. If L_0 is strictly contained in X, let P_{-1} denote the connected component of $X \setminus L_0$ that contains x_f.

LEMMA 3.4. *Let $f: X \to X$ be S-unimodal such that L_0 is strictly contained in X and is not a singleton. Let P be a non-degenerate closed interval in* $\mathrm{int}(X)$. *Suppose that either*

(i) *f is renormalizable and $P \subset L_0$, or*
(ii) *f is non-renormalizable and P is contained in the open interval bordered by x_f and the unique fixed point z_f of f in* $\mathrm{int}(X)$ *(see (C1) in Figure 1).*

For any $\varepsilon > 0$, there exists $N \geq 1$ such that for each integer $n \geq N$, there exists a pullback B of P by f^n that is contained in P_{-1} such that

$$\sup_B \left\| \frac{1}{n} S_n \vec{\phi} - \int \vec{\phi}\, d\mu_{-1} \right\| < \varepsilon. \tag{a}$$

If, moreover, $f(c) \notin P$, then the pullback B is diffeomorphic and satisfies

$$\sup_B \left| \frac{1}{n} \log |(f^n)'| - \chi^+(\mu_{-1}) \right| < \varepsilon. \tag{b}$$

Proof. In case (i), $f(c)$ is not contained in the closed interval bordered by x_f and c. Without loss of generality we may assume $x_f < c$. We define a sequence $\{x_n\}_{n=2}^{\infty}$ in (x_f, c) inductively by $x_2 \in f^{-1}(c)$ and $x_{n+1} \in f^{-1}(x_n)$ for every $n \geq 2$. From the minimum principle for negative Schwarzian unimodal maps [10], this sequence is well defined and satisfies $\lim_{n\to\infty} x_n = x_f$. The interval $[x_f, x_n]$ is mapped by f^n homeomorphically onto $[x_f, f(c)]$. Let B denote the pullback of P by f^n that is contained in $[x_f, x_n]$. If $f(c) \notin P$, then B is a diffeomorphic pullback. If n is sufficiently large, then (a) and (b) follow from the bounded distortion near x_f.

The proof for case (ii) is analogous, with c replaced by z_f. The proof of Lemma 3.4 is complete. □

3.4. *Proof of Proposition 3.1.* If $L_0 = X$ then f is non-renormalizable, and the conclusion of Proposition 3.1 was already shown in the proof of [7, Proposition 2.1]. If L_0 is a singleton, then $f(c) = c$ and any element of $\mathcal{M}(f)$ is a convex combination of μ_{-1} and the unit point mass at c. Hence, it is not difficult to show the conclusion of Proposition 3.1 by slightly modifying the argument in Case 2 below.

We assume L_0 is strictly contained in X and not a singleton. Let $\mu \in \mathcal{M}(f)$ satisfy $\int \vec{\phi}\, d\mu > \vec{\alpha}$. Let $\varepsilon > 0$ be small enough so that

$$\int \vec{\phi}\, d\mu - \vec{\varepsilon} > \vec{\alpha}. \tag{3.5}$$

In what follows we treat four cases separately. We start with the case where f is infinitely renormalizable. Arguments for the remaining cases will proceed pretty much in parallel to the infinitely renormalizable case.

Case 1: $m(f) = \infty$. Let m_*, N_* be positive integers for which the conclusion of Lemma 2.9 holds with ε replaced by $\varepsilon/3$. By the continuity of the Lyapunov exponent in Lemma 2.10, we may assume with no loss of generality that there exist constants

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

$\rho_{-1}, \dots, \rho_{m_*} \in (0, 1)$ that add up to 1, and for each $m \in \{-1, \dots, m_* - 1\}$ a measure $\mu_m \in \mathcal{M}_m(f)$ such that

$$\mu = \rho_{-1} \cdot \mu_{-1} + \rho_0 \cdot \mu_0 + \cdots + \rho_{m_*-1} \cdot \mu_{m_*-1} + \rho_{m_*} \cdot \mu_\infty.$$

For each $m \in \{0, \dots, m_* - 1\}$ we take a non-degenerate closed interval P_m so that the following statements hold: if $p_{m+1}/p_m \neq 2$ then $P_m \in \mathscr{P}_m$, $P_m \subset L_m$ and $\mu_m(P_m) > 0$; if $p_{m+1}/p_m = 2$ then $P_m \subset L_m$, P_m contains the periodic point in ∂J_{m+1} in its interior and satisfies $P_m \cap \{f(c), \dots, f^{M_{m-1}-1}(c)\} = \emptyset$ (recall (2.5)). We set $P_{m_*} = J_{m_*}$.

For each $m \in \{-1, \dots, m_* - 1\}$ and for sufficiently large $q \in \mathbb{N}$, we define a collection $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ of non-degenerate closed subintervals of P_m as follows.

- $m = -1$: $\mathscr{B}_{p_{-1}q}(P_{-1}, P_0)$ is the singleton consisting of the pullback of P_0 by $f^{p_{-1}q}$ that is obtained by applying Lemma 3.4 with $P = P_0$. Since $f(c) \notin P_0$ by the choice of P_0, this pullback is diffeomorphic by Lemma 3.4.
- $m \geq 0$: if $p_{m+1}/p_m \neq 2$ then $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ is the collection of the diffeomorphic pullbacks of P_{m+1} by $f^{p_m q}$ contained in P_m that are obtained by applying Lemma 3.3 with $\mu = \mu_m$, $P = P_m$, $P' = P_{m+1}$ and ε replaced by $\varepsilon/2$; if $p_{m+1}/p_m = 2$ then $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ is the singleton consisting of the diffeomorphic pullback of P_{m+1} by $f^{p_m q}$ that is contained in P_m and closest to the periodic point in ∂J_{m+1} among all such diffeomorphic pullbacks.

We claim that for every $-1 \leq m \leq m_* - 1$ the following inequalities hold provided q is sufficiently large:

$$\left| \frac{1}{p_m q} \log \#\mathscr{B}_{p_m q}(P_m, P_{m+1}) - h(\mu_m) \right| < \frac{\varepsilon}{2}; \tag{3.6}$$

$$\sup_{B \in \mathscr{B}_{p_m q}(P_m, P_{m+1})} \sup_B \left\| \frac{1}{p_m q} S_{p_m q} \vec{\phi} - \int \vec{\phi} \, d\mu_m \right\| < \frac{\varepsilon}{2}; \tag{3.7}$$

$$\sup_{B \in \mathscr{B}_{p_m q}(P_m, P_{m+1})} \sup_B \left| \frac{1}{p_m q} \log |(f^{p_m q})'| - \chi^+(\mu_m) \right| < \frac{\varepsilon}{2}. \tag{3.8}$$

Indeed, for $m = -1$, (3.6) is trivial since μ_{-1} is the unit point mass at the fixed point of f in ∂X, and (3.7) and (3.8) are consequences of Lemma 3.4. Let $m \geq 0$. If $p_{m+1}/p_m = 2$ then μ_m is the empirical measure on the orbit of the hyperbolic repelling periodic point in ∂J_{m+1}. Therefore, (3.6), (3.7) and (3.8) hold in these two cases. If $p_{m+1}/p_m \neq 2$ then (3.6), (3.7) and (3.8) are consequences of Lemma 3.3.

We write $n \in \mathbb{N}$ as a linear combination of non-negative integers $q_{-1}, \dots, q_{m_*-1}, r$ in the form

$$n = p_{-1}q_{-1} + p_0 q_0 + \cdots + p_{m_*-1} q_{m_*-1} + r,$$

with $|p_m q_m - \rho_m n| \leq 1$ for $-1 \leq m \leq m_* - 1$ and $|r - \rho_{m_*} n| \leq 1$. In what follows we assume n is sufficiently large so that $r \geq N_*$, and for each $-1 \leq m \leq m_* - 1$, $\mathscr{B}_{p_m q_m}(P_m, P_{m+1})$ satisfies (3.6), (3.7) and (3.8). Lemma 2.9(b) gives

$$\sup_{K_{m_*}} \left\| \frac{1}{r} S_r \vec{\phi} - \int \vec{\phi} \, d\mu_\infty \right\| < \frac{\varepsilon}{2}. \tag{3.9}$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Let $\mathscr{A}_n$ denote the collection of the pullbacks of P_{m_*} by maps $h_{m_*-1} \circ \cdots \circ h_{-1}$, where each h_m is a diffeomorphism of the form $f^{p_m q_m}|_B$ with $B \in \mathscr{B}_{p_m q_m}(P_m, P_{m+1})$. By $\#\mathscr{A}_n = \prod_{m=-1}^{m_*-1} \#\mathscr{B}_{p_m q_m}(P_m, P_{m+1})$, (3.6) and $h(\mu_\infty) = 0$, we have

$$\left| \frac{1}{n} \log \#\mathscr{A}_n - h(\mu) \right| < \varepsilon. \tag{3.10}$$

A similar argument to the proofs of (b) and (c) in Lemma 3.3 based on (3.7) and (3.9) shows that

$$\sup_{A \in \mathscr{A}_n} \sup_A \left\| \frac{1}{n} S_n \vec{\phi} - \int \vec{\phi}\, d\mu \right\| < \varepsilon. \tag{3.11}$$

From $\chi^+(\mu_\infty) = 0$, $\chi^+(\mu) = \sum_{m=-1}^{m_*-1} \rho_m \chi^+(\mu_m)$ by Lemma 2.1 and (3.8), we obtain

$$\sup_{A \in \mathscr{A}_n} \sup_A \left| \frac{1}{n-r} \log |(f^{n-r})'| - \chi^+(\mu) \right| < \frac{\varepsilon}{2}. \tag{3.12}$$

The inequality in Proposition 3.1(a) follows from (3.10). From (3.5) and (3.11), the elements of $\mathscr{A}_n$ are contained in $A_n(\vec{\phi}, \vec{\alpha})$. The inequality in Proposition 3.1(b) follows from (3.12).

Case 2: $m(f) < \infty$ and $\mathrm{int}(J_{m(f)})$ contains a two-sided attracting periodic point (Remark 2.3(C-I)). Recall that $\delta_{O(f)}$ denotes the element of $\mathcal{M}(f)$ supported on $O(f)$. With no loss of generality we may assume there exist constants $\rho_{-1}, \ldots, \rho_{m(f)} \in (0, 1)$ adding up to 1 and for each $m \in \{-1, \ldots, m(f) - 1\}$ a measure $\mu_m \in \mathcal{M}_m(f)$ such that

$$\mu = \rho_{-1} \cdot \mu_{-1} + \rho_0 \cdot \mu_0 + \cdots + \rho_{m(f)} \cdot \delta_{O(f)}.$$

For each $m \in \{0, \ldots, m(f)\}$ we take a non-degenerate closed interval P_m so that the following assertions hold: if $m < m(f)$ and $p_{m+1}/p_m \neq 2$ then $P_m \in \mathscr{P}_m$, $P_m \subset L_m$ and $\mu_m(P_m) > 0$; if $m < m(f)$ and $p_{m+1}/p_m = 2$ then $P_m \subset L_m$, P_m contains the periodic point in ∂J_{m+1} in its interior and satisfies $P_m \cap \{f(c), \ldots, f^{M_{m-1}-1}(c)\} = \emptyset$; $P_{m(f)}$ is contained in the connected component of the immediate basin of the attracting periodic orbit that contains c. In the case of $m(f) = 0$, we take $P_{m(f)}$ to be a closed interval contained in the open interval bordered by x_f and the unique fixed point in z_f.

For each $m \in \{-1, \ldots, m(f) - 1\}$ and for sufficiently large $q \in \mathbb{N}$, we define a collection $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ of non-degenerate closed subintervals of P_m as follows.

- $m = -1$: $\mathscr{B}_{p_{-1} q}(P_{-1}, P_0)$ is the singleton consisting of the pullback of P_0 by $f^{p_{-1} q}$ that is obtained by applying Lemma 3.4 with $P = P_0$. Since $f(c) \notin P_0$ by the choice of P_0, this pullback is diffeomorphic by Lemma 3.4.
- $m \geq 0$: if $p_{m+1}/p_m \neq 2$ then $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ is the collection of the diffeomorphic pullbacks of P_{m+1} by $f^{p_m q}$ contained in P_m that are obtained by applying Lemma 3.3 with $\mu = \mu_m$, $P = P_m$, $P' = P_{m+1}$ and ε replaced by $\varepsilon/2$; if $p_{m+1}/p_m = 2$ then $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ is the singleton consisting of the diffeomorphic pullback of P_{m+1} by $f^{p_m q}$ that is contained in P_m and closest to the periodic point in ∂J_{m+1} among all such diffeomorphic pullbacks.

Note that the second case $m \geq 0$ occurs only if $m(f) \geq 1$.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Let $n \geq 1$ be an integer, and write it as a linear combination of positive integers $q_1, \ldots, q_{m(f)-1}$ in the form

$$n = p_{-1}q_{-1} + \cdots + p_{m(f)-1}q_{m(f)-1} + r,$$

with $|p_m q_m - \rho_m n| \leq 1$ for $-1 \leq m \leq m(f) - 1$ and $|r - \rho_{m(f)} n| \leq 1$. In what follows we assume n is sufficiently large so that for each $m \in \{-1, \ldots, m(f) - 1\}$, $\mathscr{B}_{p_m q_m}(P_m, P_{m+1})$ satisfies (3.6)′, (3.7)′ and (3.8)′ which are exactly the same as (3.6), (3.7) and (3.8) respectively under the different assumption on f, and

$$\sup_{P_{m(f)}} \left\| \frac{1}{r} S_r \vec{\phi} - \int \vec{\phi}\, d\delta_{O(f)} \right\| < \frac{\varepsilon}{2}. \tag{3.13}$$

Let $\mathscr{A}_n$ denote the collection of the pullbacks of $P_{m(f)}$ by maps $h_{m(f)-1} \circ \cdots \circ h_0 \circ h_{-1}$ where h_m for $-1 \leq m \leq m(f) - 1$ is a diffeomorphism of the form $f^{p_m q}|_B$ with $B \in \mathscr{B}_{p_m q}(P_m, P_{m+1})$. From $\#\mathscr{A}_n = \prod_{m=-1}^{m(f)-1} \#\mathscr{B}_{p_m q_m}(P_m, P_{m+1})$ and (3.6)′ we obtain Proposition 3.1(a). From (3.5), (3.7)′ and (3.13), the elements of $\mathscr{A}_n$ are contained in $A_n(\vec{\phi}, \vec{\alpha})$. Since the measure $\delta_{O(f)}$ is supported on $O(f)$, we have $\chi^+(\delta_{O(f)}) = 0$ from the definition (1.1) of the Lyapunov exponent and $\chi^+(\mu) = \sum_{m=-1}^{m(f)-1} \rho_m \chi^+(\mu_m)$ by Lemma 2.1. Hence, (3.8)′ implies Proposition 3.1(b).

Case 3: $m(f) < \infty$ and $\partial J_{m(f)}$ contains a one-sided attracting periodic point (Remark 2.3(C-II)). This case is treated by a slight modification of the argument for Case 2.

Case 4: $m(f) < \infty$ and f has no attracting periodic orbit. We may assume there exist constants $\rho_{-1}, \ldots, \rho_{m(f)} \in (0, 1)$ adding up to 1, and for each $m \in \{-1, \ldots, m(f)\}$ a measure $\mu_m \in \mathcal{M}_m(f)$ such that

$$\mu = \rho_{-1} \cdot \mu_{-1} + \rho_0 \cdot \mu_0 + \cdots + \rho_{m(f)} \cdot \mu_{m(f)}.$$

Set $g = f^{p_{m(f)}}|_{L_{m(f)}}$, and let ν denote the normalized restriction of $\mu_{m(f)}$ to $L_{m(f)}$. Let $h(g, \nu)$, $\chi^+(g, \nu)$ denote the entropy and Lyapunov exponent of the measure ν with respect to g. By Lemma 2.1 we have $\chi^+(g, \nu) = \int \log |Dg| d\nu$. As in §2.3, g is topologically exact. From the proof of [7, Proposition 2.1], for all sufficiently large $q \geq 1$ there exist a closed interval $P_{m(f)}$ in $L_{m(f)}$ and a finite collection $\mathscr{B}_{p_{m(f)}q}(P_{m(f)}, P_{m(f)})$ of diffeomorphic pullbacks of $P_{m(f)}$ by g^q that are contained in $P_{m(f)}$ such that

$$\begin{aligned} \#\mathscr{B}_{p_{m(f)}q}(P_{m(f)}, P_{m(f)}) &> \exp\left(\left(h(g, \nu) - \frac{\varepsilon}{2}\right)q\right) \\ &= \exp\left(\left(p_{m(f)} h(\mu_{m(f)}) - \frac{\varepsilon}{2}\right)q\right), \end{aligned} \tag{3.14}$$

and for every $B \in \mathscr{B}_{p_{m(f)}q}(P_{m(f)}, P_{m(f)})$,

$$\sup_B \left\| \frac{1}{q} \sum_{k=0}^{q-1} \vec{\phi} \circ g^k - \int \vec{\phi}\, d\nu \right\| = \sup_B \left\| \frac{1}{q} S_{p_{m(f)}q} \vec{\phi} - p_{m(f)} \int \vec{\phi}\, d\mu_{m(f)} \right\| < \frac{\varepsilon}{2} \tag{3.15}$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

and

$$|B| > \exp\left(-\left(\chi^+(g,\nu)+\frac{\varepsilon}{2}\right)q\right) = \exp\left(-\left(p_{m(f)}\chi^+(\mu_{m(f)})+\frac{\varepsilon}{2}\right)q\right). \tag{3.16}$$

For each $m \in \{0,\dots,m(f)-1\}$, we take a non-degenerate closed interval P_m such that the following assertions hold: if $p_{m+1}/p_m \neq 2$ then $P_m \in \mathscr{P}_m$, $P_m \subset L_m$ and $\mu_m(P_m) > 0$: if $p_{m+1}/p_m = 2$ then $P_m \subset L_m$, P_m contains the periodic point in ∂J_{m+1} in its interior and satisfies $P_m \cap \{f(c),\dots,f^{M_{m-1}-1}(c)\} = \emptyset$.

For each $m \in \{-1,\dots,m(f)-1\}$ and for sufficiently large $q \in \mathbb{N}$, we define a collection $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ of non-degenerate closed subintervals of P_m as follows.

- $m=-1$: $\mathscr{B}_{p_{-1}q}(P_{-1}, P_0)$ is the singleton consisting of the pullback of P_0 by $f^{p_{-1}q}$ that is obtained by applying Lemma 3.4 with $P = P_0$. Since $f(c) \notin P_0$ by the choice of P_0, this pullback is diffeomorphic by Lemma 3.4.
- $m \geq 0$: if $p_{m+1}/p_m \neq 2$ then $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ is the collection of the diffeomorphic pullbacks of P_{m+1} by $f^{p_m q}$ contained in P_m obtained by applying Lemma 3.3 with $\mu = \mu_m$, $P = P_m$, $P' = P_{m+1}$ and ε replaced by $\varepsilon/2$; if $p_{m+1}/p_m = 2$ then $\mathscr{B}_{p_m q}(P_m, P_{m+1})$ is the singleton consisting of the diffeomorphic pullback of P_{m+1} by $f^{p_m q}$ that is contained in P_m and closest to the periodic point in ∂J_{m+1} among all such diffeomorphic pullbacks.

Note that the second case $m \geq 0$ occurs only if $m(f) \geq 1$.

Let $n \geq 1$ be an integer that is written as a linear combination of non-negative integers $q_{-1},\dots,q_{m(f)}, r$ in the form

$$n = p_{-1}q_{-1} + \cdots + p_{m(f)-1}q_{m(f)-1} + p_{m(f)}q_{m(f)} + r,$$

with $|p_m q_m - \rho_m n| \leq 1$ for $-1 \leq m \leq m(f)$ and $0 \leq r < p_{m(f)}$. We assume n is sufficiently large so that for each m, $\mathscr{B}_{p_m q_m}(P_m, P_{m+1})$ satisfies $(3.6)''$, $(3.7)''$ and $(3.8)''$ which are exactly the same as (3.6), (3.7) and (3.8) respectively under the different assumption on f, and $\mathscr{B}_{p_{m(f)}q_{m(f)}}(P_{m(f)}, P_{m(f)})$ satisfying (3.14), (3.15) and (3.16) exists. We set $P_{m(f)+1} = P_{m(f)}$ for convenience. Define $\mathscr{A}_n$ to be the collection of the pullbacks of $P_{m(f)}$ by maps $h_{m(f)} \circ \cdots \circ h_0 \circ h_{-1}$, where each h_m, $-1 \leq m \leq m(f)$ is a diffeomorphism of the form $f^{p_m q_m}|_B$ with $B \in \mathscr{B}_{p_m q_m}(P_m, P_{m+1})$. From $\#\mathscr{A}_n = \prod_{m=-1}^{m(f)} \#\mathscr{B}_{p_m q_m}(P_m, P_{m+1})$, $(3.6)''$ and (3.14) we obtain the inequality in Proposition 3.1(a). From (3.5), $(3.7)''$ and (3.15), the elements of $\mathscr{A}_n$ are contained in $A_n(\vec{\phi}, \vec{\alpha})$. By Lemma 2.1 we have $\chi^+(\mu) = \sum_{m=-1}^{m(f)} \rho_m \chi^+(\mu_m)$. From $(3.8)''$ and (3.16) we obtain the inequality in Proposition 3.1(b). The proof of Proposition 3.1 is complete.

4. *Large deviations upper bound*

In this section we complete the proof of the upper bound in Theorem A. In §4.1 we show that this follows from a key upper bound stated in Proposition 4.1. The rest of this section is dedicated to a proof of Proposition 4.1.

4.1. *Key upper bound.*

Let $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$. Define

$$\bar{A}_n(\vec{\phi}, \vec{\alpha}) = \left\{x \in X : \int \vec{\phi}\, d\delta_x^n \geq \vec{\alpha}\right\}.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

PROPOSITION 4.1. *Let $f: X \to X$ be an S-unimodal map with a non-flat critical point. Let $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$. For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $\bar{A}_n(\vec{\phi}, \vec{\alpha}) \neq \emptyset$, then there exists a measure $\mu \in \mathcal{M}(f)$ such that*

$$|\bar{A}_n(\vec{\phi}, \vec{\alpha})| \leq \exp((F(\mu) + \varepsilon)n) \quad \textit{and} \tag{a}$$

$$\int \vec{\phi}\, d\mu > \vec{\alpha} - \vec{\varepsilon}. \tag{b}$$

We finish the proof of the upper bound in Theorem A assuming Proposition 4.1.

Proof of the upper bound in Theorem A. Let $\mathcal{C}$ be a non-empty closed subset of $\mathcal{M}$. Let $\mathcal{G}$ be an arbitrary open set containing $\mathcal{C}$. Since $\mathcal{M}$ is metrizable and $\mathcal{C}$ is compact, we can choose $\varepsilon > 0$ and finitely many closed sets $\mathcal{C}_1, \ldots, \mathcal{C}_s$ of the form $\mathcal{C}_i = \{\mu \in \mathcal{M} : \int \vec{\phi}_i\, d\mu \geq \vec{\alpha}_i\}$ with $\ell_i \in \mathbb{N}$, $\vec{\phi}_i \in C(X)^{\ell_i}$, $\vec{\alpha}_i \in \mathbb{R}^{\ell_i}$ for $1 \leq i \leq s$ so that

$$\mathcal{C} \subset \bigcup_{i=1}^{s} \mathcal{C}_i \subset \bigcup_{i=1}^{s} \mathcal{C}_i(\varepsilon) \subset \mathcal{G}, \tag{4.1}$$

where $\mathcal{C}_i(\varepsilon) = \{\mu \in \mathcal{M} : \int \vec{\phi}_i\, d\mu > \vec{\alpha}_i - \vec{\varepsilon}\}$. Since $F(\mu) \leq -I(\mu)$ for $\mu \in \mathcal{M}$, Proposition 4.1 implies

$$\limsup_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta_x^n \in \mathcal{C}_i\}| \leq -\inf_{\mathcal{C}_i(\varepsilon)} I + \varepsilon \quad \text{for } 1 \leq i \leq s.$$

These inequalities and (4.1) give

$$\limsup_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta_x^n \in \mathcal{C}\}| \leq \max_{1\leq i\leq s} \left(-\inf_{\mathcal{C}_i(\varepsilon)} I\right) + \varepsilon \leq -\inf_{\mathcal{G}} I + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\mathcal{G}$ is an arbitrary open set containing $\mathcal{C}$, it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta_x^n \in \mathcal{C}\}| \leq \inf_{\mathcal{G}\supset\mathcal{C}} \left(-\inf_{\mathcal{G}} I\right) = -\inf_{\mathcal{C}} I,$$

as required. The last equality is due to the lower semicontinuity of I. □

Standing hypotheses from §4.2 to §4.4. The rest of this section is entirely dedicated to the proof of Proposition 4.1. We assume $f: X \to X$ is an S-unimodal map with a non-flat critical point c, and $\ell \in \mathbb{N}$, $\vec{\phi} \in C(X)^\ell$, $\vec{\alpha} \in \mathbb{R}^\ell$ in the statement of Proposition 4.1 are fixed till the end of §4.4.

4.2. *Escape estimate in a single renormalization cycle.* Suppose $m(f) \geq 1$, and let m be an integer with $0 \leq m \leq m(f) - 1$. For $\vec{\beta} \in \mathbb{R}^\ell$, $P \subset K_{m,m+1}$ and $n \in \mathbb{N}$, define

$$A_n(\vec{\phi}, \vec{\beta}, P) = \bar{A}_n(\vec{\phi}, \vec{\beta}) \cap P \cap \bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1}).$$

This subsection is dedicated to proving of the following lemma.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

LEMMA 4.2. *Suppose $m(f) \geq 1$ and let $0 \leq m \leq m(f) - 1$ be an integer. For any $\varepsilon > 0$ there exists $N \geq 1$ such that if $\vec{\beta} \in \mathbb{R}^\ell$, $P \in \tilde{\mathscr{P}}_m$, $n \geq N$ satisfy $P \subset K_{m,m+1}$ and $A_n(\vec{\phi}, \vec{\beta}, P) \neq \emptyset$, then there exists a measure $\mu \in \mathcal{M}(f)$ such that*

$$|A_n(\vec{\phi}, \vec{\beta}, P)| \leq \exp((F(\mu) + \varepsilon)n)|P| \quad \textit{and} \tag{a}$$

$$\int \vec{\phi}\, d\mu > \vec{\beta} - \vec{\varepsilon}. \tag{b}$$

Proof. Let $\varepsilon > 0$ and let $P \in \tilde{\mathscr{P}}_m$. Since $P \subset K_{m,m+1}$, if $p_{m+1}/p_m = 2$ then $f(P)$ contains a point from the orbit of the periodic point in ∂J_m. Then we can verify (a) and (b) easily, taking the measure μ to be the equidistribution on this periodic orbit. For the rest of this proof, we suppose $p_{m+1}/p_m \neq 2$, and split the proof into two cases, in which either $P \in \mathscr{P}_m$ or $P \in \tilde{\mathscr{P}}_m \setminus \mathscr{P}_m$.

Case 1: $P \in \mathscr{P}_m$. We perform an escape estimate relative to P. Let $n \in \mathbb{N}$ and suppose $A_n(\vec{\phi}, \vec{\beta}, P) \neq \emptyset$. Let $\mathscr{P}_{m,n}(P)$ denote the collection of the diffeomorphic pullbacks of elements of $\mathscr{P}_m$ by f^n that are contained in P and intersect $\bar{A}_n(\vec{\phi}, \vec{\beta})$. Note that every element of $\mathscr{P}_{m,n}(P)$ is contained in $\bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1})$, and the union of elements of $\mathscr{P}_{m,n}(P)$ contains $A_n(\vec{\phi}, \vec{\beta}, P)$.

SUBLEMMA 4.3. *For any $\varepsilon > 0$ there exists $N \geq 1$ such that for every $Q \in \mathscr{P}_m$, every $n \geq N$ and every diffeomorphic pullback W of Q by f^n contained in P, there exist $k \in \mathbb{Z}$ with $|k| \leq p_m - 1$ and a diffeomorphic pullback Y of P by f^{n+k+M_m} that is contained in W and satisfies*

$$|Y| \geq \exp(-\varepsilon n)|W|.$$

Proof. Since P, $Q \in \mathscr{P}_m$, there exist k_P, $k_Q \in \{0, \ldots, p_m - 1\}$ such that $f^{k_P}(P)$ and $f^{k_Q}(Q)$ are elements of $\mathscr{P}_m$ contained in L_m. By the definition of M_m in (2.5), there exists a diffeomorphic pullback W_P of $f^{k_P}(P)$ by f^{M_m} that is contained in $f^{k_Q}(Q)$. Let W'_P be the diffeomorphic pullback of W_P by f^{k_Q} contained in Q. This is a diffeomorphic pullback of P by $f^{M_m+k_Q-k_P}$. Let Y be the pullback of W'_P by $f^n|_W$. Then Y is a diffeomorphic pullback of P by $f^{n+M_m+k_Q-k_P}$.

By Proposition 2.7 with ε replaced by $\varepsilon/2$, if n is sufficiently large we have

$$\begin{aligned}\frac{|Y|}{|W|} &\geq \exp\left(-\frac{\varepsilon n}{2}\right)\frac{|f^n(Y)|}{|f^n(W)|} = \exp\left(-\frac{\varepsilon n}{2}\right)\frac{|W'_P|}{|Q|} \\ &\geq \exp\left(-\frac{\varepsilon n}{2}\right)C\frac{|W_P|}{|f^{k_Q}(Q)|} \geq \exp(-\varepsilon n)\end{aligned}$$

as required, where $C > 0$ is a constant independent of P and Q. □

In view of Sublemma 4.3, for each $W \in \mathscr{P}_{m,n}(P)$ we fix an integer $k(W)$ with $|k(W)| \leq p_m - 1$ and a diffeomorphic pullback Y_W of P by $f^{n+k(W)+M_m}$ that is contained in W and satisfies $|Y_W| \geq \exp(-\varepsilon n/3)|W|$. Set

$$\mathscr{P}_{m,n,k}(P) = \{W \in \mathscr{P}_{m,n}(P) \colon k(W) = k\}.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

If n is sufficiently large, then for all $-p_m + 1 \le k \le p_m - 1$ we have

$$\bigcup_{W \in \mathscr{P}_{m,n,k}(P)} Y_W \subset A_{n+k+M_m}\left(\vec{\phi}, \vec{\beta} - \frac{1}{2}\vec{\varepsilon}\right) \tag{4.2}$$

and

$$|A_n(\vec{\phi}, \vec{\alpha}, P)| \le \sum_{W \in \mathscr{P}_{m,n}(P)} |W| \le \exp\left(\frac{\varepsilon n}{3}\right) \sum_{W \in \mathscr{P}_{m,n}(P)} |Y_W|. \tag{4.3}$$

Choose $\underline{k} \in \{-p_m + 1, \ldots, p_m - 1\}$ such that

$$\sum_{W \in \mathscr{P}_{m,n}(P)} |Y_W| \le (2p_m - 1) \sum_{W \in \mathscr{P}_{m,n,\underline{k}}(P)} |Y_W|, \tag{4.4}$$

and set $\tilde{n} = n + \underline{k} + M_m$. The restriction of $f^{\tilde{n}}$ to $\bigcup_{W \in \mathscr{P}_{m,n,\underline{k}}(P)} Y_W$ induces a fully branched expanding Markov map onto P with finitely many branches. Let $\Delta \subset \Gamma_m$ denote its maximal invariant set, namely

$$\Delta = \bigcap_{j=0}^{\infty} (f^{\tilde{n}})^{-j}\left(\bigcup_{W \in \mathscr{P}_{m,n,\underline{k}}(P)} Y_W\right).$$

The map $f^{\tilde{n}}|_\Delta \colon \Delta \to \Delta$ is topologically conjugate to the full shift $\sigma \colon \Sigma \to \Sigma$ over the finite alphabet $\mathscr{P}_{m,n,\underline{k}}(P)$. We write $\pi \colon \Sigma \to \Delta$ for the conjugacy map, and define the induced potential $\Phi \colon \Sigma \to \mathbb{R}$ by $\Phi(\omega) = -\log |(f^{\tilde{n}})'(\pi(\omega))|$. From Lemma 2.6, Φ is continuous with respect to the shift metric. The variational principle [**2**, pp. 40, 2.17] gives

$$\sup_{\tilde{\nu} \in \mathcal{M}(\sigma)} \left(h(\sigma, \tilde{\nu}) + \int \Phi \, d\tilde{\nu}\right) = \lim_{j \to \infty} \frac{1}{j} \log\left(\sum_{\omega \in \sigma^{-j}(\omega')} \exp \sum_{i=0}^{j-1} \Phi(\sigma^i \omega)\right) \tag{4.5}$$

for any fixed $\omega' \in \Sigma$, where $\mathcal{M}(\sigma)$ denotes the space of σ-invariant Borel probability measures endowed with the weak* topology, and $h(\sigma, \tilde{\nu})$ denotes the measure-theoretic entropy of $\tilde{\nu}$ with respect to σ. By the distortion estimates in Proposition 2.7, if $n \ge 1$ is sufficiently large then for any $W \in \mathscr{P}_{m,n,\underline{k}}(P)$ and any $\omega \in \Sigma$ such that $\pi(\omega) \in Y_W$ we have

$$\exp(\Phi(\omega)) \ge \exp\left(-\frac{\varepsilon n}{2}\right) \frac{|Y_W|}{|P|}.$$

Hence, the series inside the logarithm in (4.5) is bounded from below as follows:

$$\begin{aligned} \sum_{\omega \in \sigma^{-j}(\omega')} \exp\left(\sum_{i=0}^{j-1} \Phi(\sigma^i \omega)\right) &\ge \left(\inf_{\omega' \in \Sigma} \sum_{\omega \in \sigma^{-1}(\omega')} \exp(\Phi(\omega))\right)^j \\ &\ge \left(\exp\left(-\frac{\varepsilon n}{2}\right) \sum_{W \in \mathscr{P}_{m,n,\underline{k}}(P)} \frac{|Y_W|}{|P|}\right)^j. \end{aligned}$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Taking logarithms of both sides, dividing by j and letting $j \to \infty$, we have

$$\lim_{j\to\infty} \frac{1}{j} \log \left(\sum_{\omega\in\sigma^{-j}(\omega')} \exp\left(\sum_{i=0}^{j-1} \Phi(\sigma^i\omega)\right)\right) \geq \log \sum_{W\in\mathscr{P}_{m,n,\underline{k}}(P)} \frac{|Y_W|}{|P|} - \frac{\varepsilon n}{2}.$$

Plugging this inequality into (4.5) yields

$$\sup_{\tilde{\nu}\in\mathcal{M}(\sigma)} \left(h(\sigma,\tilde{\nu}) + \int \Phi\, d\tilde{\nu}\right) \geq \log \sum_{W\in\mathscr{P}_{m,n,\underline{k}}(P)} \frac{|Y_W|}{|P|} - \frac{\varepsilon n}{2}. \tag{4.6}$$

Since $\mathcal{M}(\sigma)$ is compact in the weak* topology and the entropy function on it is upper semicontinuous, there exists a measure $\tilde{\mu} \in \mathcal{M}(\sigma)$ that attains the supremum in (4.6). The measure $\tilde{\mu}\circ\pi^{-1}$ on Δ is $f^{\tilde{n}}$-invariant and its spread

$$\mu = \frac{1}{\tilde{n}} \sum_{W\in\mathscr{P}_{m,n,\underline{k}}(P)} \sum_{j=0}^{\tilde{n}-1} (\tilde{\mu}|_{Y_W}) \circ (f^j\circ\pi)^{-1}$$

belongs to $\mathcal{M}_m(f)$. Although $f^{\tilde{n}}$ may not be the first return map to Δ, Abramov's formula, connecting the entropies of μ and $\tilde{\mu}\circ\pi^{-1}$, and Kac's formula, connecting the integrals of $-\log|f|$ and Φ, still hold [**25**, Theorem 2.3] and we have

$$h(\sigma,\tilde{\mu}) + \int \Phi\, d\tilde{\mu} = F(\mu)\tilde{n}. \tag{4.7}$$

From (4.6) and (4.7) we have

$$\sum_{W\in\mathscr{P}_{m,n,\underline{k}}(P)} |Y_W| \leq \exp\left(F(\mu)\tilde{n} + \frac{\varepsilon n}{2}\right)|P|.$$

From this inequality, (4.3) and (4.4), for all sufficiently large $n \geq 1$ we obtain

$$\begin{aligned} |A_n(\vec{\phi},\vec{\beta},P)| &\leq (2p_m - 1)\exp\left(\frac{\varepsilon n}{3}\right) \sum_{W\in\mathscr{P}_{m,n,\underline{k}}(P)} |Y_W| \\ &\leq \exp((F(\mu)+\varepsilon)n)|P|, \end{aligned}$$

as required in Lemma 4.2(a). The inequality in Lemma 4.2(b) follows from (4.2).

Case 2: $P \in \tilde{\mathscr{P}}_m \setminus \mathscr{P}_m$. Let z denote the periodic point of period p_m in ∂J_m. Then z is hyperbolic repelling, and f^{p_m} maps the point in $\partial J_m \setminus \{z\}$ to z. For simplicity we assume $z \in P$. Otherwise, $z \in f^{p_m}(P)$ holds and so the argument is analogous.

Let $n \in \mathbb{N}$ satisfy $A_n(\vec{\phi},\vec{\beta},P) \neq \emptyset$. Our strategy is to use the simplest version of the coarse graining to be formally introduced in §4.3: we begin by splitting $A_n(\vec{\phi},\vec{\beta},P)$ into two subsets, one consisting of points which remain in $\bigcup_{k=0}^{p_m-1} f^k(P)$ almost until time n, and the complement of this set. For the first set, the influence of the dynamics near the set Γ_m is negligible, and a local analysis near the orbit of z suffices for the estimate of its Lebesgue measure. The second set is influenced by the dynamics near the orbit of z and the dynamics near Γ_m. We estimate the Lebesgue measure of each set separately, and unify the estimates at the end to obtain the desired one in Lemma 4.2.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

For each $t \in \mathbb{N}$, we put

$$V_t = P \cap \Big(f^{-t}\Big(\bigcup_{Q \in \mathscr{P}_m} Q \Big) \setminus \bigcup_{k=0}^{t-1} f^{-k}\Big(\bigcup_{Q \in \mathscr{P}_m} Q \Big) \Big),$$

and split $|A_n(\vec{\phi}, \vec{\beta}, P)| = I + II$ where

$$I = \sum_{t=1}^{n} |A_n(\vec{\phi}, \vec{\beta}, V_t)| \quad \text{and} \quad II = \sum_{t=n+1}^{\infty} |A_n(\vec{\phi}, \vec{\beta}, V_t)|.$$

The set V_t is non-empty if and only if t is an integer multiple of p_m. If $V_t \neq \emptyset$ then $f^t(V_t)$ contains some element of $\mathscr{P}_m$, and thus

$$|f^t(V_t)| \geq \min\{|Q| \colon Q \in \mathscr{P}_m\} > 0. \tag{4.8}$$

Since the restriction of f^t to the smallest closed interval that contains V_t and z extends to a diffeomorphism on an open interval, the distortion of f^t on the former interval is bounded by a constant $C > 0$ that depends only on f. Then we have

$$|V_t| \leq C \exp(F(\delta_z^{p_m})t)|P|. \tag{4.9}$$

By (4.9) we obtain

$$II \leq \sum_{t=n+1}^{\infty} |V_t| \leq \frac{C \exp(F(\delta_z^{p_m})(n+1))}{1 - \exp(F(\delta_z^{p_m})p_m)}|P|. \tag{4.10}$$

In order to estimate the sum I, for each $t \in \{1, \ldots, n\}$ let $M_{2,\ell}(t, \mathbb{Z}_\varepsilon)$ denote the set of $2 \times \ell$ matrices $(\beta_{ij})_{i=0,1,1 \leq j \leq \ell}$ with entries in $\{(\varepsilon/3)a \colon a \in \mathbb{Z}\}$ for which the following inequalities hold:

$$\beta_{0j}t + \beta_{1j}(n-t) > \Big(\beta_j - \frac{\varepsilon}{3}\Big)n \quad \text{for } 1 \leq j \leq \ell, \tag{4.11}$$

$$\inf \phi_j - \frac{\varepsilon}{3} < \beta_{ij} \leq \sup \phi_j \quad \text{for } i = 0, 1 \text{ and } 1 \leq j \leq \ell. \tag{4.12}$$

For $\mathbf{B} = (\beta_{ij}) \in M_{2,\ell}(t, \mathbb{Z}_\varepsilon)$ and $i = 0, 1$, put $\vec{\beta}_i = (\beta_{i1}, \ldots, \beta_{i\ell})$ and define

$$A_n(t, \mathbf{B}) = \Big\{ x \in V_t \colon \frac{1}{t} S_t \vec{\phi}(x) \geq \vec{\beta}_0 \text{ and } \frac{1}{n-t} S_{n-t} \vec{\phi}(f^t(x)) \geq \vec{\beta}_1 \Big\}.$$

We claim that

$$A_n(\vec{\phi}, \vec{\beta}, V_t) \subset \bigcup_{\mathbf{B} \in M_{2,\ell}(t, \mathbb{Z}_\varepsilon)} A_n(t, \mathbf{B}). \tag{4.13}$$

Indeed, for each $x \in A_n(\vec{\phi}, \vec{\beta}, V_t)$ there exists $\mathbf{B} = (\beta_{ij}) \in M_{2,\ell}(t, \mathbb{Z}_\varepsilon)$ such that for $1 \leq j \leq \ell$ we have

$$\beta_{0j} \leq \frac{1}{t} S_t \phi_j(x) < \beta_{0j} + \frac{\varepsilon}{3} \quad \text{and} \quad \beta_{1j} \leq \frac{1}{n-t} S_{n-t} \phi_j(f^t(x)) < \beta_{1j} + \frac{\varepsilon}{3}.$$

This implies (4.11) and (4.12), and hence $x \in A_n(t, \mathbf{B})$, which verifies (4.13).

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Let $\delta > 0$ be such that

$$\|\vec{\phi}(x) - \vec{\phi}(y)\| < \frac{\varepsilon}{7} \quad \text{for } x, y \in X \text{ with } |x - y| < \delta. \tag{4.14}$$

In view of what was proved in Case 1 and Lemma 2.6, for the rest of the proof in Case 2 we assume $N \geq 1$ is a sufficiently large integer for which the following assertions hold.

(D1) If $n \geq N$ then the inequalities in (a) and (b) of Lemma 4.2 hold for every element of $\mathscr{P}_m$ with ε replaced by $\varepsilon/4$.

(D2) For every connected component W of $\bigcap_{k=0}^{N-1} f^{-k}(K_{m,m+1})$ we have $|W| < \delta$.

To finish the proof in Case 2, for all sufficiently large n we establish the inequalities in (a) and (b) of Lemma 4.2 for every element of $\tilde{\mathscr{P}}_m \setminus \mathscr{P}_m$.

Let $n \geq N$ and let $t \in \{1, \ldots, n - N + 1\}$. For each matrix $\mathbf{B} = (\beta_{ij}) \in M_{2,\ell}(t, \mathbb{Z}_\varepsilon)$ with $A_n(t, \mathbf{B}) \neq \emptyset$, we estimate the Lebesgue measure $|A_n(t, \mathbf{B})|$ from above. Let $\mathscr{P}_m(t, \mathbf{B})$ denote the collection of the diffeomorphic pullbacks of elements of $\mathscr{P}_m$ by f^{n-t} that intersect $f^t(A_n(t, \mathbf{B}))$. The elements of $\mathscr{P}_m(t, \mathbf{B})$ intersect $\bar{A}_{n-t}(\vec{\phi}, \vec{\beta}_1)$ and their union contains $f^t(A_n(t, \mathbf{B}))$, because

$$f^t(A_n(t, \mathbf{B})) \subset \bar{A}_{n-t}(\vec{\phi}, \vec{\beta}_1) \cap \bigcap_{k=0}^{n-t-1} f^{-k}(K_{m,m+1}).$$

By (D1), for each $Q \in \mathscr{P}_m(t, \mathbf{B})$ there exists $\mu_{t,Q} \in \mathcal{M}_m(f)$ such that

$$|\bar{A}_{n-t}(\vec{\phi}, \vec{\beta}_1) \cap Q| \leq \exp\left(\left(F(\mu_{t,Q}) + \frac{\varepsilon}{4}\right)(n-t)\right)|Q| \quad \text{and} \tag{4.15}$$

$$\int \vec{\phi}\, d\mu_{t,Q} > \vec{\beta}_1 - \frac{1}{4}\vec{\varepsilon}. \tag{4.16}$$

Pick a measure $\mu_{t,\mathbf{B}}$ in the finite set $\{\mu_{t,Q} : Q \in \mathscr{P}_m(t, \mathbf{B})\}$ that maximizes the free energy within this finite set. By (4.15) and $\sum_{Q \in \mathscr{P}_m(t,\mathbf{B})} |Q| \leq |X|$ we have

$$\begin{aligned} |f^t(A_n(t, \mathbf{B}))| &\leq \sum_{Q \in \mathscr{P}_m(t,\mathbf{B})} |\bar{A}_{n-t}(\vec{\phi}, \vec{\beta}_1) \cap Q| \\ &\leq \exp\left(\left(F(\mu_{t,\mathbf{B}}) + \frac{\varepsilon}{4}\right)(n-t)\right)|X|. \end{aligned}$$

Combining this inequality with (4.8), we obtain

$$\frac{|A_n(t, \mathbf{B})|}{|V_t|} \leq C\frac{|f^t(A_n(t, \mathbf{B}))|}{|f^t(V_t)|} \leq \frac{C|X|}{|f^t(V_t)|} \exp\left(\left(F(\mu_{t,\mathbf{B}}) + \frac{\varepsilon}{4}\right)(n-t)\right). \tag{4.17}$$

For the first inequality we have used the bounded distortion.

SUBLEMMA 4.4. *If n is sufficiently large, then we have*

$$\int \vec{\phi}\, d\delta_z^{p_m} > \vec{\beta} - \vec{\varepsilon}.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Proof. Take $x \in A_n(\vec{\phi}, \vec{\beta}, P)$. Since $x, z \in P$, the same argument as the deduction of (3.1) yields

$$\frac{1}{n} S_n \vec{\phi}(z) \geq \frac{1}{n} S_n \vec{\phi}(x) - \frac{\vec{\varepsilon}}{2} > \vec{\beta} - \frac{\vec{\varepsilon}}{2} \tag{4.18}$$

provided n is sufficiently large.

Write $n = p_m q + r$, with non-negative integers q, r satisfying $0 \leq r \leq q - 1$. Using (4.18), we have

$$\begin{aligned} \int \vec{\phi}\, d\delta_z^{p_m} &= \frac{1}{p_m q} S_{p_m q} \vec{\phi}(z) = \frac{n}{p_m q} \frac{1}{n} S_n \vec{\phi}(z) - \frac{1}{p_m q} S_{n - p_m q} \vec{\phi}(f^{p_m q}(z)) \\ &= \frac{1}{n} S_n \vec{\phi}(z) + \frac{n - p_m q}{p_m q} \frac{1}{n} S_n \vec{\phi}(z) - \frac{1}{p_m q} S_{n - p_m q} \vec{\phi}(f^{p_m q}(z)) \\ &> \vec{\beta} - \frac{\vec{\varepsilon}}{2} - \frac{2r}{p_m q} \|\vec{\phi}(z)\| > \vec{\beta} - \vec{\varepsilon} \end{aligned}$$

provided n is sufficiently large. □

Define a measure $\nu_{t,\mathbf{B}} \in \mathcal{M}(f)$ by

$$\nu_{t,\mathbf{B}} = \frac{t}{n} \cdot \delta_z^{p_m} + \left(1 - \frac{t}{n}\right) \cdot \mu_{t,\mathbf{B}}.$$

SUBLEMMA 4.5. *If n is sufficiently large, then we have*

$$\int \vec{\phi}\, d\nu_{t,\mathbf{B}} > \vec{\beta} - \vec{\varepsilon}.$$

Proof. From (4.16) we have

$$\int \vec{\phi}\, d\mu_{t,\mathbf{B}} > \vec{\beta}_1 - \frac{1}{4}\vec{\varepsilon}. \tag{4.19}$$

We first treat the case $t \sup \|\vec{\phi}\| < \varepsilon n/30$. Clearly we have

$$\left\| \frac{t}{n} \int \vec{\phi}\, d\delta_z^{p_m} \right\| < \frac{\varepsilon}{30}. \tag{4.20}$$

From (4.12) we have $\|\vec{\beta}_0\| \leq \sup \|\vec{\phi}\| + \varepsilon/3$, which gives

$$\left\| \frac{t}{n} \vec{\beta}_0 \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{30}. \tag{4.21}$$

Combining (4.19), (4.20), (4.21) and then using (4.11), we obtain

$$\begin{aligned} \int \vec{\phi}\, d\nu_{t,\mathbf{B}} &= \frac{t}{n} \int \vec{\phi}\, d\delta_z^{p_m} + \left(1 - \frac{t}{n}\right) \int \vec{\phi}\, d\mu_{t,\mathbf{B}} \\ &> -\frac{1}{30}\vec{\varepsilon} + \frac{t}{n}\vec{\beta}_0 + \frac{n - t}{n}\left(\vec{\beta}_1 - \frac{1}{4}\vec{\varepsilon}\right) - \frac{t}{n}\vec{\beta}_0 \end{aligned}$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

$$\geq -\frac{1}{30}\vec{\varepsilon} + \left(\frac{t}{n}\vec{\beta}_0 + \frac{n-t}{n}\vec{\beta}_1\right) - \frac{1}{4}\vec{\varepsilon} - \frac{t}{n}\vec{\beta}_0$$
$$> -\frac{1}{30}\vec{\varepsilon} + \vec{\beta} - \frac{1}{3}\vec{\varepsilon} - \frac{1}{4}\vec{\varepsilon} - \frac{1}{3}\vec{\varepsilon} - \frac{1}{30}\vec{\varepsilon} > \vec{\beta} - \vec{\varepsilon},$$

as required.

It remains to treat the case $t \sup \|\vec{\phi}\| \geq \varepsilon n/30$. We assume n is large enough so that $t \geq N$. Let $x \in A_n(t, \mathbf{B})$. There exists $k \in \{0, \dots, p_m - 1\}$ such that for every $0 \leq s \leq t-1$, $f^s(x)$ and $f^s(f^k(z))$ belong to the same connected component of $K_{m,m+1}$. Hence, if $s \leq t - N$ then $f^s(x)$ and $f^s(f^k(z))$ belong to the same connected component of $\bigcap_{\ell=0}^{t-s-1} f^{-\ell}(K_{m,m+1})$, and so $|f^s(x) - f^s(f^k(z))| < \delta$ by (D2). From (4.14) we have

$$\begin{aligned}\|S_t\vec{\phi}(x) - S_t\vec{\phi}(f^k(z))\| &\leq \|S_{t-N+1}\vec{\phi}(x) - S_{t-N+1}\vec{\phi}(f^k(z))\| \\ &\quad + \|S_{N-1}\vec{\phi}(f^{t-N+1}(x)) - S_{N-1}\vec{\phi}(f^{t-N+1}(f^k(z)))\| \\ &\leq \frac{\varepsilon}{7}(t-N+1) + 2(N-1)\sup\|\vec{\phi}\| < \frac{t\varepsilon}{6}\end{aligned}$$

provided n is sufficiently large, and thus

$$\frac{1}{t}S_t\vec{\phi}(f^k(z)) > \frac{1}{t}S_t\vec{\phi}(x) - \frac{1}{6}\vec{\varepsilon} \geq \vec{\beta}_0 - \frac{1}{6}\vec{\varepsilon}. \tag{4.22}$$

Since z is of period p_m, we have

$$\left\|\frac{1}{p_m}S_{p_m}\vec{\phi}(z) - \frac{1}{t}S_t\vec{\phi}(f^k(z))\right\| \leq \frac{p_m}{t}\sup\|\vec{\phi}\| < \frac{\varepsilon}{6}$$

provided n is sufficiently large, and thus

$$\frac{1}{p_m}S_{p_m}\vec{\phi}(z) > \frac{1}{t}S_t\vec{\phi}(f^k(z)) - \frac{1}{6}\vec{\varepsilon}. \tag{4.23}$$

Combining (4.22) and (4.23), we have

$$\int \vec{\phi}\, d\delta_z^{p_m} = \frac{1}{p_m}S_{p_m}\vec{\phi}(z) > \vec{\beta}_0 - \frac{1}{3}\vec{\varepsilon}. \tag{4.24}$$

Combining (4.19), (4.24) and then using (4.11), we obtain

$$\begin{aligned}\int \vec{\phi}\, d\nu_{t,\mathbf{B}} &= \frac{t}{n}\int \vec{\phi}\, d\delta_z^{p_m} + \frac{n-t}{n}\int \vec{\phi}\, d\mu_{t,\mathbf{B}} \\ &> \frac{t}{n}\left(\vec{\beta}_0 - \frac{1}{3}\vec{\varepsilon}\right) + \frac{n-t}{n}\left(\vec{\beta}_1 - \frac{1}{4}\vec{\varepsilon}\right) \\ &= \frac{t}{n}\vec{\beta}_0 + \frac{n-t}{n}\vec{\beta}_1 - \frac{1}{3}\vec{\varepsilon} - \frac{1}{4}\vec{\varepsilon} > \vec{\beta} - \vec{\varepsilon},\end{aligned}$$

as required. □

From (4.8) (4.9) and (4.17) we have

$$|A_n(t, \mathbf{B})| \leq \frac{C^2|X|}{|f^t(V_t)|}\exp\left(\left(F(\nu_{t,\mathbf{B}}) + \frac{\varepsilon}{4}\right)n\right)|P|. \tag{4.25}$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Let μ be a measure in the finite set $\{\nu_{t,\mathbf{B}}\colon \mathbf{B} \in M_{2,\ell}(t, \mathbb{Z}_\varepsilon)\} \cup \{\delta_z^{p_m}\}$ that maximizes the free energy within this finite set. By (4.12), clearly we have

$$\#M_{2,\ell}(t, \mathbb{Z}_\varepsilon) \le \prod_{j=1}^{\ell} \left(\frac{3}{\varepsilon}\Big(\sup \phi_j - \inf \phi_j\Big) + 2\right)^2. \tag{4.26}$$

By (4.13), if n is sufficiently large then (4.25) and (4.26) together imply

$$|A_n(\vec{\phi}, \vec{\beta}, V_t)| \le \sum_{\mathbf{B}\in M_{2,\ell}(t,\mathbb{Z}_\varepsilon)} |A_n(t, \mathbf{B})| \le \exp\left(\left(F(\mu) + \frac{\varepsilon}{3}\right)n\right)|P|. \tag{4.27}$$

To finish, from (4.9) and (4.27) we have

$$\begin{aligned} I &= \sum_{t=1}^{n-N} |A_n(\vec{\phi}, \vec{\beta}, V_t)| + \sum_{t=n-N+1}^{n} |A_n(\vec{\phi}, \vec{\beta}, V_t)| \\ &\le \sum_{t=1}^{n-N} |A_n(\vec{\phi}, \vec{\beta}, V_t)| + \sum_{t=n-N+1}^{n} |V_t| \\ &\le \frac{n}{p_m} \exp\left(\left(F(\mu) + \frac{\varepsilon}{3}\right)n\right)|P| + \frac{Cn}{p_m} \exp(F(\mu)(n - N + 1))|P|. \end{aligned}$$

Combining this estimate with that of the sum II in (4.10), we obtain

$$|A_n(\vec{\phi}, \vec{\beta}, P)| = I + II \le \exp((F(\mu) + \varepsilon)n)|P|$$

for all sufficiently large n, as required in Lemma 4.2(a). The inequality in Lemma 4.2(b) is a consequence of Sublemmas 4.4 and 4.5. The proof of Lemma 4.2 is complete. □

4.3. *Coarse graining decomposition into clusters.* In this subsection, for all sufficiently large $n \ge 1$ we decompose the set $\bar{A}_n(\vec{\phi}, \vec{\alpha})$ into a finite number of clusters consisting of points that share the same 'itinerary' up to time n, and share almost the same average values of $\vec{\phi}$ along the segments of orbits each contained in a single cycle.

Suppose $m(f) \ge 1$. Let $\varepsilon > 0$ and write $\mathbb{Z}_\varepsilon = \{(\varepsilon/3)a\colon a \in \mathbb{Z}\}$. If $m(f) = \infty$, then let m_*, $N_* \in \mathbb{N}$ satisfy the conclusion of Lemma 2.9 with ε replaced by $\varepsilon/2$, for any $\vec{\beta} = (\beta_1, \ldots, \beta_\ell) \in \mathbb{Z}_\varepsilon^\ell$ satisfying $\inf \phi_j - \varepsilon/3 < \beta_j \le \sup \phi_j$ for $1 \le j \le \ell$. We set

$$M = \begin{cases} m_* & \text{if } m(f) = \infty, \\ m(f) & \text{if } 1 \le m(f) < \infty. \end{cases} \tag{4.28}$$

For each $x \in X$, we define by induction two (possibly infinite) increasing sequences $\{\hat{n}_i(x)\}_{i=0}^{\hat{q}(x)}$, $\{\hat{m}_i(x)\}_{i=0}^{\hat{q}(x)}$ of non-negative integers that successively record the time and position at which the orbit of x falls into deeper renormalization cycles. Start with

$$\hat{n}_0(x) = 0 \quad \text{and} \quad \hat{m}_0(x) = \max\{0 \le m \le M\colon x \in K_m\}.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Let $i \geq 0$ and suppose $\hat{n}_i(x)$, $\hat{m}_i(x)$ are defined. If $\hat{m}_i(x) = M$, or if $\hat{m}_i(x) < M$ and $f^k(x) \in K_{\hat{m}_i(x),\hat{m}_i(x)+1}$ for every $k > \hat{n}_i(x)$, then we stop the inductive definition by setting $\hat{q}(x) = i$. Otherwise, we define

$$\hat{n}_{i+1}(x) = \min\{k > \hat{n}_i(x) \colon f^k(x) \notin K_{\hat{m}_i(x),\hat{m}_i(x)+1}\},$$
$$\hat{m}_{i+1}(x) = \max\{\hat{m}_i(x) + 1 \leq m \leq M \colon f^{\hat{n}_{i+1}(x)}(x) \in K_m\}.$$

If $\hat{m}_i(x)$, $\hat{n}_i(x)$ are defined for all $i \geq 1$, then we set $\hat{q}(x) = \infty$.

For integers $n > M$ and $0 \leq q \leq M$, we define

$$I_n(q) = \left\{ \begin{matrix} \mathbf{t} = ((n_0, m_0), (n_1, m_1), \dots, (n_q, m_q)) \in (\mathbb{Z}^2)^{q+1} \colon \\ 0 = n_0 < n_1 < \cdots < n_q < n, \\ 0 \leq m_0 < m_1 < \cdots < m_q \leq M \end{matrix} \right\}.$$

Let $\mathbf{t} = ((n_0, m_0), \dots, (n_q, m_q)) \in I_n(q)$. For convenience we set

$$n_{q+1} = n,$$

and call $n_0, n_1, \dots, n_{q+1}$ *transition times*. Further, we set

$$t_i = n_{i+1} - n_i \quad \text{for } 0 \leq i \leq q. \tag{4.29}$$

Define

$$R(\mathbf{t}) = \left\{ \begin{matrix} x \in X \colon \hat{q}(x) \geq q, (\hat{n}_i(x), \hat{m}_i(x)) = (n_i, m_i) \text{ for } 0 \leq i \leq q \\ \text{and } \hat{n}_{q+1}(x) \geq n \text{ if } \hat{q}(x) > q \end{matrix} \right\}.$$

We have

$$X = \bigcup_{q=0}^{M} \bigcup_{\mathbf{t} \in I_n(q)} R(\mathbf{t}). \tag{4.30}$$

We decompose the set $\bar{A}_n(\vec{\phi}, \vec{\alpha}) \cap R(\mathbf{t})$ with respect to coarse-grained values at scale ε of time averages of $\vec{\phi}$ between two consecutive transition times. For each $\mathbf{t} \in I_n(q)$, let $M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)$ denote the set of $(q+1) \times \ell$ matrices $\mathbf{A} = (\alpha_{ij})_{0 \leq i \leq q, 1 \leq j \leq \ell}$ with entries in $\mathbb{Z}_\varepsilon$ such that the following inequalities hold:

$$\sum_{i=0}^{q} \alpha_{ij} t_i > \left(\alpha_j - \frac{\varepsilon}{3}\right) n \quad \text{for } 1 \leq j \leq \ell, \tag{4.31}$$

$$\inf \phi_j - \frac{\varepsilon}{3} < \alpha_{ij} \leq \sup \phi_j \quad \text{for } 0 \leq i \leq q \text{ and } 1 \leq j \leq \ell. \tag{4.32}$$

For a matrix $\mathbf{A} = (\alpha_{ij}) \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)$, let $\vec{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{i\ell})$ denote the ith row of $\mathbf{A}$. Define

$$R(\mathbf{t}, \mathbf{A}) = \{x \in R(\mathbf{t}) \colon S_{t_i} \vec{\phi}(f^{n_i}(x)) \geq t_i \vec{\alpha}_i \text{ for } 0 \leq i \leq q\}.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

We claim that

$$\bar{A}_n(\vec{\phi}, \vec{\alpha}) \cap R(\mathbf{t}) \subset \bigcup_{\mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)} R(\mathbf{t}, \mathbf{A}). \tag{4.33}$$

Indeed, for each $x \in \bar{A}_n(\vec{\phi}, \vec{\alpha}) \cap R(\mathbf{t})$ there exists $\mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)$ such that for every entry α_{ij} of $\mathbf{A}$ we have

$$\alpha_{ij} \le \frac{1}{t_i} S_{t_i} \phi_j(f^{n_i}(x)) < \alpha_{ij} + \frac{\varepsilon}{3}.$$

This implies (4.31) and (4.32), which verifies (4.33).

4.4. *Lebesgue measures of clusters.* We are in a position to state and prove a main technical estimate on the Lebesgue measure of each cluster $R(\mathbf{t}, \mathbf{A})$.

PROPOSITION 4.6. *Suppose $m(f) \ge 1$. For any $\varepsilon > 0$ there exists an integer $N > M$ such that if $n \ge N$ and $0 \le q \le M$, then for every $\mathbf{t} \in I_n(q)$ and every $\mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)$ satisfying $R(\mathbf{t}, \mathbf{A}) \ne \emptyset$, there exists a measure $\mu \in \mathcal{M}(f)$ satisfying*

$$|R(\mathbf{t}, \mathbf{A})| \le \exp((F(\mu) + \varepsilon)n)|X| \quad \textit{and} \tag{a}$$

$$\int \vec{\phi}\, d\mu > \vec{\alpha} - \vec{\varepsilon}. \tag{b}$$

Proof. Let $\varepsilon > 0$, and let $\delta > 0$ be such that

$$\|\vec{\phi}(x) - \vec{\phi}(y)\| < \frac{\varepsilon}{3} \quad \text{for } x, y \in X \text{ with } |x - y| < \delta. \tag{4.34}$$

We first point out that the following three exceptional cases with no transition between renormalization cycles can be treated by much simpler versions of later arguments.

Case E1: $q = 0$, $m_0 = M$, $m(f) < \infty$ and $\mathrm{int}(J_{m(f)})$ contains a two-sided attracting periodic point (Remark 2.3(C-I)). This case is treated by a much simpler version of the argument in Case 2 in Step 2 below.

Case E2: $q = 0$, $m_0 = M$, $m(f) < \infty$ and $\partial J_{m(f)}$ contains a one-sided attracting periodic point (Remark 2.3(C-II)). This case is treated by a much simpler version of the argument in Case 3 in Step 2 below.

Case E3: $q = 0$, $m_0 = M$, $m(f) < \infty$ and f has no attracting periodic point ((A) in §2.3). This case is treated by a much simpler version of the argument in Case 4 in Step 2 below.

Hence, we refer the reader to Step 2 below, for details regarding proofs in cases E1, E2, E3.

We now move on to all the remaining cases with transitions between renormalization cycles. We are concerned with sufficiently large integers N_0, N_1 the purposes of which are as follows.

- N_0 concerns the number of iterations in one renormalization cycle, chosen depending only on δ in (4.34). In view of Lemma 2.6, we take N_0 such that for every integer $0 \le m \le M - 1$ and every connected component W of $\bigcap_{k=0}^{N_0-1} f^{-k}(K_{m,m+1})$, $|W| < \delta$ holds.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

- N_1 is a lower bound on consecutive transition times such that some estimates go through (see Lemmas 4.8 and 4.9), which is chosen depending only on $\vec{\phi}$, ε, δ, N_0, f and satisfies

$$N_1 \geq N_0. \tag{4.35}$$

The rest of the proof of Proposition 4.6 divides into three steps.

Step 1: Estimate of error bound in transition. Let $n > M$, $0 \leq q \leq M$ and let $\mathbf{t} \in I_n(q)$, $\mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)$ satisfy $R(\mathbf{t}, \mathbf{A}) \neq \emptyset$. Since we have excluded cases E1, E2, E3, we have $m_0 + 1 \leq M$ and so K_{m_0,m_0+1} is defined. Let $\mathscr{R}_0(\mathbf{A})$ denote the collection of the connected components of K_{m_0,m_0+1}. Let $R_0(\mathbf{A})$ denote the union of elements of $\mathscr{R}_0(\mathbf{A})$. We have $R_0(\mathbf{A}) = K_{m_0,m_0+1}$. For $i = 1, \ldots, q+1$ we define a collection $\mathscr{R}_i(\mathbf{A})$ of non-degenerate closed subintervals of $R_0(\mathbf{A})$ as follows.

- $\mathscr{R}_i(\mathbf{A})$ for $1 \leq i \leq q$ is the collection of the diffeomorphic pullbacks of $f^k(J_{m_i})$ for some $k \in \{0, \ldots, p_{m_i} - 1\}$ by f^{n_i} that are contained in $R_0(\mathbf{A})$ and intersect $R(\mathbf{t}, \mathbf{A})$.
- If $m_q < M$, then $\mathscr{R}_{q+1}(\mathbf{A})$ is the collection of the diffeomorphic pullbacks of $f^k(J_M)$ for some $k \in \{0, \ldots, p_M - 1\}$ by f^n that are contained in $R_0(\mathbf{A})$ and intersect $R(\mathbf{t}, \mathbf{A})$.
- If $m_q = M$ and $m(f) = \infty$, or if $m_q = M$, $m(f) < \infty$ and f has an attracting periodic point, then let $\mathscr{R}_{q+1}(\mathbf{A})$ denote the collection of the pullbacks of $f^k(J_M)$ for some $k \in \{0, \ldots, p_M - 1\}$ by f^n that are contained in $R_0(\mathbf{A})$ and intersect $R(\mathbf{t}, \mathbf{A})$. If $m_q = M$, $m(f) < \infty$ and f has no attracting periodic point, then let $\mathscr{R}'_{q+1}(\mathbf{A})$ denote the collection of the pullbacks of $f^k(L_M)$ for some $k \in \{0, \ldots, p_M - 1\}$ by f^n that are contained in $R_0(\mathbf{A})$ and intersect $R(\mathbf{t}, \mathbf{A})$. Then $\mathscr{R}_{q+1}(\mathbf{A})$ is the refinement of $\mathscr{R}'_{q+1}(\mathbf{A})$ by points in $\bigcup_{k=n_q}^{n-1} f^{-k}(c)$.

Remark 4.7. Recall that $n > n_q$, and see (2.3) to motivate the definition of $\mathscr{R}_{q+1}(\mathbf{A})$ in the case $m_q = M$. In this case, for $R \in \mathscr{R}_{q+1}(\mathbf{A})$, $f^n|_R$ is may not be a diffeomorphism. However, for every $R \in \mathscr{R}_{q+1}(\mathbf{A})$, $f^{n_q}|_R$ is a diffeomorphism. The latter will suffice for bounded distortion arguments to be developed later in the proof of Proposition 4.6: see Cases 2 and 4 in Step 2.

For $1 \leq i \leq q+1$ let $R_i(\mathbf{A})$ denote the union of elements of $\mathscr{R}_i(\mathbf{A})$. Note that

$$R(\mathbf{t}, \mathbf{A}) \subset R_{q+1}(\mathbf{A}) \subset \cdots \subset R_1(\mathbf{A}) \subset R_0(\mathbf{A}). \tag{4.36}$$

Among all transition times, for our purpose it suffices to concentrate on consecutive ones that are separated by N_1.

LEMMA 4.8. *If N_1 is sufficiently large then the following statements hold.*

(a) *For every $0 \leq i \leq q$ with $t_i > N_1$ and $m_i < M$,*

$$f^{n_i}(R_{i+1}(\mathbf{A})) \subset \bar{A}_{t_i}(\vec{\phi}, \vec{\alpha}_i - \tfrac{1}{2}\vec{\varepsilon}) \cap K_{m_i}.$$

(b) *If $t_q > N_1$, $m_q = M$ and*

(i) *$m(f) = \infty$, or*

(ii) *$m(f) < \infty$ and $J_{m(f)}$ contains a one-sided attracting periodic point, or*

(iii) *f has no attracting periodic point,*

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

then

$$f^{n_q}(R_{q+1}(\mathbf{A})) \subset \bar{A}_{t_q}(\vec{\phi}, \vec{\alpha}_q - \tfrac{1}{2}\vec{\varepsilon}) \cap K_{m_q}.$$

Proof. Let $0 \le i \le q$ and suppose $t_i > N_1$. Since the definition of $\mathscr{R}_{i+1}(\mathbf{A})$ yields $f^{n_i}(R_{i+1}(\mathbf{A})) \subset K_{m_i}$, it suffices to show that

$$f^{n_i}(R_{i+1}(\mathbf{A})) \subset \bar{A}_{t_i}(\vec{\phi}, \vec{\alpha}_i - \tfrac{1}{2}\vec{\varepsilon}). \tag{4.37}$$

To this end, for each $R \in \mathscr{R}_{i+1}(\mathbf{A})$ we fix $x_R \in f^{n_i}(R \cap R(\mathbf{t}, \mathbf{A}))$. The definition of $\mathscr{R}_{i+1}(\mathbf{A})$ and that of $R(\mathbf{t}, \mathbf{A})$ give

$$S_{t_i}\vec{\phi}(x_R) \ge t_i\vec{\alpha}_i. \tag{4.38}$$

Suppose $m_i < M$. By (4.35), $t_i > N_1$ and (4.29), we have

$$0 \le n_i + N_1 - N_0 \le n_i + t_i - N_0 - 1 = n_{i+1} - N_0 - 1,$$

and so for every $R \in \mathscr{R}_{i+1}(\mathbf{A})$,

$$\bigcup_{k=0}^{t_i - N_0 - 1} f^{n_i+k}(R) \subset \bigcap_{k=0}^{N_0-1} f^{-k}(K_{m_i, m_i+1}).$$

Moreover, the choice of N_0 yields $|f^{n_i+k}(R)| < \delta$ for $0 \le k \le t_i - N_0 - 1$. For any $x \in f^{n_i}(R)$, (4.34) gives

$$\|S_{t_i - N_0}\vec{\phi}(x) - S_{t_i - N_0}\vec{\phi}(x_R)\| \le \frac{\varepsilon}{3}(t_i - N_0). \tag{4.39}$$

For the remaining N_0 iterations, clearly we have

$$\|S_{N_0}\vec{\phi}(f^{t_i - N_0}(x)) - S_{N_0}\vec{\phi}(f^{t_i - N_0}(x_R))\| \le 2N_0 \sup \|\vec{\phi}\|. \tag{4.40}$$

Combining (4.38), (4.39), and (4.40) and dividing the result by t_i, we obtain

$$\frac{1}{t_i} S_{t_i}\vec{\phi}(x) \ge \vec{\alpha}_i - \frac{1}{2}\vec{\varepsilon}$$

provided N_1 is sufficiently large depending only on $\vec{\phi}$, ε, N_0. Since $R \in \mathscr{R}_{i+1}(\mathbf{A})$ and $x \in f^{n_i}(R)$ are arbitrary, (4.37) follows. We have verified Lemma 4.8(a).

To prove Lemma 4.8(b), suppose $t_q > N_1$ and $m_q = M$. In case (i), the same argument as in the proof of Lemma 4.8(a) remains valid to show (4.37) with $i = q$. In case (ii), since $n - n_q = t_q > N_1$, if N_1 is sufficiently large then the uniform convergence in (2.2) and equation (4.34) together imply

$$\sup_{x \in f^{n_q}(R)} \|S_{t_q}\vec{\phi}(x) - S_{t_q}\vec{\phi}(x_R)\| \le \frac{\varepsilon}{2} t_q \quad \text{for every } R \in \mathscr{R}_{q+1}(\mathbf{A}). \tag{4.41}$$

From (4.38) and (4.41) we obtain (4.37) with $i = q$.

In case (iii), $M = m(f)$ and $f^{p_M}|_{L_M}$ is topologically exact. Recall that we have chosen $\delta > 0$ so that (4.34) holds. There is $n(\delta) \in \mathbb{N}$ such that for any integer $0 \le j \le p_M - 1$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

and any interval J such that $J \subset f^j(L_M)$ and $|J| \geq \delta$, there is $r \in \{0, \dots, p_M - 1\}$ such that $f^{p_M n(\delta)+r}(J) = L_M$. We choose N_1 so that

$$N_1 \geq p_M(n(\delta) + 1).$$

Then we have $n - n_q - p_M(n(\delta) + 1) > t_q - N_1 > 0$. We claim that if $R \in \mathscr{R}_{q+1}(\mathbf{A})$ and $k \in \{0, \dots, n - n_q - p_M(n(\delta) + 1)\}$ then $|f^k(f^{n_q}(R))| < \delta$, for otherwise $f^{p_M n(\delta)+r+k}(f^{n_q}(R)) = L_M$ would hold for some $r \in \{0, \dots, p_M - 1\}$ and $p_M n(\delta) + r + k \leq n - n_q - 1$, and as a result $f^n|_R$ would not be injective, a contradiction.

The rest of the argument is similar to the proof of Lemma 4.8(a). By the above claim and (4.34), for every $R \in \mathscr{R}_{q+1}(\mathbf{A})$ and any $x \in f^{n_q}(R)$ we have

$$\|S_{n-n_q-p_M(n(\delta)+1)}\vec{\phi}(x) - S_{n-n_q-p_M(n(\delta)+1)}\vec{\phi}(x_R)\| \leq \frac{\varepsilon}{3}(n - n_q - p_M(n(\delta) + 1)). \tag{4.42}$$

For the remaining $p_M(n(\delta) + 1)$ iterations, clearly we have

$$\begin{aligned}\|S_{p_M(n(\delta)+1)}\vec{\phi}(f^{n-n_q-p_M(n(\delta)+1)}(x)) - S_{p_M(n(\delta)+1)}\vec{\phi}(f^{n-n_q-p_M(n(\delta)+1)}(x_R))\| \\ \leq 2p_M(n(\delta) + 1) \sup \|\vec{\phi}\|.\end{aligned} \tag{4.43}$$

Combining (4.38), (4.42), and (4.43) and dividing the result by t_q, we obtain

$$\frac{1}{t_q} S_{t_q}\vec{\phi}(f^{n_q}(x)) \geq \vec{\alpha}_q - \frac{1}{2}\vec{\varepsilon}$$

provided N_1 is sufficiently large. Since $R \in \mathscr{R}_{q+1}(\mathbf{A})$ and $x \in f^{n_q}(R)$ are arbitrary, (4.37) with $i = q$ follows. The proof of Lemma 4.8 is complete. □

Step 2: Estimate of $\boldsymbol{i}$*-step conditional probability.* The next lemma provides estimates of conditional probabilities at consecutive transition times separated by N_1.

LEMMA 4.9. *If N_1 is sufficiently large, then for every $0 \leq i \leq q$ with $t_i > N_1$, there exists a measure $\mu_i \in \mathcal{M}(f)$ such that*

$$\frac{|R_{i+1}(\mathbf{A})|}{|R_i(\mathbf{A})|} \leq \exp((F(\mu_i) + \varepsilon)t_i) \quad \textit{and} \tag{a}$$

$$\int \vec{\phi}\, d\mu_i > \vec{\alpha}_i - \vec{\varepsilon}. \tag{b}$$

Proof. Let $0 \leq i \leq q$ and suppose $t_i > N_1$. In view of the definition of $\mathscr{R}_i(\mathbf{A})$ in Step 1, we treat the case $m_i < M$ and the case $m_i = M$ (hence $i = q$) separately.

In the former case, let $R \in \mathscr{R}_i(\mathbf{A})$ and suppose $R \cap R_{i+1}(\mathbf{A}) \neq \emptyset$. Let $\mathscr{P}_{m_i}(\mathbf{A}, R)$ denote the collection of the diffeomorphic pullbacks of $f^k(J_{m_{i+1}})$ for some $k \in \{0, \dots, p_{m_{i+1}} - 1\}$ by f^{t_i} that are contained in $f^{n_i}(R \cap R_{i+1}(\mathbf{A}))$. The union of elements of $\mathscr{P}_{m_i}(\mathbf{A}, R)$ equals $f^{n_i}(R \cap R_{i+1}(\mathbf{A}))$. By Lemma 4.8(a), every element of $\mathscr{P}_{m_i}(\mathbf{A}, R)$ is contained in $\bar{A}_{t_i}(\vec{\phi}, \vec{\alpha}_i - (1/2)\vec{\varepsilon}) \cap K_{m_i}$ provided N_1 is sufficiently large. Moreover, for every $Q \in \mathscr{P}_{m_i}(\mathbf{A}, R)$ there exists $P \in \tilde{\mathscr{P}}_{m_i}$ such that $P \subset K_{m_i, m_i+1}$ and $Q \subset P \subset f^{n_i}(R)$. Let $\tilde{\mathscr{P}}_{m_i}(\mathbf{A}, R)$ denote the collection of elements of $\tilde{\mathscr{P}}_{m_i}$ that is

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

contained in $f^{n_i}(R)$ and contains an element of $\mathscr{P}_{m_i}(\mathbf{A}, R)$. For each $P \in \tilde{\mathscr{P}}_{m_i}(\mathbf{A}, R)$ we have

$$\bigcup_{\substack{Q \in \mathscr{P}_{m_i}(\mathbf{A},R) \\ Q \subset P}} Q \subset A_{t_i}\left(\vec{\phi}, \vec{\alpha}_i - \frac{1}{2}\vec{\varepsilon}, P\right).$$

By Lemma 4.2, there exists $\nu_{R,P} \in \mathcal{M}(f)$ such that

$$\sum_{\substack{Q \in \mathscr{P}_{m_i}(\mathbf{A},R) \\ Q \subset P}} |Q| \le \left|A_{t_i}\left(\vec{\phi}, \vec{\alpha}_i - \frac{1}{2}\vec{\varepsilon}, P\right)\right| \le \exp\left(\left(F(\nu_{R,P}) + \frac{\varepsilon}{2}\right)t_i\right)|P| \tag{4.44}$$

and

$$\int \vec{\phi}\, d\nu_{R,P} > \vec{\alpha}_i - \vec{\varepsilon}. \tag{4.45}$$

Let $\mu_i \in \mathcal{M}(f)$ be a measure in the finite set

$$\{\nu_{R,P} \colon R \in \mathscr{R}_i(\mathbf{A}),\ R \cap R_{i+1}(\mathbf{A}) \neq \emptyset,\ P \in \tilde{\mathscr{P}}_{m_i}(\mathbf{A}, R)\}$$

that maximizes the free energy within this finite set. By (4.44) we have

$$\begin{aligned} |f^{n_i}(R \cap R_{i+1}(\mathbf{A}))| &= \sum_{P \in \mathscr{P}_{m_i}(\mathbf{A},R)} |P| \\ &\le \sum_{P \in \tilde{\mathscr{P}}_{m_i}(\mathbf{A},R)} \left|A_{t_i}\left(\vec{\phi}, \vec{\alpha}_i - \frac{1}{2}\vec{\varepsilon}, P\right)\right| \\ &\le \sum_{P \in \tilde{\mathscr{P}}_{m_i}(\mathbf{A},R)} \exp\left(\left(F(\nu_{R,P}) + \frac{\varepsilon}{2}\right)t_i\right)|P| \\ &\le \exp\left(\left(F(\mu_i) + \frac{\varepsilon}{2}\right)t_i\right)|f^{n_i}(R)|. \end{aligned} \tag{4.46}$$

If $i \ge 1$, then we have $\bigcup_{k=n_j}^{n_{j+1}-1} f^k(R) \subset K_{n_j,n_j+1}$ for $0 \le j \le i-1$. By Proposition 2.7(a), the distortion of the composition $f^{n_i} = f^{t_{i-1}} \circ \cdots \circ f^{t_1} \circ f^{t_0}$ on R is bounded by the constant

$$D_i = \prod_{j=0}^{i-1} \gamma_{m_j}. \tag{4.47}$$

Hence, from (4.46) we obtain

$$\frac{|R \cap R_{i+1}(\mathbf{A})|}{|R|} \le D_i \frac{|f^{n_i}(R \cap R_{i+1}(\mathbf{A}))|}{|f^{n_i}(R)|} \le \exp((F(\mu_i) + \varepsilon)t_i)$$

provided N_1 is sufficiently large. Since $n_0 = 0$, the same upper bound remains valid for $i = 0$. Since R is an arbitrary element of $\mathscr{R}_i(\mathbf{A})$ intersecting $R_{i+1}(\mathbf{A})$, we obtain

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

$$\frac{|R_{i+1}(\mathbf{A})|}{|R_i(\mathbf{A})|} \le \max_{\substack{R\in\mathscr{R}_i(\mathbf{A})\\ R\cap R_{i+1}(\mathbf{A})\neq\emptyset}} \frac{|R\cap R_{i+1}(\mathbf{A})|}{|R|} \le \exp((F(\mu_i)+\varepsilon)t_i),$$

as required in Lemma 4.9(a). The inequality in Lemma 4.9(b) is a consequence of (4.45).

To complete the proof of Lemma 4.9, it remains to treat the case $m_i = M$. In this case we have $i = q$, and there are four subcases.

Case 1: $m(f) = \infty$. We have $M = m_*$ by (4.28). Set $\mu_q = \mu_\infty$, which is the post-critical measure. The inequality in Lemma 4.9(a) with $i = q$ follows from $R_{q+1}(\mathbf{A}) \subset R_q(\mathbf{A})$ and $F(\mu_q) = 0$. Since $t_q > N_1$ and $f^{n_q}(R_{q+1}(\mathbf{A}))$ is contained in $\tilde{A}_{t_q}(\vec{\phi}, \vec{\alpha}_q - \frac{1}{2}\vec{\varepsilon}) \cap K_M$ by Lemma 4.8(b)(i), the inequality in Lemma 4.9(b) with $i = q$ follows from Lemma 2.9(a) provided $N_1 \ge N_*$ and N_1 is sufficiently large.

Case 2: $m(f) = M$ *and* $\mathrm{int}(J_{m(f)})$ *contains a two-sided attracting periodic point (Remark 2.3(C-I)).* We have $K_M = \mathrm{cl}(B(f))$ by (2.3) and Lemma 2.2. Fix a closed subinterval J of J_M such that $c, z_f \in \mathrm{int}(J)$, $f^{p_M}(J) \subset J$ and $|K_M \setminus K_{M+1}| < \delta$ where $K_{M+1} = \bigcup_{k=0}^{p_M-1} f^k(J)$.

Recall that ν denotes the element of $\mathcal{M}_{M-1}(f)$ supported on the orbit of the hyperbolic repelling periodic point in ∂J_M. The rest of the proof is similar in spirit to the argument in Case 2 in the proof of Lemma 4.2. For each $R \in \mathscr{R}_q(\mathbf{A})$ and each integer t with $0 \le t \le t_q$, define

$$V_{R,t} = \begin{cases} f^{n_q}(R) \cap K_{M+1} & \text{if } t = 0,\\ f^{n_q}(R) \cap \left(f^{-t}(K_{M+1}) \setminus \bigcup_{k=0}^{t-1} f^{-k}(K_{M+1}) \right) & \text{if } 1 \le t \le t_q - 1,\\ f^{n_q}(R) \setminus \bigcup_{k=0}^{t_q-1} f^{-k}(K_{M+1}) & \text{if } t = t_q. \end{cases}$$

Notice that $f^{n_q}(R) = \bigcup_{t=0}^{t_q} V_{R,t}$. Then $V_{R,t}$ has at most two connected components, which are non-degenerate intervals. Let $W_{R,t}$ denote union of the diffeomorphic pullbacks of the connected components of $V_{R,t}$ by f^{n_q} that are contained in R. Define $\nu_{R,t} \in \mathcal{M}(f)$ by

$$\nu_{R,t} = \frac{t}{t_q} \cdot \nu + \left(1 - \frac{t}{t_q}\right) \cdot \delta_{O(f)}.$$

There exists a constant $C \ge 1$ depending only f, δ such that if $x \in X$ and $k \in \mathbb{N}$ satisfy $x, \dots, f^{k-1}(x) \in K_M \setminus K_{M+1}$ then $|(f^k)'x| \ge C^{-1} e^{\chi^+(\nu)k}$. Hence,

$$\frac{|V_{R,t}|}{|f^{n_q}(R)|} \le C \exp(-\chi^+(\nu)t) = C \exp(F(\nu)t). \tag{4.48}$$

Since $F(\delta_{O(f)}) = 0$ by the definition of the Lyapunov exponent (1.1), we have $F(\nu)t \le F(\nu_{R,t})t_q$. By Proposition 2.7, the distortion of f^{n_q} on R is bounded by the constant D_q in (4.47) so that (4.48) yields

$$\frac{|W_{R,t}|}{|R|} \le C D_q \exp(F(\nu_{R,t})t_q). \tag{4.49}$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Let μ_q be a measure in the finite set

$$\{\nu_{R,t}\colon R \in \mathscr{R}_q(\mathbf{A}), 0 \le t \le t_q,\ W_{R,t} \cap R_{q+1}(\mathbf{A}) \neq \emptyset\}$$

that maximizes the free energy within this finite set. Since $t_q > N_1$, from (4.49) we obtain

$$\frac{|R_{q+1}(\mathbf{A})|}{|R_q(\mathbf{A})|} \le \max_{R\in\mathscr{R}_q(\mathbf{A})} \frac{|R\cap R_{q+1}(\mathbf{A})|}{|R|} \le \max_{R\in\mathscr{R}_q(\mathbf{A})} \sum_{\substack{0\le t\le t_q\\ W_{R,t}\cap R_{q+1}(\mathbf{A})\neq\emptyset}} \frac{|W_{R,t}|}{|R|}$$

$$\le (t_q+1)\exp(F(\mu_q)t_q) \le \exp((F(\mu_q)+\varepsilon)t_q)$$

provided N_1 is sufficiently large. This yields the inequality in Lemma 4.9(a) with $i = q$.

Since $t_q > N_1$, similarly to the proof of Sublemma 4.5 one can show that if $R \in \mathscr{R}_q(\mathbf{A})$, $0 \le t \le t_q$ and $W_{R,t} \cap R_{q+1}(\mathbf{A}) \neq \emptyset$ then $\int \vec{\phi} d\nu_{R,t} > \vec{\alpha}_q - \vec{\varepsilon}$ holds, provided N_1 is sufficiently large. This yields the desired inequality in Lemma 4.9(b) with $i = q$.

Case 3: $m(f) = M$ and $\partial J_{m(f)}$ contains a one-sided attracting periodic point (Remark 2.3(C-II)). Set $\mu_q = \delta_{O(f)}$. The inequality in Lemma 4.9(a) with $i = q$ follows from $R_{q+1}(\mathbf{A}) \subset R_q(\mathbf{A})$ and $F(\mu_q) = 0$. By Lemma 4.8(b)(ii), $f^{n_q}(R_{q+1}(\mathbf{A}))$ is contained in $\bar{A}_{t_q}(\vec{\phi}, \vec{\alpha}_q - \frac{1}{2}\vec{\varepsilon}) \cap K_M$. Since $t_q > N_1$, if N_1 is sufficiently large then the uniform convergence in (2.2) implies the inequality in Lemma 4.9(b) with $i = q$.

Case 4: $m(f) = M$ and f has no attracting periodic point. Recall that $K_M = \bigcup_{k=0}^{p_M-1} f^k(L_M)$ by (2.3). Let $R \in \mathscr{R}_q(\mathbf{A})$. Lemma 4.8(b)(iii) gives

$$f^{n_q}(R \cap R_{q+1}(\mathbf{A})) \subset \bar{A}_{t_q}(\vec{\phi}, (\vec{\alpha}_q - \tfrac{1}{2})\vec{\varepsilon}) \cap K_M.$$

Since the unimodal map $f^{p_M}|_{L_M}$ is topologically exact, the Lebesgue measure of the right set is estimated as follows by [7, Proposition 4.4]: if N_1 is sufficiently large, then there exists a measure $\nu_R \in \mathcal{M}(f)$ such that

$$\frac{|\bar{A}_{t_q}(\vec{\phi}, \vec{\alpha}_q - (1/2)\vec{\varepsilon}) \cap K_M|}{|K_M|} \le \exp\left(\left(F(\nu_R) + \frac{\varepsilon}{2}\right)t_q\right) \quad \text{and} \tag{4.50}$$

$$\int \vec{\phi}\, d\nu_R > \vec{\alpha}_q - \vec{\varepsilon}. \tag{4.51}$$

For all sufficiently large N_1 we deduce that

$$\begin{aligned}\frac{|R\cap R_{q+1}(\mathbf{A})|}{|R|} &\le D_q \frac{|f^{n_q}(R\cap R_{q+1}(\mathbf{A}))|}{|f^{n_q}(R)|}\\ &\le D_q \frac{|\bar{A}_{t_q}(\vec{\phi}, \vec{\alpha}_q - (1/2)\vec{\varepsilon}) \cap K_M|}{|f^{n_q}(R)|}\\ &\le \exp((F(\nu_R)+\varepsilon)t_q).\end{aligned} \tag{4.52}$$

The first inequality is because the distortion of f^{n_q} on R is bounded by the constant D_q in (4.47) by Proposition 2.7. Since $t_q > N_1$, the last inequality follows from (4.50) and the next uniform lower bound

$$\frac{|f^{n_q}(R)|}{|K_M|} \ge \frac{\min_{0\le k\le p_M-1}|f^k(L_M)|}{|K_M|} > 0.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Let μ_q be a measure in the finite set $\{\nu_R\colon R \in \mathscr{R}_q(\mathbf{A})\}$ that maximizes the free energy within this finite set. By (4.52) we have

$$\frac{|R_{q+1}(\mathbf{A})|}{|R_q(\mathbf{A})|} \le \max_{R\in\mathscr{R}_q(\mathbf{A})} \frac{|R \cap R_{q+1}(\mathbf{A})|}{|R|} \le \exp((F(\mu_q)+\varepsilon)t_q),$$

as required in Lemma 4.9(a) with $i = q$. The inequality in Lemma 4.9(b) with $i = q$ is a consequence of (4.51). The proof of Lemma 4.9 is complete. □

Step 3: Overall estimates. Let N_1 be sufficiently large so that the conclusions of Lemma 4.8 and 4.9 hold with ε replaced by $\varepsilon/3$. Let $n > \max\{M, (q+1)N_1\}$. Put

$$s(N_1) = \{0 \le i \le q\colon t_i > N_1\}.$$

By (4.29) we have $n = \sum_{i=0}^{q} t_i$, and so $s(N_1) \ne \emptyset$. For each $i \in s(N_1)$, let μ_i be a measure in $\mathcal{M}(f)$ for which the conclusion of Lemma 4.9 holds with ε replaced by $\varepsilon/3$. Define a measure $\mu \in \mathcal{M}(f)$ by

$$\mu = \Bigg(\sum_{i\in s(N_1)} t_i\Bigg)^{-1} \sum_{i\in s(N_1)} t_i \cdot \mu_i.$$

Since $R_{i+1}(\mathbf{A}) \subset R_i(\mathbf{A})$ for $0 \le i \le q$ as in (4.36) and $(1/n)\sum_{i\in s(N_1)} t_i \to 1$ as $n \to \infty$, for all sufficiently large n we have

$$\begin{aligned}\frac{|R_{q+1}(\mathbf{A})|}{|R_0(\mathbf{A})|} &= \prod_{i=0}^{q} \frac{|R_{i+1}(\mathbf{A})|}{|R_i(\mathbf{A})|} \le \prod_{i\in s(N_1)} \frac{|R_{i+1}(\mathbf{A})|}{|R_i(\mathbf{A})|} \le \exp\Bigg(\sum_{i\in s(N_1)} \Big(F(\mu_i)+\frac{\varepsilon}{2}\Big)t_i\Bigg)\\ &\le \exp\Big(F(\mu)(n-(q+1)N_1)+\frac{\varepsilon n}{2}\Big) < \exp((F(\mu)+\varepsilon)n).\end{aligned}$$

The desired inequality in Proposition 4.6(a) follows from this estimate and the first inclusion in (4.36). By (4.31) we have

$$\begin{aligned}\sum_{i\in s(N_1)} t_i \int \vec{\phi}\, d\mu_i &\ge \sum_{i\in s(N_1)} t_i\Big(\vec{\alpha}_i - \frac{1}{3}\vec{\varepsilon}\Big)\\ &= \sum_{i=0}^{q} t_i\Big(\vec{\alpha}_i - \frac{1}{3}\vec{\varepsilon}\Big) - \sum_{i\notin s(N_1)} t_i\Big(\vec{\alpha}_i - \frac{1}{3}\vec{\varepsilon}\Big)\\ &> n\Big(\vec{\alpha} - \frac{2}{3}\vec{\varepsilon}\Big) - (q+1)N_1 \max_{0\le i\le q} \|\vec{\alpha}_i\|\vec{1}.\end{aligned}$$

This yields the desired inequality in Proposition 4.6(b) for all sufficiently large n. The proof of Proposition 4.6 is complete. □

4.5. *Proof of Proposition 4.1.* Let $f\colon X \to X$ be an S-unimodal map with a non-flat critical point. We first treat the case $m(f) \ge 1$. Let $\varepsilon > 0$. Let $N > M$ be an integer for which the conclusion of Proposition 4.6 holds with ε replaced by $\varepsilon/4$. Let $n \ge N$ satisfy

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

$\bar{A}_n(\vec{\phi}, \vec{\alpha}) \neq \emptyset$. From (4.30) and (4.33) we have

$$\bar{A}_n(\vec{\phi}, \vec{\alpha}) \subset \bigcup_{q=0}^{M} \bigcup_{\mathbf{t} \in I_n(q)} \bigcup_{\mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)} R(\mathbf{t}, \mathbf{A}).$$

Let $0 \leq q \leq M$. By Proposition 4.6, for every $\mathbf{t} \in I_n(q)$ and every $\mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)$ with $R(\mathbf{t}, \mathbf{A}) \neq \emptyset$, there exists a measure $\mu_{\mathbf{t},\mathbf{A}} \in \mathcal{M}(f)$ such that

$$|R(\mathbf{t}, \mathbf{A})| \leq \exp\left(\left(F(\mu_{\mathbf{t},\mathbf{A}}) + \frac{\varepsilon}{4}\right)n\right) \quad \text{and} \tag{4.53}$$

$$\int \vec{\phi}\, d\mu_{\mathbf{t},\mathbf{A}} > \vec{\alpha} - \frac{1}{4}\vec{\varepsilon}. \tag{4.54}$$

If n is sufficiently large, then by (4.32) we have

$$\#M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon) \leq \prod_{j=1}^{\ell} \left(\frac{3}{\varepsilon}(\sup \phi_j - \inf \phi_j) + 2\right)^{q+1} \leq \exp\left(\frac{\varepsilon n}{4}\right).$$

Moreover, we have $\#I_n(0) = M + 1$, and for $q \geq 1$,

$$\#I_n(q) \leq \binom{n}{q}\binom{M+1}{q+1} \leq n^q \binom{M+1}{q+1} \leq \exp\left(\frac{\varepsilon n}{4}\right).$$

Let $\mu \in \mathcal{M}(f)$ be a measure in the finite set

$$\bigcup_{q=0}^{M} \{\mu_{\mathbf{t},\mathbf{A}} : \mathbf{t} \in I_n(q),\ \mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon),\ R(\mathbf{t}, \mathbf{A}) \neq \emptyset\}$$

that maximizes the free energy within this finite set. Combining (4.53) with the upper bound on the number of all itineraries yields

$$\begin{aligned}
|\bar{A}_n(\vec{\phi}, \vec{\alpha})| &\leq \sum_{q=0}^{M} \sum_{\mathbf{t} \in I_n(q)} \sum_{\mathbf{A} \in M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon)} |R(\mathbf{t}, \mathbf{A})| \\
&\leq (M+1) \exp\left(\left(F(\mu) + \frac{\varepsilon}{4}\right)n\right) \max_{0 \leq q \leq M} \#I_n(q) \#M_{q+1,\ell}(\mathbf{t}, \mathbb{Z}_\varepsilon) \\
&\leq \exp((F(\mu) + \varepsilon)n),
\end{aligned}$$

as required in Proposition 4.1(a) provided n is sufficiently large. The inequality in Proposition 4.1(b) is a consequence of (4.54).

It remains to treat the case $m(f) = 0$. If $\bar{m}(f) = 1$, then f has a two-sided attracting fixed point in $\operatorname{int}(X)$. The desired upper bound follows from the argument of Case 2 in the proof of Lemma 4.9. If $\bar{m}(f) = 0$, then $f|_{L_0} : L_0 \to L_0$ is topologically exact. The desired upper bound follows from the argument of Case 4 in the proof of Lemma 4.9. The proof of Proposition 4.1 is complete.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

5. *Bimodal maps for which the level-2 LDP does not hold*

In this final section we prove Theorem B. We start with a post-critically finite map with two critical points, one periodic and the other not. After a small perturbation we obtain a map for which the level-2 LDP does not hold.

5.1. *A one-parameter family of bimodal maps.* Let $X = [0, 1]$ and $K = [1/2, 1]$. Let $f: X \to X$ be a C^3 map with negative Schwarzian derivative having only two critical points c_0 and c_1 with $0 < c_0 < \frac{1}{2} < c_1 < 1$, which are assumed to be non-degenerate: $f''(c_0)f''(c_1) \neq 0$. We assume the following conditions on f.

(F1) $f(X) \subset K$ and c_1 is a local maximal point of f.

(F2) c_1 is periodic with prime period 3.

See Figure 2. Condition (F2) implies that the complement of the basin of the super-attracting periodic orbit of c_1 in $[f^2(c_1), f(c_1)]$ is a non-trivial hyperbolic basic set Λ. Set $p = f(c_0)$. Fix a periodic point $q \in \Lambda$. Further, we make the following assumption.

(F3) p is a periodic point in Λ, and not contained in the periodic orbit of q.

Remark 5.1. From (F1), we have $f^2(X) \subset [\frac{1}{2}, f(c_1)]$ and so $f^3(X) \subset [f^2(c_1), f(c_1)]$.

We consider a C^3 parametrized family of maps $f_a: X \to X$, $a \in [-1, 1]$, such that $f_0 = f$ and the following conditions hold.

(F4) $f_a|_K$ does not depend on $a \in [-1, 1]$.

(F5) The critical point c_0 of f remains a non-degenerate critical point of f_a for $a \in [-1, 1]$, and $(d/da)(f_a(c_0))|_{a=0} \neq 0$.

Let μ_p and μ_q denote the uniform probability distributions on the periodic orbits of p and q for f, respectively. Let χ_p and χ_q denote their Lyapunov exponents. From (F4), these do not depend on the parameter a. For $x \in X$, $n \in \mathbb{N}$ and $a \in [-1, 1]$, let $\delta^n_{a,x}$ denote the uniform probability distribution on the orbit $\{f_a^j(x)\}_{j=0}^{n-1}$. Theorem B follows from the next proposition.

PROPOSITION 5.2. *There exists a parameter $a \in [-1, 1]$ arbitrarily close to 0 such that for any $\varepsilon > 0$, there exist open sets $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathcal{M}$ such that:*

$$\mu_p \in \mathcal{G}_1 \subset \mathrm{cl}(\mathcal{G}_1) \subset \mathcal{G}_2;$$

$$\liminf_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta^n_{a,x} \in \mathcal{G}_2\}| \leq -\chi_p + \varepsilon;$$

$$\limsup_{n\to\infty} \frac{1}{n} \log |\{x \in X : \delta^n_{a,x} \in \mathrm{cl}(\mathcal{G}_1)\}| \geq -\frac{\chi_p}{2}.$$

Proof of Theorem B. Let $a \in [-1, 1]$ be a parameter for which the conclusion of Proposition 5.2 holds. If there were a rate function $I: \mathcal{M} \to [0, \infty]$ for f_a, we would have

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

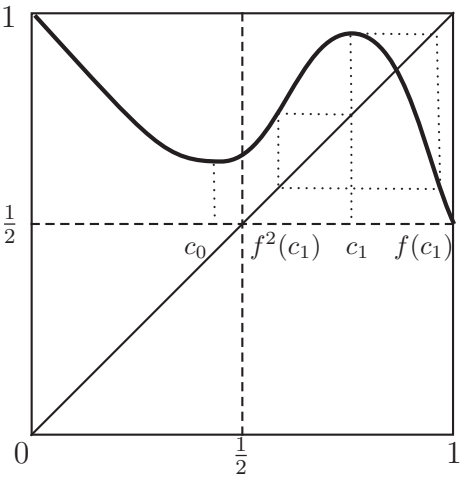


FIGURE 2. The graph of the bimodal map f_a in §5.1.

$$\limsup_{n\to\infty}\frac{1}{n}\log|\{x\in X:\delta^n_{a,x}\in \mathrm{cl}(\mathcal{G}_1)\}| \le -\inf_{\mathrm{cl}(\mathcal{G}_1)} I \le -\inf_{\mathcal{G}_2} I \le \liminf_{n\to\infty}\frac{1}{n}\log|\{x\in X:\delta^n_{a,x}\in \mathcal{G}_2\}|.$$

This contradicts the conclusion of Proposition 5.2 for sufficiently small $\varepsilon > 0$. Therefore, the level-2 LDP does not hold for f_a. □

5.2. *Abnormal and normal large deviations rates.* For $\delta \in (0, 1)$, let $O_\delta(p)$ denote the open δ-neighborhood of the periodic orbit of p and put

$$\mathcal{G}_\delta = \{\mu\in\mathcal{M}:\mu(O_\delta(p)) > 1-\delta\}.$$

This is an open neighborhood of μ_p and we have $\mathrm{cl}(\mathcal{G}_{\delta'}) \subset \mathcal{G}_\delta$ for $0 < \delta' < \delta < 1$.

The next two lemmas provide large deviations rates for neighborhoods of μ_p. For many a, we observe an abnormal rate as in Lemma 5.3. For many other a, we observe the normal rate as in Lemma 5.4.

LEMMA 5.3. *If $|a|$ is sufficiently small and $f_a^m(c_0) = p$ for some $m \ge 1$, then for any $\delta \in (0, 1)$ we have*

$$\liminf_{n\to\infty}\frac{1}{n}\log|\{x\in X:\delta^n_{a,x}\in\mathcal{G}_\delta\}| \ge -\frac{\chi_p}{2}.$$

Proof. Set $\gamma = \sqrt{|f_a''(c_0)|^{-1}\delta}$, and let $\varepsilon > 0$. The condition $|x - c_0| \le \gamma e^{-n(\chi_p+\varepsilon)/2}$ for $x \in X$ and sufficiently large $n \ge 1$ implies $|f_a(x) - f_a(c_0)| \le \delta e^{-n(\chi_p+\varepsilon)}$, and so $\delta^n_{a,x} \in \mathcal{G}_\delta$. Hence, we have

$$|\{x\in X:|x-c_0| < \gamma e^{-n(\chi_p+\varepsilon)/2}\}| = 2\gamma e^{-(\chi_p+\varepsilon)n/2} \le |\{x\in X:\delta^n_{a,x}\in\mathcal{G}_\delta\}|.$$

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

This implies that the lower limit in the lemma is bounded from below by $-(\chi_p+\varepsilon)/2$. Since $\varepsilon>0$ is arbitrary, the desired inequality follows. □

LEMMA 5.4. *For any $\varepsilon>0$ there exists $\delta_*>0$ such that for any $\delta\in(0,\delta_*]$, if $|a|$ is sufficiently small and $f_a^m(c_0)=q$ for some $m\geq 1$ then*

$$\limsup_{n\to\infty}\frac{1}{n}\log|\{x\in X:\delta_{a,x}^n\in\mathcal{G}_\delta\}|\leq-\chi_p+\varepsilon.$$

Proof. Let $\varepsilon>0$. Since f coincides with f_a on $[f^2(c_1),f(c_1)]$, we have the large deviation estimate

$$\limsup_{n\to\infty}\frac{1}{n}\log|\{x\in[f^2(c_1),f(c_1)]:\delta_{a,x}^n\in\mathcal{G}_\delta\}|\leq-\chi_p+\frac{\varepsilon}{2},\tag{5.1}$$

for $\delta>0$ small enough. This is a simplest case in the proof of Theorem A, and basically a known result. Since f is locally diffeomorphic except at the two critical points, we see from Remark 5.1 that for any open interval U that contains c_0,

$$\limsup_{n\to\infty}\frac{1}{n}\log|\{x\in X\setminus U:\delta_{a,x}^n\in\mathcal{G}_\delta\}|\leq-\chi_p+\frac{\varepsilon}{2}\tag{5.2}$$

provided $|a|$ is sufficiently small. Below we fix a small neighborhood U of c_0, and consider orbits starting from U.

Let $\rho>0$. For an integer $k\geq 0$, let

$$J_k=(q-\rho e^{-\chi_q k},q+\rho e^{-\chi_q k}).$$

We assume ρ is small enough so that $J_0\subset K$, and the following properties hold for all $k\geq 0$.

- The distortion of f^k on J_k is bounded by a constant independent of k.
- The Euclidean length of every connected component of $f^k(J_k\setminus J_{k+1})$ is bounded from below by a positive constant independent of k.
- $O_\delta(p)\cap\bigcup_{\ell=0}^k f^\ell(J_k)=\emptyset$.

Since q is a hyperbolic repelling periodic point, all these conditions can be checked by taking a C^2 linearization of f in a neighborhood of the orbit of q.

Below we suppose that $|a|$ is sufficiently small and $m\geq 1$ is the smallest integer with $f_a^m(c_0)=q$. Recall that $f_a|_K=f|_K$. The point c_0 is a non-degenerate critical point of f_a^m because its orbit does not contain c_1. For $k\geq 0$, let U_k be the connected component of $f_a^{-m}(J_k)$ that contains c_0. If $\rho>0$ is sufficiently small, there exists a constant $C_1>1$ depending on m such that $C_1^{-1}e^{-\chi_q k/2}<|U_k|<C_1e^{-\chi_q k/2}$ for $k\geq 0$. The restriction of f_a^m to each connected component of $U_k\setminus U_{k+1}$ for $k\geq 0$ is a diffeomorphism onto a connected component of $J_k\setminus J_{k+1}$ with distortion bounded uniformly over all $k\geq 0$.

From the last property of J_k, if $x\in U_k$ and $m\leq\ell\leq m+k$ then $f_a^\ell(x)\notin O_\delta(p)$. Hence, if $\delta_{a,x}^n$ belongs to $\mathcal{G}_\delta$ for some $x\in U_k$ and $n\geq 0$, we have

$$m+\#\{\ell:m+k\leq\ell\leq n-1,\ f_a^\ell(x)\in O_\delta(p)\}\geq(1-\delta)n.\tag{5.3}$$

Since the left-hand side is bounded by $n-k$, this happens only if $0\leq k\leq\delta n$.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

The length of the image $f_a^{m+k}(U_k \setminus U_{k+1}) = f_a^k(J_k)$ is bounded from below by a constant independent of k. Hence, combining (5.1) with the uniform distortion estimates for $f_a^k|_{J_k}$ and $f_a^m|_{U_k \setminus U_{k+1}}$, we conclude that there exists a constant $C_2 > 1$ depending on m such that the Lebesgue measure of the set of points in $U_k \setminus U_{k+1}$ for $0 \le k \le \delta n$ satisfying (5.3) is bounded from above by $C_2 \exp((-\chi_p + \varepsilon/2)(1-\delta)n)$ provided n is sufficiently large. Summing this bound over all integers k satisfying $0 \le k \le \delta n$ yields

$$\limsup_{n\to\infty} \frac{1}{n} \log |\{x \in U_0 : \delta_{a,x}^n \in \mathcal{G}_\delta\}| \le \left(-\chi_p + \frac{\varepsilon}{2}\right)(1-\delta) \le -\chi_p + \varepsilon.$$

The last inequality holds if $\delta > 0$ is sufficiently small. Since $\varepsilon > 0$ is arbitrary, this together with (5.2) gives the conclusion. □

5.3. *Proof of Proposition 5.2.* Let $\varepsilon > 0$. Let $\delta_* > 0$ be as in Lemma 5.4 and let $\delta, \delta' \in (0, \delta_*)$ satisfy $\delta > \delta'$. Recall that the periodic points p, q of f_0 are contained in its basic set Λ. From (F4) and (F5), for $n \ge 2$ we have

$$\frac{d}{da}(f_a^n(c_0)) = (f^{n-1})'(f_a(c_0)) \frac{d}{da}(f_a(c_0)) \ne 0 \tag{5.4}$$

provided that $|a|$ is sufficiently small and $f^\ell(f_a(c_0)) \ne c_1$ for every $0 \le \ell \le n-2$. By the transitivity of Λ, if $|a|$ is sufficiently small and $f_a^m(c_0) = p$ (respectively, $f_a^m(c_0) = q$) for some $m \ge 1$, we can find a parameter a' arbitrarily close to a such that $f_{a'}^{m'}(c_0) = q$ (respectively, $f_{a'}^{m'}(c_0) = p$) for some $m' > m$.

We construct a sequence $\{a(i)\}_{i=0}^\infty$ of parameters in $[-1, 1]$ converging to 0, a sequence $\{\Delta(i)\}_{i=0}^\infty$ of positive reals converging to 0, and strictly increasing sequences $\{m(i)\}_{i=0}^\infty$, $\{n(i)\}_{i=0}^\infty$ of positive integers with the following properties.

(a) $f_{a(i)}^{m(i)}(c_0) = p$ for i even, and $f_{a(i)}^{m(i)}(c_0) = q$ for i odd.

(b) If $a \in [-1, 1]$ and $|a - a(i)| \le \Delta(i)$ then

$$\frac{1}{n(i)} \log |\{x \in X : \delta_{a,x}^{n(i)} \in \mathcal{G}_{\delta'}\}| > -\frac{\chi_p}{2} - \frac{1}{i} \quad \text{for } i \text{ even},$$

$$\frac{1}{n(i)} \log |\{x \in X : \delta_{a,x}^{n(i)} \in \mathcal{G}_\delta\}| < -\chi_p + \varepsilon \quad \text{for } i \text{ odd}.$$

(c) For all integers i, i' with $0 \le i' < i$, $|a(i') - a(i)| < \Delta(i')$.

Then Proposition 5.2 holds for f_a with $a = \lim_{i\to\infty} a(i)$, $\mathcal{G}_1 = \mathcal{G}_{\delta'}$ and $\mathcal{G}_2 = \mathcal{G}_\delta$.

The construction is inductive. Start with $a(0) = 0$, $m(0) = n(0) = 1$ and a small number $\Delta(0) > 0$ such that $(d/da)(f_a(c_0)) \ne 0$ when $|a| < \Delta(0)$. Let $j \ge 1$ and suppose $a(i), m(i), n(i), \Delta(i)$ have been constructed for $i = 0, \dots, j-1$. As in the remark after (5.4), we take a parameter $a(j)$ close to $a(j-1)$ and an integer $m(j) > m(j-1)$ so that (a) and (c) hold for $i = j$. Further, we take a large integer $n(j) > n(j-1)$ so that one of the two conditions in (b) holds for $i = j$ and $a = a(j)$, by Lemmas 5.3 or 5.4 depending on the parity of j. Finally, we take sufficiently small $\Delta(j) > 0$ so that one of the two conditions in (b) holds for any $a \in [-1, 1]$ with $|a - a(j)| < \Delta(j)$. Iterating this construction yields the four sequences with the required properties.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

Remark 5.5. The counterexample in Theorem B can be found in polynomial maps of degree 3, by considering an appropriate one-parameter family (f_a) of polynomials of degree 3 and applying the argument parallel to the proof of Proposition 5.2 above. Below we explain the construction of the one-parameter family (f_a), leaving the details to interested readers.

First, we take a polynomial f of degree 3 so that there exist two non-degenerate critical points $c_0 < c_1$ and non-empty compact intervals K and X with $K \subsetneq X$, $c_0 \in X \setminus K$ and $c_1 \in K$, and a non-trivial hyperbolic basic set $\Lambda \subset K$, and a hyperbolic repelling periodic point $q \in \Lambda$ for which (F1), (F2), (F3) hold with $p = f(c_0)$. This is possible by the combinatorial argument on the dynamics of continuous piecewise monotone maps [**10**, Ch. II, §4–6].

Second, we consider a one-parameter family (f_a) of polynomials of degree 3 with $f_0 = f$. Clearly (F4) is not true in general, but this is not essential as we put (F4) just to make computations easier. For $a \in \mathbb{R}$ sufficiently close to 0, let $c_0(a) < c_1(a)$ denote the critical points of f_a and let $p(a)$ denote the continuation of the hyperbolic periodic point p of f_0, respectively. Instead of (F5) we require

$$\frac{d}{da}(f_a(c_0(a)) - p(a))|_{a=0} \neq 0. \tag{5.5}$$

In order to verify (5.5), we exploit the theory of Douady, Hubbard and Thurston on the dynamics of complex polynomials. Consider the two-parameter family

$$f_{\alpha,\beta}(x) = f(x) + \alpha x + \beta,$$

and regard it as a family of complex dynamical systems with complex parameters $(\alpha, \beta) \in \mathbb{C}^2$. We apply [**34**, Main Theorem 1.1] to this family, restricting (α, β) to a small neighborhood V of $(0, 0)$ in $\mathbb{C}^2$. To this end, we have to check that the family $(f_{\alpha,\beta})$ is a *normalized family* (see [**34**]), that is, no affine transformation conjugates two maps in the family $(f_{\alpha,\beta})$ with different parameters in V. This is not difficult: compute $L^{-1} \circ f_{\alpha,\beta} \circ L$ for an affine map $L(z) = Az + B$ and use the fact that the coefficients of $f_{\alpha,\beta}$ with degree 2 and 3 do not depend on the parameter (α, β). As a conclusion, we obtain that the differential of the map

$$\Psi\colon (\alpha, \beta) \in V \mapsto (f_{\alpha,\beta}(c_0(\alpha, \beta)) - p(\alpha, \beta),\ f^3_{\alpha,\beta}(c_1(\alpha, \beta)) - c_1(\alpha, \beta)) \in \mathbb{C}^2$$

at $(0, 0)$ is injective, where $c_0(\alpha, \beta)$, $c_1(\alpha, \beta)$ denote the two non-degenerate critical points of $f_{\alpha,\beta}$ and $p(\alpha, \beta)$ denotes the hyperbolic periodic point of $f_{\alpha,\beta}$ that are continuations of c_0, c_1 and p for $f_{0,0}$, respectively. Clearly one can restrict Ψ to $V \cap \mathbb{R}^2$ and see that $D\Psi_{(0,0)}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is injective. For all $a \in \mathbb{R}$ sufficiently close to 0, define $f_a = f_{\alpha(a),\beta(a)}$ by $\Psi(\alpha(a), \beta(a)) = (a, 0)$. Then (5.5) holds for the family (f_a).

For the family f_a constructed as above, we have

$$\frac{d}{da}(f_a^n(c_0(a)) - f_a^{n-1}(p(a)))\bigg|_{a=0} = (f_0^{n-1})'(f_0(c_0)) \cdot \frac{d}{da}(f_a(c_0(a)) - p(a))\bigg|_{a=0}.$$

Since $(d/da) f_a^{n-1}(p(a))|_{a=0}$ is uniformly bounded over all $n \geq 1$, (5.5) implies that $(d/da) f_a^n(c_0(a))|_{a=0}$ is comparable with $(f_0^{n-1})'(f_0(c_0))$ and grows exponentially in n.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

With this estimate and other conditions on $f = f_0$ discussed above, we can follow the argument in the proof of Proposition 5.2 to obtain the counterexample in Theorem B in the family (f_a).

Acknowledgements. We thank the referee for his or her very careful reading of the earlier versions of the paper and giving suggestions for improvements, which were indispensable for the completion of this paper. We thank Juan Rivera-Letelier for enlightening discussions throughout this project. We also thank Masayuki Asaoka and Miki U. Kobayashi for fruitful discussions. This paper was based on the authors' discussion during their visits to the CIRM Luminy (Marseille) for participation in the 'Dynamics Beyond Uniform Hyperbolicity' conference in May 2019. HT was supported by the JSPS KAKENHI 25K21999, Grant-in-Aid for Challenging Research (Exploratory). MT was supported by the JSPS KAKENHI 23K20806, Grant-in-Aid for Scientific Research (B).

References

[1] M. Benedicks and L. Carleson. On iterations of $1 - ax^2$ on $(-1, 1)$. *Ann. of Math.* **122** (1985), 1–25.
[2] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*, 2nd revised edn. Springer-Verlag, Berlin, 2008.
[3] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), 181–202.
[4] H. Bruin and G. Keller. Equilibrium states for S-unimodal maps. *Ergod. Th. & Dynam. Sys.* **18** (1998), 765–789.
[5] H. Bruin, G. Keller, T. Nowicki and S. van Strien. Wild Cantor attractors exist. *Ann. of Math.* **143** (1996), 97–130.
[6] Y. M. Chung. Large deviations on Markov towers. *Nonlinearity* **24** (2011), 1229–1252.
[7] Y. M. Chung, J. Rivera-Letelier and H. Takahasi. Large deviation principle in one-dimensional dynamics. *Invent. Math.* **218** (2019), 853–888.
[8] Y. M. Chung and H. Takahasi. Large deviation principle for S-unimodal maps with flat critical points. *J. Math. Soc. Japan* **74** (2022), 129–150.
[9] I. P. Cornfeld, S. V. Fomin and Y. G. Sinaĭ. *Ergodic Theory (Grundlehren der mathematischen Wissenschaften, 245)*. Springer-Verlag, New York, 1982.
[10] W. de Melo and S. van Strien. *One-Dimensional Dynamics* (*Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 25*). Springer-Verlag, Berlin, 1993.
[11] M. Feigenbaum. Quantitative universality for a class of nonlinear transformations. *J. Stat. Phys.* **19** (1978), 25–52.
[12] M. Feigenbaum. The universal metric properties of non-linear transformations. *J. Stat. Phys.* **21** (1979), 669–706.
[13] J. Graczyk and G. Świątek. Generic hyperbolicity in the logistic family. *Ann. of Math.* **146** (1997), 1–52.
[14] J. Guckenheimer. Sensitive dependence on initial conditions for one dimensional maps. *Commun. Math. Phys.* **70** (1979), 133–160.
[15] F. Hofbauer and G. Keller. Quadratic maps without asymptotic measure. *Commun. Math. Phys.* **127** (1990), 319–337.
[16] K. Jacobs. *Neue Methode und Ergebnisse der Ergodentheorie*. Springer-Verlag, Berlin, 1960.
[17] M. Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Commun. Math. Phys.* **81** (1981), 39–88.
[18] S. Johnson. Singular measures without restrictive intervals. *Commun. Math. Phys.* **110** (1987), 185–190.
[19] Y. Kifer. Large deviations in dynamical systems and stochastic processes. *Trans. Amer. Math. Soc.* **321** (1990), 505–524.
[20] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*, 2nd edn, Cambridge University Press, Cambridge, 2021.
[21] M. Lyubich. Non-existence of wandering intervals and structure of topological attractors of one dimensional dynamical systems: 1. The case of negative Schwarzian derivative. *Ergod. Th. & Dynam. Sys.* **9** (1989), 737–749.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press

[22] M. Lyubich. Almost every real quadratic map is either regular or stochastic. *Ann. of Math.* **156** (2002), 1–78.
[23] R. Mañé. Hyperbolicity, sinks and measure in one dimensional dynamics. *Commun. Math. Phys.* **100** (1985), 495–524.
[24] S. Orey and S. Pelikan. Deviations of trajectory averages and the defect in Pesin's formula for Anosov diffeomorphisms. *Trans. Amer. Math. Soc.* **315** (1989), 741–753.
[25] Y. Pesin and S. Senti. Equilibrium measures for maps with inducing schemes. *J. Mod. Dyn.* **3** (2008), 397–430.
[26] M. Pollicott and R. Sharp. Large deviations for intermittent maps. *Nonlinearity* **22** (2009), 2079–2092.
[27] M. Pollicott, R. Sharp and M. Yuri. Large deviations for maps with indifferent fixed points. *Nonlinearity* **11** (1998), 1173–1184.
[28] F. Rassoul-Agha and T. Seppäläinen. *A Course on Large Deviations with an Introduction to Gibbs Measures (Graduate Studies in Mathematics, 162)*. American Mathematical Society, Providence, RI, 2015.
[29] J. Rivera-Letelier. Asymptotic expansion of smooth interval maps. *Astérisque* **416** (2020), 33–63.
[30] S. Ruette. *Chaos on the Interval (University Lecture Series, 67)*. American Mathematical Society, Providence, RI, 2017.
[31] Y. G. Sinaĭ. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk.* **27** (1972), 21–64.
[32] Y. Takahashi. Entropy functional (free energy) for dynamical systems and their random perturbations. *Stochastic Analysis (Katata/Kyoto, 1982) (North-Holland Mathematical Library, 32)*. North-Holland, Amsterdam, 1984, pp. 437–467.
[33] H. Takahasi. Statistical properties of periodic points for infinitely renormalizable unimodal maps. *Nonlinearity* **35** (2022), 6399–6421.
[34] S. van Strien. Misiurewicz maps unfold generically (even if they are critically non-finite). *Fundam. Math.* **163**(1) (2000), 39–54.

https://doi.org/10.1017/etds.2026.10302 Published online by Cambridge University Press