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More variations on the Steiner-Lehmus theme

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The celebrated Steiner-Lehmus theorem states that if the internal bisectors of two angles of a triangle are equal, then the triangle is isosceles. In other words, if P is the incentre of triangle ABC, and if BP and CP meet the sides AC and AB at B' and C', respectively, then

$$BB' = CC' \Rightarrow AB = AC; \tag{1}$$

see Figure 1(a). Of course, the converse of (1) follows trivially from symmetry.

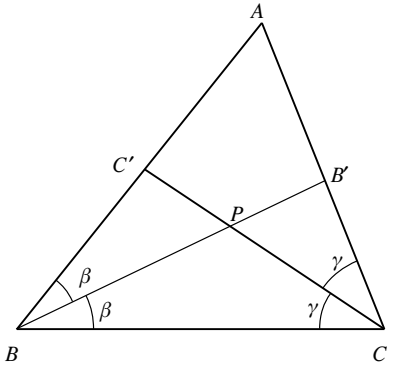


FIGURE 1(a):
Illustrating the statement of the Steiner-Lehmus theorem

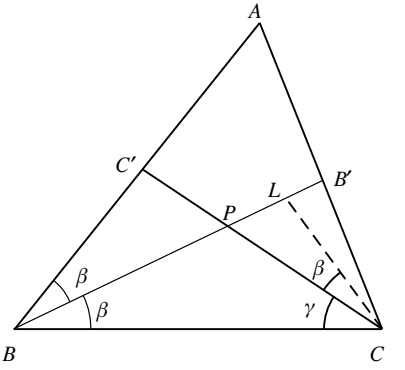


FIGURE 1(b):
Illustrating a proof of the Steiner-Lehmus theorem

Proofs of the Steiner-Lehmus theorem abound in the literature. A fairly comprehensive list of references that include most of the pre-2008 literature on the theorem can be found in [1], where a short proof of the theorem is given. Another very short proof has recently appeared in [2]. A simplified version of the proof in [2] that also proves a stronger form of the Steiner-Lehmus theorem is included in Theorem 1 below, which also borrows from

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[3]. For the convenience of the reader, we reproduce here a most elegant proof that appeared in [4] and that was reproduced in the selection [5, pp. 127-128]. In this proof, we start with Figure 1, and assume that $C > B$, i.e. $\gamma > \beta$, and we show that $BB' > CC'$. To achieve this, we take L on the line segment PB' such that $\angle PCL = \beta$. Since $\angle C'BL = \beta$, it follows that the quadrilateral $C'BCL$ is cyclic. Also

$$\angle C'BC < \angle BCL = \beta + \gamma = \frac{B + C}{2} < 90°.$$

Therefore $\angle C'BC$ and $\angle BCL$ are acute, $\angle C'BC < \angle BCL$, and they are subtended by the chords $C'C$ and BL. Therefore $CC' < BL$. Since $BL < BB'$, it follows that $CC' < BB'$, as desired.

Before stating and proving Theorem 1, we prove a lemma that will be used later in two of the proofs. The lemma was designed by R. Breusch to solve the *Monthly* problem [6], and therefore we shall refer to it as Breusch's lemma. We shall also follow Euler's notation, and denote the side lengths and the angles of a triangle XYZ by x, y, z, X, Y, Z, in the standard order. Also, the area and perimeter of XYZ will be denoted by $[XYZ]$ and $\mathrm{per}\,(XYZ)$ respectively.

Breusch's lemma: If ABC is a triangle in which $B = 2\beta$ and $C = 2\gamma$, then

$$\frac{\mathrm{per}\,(ABC)}{a} = \frac{2}{1 - \tan\beta \tan\gamma}.$$

Consequently, if ABC and $A'B'C'$ are triangles in which $a = a'$, $B = 2\beta$, $C = 2\gamma$, $B' = 2\beta'$ and $C' = 2\gamma'$, then

$$\mathrm{per}\,(ABC) > \mathrm{per}\,(A'B'C') \Leftrightarrow \tan\beta \tan\gamma > \tan\beta' \tan\gamma'.$$

Proof:

$$\frac{\mathrm{per}\,(ABC)}{a} = 1 + \frac{b + c}{a} = 1 + \frac{\sin B + \sin C}{\sin A} = 1 + \frac{\sin 2\beta + \sin 2\gamma}{\sin(2\beta + 2\gamma)}$$

$$= 1 + \frac{2\sin(\beta + \gamma)\cos(\beta - \gamma)}{2\sin(\beta + \gamma)\cos(\beta + \gamma)} = 1 + \frac{\cos\beta\cos\gamma + \sin\beta\sin\gamma}{\cos\beta\cos\gamma - \sin\beta\sin\gamma}$$

$$= \frac{2\cos\beta\cos\gamma}{\cos\beta\cos\gamma - \sin\beta\sin\gamma} = \frac{2}{1 - \tan\beta\tan\gamma},$$

as desired. The last statement follows immediately.

Theorem 1: In triangle ABC, the internal angle bisectors of B and C meet AC and AB at B' and C', respectively, as shown in Figure 1.

(a) The statement $AB = AC$ is equivalent to each of the statements

(i) $BB' = CC'$, (ii) $[CBC'] = [BCB']$, (iii) $\mathrm{per}\,(CBC') = \mathrm{per}\,(BCB')$,

(iv) $AC' = AB'$, (v) $C'B = B'C$. (2)

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(b) The statement $AB > AC$ is equivalent to each of the statements

(i) $BB' > CC'$, (ii) $[CBC'] > [BCB']$, (iii) $\operatorname{per}(CBC') < \operatorname{per}(BCB')$

(iv) $AC' > AB'$, (v) $C'B > B'C$. (3)

(c) $PB' = PC'$ if, and only if, $AB = AC$ or $A = 60°$.

Proof: We refer to Figure 1.

Clearly, if $AB = AC$, then the equalities in (2) hold by symmetry. So it is enough to show that if $AB > AC$, then the inequalities in (3) hold. The rest of (a) and (b) follows by contradiction, as explained in Note 1 below.

So we suppose that $AB > AC$, i.e. $c > b$ and $C > B$, and we prove that the inequalities in (3) hold.

Let $B = 2\beta$, $C = 2\gamma$, as shown in Figure 1.

(b)(i) The angle bisector theorem states that

$$\frac{AB'}{B'C} = \frac{c}{a}, \text{ or, equivalently, } \frac{AB'}{AC} = \frac{c}{a+c}.$$

Therefore

$$AB' = \frac{bc}{a+c}, \qquad B'C = \frac{ba}{a+c}. \tag{4}$$

Applying the law of sines to triangle ABB', and using (4), we obtain

$$\frac{BB'}{\sin A} = \frac{AB'}{\sin\beta} = \left(\frac{bc}{a+c}\right)\left(\frac{2\cos\beta}{2\sin\beta\cos\beta}\right) = \frac{1}{\frac{a}{c}+1}\,\frac{2b}{\sin B}\cos\beta.$$

Similarly

$$\frac{CC'}{\sin A} = \frac{1}{\frac{a}{b}+1}\,\frac{2c}{\sin C}\cos\gamma.$$

Since $c > b$, it follows that $\frac{a}{c} < \frac{a}{b}$ and $\cos\beta > \cos\gamma$, and therefore

$$\frac{1}{\frac{a}{c}+1}\,\frac{2b}{\sin B}\cos\beta > \frac{1}{\frac{a}{b}+1}\,\frac{2c}{\sin C}\cos\gamma.$$

Hence

$$\frac{BB'}{\sin A} > \frac{CC'}{\sin A}, \text{ i.e. } BB' > CC'.$$

Thus we have proved that if $c > b$, then $BB' > CC'$, as desired.

(b)(ii) Next we show that $\operatorname{per}(B'BC) > \operatorname{per}(C'CB)$. Letting $\beta = 2\eta$ and $\gamma = 2\zeta$, and applying Breusch's lemma, we obtain

$$\operatorname{per}(B'BC) > \operatorname{per}(C'CB) \Leftrightarrow \tan\eta\tan 2\zeta > \tan\zeta\tan 2\eta.$$

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But

$$\tan\eta\tan 2\zeta = \tan\eta\frac{2\tan\zeta}{1-\tan^2\zeta} = \frac{2\tan\eta\tan\zeta}{1-\tan^2\zeta} > \frac{2\tan\eta\tan\zeta}{1-\tan^2\eta} = \tan\zeta\tan 2\eta.$$

Therefore $\text{per}\,(B'BC) > \text{per}\,(C'CB)$.

(b)(iii) Next, we show that $[B'BC] < [C'BC]$.

$$\frac{[B'BC]}{[C'BC]} = \frac{a(B'C)\sin C}{a(C'B)\sin B} = \frac{(B'C)c}{(C'B)b}$$

$$= \left(\frac{B'C}{b}\right)\left(\frac{c}{C'B}\right) = \left(\frac{a}{a+c}\right)\left(\frac{a+b}{a}\right) \text{ (by (4))}$$

$$= \frac{a+b}{a+c} < 1.$$

Therefore $[B'BC] < [C'BC]$.

Finally, (iv) and (v) follow from (4) and the analogous identities

$$AC' = \frac{bc}{a+b}, \qquad C'B = \frac{ca}{a+b}.$$

Thus if $c > b$, then the inequalities in (3) hold. It remains to prove (c). For this, we join AP and use the fact that the internal angle bisectors of a triangle are concurrent to conclude that AP bisects A, i.e. $\angle PAB' = \angle PAC' = \alpha$, say. Applying the sine law to triangles PAB' and PAC', we obtain

$$\frac{PB'}{\sin\alpha} = \frac{PA}{\sin(2\gamma+\beta)}, \qquad \frac{PC'}{\sin\alpha} = \frac{PA}{\sin(2\beta+\gamma)}.$$

Therefore

$$PB' = PC' \Leftrightarrow \sin(2\gamma+\beta) = \sin(2\beta+\gamma)$$

$$\Leftrightarrow (2\gamma+\beta) = (2\beta+\gamma), \text{ in which case } \beta=\gamma \text{ and } B=C,$$

$$\text{or } (2\gamma+\beta)+(2\beta+\gamma) = 180°, \text{ in which case } \beta+\gamma = 60°,$$

$$\text{and hence } A = 60°.$$

This proves (c) and completes the proof of the theorem.

Note 1: In the proof of Theorem 1 above, we have shown that if $AB > AC$, then the inequalities in (2) hold, and we remarked that the converse follows by contradiction. To illustrate this, notice that an implication such as

$$AB > AC \Rightarrow BB' > CC' \tag{5}$$

does indeed yield the converse implication

$$BB' > CC' \Rightarrow AB > AC. \tag{6}$$

For if $BB' > CC'$, then AB can neither be equal to AC (because this would

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imply that $BB' = CC'$ by symmetry), nor less than AC (because this would imply that $BB' < CC'$ by (5)). Thus (5) yields (6). Similarly (5) implies that if $BB' = CC'$, then $AB = AC$.

This argument will be used freely in the proofs of the remaining theorems.

Several variations on the Steiner-Lehmus theme can also be considered. For example, one may take B' and C' to be the points where the angle bisectors BP and CP meet the external angle bisectors of C and B respectively; see Figure 2. One may also take B' and C' to be the points where BP and CP meet the line drawn from A parallel to BC; see Figure 3. One may also take B' and C' to be the points where BP and CP meet the circumcircle of ABC; see Figure 4. These three configurations are treated in Theorems 2, 3 and 4 respectively.

Theorem 2: In triangle ABC, the internal angle bisectors of B and C meet the external angle bisectors of C and B at B' and C' respectively, as shown in Figure 2 below.

(a) The statement $AB = AC$ is equivalent to each of the statements

(i) $CC' = BB'$, (ii) $[CBC'] = [BCB']$, (iii) $\mathrm{per}(CBC') = \mathrm{per}(BCB')$,
(iv) $BC' = CB'$, (v) $[FAC'] = [EAB']$, (vi) $\mathrm{per}(FAC') = \mathrm{per}(EAB')$.

(b) The statement $AB > AC$ is equivalent to each of the statements

(i) $CC' > BB'$, (ii) $[CBC'] > [BCB']$, (iii) $\mathrm{per}(CBC') > \mathrm{per}(BCB')$,
(iv) $BC' > CB'$, (v) $[FAC'] > [EAB']$, (vi) $\mathrm{per}(FAC') > \mathrm{per}(EAB')$. (7)

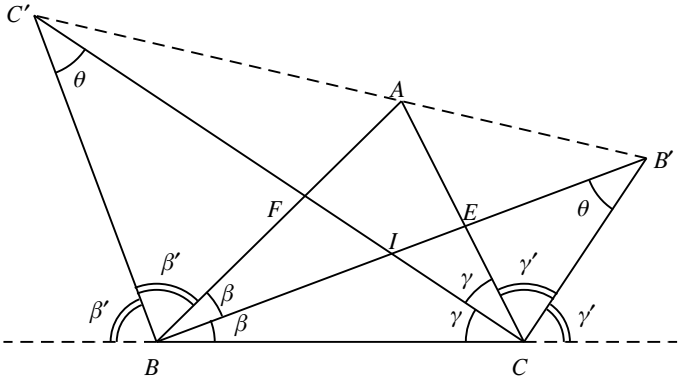


FIGURE 2: Illustrating Theorem 2

Proof: In view of Note 1, it is enough to prove that if $AB > AC$ then the inequalities in (7) hold. Notice also that the line $B'C'$ in Figure 2 passes through A because B' is the B-excentre and C' is the C-excentre of ABC.

So we suppose that $AB > AC$, i.e. $C > B$, and we are to prove that the inequalities in (7) hold. Let $B = 2\beta$, $C = 2\gamma$, $180° - B = 2\beta'$, $180° - C = 2\gamma'$,

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$\theta_1 = \angle CB'B$, $\theta_2 = \angle CC'B$, as shown in Figure 4. Since

$$\beta + \beta' = \gamma + \gamma' = 90°,$$

it follows that the quadrilateral $BCB'C'$ is cyclic. Hence $\theta_1 = \theta_2 = \theta$, say. Applying the law of sines to triangles BCB' and BCC', we see that $BC / \sin\theta$ is equal to each of the quantities

$$\frac{BB'}{\sin(\gamma + 90°)}, \qquad \frac{CB'}{\sin\beta}, \qquad \frac{CC'}{\sin(\beta + 90°)}, \qquad \frac{C'B}{\sin\gamma},$$

and therefore

$$\frac{CC'}{\cos\beta} = \frac{B'B}{\cos\gamma}, \qquad \frac{CB'}{\sin\beta} = \frac{C'B}{\sin\gamma}. \tag{8}$$

Since $0 < \beta < \gamma < 90°$, it follows that $\cos\beta > \cos\gamma > 0$ and $\sin\gamma > \sin\beta > 0$. By (8), we conclude that

$$C'C > B'B, \qquad C'B > B'C. \tag{9}$$

Therefore

$$[C'CB] = \frac{(C'C)(CB)\sin\gamma}{2} > \frac{(B'B)(BC)\sin\beta}{2} = [B'BC], \tag{10}$$

$$\operatorname{per}(C'CB) = C'C + CB + BC' > B'B + BC + CB' = \operatorname{per}(B'BC). \tag{11}$$

Now (b)(i), (b)(ii), (b)(iii), and (b)(iv) follow from (9), (10), and (11).

To prove (b)(v) and (b)(vi), we first apply the exterior angle theorem to triangles BAB' and CAC' to obtain

$$\angle BAC' = \angle CAB' = \beta + \gamma.$$

Then we show that

$$FC' > EB' \ldots (12a), \qquad FA > EA \ldots (12b), \qquad AC' > AB' \ldots (12c).$$

These would imply that $\operatorname{per}(FAC') > \operatorname{per}(EAB')$, and also

$$\frac{[FAC']}{[EAB']} = \frac{(FA)(AC')}{(EA)(AB')} > 1,$$

and hence (b)(v) and (b)(vi).

By (b)(i), we have $CC' > BB'$. By Theorem 1, we have $CF < BE$. Subtracting, we get (12a).

By the angle bisector theorem, we have

$$FA = \frac{bc}{b + a} > \frac{bc}{c + a} = EA.$$

and therefore $FA > EA$, thus proving (12b).

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Finally, applying the sine law to triangles BAC' and CAB', we obtain

$$\frac{C'A}{\sin\beta'} = \frac{BC'}{\sin(\beta + \gamma)}, \qquad \frac{B'A}{\sin\gamma'} = \frac{CB'}{\sin(\beta + \gamma)},$$

and therefore

$$\frac{C'A}{B'A} = \left(\frac{BC'}{CB'}\right)\left(\frac{\sin\beta'}{\sin\gamma'}\right).$$

By (b)(iv), $CB' > BC'$. Also, $\sin\gamma' < \sin\beta'$, because $\gamma' < \beta' < 90°$. Therefore $C'A > B'A$, and hence (12c).

This completes the proof.

Theorem 3: In triangle ABC, the internal angle bisectors of B and C meet the line drawn from A parallel to BC at B' and C' respectively, as shown in Figure 3 below.

(a) The statement $AB = AC$ is equivalent to each of the statements

(i) $BB' = CC'$, (ii) $EB' = FC'$,

(iii) $[EAB'] = [FAC']$, (iv) $\text{per}(EAB') = \text{per}(FAC')$.

(b) The statement $AB > AC$ is equivalent to each of the statements

(i) $BB' > CC'$, (ii) $EB' > FC'$,

(iii) $[EAB'] > [FAC']$, (iv) $\text{per}(EAB') > \text{per}(FAC')$. (13)

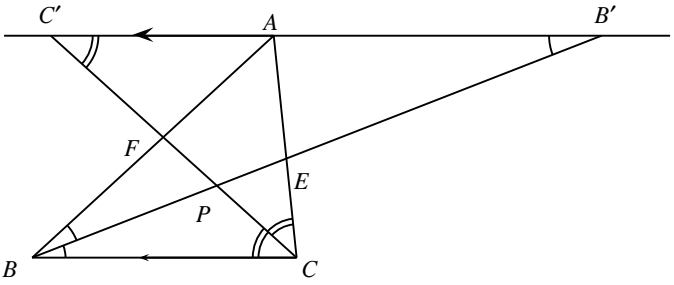


FIGURE 3: Illustrating Theorem 3

Proof: Again, it is enough, by Note 1, to prove that if $AB > AC$ then the inequalities in (13) hold. So we suppose that $AB > AC$, and thus $C > B$.

We shall use the obvious facts that $AB' = AB = c$ and $AC' = AC = b$. Then triangles $AB'E$ and CBE are similar with similarity factor c/a, and triangles $AC'F$ and BCF are similar with similarity factor b/a. Letting $u = BE$ and $v = CF$, we know from Theorem 1 that

$$BE > CF, \qquad \text{per}(EBC) > \text{per}(FBC), \qquad [EBC] < [FBC].$$

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Then

$$\frac{EB'}{EB} = \frac{c}{a} > \frac{b}{a} = \frac{FC'}{FC}.$$

Since $EB > FC$, it follows that $EB' > FC'$, and hence $BB' > CC'$. This proves (b)(i). To prove (b)(ii), we use the angle bisector theorem and the sine law for areas to obtain

$$CE = \frac{ab}{a+c} \text{ and } [BCE] = \left(\frac{a \sin C}{2}\right)\left(\frac{ab}{a+c}\right) = \frac{a^2 b \sin C}{2(a+c)}.$$

Since

$$\frac{[AB'E]}{[BCE]} = \frac{c^2}{a^2},$$

it follows that

$$[AB'E] = \frac{c^2}{a^2}\frac{a^2 b \sin C}{2(a+c)} = \frac{bc^2 \sin C}{2(a+c)}.$$

A similar formula holds for $[AC'F]$. Therefore

$$\frac{[AB'E]}{[AC'F]} = \frac{bc^2 \sin C\,(a+b)}{b^2 c \sin B\,(a+c)} = \frac{c^2(a+b)}{b^2(a+c)}.$$

This is greater than 1, because

$$c^2(a+b) - b^2(a+c) = a\left(c^2 - b^2\right) + cb(c-b) > 0.$$

This proves (b)(ii).

Finally,

$$\frac{\text{per}\,(AB'E)}{\text{per}\,(BCE)} = \frac{c}{a} > \frac{b}{a} = \frac{\text{per}\,(AC'F)}{\text{per}\,(BCF)}.$$

Since $\text{per}\,(BCE) > \text{per}\,(BCF)$, it follows that $\text{per}\,(AB'E) > \text{per}\,(AC'F)$. This proves (b)(iii), and the proof is complete.

Note 2: In Theorem 3, the implication $EB' = FC' \Rightarrow AB = AC$ was proved in [7], but the proof there is more involved than the one we have just presented. The implication $[EAB'] = [FAC'] \Rightarrow AB = AC$ appears in [8].

It is also interesting to note that certain steps in the proof of Theorem 2 use the Steiner-Lehmus theorem.

Theorem 4: In triangle ABC, the internal angle bisectors of B and C meet AC and AB at E and F, and meet the circumcircle of ABC at B' and C' respectively, as shown in Figure 4 below.

(a) (i) $BB' = CC'$ if, and only if, $AB = AC$ or $A = 60°$,
(ii) If $A < 60°$, then $AB > AC$ if, and only if, $BB' < CC'$,
(iii) If $A > 60°$, then $AB > AC$ if, and only if, $BB' > CC'$.

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(b) The statement $AB = AC$ is equivalent to each of the statements

(i) $C'B = B'C$, (ii) $[C'BC] = [B'BC]$, (iii) $\mathrm{per}(C'BC) = \mathrm{per}(B'BC)$,

(iv) $[FAC'] = [EAB']$, (v) $\mathrm{per}(FAC') = \mathrm{per}(EAB')$.

(c) The statement $AB > AC$ is equivalent to each of the statements

(i) $C'B > B'C$, (ii) $[C'BC] > [B'BC]$, (iii) $\mathrm{per}(C'BC) > \mathrm{per}(B'BC)$,

(iv) $[FAC'] > [EAB']$, (v) $\mathrm{per}(FAC') > \mathrm{per}(EAB')$. (14)

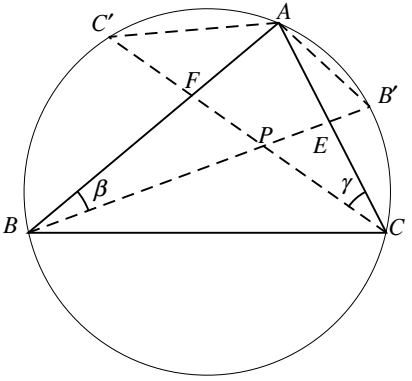


FIGURE 4: Illustrating Theorem 4

Proof: To prove (a)(i), we use the fact that if U and V are angles in a circle, then the chords which subtend them are equal in length if, and only if, $U = V$ or $U + V = 180°$. Thus

$$\begin{aligned} BB' = CC' &\Leftrightarrow (\angle BAB' = \angle CAC') \text{ or } (\angle BAB' + \angle CAC' = 180°) \\ &\Leftrightarrow (A + \beta = A + \gamma) \text{ or } (A + \beta + A + \gamma = 180°) \\ &\Leftrightarrow (\beta = \gamma) \text{ or } (2A + (90° - \tfrac{1}{2}A) = 180°) \\ &\Leftrightarrow (\beta = \gamma) \text{ or } (\tfrac{3}{2}A = 90°) \\ &\Leftrightarrow (B = C) \text{ or } (A = 60°). \end{aligned}$$

To prove (a)(ii) or (a)(iii), we use the fact that if U and V are peripheral angles in a circle with chords of length u and v, and if $U + V \leqslant 180°$, then $U < V$ if, and only if, $u < v$. Thus if $A < 60°$, then

$$\angle BAB' + \angle CAC' = 2A + \beta + \gamma = \tfrac{3}{2}A + 90° < 180°,$$

and therefore

$$BB' < CC' \Leftrightarrow \angle BAB' < \angle CAC' \Leftrightarrow A + \beta < A + \gamma \Leftrightarrow B < C \Leftrightarrow AB > AC.$$

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Similarly, if $A > 60°$ then

$$\angle BCB' + \angle CBC' < 180°,$$

and therefore

$$BB' < CC' \Leftrightarrow AB < AC.$$

To prove (b) and (c), it is enough, by Note 1, to prove that if $AB > AC$, then the inequalities in (14) hold. So we assume that $AB > AC$.

To prove (c)(i), notice that the chords $C'B$ and CB' subtend the acute angles γ and β respectively, and that $\gamma > \beta$. Therefore $C'B > B'C$.

For (c)(ii), we have

$$\begin{aligned}\frac{[B'BC]}{[C'BC]} &= \frac{a(B'C)\sin(C+\beta)}{a(C'B)\sin(B+\gamma)} = \left(\frac{B'C}{C'B}\right)\left(\frac{\sin(C+\beta)}{\sin(B+\gamma)}\right)\\ &= \left(\frac{\sin\beta}{\sin\gamma}\right)\left(\frac{\sin(C+\beta)}{\sin(B+\gamma)}\right) = \frac{\sin\beta\sin(C+\beta)}{\sin\gamma\sin(B+\gamma)}\\ &= \frac{\cos C - \cos(C+B)}{\cos B - \cos(C+B)} = \frac{\cos C + \cos A}{\cos B + \cos A} < 1.\end{aligned}$$

Therefore $[B'BC'] < [C'BC]$.

For (c)(iii), let $B = 2\beta = 4\eta$ and $C = 2\gamma = 4\zeta$. Then it follows from Breusch's lemma that

$$\operatorname{per}(B'BC) < \operatorname{per}(C'CB) \Leftrightarrow \tan\eta\tan(\eta+2\zeta) < \tan\zeta\tan(\eta+2\zeta).$$

But

$$\begin{aligned}\tan\eta\tan(\eta+2\zeta) &= \frac{\sin\eta\sin(\eta+2\zeta)}{\cos\eta\cos(\eta+2\zeta)} = \frac{\cos(2\zeta)-\cos(2\eta+2\zeta)}{\cos(2\zeta)+\cos(2\eta+2\zeta)}\\ &= \frac{t-1}{t+1}, \text{ where } t = \frac{\cos(2\zeta)}{\cos(2\eta+2\zeta)}\\ &< \frac{s-1}{s+1}, \text{ where } s = \frac{\cos(2\eta)}{\cos(2\eta+2\zeta)}\\ &= \tan\zeta\tan(\zeta+2\eta).\end{aligned}$$

Therefore $\operatorname{per}(B'BC) < \operatorname{per}(C'CB)$.

It remains to prove (c)(iv) and (c)(v). For these, notice that

$$\angle FAC' = \gamma > \beta = \angle EAB', \quad \angle C'FA = A + \gamma > A + \beta = \angle B'EA.$$

Hence there exists a point Q inside triangle $AC'F$ for which triangles AQF and $AB'E$ are similar with scale factor $\dfrac{AF}{AE}$, which is greater than 1 by Theorem 1(a)(iv). Therefore if f stands for the area or the perimeter, then

$$f(AB'E) < f(AQF) < f(AC'F),$$

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as desired. Notice that the inequality per $(AQF) < \text{per}\,(AC'F)$ follows from Proposition 21 of Book I of Euclid's *Elements* (or more fancifully, from Breusch's lemma).

This completes the proof.

We end this Article by adding that other variations on the Steiner-Lehmus theme can be obtained by taking P in the configurations above to be, not the incentre, but any other centre, such as the circumcentre, the orthocentre, the centroid, the Fermat-Torricelli centre, the Gergonne centre, or the Nagel centre. By doing so, and by asking questions similar to those treated in the theorems above, one obtains hundreds of problems that are expected to have a range of difficulty wide enough to accommodate quizzes, examinations, contests and competitions, and problem proposals for journals with problem corners. Several of such problems have already appeared in the literature.

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