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PAPERS

Optical inversion using plasmonic contrast agents

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Abstract

We describe a new method to reconstruct the permittivity distribution, of an object to image, from the remotely measured electromagnetic field. We propose to use the remote fields measured before and after injecting locally in the medium plasmonic nanoparticles. Such a technique is known in the framework of imaging using contrast agents where, in optical imaging, the nanoparticles play the role of these contrast agents. The plasmonic nanoparticles are known to enjoy resonant effects, as enhancing the applied incident field, while excited at certain particular frequencies called plasmonic resonances. These resonant frequencies encode the values of the unknown permittivity at the location of the injected nanoparticles. The imaging methods we propose mainly use this resonant effect. We show that the imaging functional build up from contrasting the fields before and after injecting the nanoparticles, measured at one single back-scattered direction, and in an explicit band of incident frequencies, reaches its maximum values, in terms of the incident frequency, precisely at the mentioned plasmonic resonances. Such a behaviour allows us to recover these plasmonic resonances from which we recover the point-wise values of the permittivity distribution. In this work, we describe the method and provide the mathematical justification of this resonant effect and its use for the optical inversion using plasmonic nanoparticles as contrast agents.

1. Introduction and statement of the results

In quantitative optical imaging, the goal is to estimate the optical properties, as the permittivity function, from the optical signal response of the object to image after excitation with coherent or incoherent incident electromagnetic fields, see [3, 10, 32, 35, 56]. The optical response is measured at receivers located away from the object to image. Such inversion requires multiple emitters-receivers, i.e. many measurements, and it is known to be highly unstable [39] and hence one is unable to recover low contrasting permittivity functions using such remote measurements. To overcome such difficulties, it is proposed in the engineering literature, see [26, 36–38, 43, 50–52], to perturb the medium with small-scaled inhomogeneities to create such missing contrast to render the imaging more accurate. In optics, such contrast agents are given by nanoparticles which enjoy appropriate contrasting properties. We have two classes of such nanoparticles: dielectric and plasmonic nanoparticles. The dielectric nanoparticles are highly localized as they are nano-scaled and have high contrast permittivity. Under these scales, we can choose the incident frequency so that we excite the dielectric resonances, which are related to the eigenvalues of the vector Newtonian operator, see [5, 13] for a mathematical justification. Moreover, in practice, optically induced waves by dielectric nanoparticles have been examined and extensively studied, see for instance [42, 57, 59, 60]. The main feature of the plasmonic nanoparticles is that they enjoy negative values of the real part of their permittivity if we choose incident frequencies close to the plasmonic frequencies of the nanoparticle. With such negative permittivity, we can excite the plasmonic

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resonances, which are related to the eigenvalues of the Magnetization operator, see [4, 6, 25]. These plasmonic nanoparticles show an ability to manipulate light at the nano-scale size due to their support for these resonant optical modes, see [11, 14, 20, 34, 40, 44, 45, 47, 58] for an extensive description and studies of their properties. Such unique properties are the reasons behind the increase of interest in the study of plasmonic nanoparticles, in particular those made of gold or silver, as demonstrated in [47]. These optical properties of plasmonic nanoparticles have been utilized, in the engineering community, in numerous applications, including bio-sensors, thermo-therapy and solar cells, etc. To learn more on these two types of nanoparticles about their common as well as different properties, the interested reader can see the references [42, 59, 60]. At the mathematical side, let us mention the works [7, 8], where a method for reconstructing the shapes of molecules using the plasmonic resonances generated by the mixture of molecules and nanoparticles, as measurements, was introduced.

The present work is a continuation of our efforts in proposing an original framework to mathematically modelling and analysing imaging modalities using contrast agents. These contrast agents can be bubbles, nanoparticles, elastic inclusions, etc, and the imaging modalities could be the ultrasound imaging applying bubbles, optical and photo-acoustic imaging with nanoparticles and elastography using inclusions, to cite a few. In the subsequent discussions, we provide an overview of our results in this direction and we referee the interested reader to [27] for more details. Here, we mainly focus on optical imaging using plasmonic nanoparticles. To describe the material properties of these nanoparticles, we use the Lorentz model where the permeability μ is kept constant as the one of the homogeneous background while the permittivity has the form:

$$\epsilon_p(\omega) = \epsilon_\infty \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right) \tag{1.1}$$

where ω_p is the electric plasma frequency, ω_0 is the undamped frequency and γ is the electric damping frequency, which is assumed to be small, i.e., $0 \leq \gamma \ll 1$, and its order of smallness will be discussed later, see (3.37). We refer [2, Formula (4)] or [19, Formula (1.3)] for the derivation of this model. It is observed that if we choose the incident frequency ω so that ω^2 is larger than ω_0^2, then the real part becomes negative. For such choices of the incident frequency, the nanoparticle behaves as a plasmonic nanoparticle. In the current work, we focus on the use of these properties for quantitative imaging. As mentioned before, these nanoparticles are used as contrast agents. Few scenarios and assumptions on the distribution of the contrast agents can be considered. These scenarios are of course not exhaustive as the reality might be more complicated. Nevertheless, we state the following classes of distributions under which we can perform rigorous analysis.

(1) The contrast agents are injected in isolation, i.e. they are injected one after another. In addition, we need to assume them well separated to insure weak multiple scattering between them.

(2) They are injected as a cluster, i.e. all-at-once. In this case, we have full strong multiple scattering between them. Here also, we can handle the following situations.
   (a) The distribution is regular, i.e. periodic or following a given density of distribution. In this case, we can expect deterministic estimations.
   (b) The distribution is random. In this case, we aim for probabilistic estimations.

The approach we propose to analyse these families of imaging modalities using contrast agents can be summarized as follows. Contrasting the measured fields collected before and after injecting the small agents, we propose the following solutions.

(1) Case when we inject the contrast agents one after another, under the weak scattering assumption.
   (a) We can recover the *related resonances*. From these resonances, we derive the values of the 'high order' coefficient (i.e. the mass density in acoustics). This idea works in the time-harmonic regimes.
   (b) We can recover the internal values of *the travel time function*. Using the Eikonal equation, we recover the speed of propagation. This idea works in time-domain regimes.

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(c) In addition, we can recover the internal values of the *total fields* generated only by the background medium. This allows to recover the lower order coefficients (as the bulk modulus in acoustics). This idea works for both the time-harmonic and the time-domain regimes.

These ideas were applied to the time-harmonic as well as the time domain imaging for acoustics, i.e. ultrasound imaging using bubble as contrast agents, see [30, 54, 55].

(2) Case when we inject the contrast agents *all-at-once*, as a cluster.

(a) using the resonant character of the contrast agents, we can *linearize* the measured boundary-map, i.e. the Dirichlet–Neumann map.

(b) then we solve the linearized inverse problem.

This idea was applied in [28] to the Calderón problem using resonant perturbations modelling the acoustic imaging with droplets as contrast agents.

So far, this approach was applied to acoustic waves based imaging, i.e. for the ultrasound imaging modality using bubbles or droplets as contrast agents and also photo-acoustic imaging using nanoparticles, see [25, 29].

Our goal in the current work is to extend this approach to the electromagnetic waves based imaging as the case of quantitative optical imaging using nanoparticles. To describe the mathematical model behind the optical experiment, we set the electric field $E(\cdot)$ to be solution of the following system

$$\begin{cases} \nabla \times \nabla \times (E) - \omega^2\, \varepsilon\, \mu\, E = 0, & \text{in } \mathbb{R}^3, \\ E := E^s + E^i, & \text{in } \mathbb{R}^3, \\ \lim\limits_{|x|\to+\infty} |x|\, (\nabla \times (E^s(x)) \times x - E^s(x)) = 0, \end{cases} \tag{1.2}$$

with the last condition known as the Silver–Müller radiation conditions. Here, ω is the incident frequency, μ is the permeability parameter, which will be taken to be constant in the whole space $\mathbb{R}^3$, and $\varepsilon(\cdot)$ is the permittivity function defined as

$$\varepsilon(x) := \begin{cases} \epsilon_\infty & \text{in} \quad \mathbb{R}^3 \setminus \Omega, \\ \epsilon_0(x) & \text{in} \quad \Omega \setminus D, \\ \epsilon_p(\omega) & \text{in} \quad D, \end{cases} \tag{1.3}$$

with ϵ_∞ being a positive constant used to represent the permittivity of the background (outside Ω), $\epsilon_p(\omega)$ is given by (1.1) and the permittivity $\epsilon_0(\cdot)$ is variable and it is supposed to be smooth of class $\mathcal{C}^1$ inside Ω. Besides, we assume that $Im(\epsilon_0(\cdot))$ is small such that

$$\|Im(\epsilon_0(\cdot))\|_{\mathbb{L}^\infty(\Omega)} = \mathcal{O}(\gamma), \tag{1.4}$$

where γ is the electric damping frequency parameter of the Lorentz model, see (1.1). The domain D is given as a collection of $\aleph$ connected and $\mathcal{C}^2$-smooth nanoparticles D_i's, i.e. $D := \bigcup_{i=1}^{\aleph} D_i$. In addition $D \subset \Omega$, where Ω is a bounded and $\mathcal{C}^2$-smooth domain in $\mathbb{R}^3$. Related to the permittivity function given by (1.3), we set the index of refraction $\boldsymbol{n}$, in $\mathbb{R}^3$, given by

$$\boldsymbol{n} := \begin{cases} \sqrt{\epsilon_p\, \mu} & \text{in } D \\ \boldsymbol{n}_0 & \text{in } \mathbb{R}^3 \setminus D \end{cases} \qquad \text{and} \qquad \boldsymbol{n}_0 := \begin{cases} \sqrt{\epsilon_0(\cdot)\, \mu} & \text{in } \Omega \\ \sqrt{\epsilon_\infty\, \mu} & \text{in } \mathbb{R}^3 \setminus \Omega \end{cases}. \tag{1.5}$$

Each nanoparticle D_j, for $1 \le j \le \aleph$, is taken of the form $D_j := a\, B_j + z_j$ where z_j models its location and a signifies its relative radius with B_j as $\mathcal{C}^2$-smooth domain of maximum radius 1 such that $B_j \subset B(0, 1)$, where $B(0, 1)$ is the unit ball centred at the origin. The parameter a is defined by

$$a := \max_{1\le j\le \aleph} \operatorname{diam}(D_j),$$

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and we denote d as the minimal distance between any two of the distributed nanoparticles, i.e.

$$d = \min_{\substack{1\le i,j\le \aleph \\ i\neq j}} \operatorname{dist}(D_i, D_j).$$

The parameters d and $\aleph$ are linked to the parameter a through the following behaviours

$$d \sim a^t \quad \text{and} \quad \aleph \sim [a^{-s}], \quad \text{with} \quad 0 < t, s < 1,$$

where $[\,\cdot\,]$ stands for the entire part function. Moreover, for short notations, we denote d_{mj} as the distance between z_m and z_j, where z_m (respectively, z_j) is the centre of D_m (respectively, D_j), i.e.

$$d_{mj} := |z_m - z_j|, \quad \text{with} \quad 1 \le m, j \le \aleph, \quad m \neq j.$$

In the sequel, without losing generalities, we assume that

$$B_j \equiv B \subset B(0,1), \qquad \text{for} \quad 1 \le j \le \aleph.$$

The problem (1.2) is well-posed in appropriate Sobolev spaces, see, for example, [41, Theorem 2.1]. We assume that the incident field $E^{Inc}(\cdot,\cdot,\cdot)$ is given by

$$E^{Inc}(x,\theta,q) = (\theta \times q)\, e^{i\,k\,x\cdot\theta}, \quad \text{for} \quad \theta, q \in \mathbb{S}^2 \quad \text{and} \quad x \in \mathbb{R}^3, \tag{1.6}$$

with $\mathbb{S}^2$ being the unit sphere, θ is the incident direction vector and q is the polarization vector such that $q\cdot\theta = 0$. The incident field $E^{Inc}(\cdot,\cdot,\cdot)$ is the solution to

$$\nabla_x \times \nabla_x \times (E^{Inc}(x,\theta,q)) - k^2\, E^{Inc}(x,\theta,q) = 0, \quad x \in \mathbb{R}^3,$$

where the wave number k is given by the positive constant $k = \omega\sqrt{\epsilon_\infty\,\mu}$. Besides, the scattered wave $E^s(\cdot,\cdot,\cdot)$ has the following asymptotic behaviour,

$$E^s(\hat{x},\theta,q) = \frac{e^{i\,k\,|x|}}{|x|}\left(E^\infty(\hat{x},\theta,q) + \mathcal{O}\left(\frac{1}{|x|}\right)\right), \quad |x| \to +\infty,$$

where $E^\infty(\hat{x},\theta,q)$ is the corresponding electromagnetic far-field pattern of (1.2) in the propagation direction $\hat{x}$.

Motivated by the use of integral equations to represent solutions to Maxwell's equations, and for future use, we introduce the Newtonian operator $N_B^k(\cdot)$ and the Magnetization operator $\nabla M_B^k(\cdot)$, both acting on vector fields

$$N_B^k(F)(x) := \int_B \Phi_k(x,y)\, F(y)dy \quad \text{and} \quad \nabla M_B^k(F)(x) := \nabla \int_B \nabla_y \Phi_k(x,y)\cdot F(y)dy, \tag{1.7}$$

where

$$\Phi_k(x,y) := \frac{e^{i\,k\,|x-y|}}{4\pi\,|x-y|}, \quad x \neq y, \tag{1.8}$$

is the fundamental solution to the Helmholtz equation in the entire space. Particularly, for $k = 0$, we obtain

$$N_B(F)(x) := \int_B \frac{1}{4\pi\,|x-y|}F(y)dy \quad \text{and} \quad \nabla M_B(F)(x) := \nabla \int_B \nabla_y \left(\frac{1}{4\pi\,|x-y|}\right)\cdot F(y)dy. \tag{1.9}$$

We will observe later that the introduced operators, see (1.7), appear after taking convolution of vector fields with the Green's kernel $G_k(\cdot,\cdot)$ associated with the problem (1.2). More precisely, $G_k(\cdot,\cdot)$ is the solution, in the distributional sense, to

$$\nabla_y \times \nabla_y \times G_k(x,y) - \omega^2\, \boldsymbol{n}_0^2(y)\, G_k(x,y) = \delta(x,y)\, I, \quad x, y \in \mathbb{R}^3, \tag{1.10}$$

such that each column of $G_k(x,\cdot)$ satisfies the outgoing radiation condition

$$\lim_{|x|\to+\infty} |x| \left(\nabla_y \times G_k(x,y) \times \frac{x}{|x|} - i\,k\,G_k(x,y)\right) = 0.$$

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When dealing with vector fields, it is crucial to recall the Helmholtz decomposition for $\mathbb{L}^2(B)$-space given by

$$(\mathbb{L}^2(B))^3 = \mathbb{H}_0(\text{div } = 0)(B) \oplus \mathbb{H}_0(Curl = 0)(B) \oplus \nabla\mathcal{H}armonic(B) \tag{1.11}$$

where

$$\mathbb{H}_0(\text{div } = 0)(B) := \{u \in (\mathbb{L}^2(B))^3 \big| \nabla \cdot (u) = 0, \nu \cdot u = 0 \text{ on } \partial B\},$$
$$\mathbb{H}_0(Curl = 0)(B) := \{u \in (\mathbb{L}^2(B))^3 \big| \nabla \times u = 0, \nu \times u = 0 \text{ on } \partial B\}$$

and

$$\nabla\mathcal{H}armonic(B) := \{u \in (\mathbb{L}^2(B))^3 \big| u = \nabla\phi, \ \ \Delta\phi = 0 \text{ in } B\},$$

see for instance [17, Chapter IX, Table I, Page 314].

Of all possible $(\mathbb{L}^2(B))^3$-space decompositions, (1.11) is the most natural one, as we know that $N|_{\mathbb{H}_0(\text{div}=0)(B)}$ and $N|_{\mathbb{H}_0(Curl=0)(B)}$ generate a complete orthonormal bases $\left(\lambda_n^{(1)}(B); e_n^{(1)}\right)_{n\in\mathbb{N}}$ and $\left(\lambda_n^{(2)}(B); e_n^{(2)}\right)_{n\in\mathbb{N}}$, respectively. In addition, it is known that ∇M: $\nabla\mathcal{H}armonic(B) \to \nabla\mathcal{H}armonic(B)$ has a complete basis $\left(\lambda_n^{(3)}(B); e_n^{(3)}\right)_{n\in\mathbb{N}}$, see [25, Proposition 5.1]. For an in-depth study of the properties of the Magnetization operator, the reader can refer to [1, 18, 21–23] and [53].

In the sequel, we fix an $n_0 \in \mathbb{N}$, and we let the used incident frequency ω to be of the form

$$\omega^2 = \omega^2_{P_\ell, n_0 j} \pm C\, a^h, \quad h \in (0,1) \quad \text{and} \quad 1 \le j \le \aleph,$$

where $\omega^2_{P_\ell,n_0 j}$ is the plasmonic resonance related to the eigenvalue $\lambda_{n_0}^{(3)}(B)$, see (3.36), and C is a constant independent on the parameter a.

We are now in a position to state the primary outcome of this work.

Theorem 1.1. *Under the regularity assumptions on $\Omega, D, \epsilon_0(\cdot)$ and μ described above, we have the following expansions.*

(1) *For the scattered fields, with x is away from D,*

$$\begin{aligned}
&(E^s - V^s)(x,\theta) \\
&= -\mu\, a^3\, \omega^2 \sum_{j=1}^{\aleph} \frac{\epsilon_0(z_j)\left(\epsilon_0(z_j) - \Lambda_{n_0,j}(\omega)\right)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} \left\langle V(z_j,\theta,q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle G_k(x,z_j) \cdot \int_B e_{n_0}^{(3)}(y)\, dy \\
&\quad + \mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right).
\end{aligned} \tag{1.12}$$

(2) *For the far fields*

$$\begin{aligned}
&\langle (E^\infty - V^\infty)(\hat{x}), (\hat{x} \times q)\rangle \\
&= \frac{\mu\, a^3}{4\pi} \sum_{j=1}^{\aleph} \frac{\omega^2_{P_\ell,n_0 j}\, \epsilon_0(z_j)\left(\epsilon_0(z_j) - \Lambda_{n_0,j}\left(\omega_{P_\ell,n_0,j}\right)\right)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0 j}(\omega)} \\
&\quad \times \left\langle V(z_j,\theta,q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle \left\langle V(z_j,-\hat{x},q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle + \mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right).
\end{aligned} \tag{1.13}$$

In particular, the far field, in the back-scattered direction $\hat{x} = -\theta$, has the approximation

$$\begin{aligned}
&\langle (E^\infty - V^\infty)(-\theta), (\theta \times q)\rangle \\
&= -\frac{\mu\, a^3}{4\pi} \sum_{j=1}^{\aleph} \frac{\omega^2_{P_\ell,n_0 j}\, \epsilon_0(z_j)\left(\epsilon_0(z_j) - \Lambda_{n_0,j}\left(\omega_{P_\ell,n_0,j}\right)\right)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0 j}(\omega)} \left(\left\langle V(z_j,\theta,q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle\right)^2 \\
&\quad + \mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right).
\end{aligned} \tag{1.14}$$

The above expansions are valid under the conditions $0 \le h, t, s < 1$ such that

$$h < \min\left(1 - \frac{s}{2}; -\ s + 3\,(1-t)\right), \tag{1.15}$$

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with $\hat{x}, q \in \mathbb{S}^2$ *and* $\Lambda_{n_0,j}(\cdot)$ *is the dispersion function given by*

$$\Lambda_{n_0,j}(\omega) := \big(\epsilon_0(z_j) - \lambda^{(3)}_{n_0}(B)(\epsilon_0(z_j) - \epsilon_p(\omega))\big), \tag{1.16}$$

where $\epsilon_p(\omega)$ *is the Lorentz model for the permittivity given by (1.1).*

The field $V(\cdot,\cdot,\cdot)$ *(as its related scattered and far-field pattern) is the solution of the problem (1.2) in the absence of the nanoparticles.*

In (1.13), the expression

$$\beth\left(j,\theta,q,-\hat{x},n_0\right) := \left\langle V(z_j,\theta,q), \int_B e^{(3)}_{n_0}(y)\,dy \right\rangle \left\langle V\left(z_j,-\hat{x},q\right), \int_B e^{(3)}_{n_0}(y)\,dy \right\rangle,$$

can be rewritten as

$$\beth\left(j,\theta,q,-\hat{x},n_0\right) = \left\langle V(z_j,\theta,q), \left(\int_B e^{(3)}_{n_0}(y)\,dy \otimes \int_B e^{(3)}_{n_0}(y)\,dy\right)\cdot V\left(z_j,-\hat{x},q\right)\right\rangle, \tag{1.17}$$

which should be understood in the following sense

$$\beth\left(j,\theta,q,-\hat{x},n_0\right) = \sum_m \left\langle V(z_j,\theta,q), \left(\int_B e^{(3)}_{n_0,m}(y)\,dy \otimes \int_B e^{(3)}_{n_0,m}(y)\,dy\right)\cdot V\left(z_j,-\hat{x},q\right)\right\rangle,$$

where $e^{(3)}_{n_0,m}(\cdot)$ are the eigenfunctions of the Magnetization operator ∇M_B such that

$$\nabla M_B\big(e^{(3)}_{n_0,m}\big) = \lambda^{(3)}_{n_0}(B)\, e^{(3)}_{n_0,m}, \quad \text{in} \quad B.$$

In the case of a unit ball, i.e. $B = B(0,1)$, an explicit computation of $\int_B e^{(3)}_1(y)\,dy$ has been given in [24, Section 4.5.3], and it was established that

$$\int_B e^{(3)}_1(y)\,dy \otimes \int_B e^{(3)}_1(y)\,dy = \frac{4\pi}{27}\, I_3.$$

Hence, (1.17) becomes,

$$\beth\left(j,\theta,q,-\hat{x},n_0=1\right) = \frac{4\pi}{27}\,\left\langle V(z_j,\theta,q), V\left(z_j,-\hat{x},q\right)\right\rangle. \tag{1.18}$$

Consequently, for the case of a unit ball and $n_0 = 1$, the formula (1.13) becomes

$$\begin{aligned}\left\langle (E^\infty - V^\infty)(\hat{x}), (\hat{x}\times q)\right\rangle = \frac{\mu\, a^3}{27}\sum_{j=1}^{\aleph} \frac{\omega^2_{P_\ell,1,j}\,\epsilon_0(z_j)\left(\epsilon_0(z_j) - \Lambda_{1,j}\big(\omega_{P_\ell,1,j}\big)\right)}{\lambda^{(3)}_1(B)\,\Lambda_{1,j}(\omega)} \left\langle V(z_j,\theta,q), V\left(z_j,-\hat{x},q\right)\right\rangle \\ + \mathcal{O}\Big(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\Big).\end{aligned}$$

The expansion (1.14) describes the field generated by the collection of nanoparticles neglecting the mutual interaction between them, i.e. the Born approximation. This Born approximation is valid under the condition (1.15). Avoiding the mutual interaction between the nanoparticles makes it easier to state an imaging functional for the optical inversion. The details are provided in Section 2. The condition (1.15) can be relaxed. In this case, the approximation (1.14) becomes the Foldy-approximation (instead of the Born approximation) which involves the multiples scattering effects. Recall that the number of injected nanoparticles is $\aleph \sim a^{-s}$, where s satisfies (1.15). Therefore, this set of nanoparticles is dense in Ω. Indeed, as $\aleph \sim d^{-3} \sim a^{-3t}$, recalling that $d \sim a^t$, then $s = 3t$ and hence (1.15) becomes $t < \frac{1}{2} - \frac{h}{6}$. Since $h < 1$, then we need the condition $t < \frac{1}{3}$. Therefore, the nanoparticles can be distributed with a minimum distance between them of the order at least $d \ll a^{\frac{1}{3}}$.

The remaining part of the manuscript is divided as follows. In Section 2, we provide a detailed explanation of how the results in Theorem 1.1 can be used to propose an algorithm in optical imaging to reconstruct the permittivity function $\epsilon_0(\cdot)$, in the bounded medium Ω. In Section 3, we prove Theorem 1.1 while postponing the justification for the Mixed Electromagnetic Reciprocity Relation and an a priori estimate related to the total electric field to the next section. In Section A, which will be

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given as an appendix, we show the Mixed Electromagnetic Reciprocity Relation in Subsection A.1, and derive an a-priori estimate of the electric total field in Subsection A.2. Furthermore, Subsection A.3 and Subsection A.4 are included to finish Subsection A.1.

2. Application to the optical imaging using plasmonic contrast agents

We develop a quantitative imaging procedure that can reconstruct the permittivity function $\epsilon_0(\cdot)$ within the bounded medium Ω, to be imaged, using the expansion given in Theorem 1.1, formula (1.14). To do this, we start by recalling (1.14) that

$$\begin{aligned}
&\langle (E^\infty - V^\infty)(-\theta, \theta, q, \omega), (\theta \times q)\rangle \\
&= -\frac{\mu\, a^3}{4\,\pi} \sum_{j=1}^{\aleph} \frac{\omega^2_{P_\ell, n_0 j}\, \epsilon_0(z_j)\left(\epsilon_0(z_j) - \Lambda_{n_0 j}\left(\omega_{P_\ell, n_0 j}\right)\right)}{\lambda^{(3)}_{n_0}(B)\, \Lambda_{n_0 j}(\omega)} \left(\left\langle V(z_j, \theta, q), \int_B e^{(3)}_{n_0}(y)\, dy \right\rangle\right)^2 \\
&\quad + \mathcal{O}\left(a^{\min\left((3-s), \left(6-3t-2h-\frac{3s}{2}\right), (4-h-s)\right)}\right),
\end{aligned}$$

which, by keeping only its dominant term, becomes

$$\langle (E^\infty - V^\infty)\,(-\theta, \theta, q, \omega), (\theta \times q)\rangle \simeq -\frac{\mu\, a^3}{4\,\pi\, \lambda^{(3)}_{n_0}(B)} \sum_{j=1}^{\aleph} \mathcal{J}(\omega, z_j), \tag{2.1}$$

where the imaging functional $\mathcal{J}(\cdot, \cdot)$, depending on both the frequency parameter ω and the location points $\{z_j\}_{j=1}^{\aleph}$, is given by

$$\mathcal{J}(\omega, z_j) := \frac{\omega^2_{P_\ell, n_0 j}\, \epsilon_0(z_j)\left(\epsilon_0(z_j) - \Lambda_{n_0 j}\left(\omega_{P_\ell, n_0 j}\right)\right)}{\Lambda_{n_0 j}(\omega)} \left(\left\langle V(z_j, \theta, q), \int_B e^{(3)}_{n_0}(y)\, dy \right\rangle\right)^2, \tag{2.2}$$

with $\Lambda_{n_0 j}(\,\cdot\,)$ being the dispersion equation given by (1.16). It is evident from the imaging functional expression, see (2.2), that the reconstruction of the permittivity function $\epsilon_0(\cdot)$ on the location points, i.e. $\{\epsilon_0(z_j)\}_{j=1}^{\aleph}$, can be derived from the reconstruction of the imaging functional on the same location points, i.e. $\{\mathcal{J}(\omega, z_j)\}_{j=1}^{\aleph}$, with

$$\omega \in \mathbf{I} := \left(\omega_0\,,\, \sqrt{\omega_0^2 + \frac{\omega_p^2}{\lambda^{(3)}_{n_0}(B)}}\right).$$

More details will be provided later in Subsection 3.3. Furthermore, the contrast between the two far fields $V^\infty(\cdot)$ and $E^\infty(\cdot)$, given by (2.1), measured in the back-scattered direction, i.e. $\hat{x} = -\theta$, provides us with

$$\mathcal{F}(\omega, \aleph) := \sum_{j=1}^{\aleph} \mathcal{J}(\omega, z_j), \tag{2.3}$$

up to a known multiplicative constant given by $-\dfrac{\mu\, a^3}{4\,\pi\, \lambda^{(3)}_{n_0}(B)}$. Unfortunately, we cannot determine the $\aleph$ imaging functional $\mathcal{J}(\omega, z_j)$, for $1 \le j \le \aleph$, from the mean (sum) of the imaging functional $\mathcal{F}(\omega, \aleph)$, given by (2.3), as the problem is not uniquely solvable. Hence, the reconstruction of the permittivity function $\epsilon_0(\cdot)$ on the location points $\{z_j\}_{j=1}^{\aleph}$ cannot be done, in a straightforward manner, as explained above. In order to solve this issue, we propose employing an iterative method that requires knowledge of $\{\mathcal{F}(\omega, \ell)\}_{\ell=1}^{\aleph}$, given by (2.3). In this case, each imaging functional $\mathcal{J}(\omega, z_j)$ can be obtained as

$$\begin{aligned}
\mathcal{J}\,(\omega, z_1) &= \mathcal{F}(\omega, 1) = \text{Initial Reconstructed Term}, \\
\mathcal{J}(\omega, z_j) &= \mathcal{F}(\omega, j) - \mathcal{F}(\omega, j-1), \quad \text{for} \quad 2 \le j \le \aleph.
\end{aligned}$$

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Based on the explanations mentioned above, our proposed imaging procedures follow as below.

(1) Step 1). Reconstructing $\mathcal{J}(\omega, z_1)$.
   (a) Collect the far field before injecting any plasmonic nanoparticle inside Ω, in the back-scattered direction at a single incident wave θ, i.e. $V^\infty(-\theta, \theta, q, \omega)$, where $\omega \in \mathbf{I}$.
   (b) Collect the far field after injecting the first plasmonic nanoparticle D_1, in the back-scattered direction at a single incident wave θ, i.e. $E^\infty(-\theta, \theta, q, \omega)$, where $\omega \in \mathbf{I}$.

   At this stage, since we have inserted only one single plasmonic nanoparticle inside Ω, the equation (2.1) will collapse to the following single term

$$\langle (E^\infty - V^\infty)(-\theta, \theta, q, \omega), (\theta \times q) \rangle \simeq -\frac{\mu\, a^3}{4\,\pi\, \lambda_{n_0}^{(3)}(B)}\, \mathcal{J}(\omega, z_1),$$

   where the left-hand side is a known (measured) term, see for instance (1a) and (1b). In addition, as stated previously, the constant $-\dfrac{\mu\, a^3}{4\,\pi\, \lambda_{n_0}^{(3)}(B)}$ appearing on the right-hand side is known. Consequently, we deduce the reconstruction of the first imaging functional

$$\omega \longrightarrow \mathcal{J}(\omega, z_1), \quad \text{with} \quad \omega \in \mathbf{I}. \tag{2.4}$$

(2) Step 2). Reconstructing $\epsilon_0(z_1)$.
   As we have reconstructed the first imaging functional $\mathcal{J}(\omega, z_1)$, see for instance (2.4), and by recalling its analytical expression given by (2.2),

$$\mathcal{J}(\omega, z_1) = \frac{\omega_{P_\ell, n_0, 1}^2\, \epsilon_0(z_1)\, \big(\epsilon_0(z_1) - \Lambda_{n_0, j}\big(\omega_{P_\ell, n_0, 1}\big)\big)\, \Big(\big\langle V(z_1, \theta, q), \int_B e_{n_0}^{(3)}(y)\, dy \big\rangle\Big)^2}{\Lambda_{n_0,1}(\omega)},$$

   which can be rewritten, by introducing the proportional symbol $\propto$, like

$$\mathcal{J}(\omega, z_1) \propto \frac{1}{\Lambda_{n_0,1}(\omega)}, \quad \omega \in \mathbf{I}, \tag{2.5}$$

   since, with respect to the parameter a, we know that

$$\omega_{P_\ell, n_0, 1}^2\, \epsilon_0(z_1)\, \big(\epsilon_0(z_1) - \Lambda_{n_0, j}\big(\omega_{P_\ell, n_0, 1}\big)\big)\, \left(\left\langle V(z_1, \theta, q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle\right)^2 \sim 1.$$

   Clearly, from (2.5), we observe that the argmax (i.e. the input point at which a function's output value is maximized) of the function $\mathcal{J}(\cdot, z_1)$ is exactly the "quasi-root" of the first dispersion equation $\Lambda_{n_0,1}(\cdot)$, given by (1.16), and vice versa.[1] Here, "quasi-root" is referred to as the root of the real part of the disperison equation $\Lambda_{n_0,1}(\cdot)$, noting that it contains a small imaginary part. Thus, we can estimate the first plasmonic resonance $\omega_{P_\ell, n_0, 1}$. Moreover, by solving (1.16), we deduce

$$\epsilon_0(z_1) = \epsilon_p\big(\omega_{P_\ell, n_0, 1}\big)\, \frac{\lambda_{n_0}^{(3)}(B)}{\big(\lambda_{n_0}^{(3)}(B) - 1\big)}.$$

   This justifies the reconstruction of $\epsilon_0(z_1)$.

(3) Step 3). Reconstructing $\mathcal{J}(\omega, z_j)$ and $\epsilon_0(z_j)$, for $2 \leq j \leq \aleph$.
   As explained through Step 1, by injecting the first plasmonic nanoparticle D_1 and measuring the contrast between the far fields that we obtain before injecting D_1 and after injecting D_1, we can reconstruct the first imaging functional $\mathcal{J}(\cdot, z_1)$, see for instance (2.4). After that, we inject the second plasmonic nanoparticle D_2 and we return to (2.1), to obtain

$$\langle (E^\infty - V^\infty)(-\theta, \theta, q, \omega), (\theta \times q) \rangle \simeq -\frac{\mu\, a^3}{4\,\pi\, \lambda_{n_0}^{(3)}(B)} \sum_{j=1}^{2} \mathcal{J}(\omega, z_j),$$

[1] The quasi-root of the function $\Lambda_{n_0,1}(\cdot)$ corresponds to the argmax of the function $\mathcal{J}(\cdot, z_1)$.

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or, equivalently,

$$\mathcal{J}(\omega, z_2) \simeq -\mathcal{J}(\omega, z_1) - \frac{4\pi\, \lambda_{n_0}^{(3)}(B)}{\mu\, a^3} \left\langle (E^{\infty} - V^{\infty})(-\theta, \theta, q, \omega), (\theta \times q) \right\rangle .$$

The right-hand side of the formula above is already known (measured). Therefore, we deduce the reconstruction of the second imaging functional, i.e.

$$\omega \longrightarrow \mathcal{J}(\omega, z_2), \quad \text{with} \quad \omega \in \mathbf{I}. \tag{2.6}$$

Additionally, by using the same arguments as those used in Step (2), we can determine the reconstruction of $\epsilon_0(z_2)$ from $\mathcal{J}(\cdot, z_2)$, see for instance (2.6). Hence, we have reconstructed

$$(\mathcal{J}(\cdot, z_2); \epsilon_0(z_2)).$$

Finally, through an iterative process, as explained in Steps (1–3), we can reconstruct

$$(\mathcal{J}(\cdot, z_j); \epsilon_0(z_j))_{j=2}^{\aleph}.$$

(4) Step 4). Reconstructing $\epsilon_0(\cdot)$ within Ω.
To reconstruct the permittivity function $\epsilon_0(\cdot)$, inside Ω, or at least approximate it, from its already reconstructed point-wise values $\{\epsilon_0(z_j)\}_{j=1}^{\aleph}$, we can use the *Dual Reciprocity Method*. Other interpolation methods can also be used. For a more detailed study of the *Dual Reciprocity Method* and its application, [16, 31] and [48] are recommended. In what follows, we provide a detailed explanation for Step 4) to accomplish the imaging procedures.

We start by assuming that $\epsilon_0(\cdot)$ is smooth enough, and we approximate it by a linear combination of a finite number of basis functions, i.e.

$$\epsilon_0(z) \simeq \sum_{k=1}^{\aleph} \beta_k f_k(z), \quad z \in \Omega, \tag{2.7}$$

with $(\beta_1, \cdots, \beta_{\aleph}) \in \mathbb{C}^{\aleph}$ being a vector to be determined, and $(f_1(\cdot), \cdots, f_{\aleph}(\cdot))$ are suitably chosen basis functions. The accuracy of the approximation (2.7) depends on the choice of $\{f_k(\cdot)\}_{k=1}^{\aleph}$ and the number $\aleph$. The *Dual Reciprocity Method* has been applied by using various types of basis functions in different literature. Among them, without being exhaustive, we can list

(a) A linear basis function given by $f_k(\cdot) = 1 + |\cdot - x_k|$;
(b) A Gaussian basis function given by $f_k(\cdot) = e^{|\cdot - x_k|^2}$;
(c) A thin plate spline basis function given by $f_k(\cdot) = |\cdot - x_k|^2 \ln(|\cdot - x_k|)$,

where the set of points $\{x_k\}_{k=1}^{\aleph}$, called collocation points, is generated randomly inside Ω. Thus, we know the set of points $\{x_k\}_{k=1}^{\aleph}$.

Now, by choosing a type for the basis functions, generating randomly a set of collocations points $\{x_k\}_{k=1}^{\aleph}$ inside Ω and evaluating both sides of (2.7) at the a priori assumed known location $\{z_j\}_{j=1}^{\aleph}$, we obtain the following algebraic system,

$$\begin{pmatrix} f_1(z_1) & \cdots & f_{\aleph}(z_1) \\ \vdots & \ddots & \vdots \\ f_1(z_{\aleph}) & \cdots & f_{\aleph}(z_{\aleph}) \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{\aleph} \end{pmatrix} = \begin{pmatrix} \epsilon_0(z_1) \\ \vdots \\ \epsilon_0(z_{\aleph}) \end{pmatrix}. \tag{2.8}$$

Thanks to Steps (1–3), the right-hand side of (2.8) is a known vector. Moreover, by its construction, the matrix appearing on the left-hand side of (2.8) can be computed explicitly. Then, by solving (2.8), we can determine the unknown vector $(\beta_1, \cdots, \beta_{\aleph})$. The final stage involves inserting the obtained vector into (2.7), to come up with an approximation of the permittivity function $\epsilon_0(\cdot)$ inside Ω.

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3. Proof of the main result, i.e. Theorem 1.1

We divide this section into four subsections. In the first subsection, we will focus on deriving the algebraic system related to the solution of the electromagnetic problem (1.2). In the second subsection, we derive the dominant term related to the optical electric field and show its dependence on the eigen-system related to the Magnetization operator, the permittivity of the medium to be imaged, i.e $\epsilon_0(\cdot)$ and the permittivity of the used plasmonic nanoparticles, i.e. $\epsilon_p(\omega)$. In the third subsection, we shall use the results from the first and the second subsections to estimate the scattered fields connected to the electric field. In the last subsection, we will use the derived result on the scattered fields to estimate the corresponding far-fields. To clear the structure, before moving forward to the main proof, we shall briefly summarize its key steps as follows.

- Step 1: We derive the algebraic system satisfied by the solution of (1.2). The expression of the algebraic system is given by (3.15).
- Step 2: On basis of Step 1, under appropriate conditions for the used parameters, we deduce the dominant term related to the solution of (1.2). The formula is given by (3.29).
- Step 3: We provide the expression of the contrast between the scattered fields (solution of (1.2)), before and after inserting the plasmonic nanoparticles, by using the dominant terms obtained in Step 2. See the expression (3.40).
- Step 4: From the precise form of the contrast between the scattered fields which has been deduced in Step 3, we formulate the corresponding far-fields contrast representation. This will be achieved after proving the Mixed Electromagnetic Reciprocity Relation, see Proposition 3.1.

3.1. Deriving the algebraic system satisfied by the solution of (1.2)

As shown in [25, Section 3.2], the solution of (1.2) can be written as the solution to the following Lippmann–Schwinger system of equations

$$u_1(x) + \omega^2 \int_D G_k(x,y)\cdot u_1(y)\left(\boldsymbol{n}_0^2(y) - \boldsymbol{n}^2(y)\right) dy = u_0(x), \quad x \in \mathbb{R}^3, \tag{3.1}$$

where $u_0(\cdot) := V$ (respectively. $u_1(\cdot) := E$) denotes the electromagnetic field before (respectively. after) injecting the nanoparticles, $D = \bigcup_{j=1}^{\aleph} D_j$, with $\aleph \sim a^{-s}$, $s > 0$, and $G_k(\cdot,\cdot)$ is solution to (1.10). To give sense of the integral appearing on the left-hand side of (3.1), i.e.

$$Int(x) := \int_D G_k(x,y)\cdot u_1(y)\left(\boldsymbol{n}_0^2(y) - \boldsymbol{n}^2(y)\right) dy, \quad x \in \mathbb{R}^3, \tag{3.2}$$

we start by recalling that

$$G_k(x,\cdot) = \Pi_k(x,\cdot) + \Gamma_1(x,\cdot), \tag{3.3}$$

see [25, Subsection 3.2], where

$$\Pi_k(x,y) := \frac{1}{k^2} \nabla_y \nabla_y \Phi_k(x,y) + \Phi_k(x,y)\, I_3, \tag{3.4}$$

and $\Gamma_1(x,\cdot) \in \mathbb{L}^{\frac{3}{2}-\delta}(\Omega)$ stands for the remainder term of the Green's kernel, with δ being a small positive real number. Then, using (3.3), (3.4) and the definition of both the Magnetization operator and the

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Newtonian operator in (1.7), (3.2) should be understood as

$$Int(x) = -\frac{1}{k^2}\,\nabla M_D^k\left(u_1(\cdot)\left(\boldsymbol{n}_0^2(\cdot)-\boldsymbol{n}^2(\cdot)\right)\right)(x) + N_D^k\left(u_1(\cdot)\left(\boldsymbol{n}_0^2(\cdot)-\boldsymbol{n}^2(\cdot)\right)\right)(x) + \int_D \Gamma_1(x,y)\cdot u_1(y)\left(\boldsymbol{n}_0^2(y)-\boldsymbol{n}^2(y)\right)dy, \quad x\in\mathbb{R}^3. \tag{3.5}$$

It is clear that the first term and the second term on the right-hand side of (3.5) are well defined. In addition, the existence of the third term on the right-hand side of (3.5) can be deduced by using the fact that $\Gamma_1(x,\cdot)\in\mathbb{L}^{\frac{3}{2}-\delta}(\Omega)$. This provides an explanation of how (3.2) can be interpreted. Therefore, from now on whenever we use the notation (3.2), it should be understood as (3.5).

Recalling the definition of the index of refraction $\boldsymbol{n}(\cdot)$, see (1.5), we rewrite (3.1) as

$$u_1(x)+\omega^2\,\mu\int_D G_k(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy = u_0(x), \quad x\in\mathbb{R}^3, \tag{3.6}$$

where $\epsilon_p(\cdot)$ is the given permittivity of the injected nanoparticles, through the Lorentz model, by (1.1). Then, by restricting $x\in D_m$, we rewrite (3.6) as

$$u_1(x)+\omega^2\,\mu\int_{D_m} G_k(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy+\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\int_{D_j} G_k(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy = u_0(x).$$

By expanding the function $(\epsilon_0(\cdot)-\epsilon_p(\omega))$ near the centres, we obtain

$$\begin{aligned} &u_1(x)+\omega^2\,\mu\,(\epsilon_0(z_m)-\epsilon_p(\omega))\int_{D_m} G_k(x,y)\cdot u_1(y)\,dy+\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\int_{D_j} G_k(x,y)\cdot u_1(y)\,dy\\ &= u_0(x)-\omega^2\,\mu\int_{D_m} G_k(x,y)\cdot u_1(y)\int_0^1 \nabla\epsilon_0(z_m+t(y-z_m))\cdot(y-z_m)\,dt\,dy\\ &\quad-\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\int_{D_j} G_k(x,y)\cdot u_1(y)\int_0^1 \nabla\epsilon_0(z_j+t(y-z_j))\cdot(y-z_j)\,dt\,dy. \end{aligned} \tag{3.7}$$

Thanks to [25, Theorem 2.1, Formula (2.3)], regarding the Green's kernel $G_k(\cdot,\cdot)$, we have the following expansion

$$G_k(x,y) = \frac{1}{\omega^2\,\mu\,\epsilon_0(y)}\,\nabla_x\nabla_x\Phi_0(x,y)+\Gamma_2(x,y), \quad x\neq y, \tag{3.8}$$

where the term $\Gamma_2(\cdot,\cdot)$[2] is given by

$$\Gamma_2(x,y) = \frac{-1}{\omega^2\,\mu\,\left(\epsilon_0(y)\right)^2}\nabla_x\nabla_x M\left(\Phi_0(\cdot,y)\nabla\epsilon_0(y)\right)(x)+\Xi(x,z), \tag{3.9}$$

with $\nabla M(\cdot)$ being the Magnetization operator defined by (1.9) and $\Xi(\cdot,z)\in\mathbb{L}^{3-\delta}(D)$, where δ is a small positive parameter. The decomposition (3.8) here can be utilized to comprehend (3.2). Then, by denoting

$$\tau_m := \frac{(\epsilon_0(z_m)-\epsilon_p(\omega))}{\epsilon_0(z_m)},$$

[2]Both of $\Gamma_2(\cdot,\cdot)$ given in (3.8) and $\Gamma_1(\cdot,\cdot)$ given in (3.3) represent the remainder terms of the Green's kernel $G_k(\cdot,\cdot)$. The kernel $\Gamma_1(\cdot,\cdot)$ is associated with the dominant term $\Pi_k(\cdot,\cdot)$, while the kernel $\Gamma_2(\cdot,\cdot)$ is associated with the dominant term $\frac{1}{\omega^2\,\mu\,\epsilon_0(\cdot)}\,\nabla\nabla\Phi_0(\cdot,\cdot)$.

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using (3.8) and (3.9), we derive from (3.7) the following equation

$$\left[I-\tau_m\,\nabla M_{D_m}\right](u_1)(x)+\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\tau_j\,\epsilon_0(z_j)\int_{D_j}G_k(x,y)\cdot u_1(y)\,dy=u_0(x)+Err_0(x), \tag{3.10}$$

where

$$\begin{aligned}Err_0(x):=&-\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\int_{D_j}G_k(x,y)\cdot u_1(y)\int_0^1\nabla(\epsilon_0)(z_j+t(y-z_j))\cdot(y-z_j)\,dt\,dy\\ &-\omega^2\,\mu\int_{D_m}G_k(x,y)\cdot u_1(y)\int_0^1\nabla(\epsilon_0)(z_m+t(y-z_m))\cdot(y-z_j)\,dt\,dy\\ &-\omega^2\,\mu\,(\epsilon_0(z_m)-\epsilon_p(\omega))\int_{D_m}\Gamma_2(x,y)\cdot u_1(y)\,dy\\ &+(\epsilon_0(z_m)-\epsilon_p(\omega))\,\nabla M_{D_m}\left(u_1(\cdot)\int_0^1\nabla\left(\epsilon_0^{-1}\right)(z_m+t(\cdot-z_m))\cdot(\cdot-z_m)\,dt\right)(x).\end{aligned} \tag{3.11}$$

We have the invertibility of the operator $[I-\tau_m\,\nabla M_{D_m}]$ due to the fact that $\tau_m^{-1}\notin\sigma\left(\nabla M_{D_m}\right)\cup\{1\}$, where $\sigma\,(\cdot)$ stands for the spectrum set. Successively, by taking the inverse of the operator $[I-\tau_m\,\nabla M_{D_m}]$ on the both sides of the equation (3.10), integrating over the domain D_m and using the definition

$$W_m(\cdot):=\left[I-\tau_m\,\nabla M_{D_m}\right]^{-1}(I)(\cdot),\quad\text{in } D_m,$$

we obtain

$$\begin{aligned}&\int_{D_m}u_1(x)\,dx+\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\tau_j\,\epsilon_0(z_j)\int_{D_m}W_m(x)\cdot\int_{D_j}G_k(x,y)\cdot u_1(y)\,dy\,dx\\ &\qquad=\int_{D_m}W_m(x)\cdot u_0(x)\,dx+\int_{D_m}W_m(x)\cdot Err_0(x)\,dx.\end{aligned} \tag{3.12}$$

Next, by setting

$$\mathcal{C}_m:=\int_{D_m}W_m(x)\,dx,$$

and expanding both the Green's kernel $G_k(\cdot,\cdot)$ and the vector field $u_0(\cdot)$ near the centres, we derive from (3.12) the following algebraic system

$$\mathcal{C}_m^{-1}\cdot\int_{D_m}u_1(x)\,dx+\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\tau_j\,\epsilon_0(z_j)\,G_k(z_m,z_j)\cdot\mathcal{C}_j\cdot\mathcal{C}_j^{-1}\cdot\int_{D_j}u_1(y)\,dy=u_0(z_m)+\mathcal{C}_m^{-1}\cdot Error_{1,m}, \tag{3.13}$$

where

$$\begin{aligned}Error_{1,m}:=&-\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\tau_j\,\epsilon_0(z_j)\int_{D_m}W_m(x)\cdot\int_{D_j}\int_0^1\nabla G_k(x,z_j+t(y-z_j))\cdot(y-z_j)\,dt\cdot u_1(y)\,dy\,dx\\ &-\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\tau_j\,\epsilon_0(z_j)\int_{D_m}W_m(x)\cdot\int_0^1\nabla G_k(z_m+t(x-z_m),z_j)\cdot(x-z_m)\,dt\,dx\cdot\int_{D_j}u_1(y)\,dy\\ &+\int_{D_m}W_m(x)\cdot\int_0^1\nabla u_0(z_m+t(x-z_m))\cdot(x-z_m)\,dt\,dx+\int_{D_m}W_m(x)\cdot Err_0(x)\,dx.\end{aligned} \tag{3.14}$$

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Moreover, for (3.13), we correspond to the following matrix form,

$$\begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_{\aleph} \end{pmatrix} + \omega^2\,\mu\,\mathcal{M}\cdot\begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_{\aleph} \end{pmatrix} = \begin{pmatrix} u_0(z_1) \\ u_0(z_2) \\ \vdots \\ u_0(z_{\aleph}) \end{pmatrix} + \begin{pmatrix} \mathcal{C}_1^{-1}\cdot Error_{1,1} \\ \mathcal{C}_2^{-1}\cdot Error_{1,2} \\ \vdots \\ \mathcal{C}_{\aleph}^{-1}\cdot Error_{1,\aleph} \end{pmatrix}, \tag{3.15}$$

where $\mathcal{M}$ is the matrix given by

$$\mathcal{M} := \begin{pmatrix} 0 & G_k(z_1,z_2)\cdot\tau_2\epsilon_0(z_2)\,\mathcal{C}_2 & \cdots & G_k(z_1,z_{\aleph})\cdot\tau_{\aleph}\epsilon_0(z_{\aleph})\,\mathcal{C}_{\aleph} \\ G_k(z_2,z_1)\cdot\tau_1\,\epsilon_0(z_1)\mathcal{C}_1 & 0 & \cdots & G_k(z_2,z_{\aleph})\cdot\tau_{\aleph}\epsilon_0(z_{\aleph})\,\mathcal{C}_{\aleph} \\ \vdots & \vdots & \ddots & \vdots \\ G_k(z_{\aleph},z_1)\cdot\tau_1\epsilon_0(z_1)\,\mathcal{C}_1 & G_k(z_{\aleph},z_2)\cdot\tau_2\epsilon_0(z_2)\,\mathcal{C}_2 & \cdots & 0 \end{pmatrix},$$

$Error_{1,m}$ is given by (3.14), and

$$\mathcal{Q}_m := \mathcal{C}_m^{-1}\cdot\int_{D_m} u_1(x)\,dx, \quad 1\le m\le\aleph. \tag{3.16}$$

3.2. Deriving the dominant term of the optical field

To extract the dominant term related to the optical field, we start by proving the invertibility of the algebraic system (3.15). To achieve this, it is sufficient to derive a condition such that the matrix appearing on the L.H.S of (3.15) is a diagonal dominant block matrix, i.e.

$$\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\left|G_k(z_m,z_j)\cdot\tau_j\epsilon_0(z_j)\,\mathcal{C}_j\right| < |I_3|\,. \tag{3.17}$$

To do this, we have

$$\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\left|G_k(z_m,z_j)\cdot\tau_j\epsilon_0(z_j)\,\mathcal{C}_j\right| \lesssim \sum_{\substack{j=1\\ j\neq m}}^{\aleph}\left|G_k(z_m,z_j)\right|\,\left|\tau_j\right|\,\left|\epsilon_0(z_j)\right|\,\left|\mathcal{C}_j\right| \lesssim \sum_{\substack{j=1\\ j\neq m}}^{\aleph}\left|G_k(z_m,z_j)\right|\,\left|\mathcal{C}_j\right|.$$

In addition, thanks to [25, Proposition 2.2], we know that

$$\mathcal{C}_j = \mathcal{O}(a^{3-h}), \tag{3.18}$$

and this implies

$$\omega^2\,\mu\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\left|G_k(z_m,z_j)\cdot\tau_j\epsilon_0(z_j)\,\mathcal{C}_j\right| \lesssim a^{3-h}\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\frac{1}{d_{mj}^3} = \mathcal{O}(a^{3-h}\,d^{-3}\,\aleph) = \mathcal{O}(a^{3-h-3t-s}).$$

Hence, under the condition

$$3-h-3t-s\;>\;0, \tag{3.19}$$

the condition (3.17) will be satisfied. By using Born series and keeping the dominant terms, we get

$$\begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_{\aleph} \end{pmatrix} = \begin{pmatrix} u_0(z_1) \\ u_0(z_2) \\ \vdots \\ u_0(z_{\aleph}) \end{pmatrix} + \begin{pmatrix} Error_{2,1} \\ Error_{2,2} \\ \vdots \\ Error_{2,\aleph} \end{pmatrix}, \tag{3.20}$$

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where

$$\begin{pmatrix} Error_{2,1} \\ Error_{2,2} \\ \vdots \\ Error_{2,\aleph} \end{pmatrix} := \begin{pmatrix} \mathcal{C}_1^{-1} \cdot Error_{1,1} \\ \mathcal{C}_2^{-1} \cdot Error_{1,2} \\ \vdots \\ \mathcal{C}_\aleph^{-1} \cdot Error_{1,\aleph} \end{pmatrix} - \omega^2 \, \mu \, \mathcal{M} \cdot \begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_\aleph \end{pmatrix},$$

which can be rewritten as

$$\begin{pmatrix} Error_{2,1} \\ Error_{2,2} \\ \vdots \\ Error_{2,\aleph} \end{pmatrix} = \begin{pmatrix} \mathcal{C}_1^{-1} \cdot Error_{1,1} - \omega^2 \, \mu \, \sum_{\substack{j=1 \\ j\neq 1}}^{\aleph} G_k(z_1, z_j) \cdot \tau_j \epsilon_0(z_j) \, \mathcal{C}_j \, \mathcal{Q}_j \\ \mathcal{C}_2^{-1} \cdot Error_{1,2} - \omega^2 \, \mu \, \sum_{\substack{j=1 \\ j\neq 2}}^{\aleph} G_k(z_2, z_j) \cdot \tau_j \epsilon_0(z_j) \, \mathcal{C}_j \, \mathcal{Q}_j \\ \vdots \\ \mathcal{C}_\aleph^{-1} \cdot Error_{1,\aleph} - \omega^2 \, \mu \, \sum_{\substack{j=1 \\ j\neq \aleph}}^{\aleph} G_k(z_\aleph, z_j) \cdot \tau_j \epsilon_0(z_j) \, \mathcal{C}_j \, \mathcal{Q}_j \end{pmatrix}.$$

Next, we need to estimate the term $Error_{2,k}$, for $k = 1, \cdots, \aleph$. To do this, we recall that

$$Error_{2,k} := \mathcal{C}_1^{-1} \cdot Error_{1,k} - \omega^2 \, \mu \sum_{\substack{j=1 \\ j\neq k}}^{\aleph} G_k(z_k, z_j) \cdot \tau_j \epsilon_0(z_j) \, \mathcal{C}_j \, \mathcal{Q}_j$$

$$Error_{2,k} \overset{(3.16)}{=} \mathcal{C}_1^{-1} \cdot Error_{1,k} - \omega^2 \, \mu \sum_{\substack{j=1 \\ j\neq k}}^{\aleph} \tau_j \, \epsilon_0(z_j) \, G_k(z_k, z_j) \cdot \int_{D_j} u_1(x) \, dx$$

$$\begin{aligned} \left|Error_{2,k}\right| &\lesssim \left|\mathcal{C}_1^{-1}\right| \, \left|Error_{1,k}\right| + \sum_{\substack{j=1 \\ j\neq k}}^{\aleph} \left|G_k(z_k, z_j)\right| \, \|1\|_{\mathbb{L}^2(D_j)} \, \|u_1\|_{\mathbb{L}^2(D_j)} \\ &\overset{(3.18)}{\lesssim} a^{h-3} \, \left|Error_{1,k}\right| + a^{\frac{3}{2}} \sum_{\substack{j=1 \\ j\neq k}}^{\aleph} \frac{1}{d_{kj}^3} \|u_1\|_{\mathbb{L}^2(D_j)} \\ &\lesssim a^{h-3} \, \left|Error_{1,k}\right| + a^{\frac{3}{2}} \, \|u_1\|_{\mathbb{L}^2(D)} \left(\sum_{\substack{j=1 \\ j\neq k}}^{\aleph} \frac{1}{d_{kj}^6} \right)^{\frac{1}{2}} \\ &\lesssim a^{h-3} \, \left|Error_{1,k}\right| + a^{\frac{3}{2}} \, \|u_1\|_{\mathbb{L}^2(D)} \, d^{-3} \, \aleph^{\frac{1}{2}}. \end{aligned}$$

Knowing that

$$\|u_1\|_{\mathbb{L}^2(D)} = \mathcal{O}\left(a^{\frac{3}{2}-h-\frac{s}{2}}\right), \tag{3.21}$$

see Proposition A.1 for its justification, we deduce that

$$\left|Error_{2,k}\right| \lesssim a^{h-3} \, \left|Error_{1,k}\right| + a^{3-h-3t-s}. \tag{3.22}$$

To accomplish the estimation of (3.22), we need to estimate $\left|Error_{1,k}\right|$. For (3.14), we take the modulus on its both sides to get

$$\begin{aligned} |Error_{1,m}| \lesssim \|W_m\|_{\mathbb{L}^2(D_m)} \Bigg[& \|1\|_{\mathbb{L}^2(D_m)} \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \|u_1\|_{\mathbb{L}^2(D_j)} \frac{1}{d_{mj}^4} \left[\int_{D_j} |y - z_j|^2 dy \right]^{\frac{1}{2}} \\ & + \|1\|_{\mathbb{L}^2(D_m)} \left[\int_{D_m} |x - z_m|^2 \, dx \right]^{\frac{1}{2}} \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \frac{1}{d_{mj}^4} \|u_1\|_{\mathbb{L}^2(D_j)} + \left[\int_{D_m} |x - z_m|^2 \, dx \right]^{\frac{1}{2}} + \|Err_0\|_{\mathbb{L}^2(D_m)} \Bigg], \end{aligned}$$

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where we have used the fact that $u_0(\cdot) \in \mathcal{C}^1$ and the singularity of the Green's kernel $G_k(\cdot,\cdot)$ is of order 3. Besides, by the use of

$$\int_{D_j} |x - z_j|^2 \, dx = \mathcal{O}(a^5), \quad 1 \le j \le \aleph,$$

we obtain

$$\begin{aligned} |Error_{1,m}| &\lesssim \|W_m\|_{\mathbb{L}^2(D_m)} \left[a^4 \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \|u_1\|_{\mathbb{L}^2(D_j)} \frac{1}{d_{mj}^4} + a^{\frac{5}{2}} + \|Err_0\|_{\mathbb{L}^2(D_m)} \right] \\ &\lesssim \|W_m\|_{\mathbb{L}^2(D_m)} \left[a^4 \, d^{-4} \, \|u_1\|_{\mathbb{L}^2(D)} \, \aleph^{\frac{1}{2}} + a^{\frac{5}{2}} + \|Err_0\|_{\mathbb{L}^2(D_m)} \right] \\ &\overset{(3.21)}{\lesssim} \|W_m\|_{\mathbb{L}^2(D_m)} \left[a^{\frac{11}{2} - h - 4t - s} + a^{\frac{5}{2}} + \|Err_0\|_{\mathbb{L}^2(D_m)} \right]. \end{aligned} \tag{3.23}$$

To estimate of $\|Err_0\|_{\mathbb{L}^2(D_m)}$, we go back to (3.11) and take the $\|\cdot\|_{\mathbb{L}^2(D_m)}$-norm on the both sides of the equation to obtain

$$\begin{aligned} \|Err_0\|_{\mathbb{L}^2(D_m)} \lesssim & \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \left\| \int_{D_j} G_k(\cdot, y) \cdot u_1(y) \int_0^1 \nabla(\epsilon_0)(z_j + t(y - z_j)) \cdot (y - z_j) \, dt \, dy \right\|_{\mathbb{L}^2(D_m)} \\ & + \left\| \int_{D_m} G_k(\cdot, y) \cdot u_1(y) \int_0^1 \nabla(\epsilon_0)(z_m + t(y - z_m)) \cdot (y - z_j) \, dt \, dy \right\|_{\mathbb{L}^2(D_m)} \\ & + \left\| \int_{D_m} \Gamma_2(\cdot, y) \cdot u_1(y) \, dy \right\|_{\mathbb{L}^2(D_m)} + \left\| \nabla M_{D_m} \left(u_1 \int_0^1 \nabla(\epsilon_0^{-1})(z_m + t(\cdot - z_m)) \cdot (\cdot - z_m) \, dt \right) \right\|_{\mathbb{L}^2(D_m)}, \end{aligned}$$

and, by using the smoothness of the function $\epsilon_0(\cdot)$, the dominant term related to the Green's kernel, see (3.8), and the dominant term related to the kernel $\Gamma_2(\cdot,\cdot)$, see (3.9), we reduce the previous estimation to

$$\begin{aligned} \|Err_0\|_{\mathbb{L}^2(D_m)} \lesssim & \, a \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \|u_1\|_{\mathbb{L}^2(D_j)} \left[\int_{D_m} \int_{D_j} |G_k(x, y)|^2 \, dy \, dx \right]^{\frac{1}{2}} \\ & + \left\| \nabla M_{D_m} \left(\frac{u_1(\cdot)}{\epsilon_0(\cdot)} \int_0^1 \nabla(\epsilon_0)(z_m + t(\cdot - z_m)) \cdot (\cdot - z_j) \, dt \right) \right\|_{\mathbb{L}^2(D_m)} \\ & + \left\| \int_{D_m} \nabla\nabla M\big(\Phi_0(\cdot, y) \nabla \epsilon_0^{-1}(y)\big)(\cdot) \cdot u_1(y) \, dy \right\|_{\mathbb{L}^2(D_m)} \\ & + \left\| \nabla M_{D_m} \left(u_1(\cdot) \int_0^1 \nabla\big(\epsilon_0^{-1}\big)(z_m + t(\cdot - z_m)) \cdot (\cdot - z_m) \, dt \right) \right\|_{\mathbb{L}^2(D_m)}. \end{aligned}$$

Moreover, since $\left\| \nabla M_{D_m} \right\|_{\mathcal{L}\left(\mathbb{L}^2(D_m);\mathbb{L}^2(D_m)\right)} = 1$, see [25, Lemma 5.5], and using the fact that from singularity point of view we have

$$\nabla\nabla M \left(\Phi_0(\cdot, y) \nabla \epsilon_0^{-1}(y) \right)(\cdot) \sim \nabla \left(\Phi_0(\cdot, y) \, \nabla \epsilon_0^{-1}(y) \right),$$

we deduce the following estimation

$$\begin{aligned} \|Err_0\|_{\mathbb{L}^2(D_m)} \lesssim & \, a \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \|u_1\|_{\mathbb{L}^2(D_j)} \left[\int_{D_m} \int_{D_j} |G_k(x, y)|^2 \, dy \, dx \right]^{\frac{1}{2}} + a \, \|u_1\|_{\mathbb{L}^2(D_m)} \\ & + \left\| \int_{D_m} \nabla\big(\Phi_0(\cdot, y) \nabla \epsilon_0^{-1}(y)\big) \cdot u_1(y) \, dy \right\|_{\mathbb{L}^2(D_m)} + a \, \|u_1\|_{\mathbb{L}^2(D_m)}. \end{aligned}$$

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On the R.H.S, to evaluate the first term we expand the Green's kernel $G_k(\cdot,\cdot)$ near the centres, together with the explicit computation for the third term, we end up with the following estimation

$$\|Err_0\|_{\mathbb{L}^2(D_m)} \lesssim a^4 \sum_{\substack{j=1\\ j\neq m}}^{\aleph} \|u_1\|_{\mathbb{L}^2(D_j)} |G_k(z_m,z_j)| + a\,\|u_1\|_{\mathbb{L}^2(D_m)} + \left\|\nabla N_{D_m}\left(\nabla \epsilon_0^{-1}\cdot u_1\right)\right\|_{\mathbb{L}^2(D_m)}.$$

Now, by using the fact that

$$\left\|\nabla N_{D_m}\right\|_{\mathcal{L}\left(\mathbb{L}^2(D_m),\mathbb{L}^2(D_m)\right)} = \mathcal{O}(a) \quad \text{and} \quad |G_k(z_m,z_j)| \sim \frac{1}{d_{mj}^3},$$

we obtain

$$\begin{aligned}
\|Err_0\|_{\mathbb{L}^2(D_m)} &\lesssim a^4 \sum_{\substack{j=1\\ j\neq m}}^{\aleph} \|u_1\|_{\mathbb{L}^2(D_j)} \frac{1}{d_{mj}^3} + a\,\|u_1\|_{\mathbb{L}^2(D_m)} \\
&\lesssim a^4 \left(\sum_{\substack{j=1\\ j\neq m}}^{\aleph} \|u_1\|^2_{\mathbb{L}^2(D_j)}\right)^{\frac{1}{2}} \left(\sum_{\substack{j=1\\ j\neq m}}^{\aleph} \frac{1}{d_{mj}^6}\right)^{\frac{1}{2}} + a\,\|u_1\|_{\mathbb{L}^2(D_m)} \\
&\lesssim \left(a^4\, d^{-3}\, \aleph^{\frac{1}{2}} + a\right) \|u_1\|_{\mathbb{L}^2(D)} \\
&\overset{(3.21)}{=} \mathcal{O}\left(a^{\frac{5}{2}-h-\frac{s}{2}}\right) + \mathcal{O}\left(a^{\frac{11}{2}-3t-s-h}\right) \overset{(3.19)}{=} \mathcal{O}\left(a^{\frac{5}{2}-h-\frac{s}{2}}\right).
\end{aligned} \tag{3.24}$$

By returning to (3.23) and using (3.24), we deduce that

$$\left|Error_{1,m}\right| \lesssim \|W_m\|_{\mathbb{L}^2(D_m)}\; a^{\frac{5}{2}-h+\min\left(3-4t-s;-\frac{s}{2}\right)}.$$

In addition, thanks to [25, Proposition 2.2], we know that $\|W_m\|_{\mathbb{L}^2(D_m)} = \mathcal{O}(a^{\frac{3}{2}-h})$, and this implies

$$\left|Error_{1,m}\right| = \mathcal{O}\left(a^{(4-2h)+\min\left(3-4t-s;-\frac{s}{2}\right)}\right).$$

Finally, by plugging the previous estimation into (3.22), we obtain

$$\left|Error_{2,m}\right| = \mathcal{O}\left(a^{\min\left((3-3t-h-s);(1-h-\frac{s}{2})\right)}\right). \tag{3.25}$$

Hence, by gathering (3.20) and (3.25), we conclude that

$$\mathcal{Q}_m = u_0(z_m) + \mathcal{O}\left(a^{\min\left((3-3t-h-s);(1-h-\frac{s}{2})\right)}\right).$$

Consequently, by using (3.16) and (3.18), we obtain

$$\int_{D_m} u_1(x)\,dx = \mathcal{C}_m \cdot u_0(z_m) + \mathcal{O}\left(a^{\min\left((6-3t-2h-s);(4-2h-\frac{s}{2})\right)}\right).$$

Furthermore, from [25, Proposition 2.2], we know that

$$\mathcal{C}_m = \frac{a^3\,\epsilon_0(z_m)}{\left(\epsilon_0(z_m) - \lambda_{n_0}^{(3)}(B)\,(\epsilon_0(z_m)-\epsilon_p(\omega))\right)} \int_B e_{n_0}^{(3)}(x)\,dx \otimes \int_B e_{n_0}^{(3)}(x)\,dx + \mathcal{O}(a^3). \tag{3.26}$$

This implies,

$$\int_{D_m} u_1(x)\,dx = \frac{a^3\,\epsilon_0(z_m)\,\left\langle u_0(z_m), \int_B e_{n_0}^{(3)}(x)\,dx\right\rangle}{(\epsilon_0(z_m) - (\epsilon_0(z_m)-\epsilon_p(\omega))\,\lambda_{n_0}^{(3)}(B))} \int_B e_{n_0}^{(3)}(x)\,dx + \mathcal{O}\left(a^{\min\left(3;(6-3t-2h-s);(4-2h-\frac{s}{2})\right)}\right), \tag{3.27}$$

where h is taken to be

$$h < \min\left(1-\frac{s}{2}; -s+3(1-t)\right). \tag{3.28}$$

It is important to note that if $\omega^2 - \omega^2_{p\ell,n_0,m} \sim a^h$ is satisfied, we get $(\epsilon_0(z_m) - (\epsilon_0(z_m)-\epsilon_p(\omega))\lambda_{n_0}^{(3)}(B)) \sim a^h$, see (3.39), thus the first term on the R.H.S of (3.27) will be of order $\sim a^{3-h}$. Furthermore, the condition (3.28) is added to ensure that the R.H.S of (3.27) is well defined and, of course, to also satisfy (3.19).

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In the subsequent discussions, we will use the notation $V := u_0$, respectively $E := u_1$ for the solutions to (1.2) signifying the electric field respectively before and after injecting the nanoparticles D inside Ω. Taking into account the introduced notation, we rewrite (3.27) as

$$\int_{D_m} E(x)\,dx = \frac{a^3\,\epsilon_0(z_m)\left\langle V(z_m,\theta,q),\int_B e_{n_0}^{(3)}(x)\,dx\right\rangle}{\left(\epsilon_0(z_m)-(\epsilon_0(z_m)-\epsilon_p(\omega))\,\lambda_{n_0}^{(3)}(B)\right)}\int_B e_{n_0}^{(3)}(x)\,dx + \mathcal{O}\big(a^{\min(3;(6-3t-2h-s);(4-2h-\frac{s}{2}))}\big). \tag{3.29}$$

3.3. Estimation of the scattered fields

To derive the estimate of the scattered field, we take x away from D in (3.6),

$$E(x)+\omega^2\,\mu\int_D G_k(x,y)\cdot E(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy = V(x).$$

In addition, as $E = E^I + E^s$ and $V = V^I + V^s$, with $V^I = E^I$, we obtain

$$\begin{aligned} E^s(x)-V^s(x) &= -\omega^2\,\mu\int_D G_k(x,y)\cdot E(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy \\ &= -\omega^2\,\mu\sum_{j=1}^{\aleph}\int_{D_j} G_k(x,y)\cdot E(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy, \end{aligned} \tag{3.30}$$

which, by Taylor expansion for the permittivity function $\epsilon_0(\cdot)$, gives us

$$E^s(x)-V^s(x) = -\,\omega^2\,\mu\sum_{j=1}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\int_{D_j} G_k(x,y)\cdot E(y)\,dy \;-\; T_1(x),$$

where

$$T_1(x) := \omega^2\,\mu\sum_{j=1}^{\aleph}\int_{D_j} G_k(x,y)\cdot E(y)\int_0^1 \nabla\epsilon_0(z_j+t(y-z_j))\cdot(y-z_j)\,dt\,dy.$$

We need to estimate the term $T_1(\cdot)$. We have

$$\begin{aligned} |T_1(x)| &\lesssim \sum_{j=1}^{\aleph}\int_{D_j}\left|G_k(x,y)\cdot E(y)\int_0^1 \nabla\epsilon_0(z_j+t(y-z_j))\cdot(y-z_j)\,dt\right|dy \\ &\lesssim \sum_{j=1}^{\aleph}\|G_k(x,\cdot)\|_{\mathbb{L}^2(D_j)}\,\|E\|_{\mathbb{L}^2(D_j)}\left\|\int_0^1 \nabla\epsilon_0(z_j+t(\cdot-z_j))\cdot(\cdot-z_j)\,dt\right\|_{\mathbb{L}^\infty(D_j)} \\ &\lesssim a\sum_{j=1}^{\aleph}\|G_k(x,\cdot)\|_{\mathbb{L}^2(D_j)}\,\|E\|_{\mathbb{L}^2(D_j)} \lesssim a\left(\sum_{j=1}^{\aleph}\|G_k(x,\cdot)\|^2_{\mathbb{L}^2(D_j)}\right)^{\frac{1}{2}}\|E\|_{\mathbb{L}^2(D)}, \end{aligned}$$

which, by knowing that x is away from D, such that the Green's kernel $G_k(x,\cdot)$ is smooth, allows us to get

$$|T_1(x)| = \mathcal{O}\big(a^{\frac{5-s}{2}}\,\|E\|_{\mathbb{L}^2(D)}\big).$$

Then,

$$E^s(x)-V^s(x) = -\,\omega^2\,\mu\sum_{j=1}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\int_{D_j} G_k(x,y)\cdot E(y)\,dy + \mathcal{O}\big(a^{\frac{5-s}{2}}\,\|E\|_{\mathbb{L}^2(D)}\big).$$

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In addition, by Taylor expansion for $G_k(x,\cdot)$, we obtain

$$E^s(x)-V^s(x)=-\omega^2\,\mu\sum_{j=1}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\,G_k(x,z_j)\cdot\int_{D_j}E(y)\,dy - T_2(x)+\mathcal{O}\big(a^{\frac{5-s}{2}}\ \|E\|_{\mathbb{L}^2(D)}\big),$$

where

$$T_2(x):=\omega^2\,\mu\sum_{j=1}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\int_{D_j}\int_0^1\nabla G_k(x,z_j+t(y-z_j))\cdot(y-z_j)\,dt\cdot E(y)\,dy.$$

We estimate the above term as

$$|T_2(x)|\lesssim\sum_{j=1}^{\aleph}|\epsilon_0(z_j)-\epsilon_p(\omega)|\left|\int_{D_j}\int_0^1\nabla G_k(x,z_j+t(y-z_j))\cdot(y-z_j)\,dt\cdot E(y)\,dy\right|,$$

which, by using the fact that $\big|\epsilon_0(z_j)-\epsilon_p(\omega)\big|\sim 1$ and the Hölder inequality, gives us

$$\begin{aligned}|T_2(x)|&\lesssim\sum_{j=1}^{\aleph}\left\|\int_0^1\nabla G_k(x,z_j+t(\cdot-z_j))\cdot(\cdot-z_j)\,dt\right\|_{\mathbb{L}^2(D_j)}\|E\|_{\mathbb{L}^2(D_j)}\\&\lesssim\left(\sum_{j=1}^{\aleph}\left\|\int_0^1\nabla G_k(x,z_j+t(\cdot-z_j))\cdot(\cdot-z_j)\,dt\right\|^2_{\mathbb{L}^2(D_j)}\right)^{\frac12}\|E\|_{\mathbb{L}^2(D)}.\end{aligned}$$

Again, using the smoothness of $\nabla G_k(x,\cdot)$, for x away from D, allows us to deduce that

$$|T_2(x)|=\mathcal{O}\big(a^{\frac{5-s}{2}}\ \|E\|_{\mathbb{L}^2(D)}\big).$$

Then,

$$\begin{aligned}E^s(x)-V^s(x)&=-\omega^2\,\mu\sum_{j=1}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\,G_k(x,z_j)\cdot\int_{D_j}E(y)\,dy+\mathcal{O}\big(a^{\frac{5-s}{2}}\ \|E\|_{\mathbb{L}^2(D)}\big)\\&\overset{(3.21)}{=}-\omega^2\,\mu\sum_{j=1}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\,G_k(x,z_j)\cdot\int_{D_j}E(y)\,dy+\mathcal{O}(a^{4-h-s}).\end{aligned}\tag{3.31}$$

In addition, by plugging (3.29) into (3.31), we derive that

$$\begin{aligned}&E^s(x)-V^s(x)\\&=-\mu\,a^3\,\omega^2\sum_{j=1}^{\aleph}\frac{\epsilon_0(z_j)\,(\epsilon_0(z_j)-\epsilon_p(\omega))}{\Lambda_{n_0,j}(\omega)}\left\langle V(z_j,\theta,q),\int_B e^{(3)}_{n_0}(y)\,dy\right\rangle G_k(x,z_j)\cdot\int_B e^{(3)}_{n_0}(y)\,dy\\&\quad+\mathcal{O}\big(a^{\min\left((3-s);\left(6-3t-2h-\frac{3s}{2}\right)\right)}\big),\end{aligned}\tag{3.32}$$

where $\Lambda_{n_0,j}(\omega)$ is a family of dispersion equations given by

$$\Lambda_{n_0,j}(\omega):=\big(\epsilon_0(z_j)-\lambda^{(3)}_{n_0}(B)\,(\epsilon_0(z_j)-\epsilon_p(\omega))\big).\tag{3.33}$$

Next, we solve the real part of the above dispersion equation. More precisely, we set $\omega_{P_\ell,n_0,j}$ to be solution to

$$Re\big(\Lambda_{n_0,j}\big(\omega_{P_\ell,n_0,j}\big)\big)=\big(Re(\epsilon_0(z_j))-\lambda^{(3)}_{n_0}(B)\,\big(Re(\epsilon_0(z_j))-Re\big(\epsilon_p\big(\omega_{P_\ell,n_0,j}\big)\big)\big)\big)=0,\tag{3.34}$$

which gives us

$$Re\big(\epsilon_p\big(\omega_{P_\ell,n_0,j}\big)\big)=Re(\epsilon_0(z_j))\,\frac{\big(\lambda^{(3)}_{n_0}(B)-1\big)}{\lambda^{(3)}_{n_0}(B)}.\tag{3.35}$$

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The existence and uniqueness of the solution $\omega_{P_\ell,n_0,j}$ to (3.34), or equivalently to (3.35), have already been discussed in [25, Lemma 5.7]. We have,

$$\omega^2_{P_\ell,n_0,j} = \frac{1}{2}\left[2\,\omega_0^2 - \gamma^2 + \frac{\omega_p^2\,\lambda^{(3)}_{n_0}(B)\,\epsilon_\infty}{\left[\mathrm{Re}\,(\epsilon_0(z_j))\,\left(1-\lambda^{(3)}_{n_0}(B)\right)+\lambda^{(3)}_{n_0}(B)\,\epsilon_\infty\right]} + \sqrt{\Delta}\right], \tag{3.36}$$

where Δ is such that

$$\Delta = \left(\frac{\omega_p^2\,\lambda^{(3)}_{n_0}(B)\,\epsilon_\infty}{\left[\mathrm{Re}\,(\epsilon_0(z_j))\,\left(1-\lambda^{(3)}_{n_0}(B)\right)+\lambda^{(3)}_{n_0}(B)\,\epsilon_\infty\right]}\right)^2$$
$$-\gamma^2\left(4\,\omega_0^2-\gamma^2+\frac{2\,\omega_p^2\,\epsilon_\infty\,\lambda^{(3)}_{n_0}(B)}{\left[\mathrm{Re}\,(\epsilon_0(z_j))\,\left(1-\lambda^{(3)}_{n_0}(B)\right)+\lambda^{(3)}_{n_0}(B)\,\epsilon_\infty\right]}\right).$$

Furthermore, the frequency $\omega_{P_\ell,n_0,j}$ satisfies

$$\Lambda_{n_0,j}\left(\omega_{P_\ell,n_0,j}\right) = Im(\epsilon_0(z_j))\left(1-\lambda^{(3)}_{n_0}(B)\right)+\lambda^{(3)}_{n_0}(B)\,Im\left(\epsilon_p\left(\omega_{P_\ell,n_0,j}\right)\right)$$
$$\left|\Lambda_{n_0,j}\left(\omega_{P_\ell,n_0,j}\right)\right| \lesssim \|Im\,(\epsilon_0)\|_{\mathbb{L}^\infty(\Omega)} + \left|Im\left(\epsilon_p\left(\omega_{P_\ell,n_0,j}\right)\right)\right| \underset{(1.4)}{\overset{(1.1)}{=}} \mathcal{O}(\gamma),$$

Consequently, if we assume that

$$\gamma = \mathcal{O}(a^h), \tag{3.37}$$

we derive from (3.37) the coming estimation

$$\Lambda_{n_0,j}\left(\omega_{P_\ell,n_0,j}\right) = \mathcal{O}(a^h).$$

In addition, if we let ω such that

$$\omega^2 = \omega^2_{P_\ell,n_0,j} \pm a^h, \tag{3.38}$$

we get

$$\begin{aligned}\Lambda_{n_0,j}(\omega) &\overset{(3.33)}{=} \epsilon_0(z_j) - \lambda^{(3)}_{n_0}(B)\,(\epsilon_0(z_j)-\epsilon_p(\omega))\\ &= Re(\epsilon_0(z_j)) - \lambda^{(3)}_{n_0}(B)\,Re\left(\epsilon_0(z_j)-\epsilon_p\left(\omega_{P_\ell,n_0,j}\right)\right) - \lambda^{(3)}_{n_0}(B)\,Re\left(\epsilon_p\left(\omega_{P_\ell,n_0,j}\right)-\epsilon_p(\omega)\right)\\ &\quad + i\,Im(\epsilon_0(z_j)) - \lambda^{(3)}_{n_0}(B)\,i\,(Im(\epsilon_0(z_j)) - Im(\epsilon_p(\omega)))\\ &\overset{(3.34)}{=} \lambda^{(3)}_{n_0}(B)\,Re\left(\epsilon_p(\omega)-\epsilon_p\left(\omega_{P_\ell,n_0,j}\right)\right) + i\,Im(\epsilon_0(z_j))\left(1-\lambda^{(3)}_{n_0}(B)\right) + i\,\lambda^{(3)}_{n_0}(B)\;Im(\epsilon_p(\omega)).\end{aligned}$$

Hence, by taking the modulus, we obtain

$$\begin{aligned}\left|\Lambda_{n_0,j}(\omega)\right| &\lesssim \left|Re\left(\epsilon_p(\omega)-\epsilon_p\left(\omega_{P_\ell,n_0,j}\right)\right)\right| + \|Im\,(\epsilon_0)\|_{\mathbb{L}^\infty(\Omega)} + |Im(\epsilon_p(\omega))|\\ &\underset{(1.4)}{\overset{(1.1)}{\lesssim}}, \left|Re\left(\epsilon_p(\omega)-\epsilon_p\left(\omega_{P_\ell,n_0,j}\right)\right)\right| + \gamma \overset{(1.1)}{\lesssim} \left|\omega^2-\omega^2_{P_\ell,n_0,j}\right| + \gamma,\end{aligned}$$

and, by using (3.37) and (3.38), we deduce that

$$\Lambda_{n_0,j}(\omega) = \mathcal{O}(a^h), \tag{3.39}$$

with ω satisfying (3.38). Now, by returning to (3.32) and using (3.33), we derive the following relation on the scattered fields

$$\begin{aligned}&E^s(x) - V^s(x)\\ &= -\mu\,a^3\,\omega^2\sum_{j=1}^{\aleph}\frac{\epsilon_0(z_j)\left(\epsilon_0(z_j)-\Lambda_{n_0,j}(\omega)\right)}{\lambda^{(3)}_{n_0}(B)\,\Lambda_{n_0,j}(\omega)}\left\langle V(z_j,\theta,q),\int_B e^{(3)}_{n_0}(y)\,dy\right\rangle G_k(x,z_j)\cdot\int_B e^{(3)}_{n_0}(y)\,dy\\ &\quad + \mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right).\end{aligned} \tag{3.40}$$

This proves (1.12).

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3.4. Estimation of the far fields

For the L.H.S of (3.40), in order to get the corresponding far-field expressions, we return to (3.30) that

$$E^s(x) - V^s(x) = -\omega^2 \mu \sum_{j=1}^{\aleph} \int_{D_j} G_k(x,y) \cdot E(y) \left(\epsilon_0(y) - \epsilon_p(\omega)\right) dy. \tag{3.41}$$

Furthermore, thanks to [15, Theorem 6.9], we know that the following asymptotic relation holds

$$G_k(x,y) = \frac{e^{i\,k\,|x|}}{|x|} \left(G_k^{\infty}(\hat{x},y) + \mathcal{O}\left(\frac{1}{|x|}\right)\right), \quad |x| \to +\infty. \tag{3.42}$$

Then, by plugging (3.42) into (3.41), we obtain

$$\begin{aligned} E^s(x) - V^s(x) = \frac{e^{i\,k\,|x|}}{|x|} \Bigg[& -\omega^2 \mu \sum_{j=1}^{\aleph} \int_{D_j} G_k^{\infty}(\hat{x},y) \cdot E(y) \left(\epsilon_0(y) - \epsilon_p(\omega)\right) dy \\ & + \mathcal{O}\left(\frac{1}{|x|} \sum_{j=1}^{\aleph} \int_{D_j} E(y) \left(\epsilon_0(y) - \epsilon_p(\omega)\right) dy \right) \Bigg]. \end{aligned}$$

By estimating the second term on the R.H.S, we obtain

$$\begin{aligned} T_0(x) &:= \frac{1}{|x|} \sum_{j=1}^{\aleph} \int_{D_j} E(y) \left(\epsilon_0(y) - \epsilon_p(\omega)\right) dy \\ |T_0(x)| &\leq \frac{1}{|x|} \sum_{j=1}^{\aleph} \left| \int_{D_j} E(y) \left(\epsilon_0(y) - \epsilon_p(\omega)\right) dy \right| \\ &\leq \frac{1}{|x|} \sum_{j=1}^{\aleph} \|E\|_{\mathbb{L}^2(D_j)} \ \|\epsilon_0(\cdot) - \epsilon_p(\omega)\|_{\mathbb{L}^{\infty}(D_j)} \ |D_j|^{\frac{1}{2}}, \end{aligned}$$

which, by knowing that $\left\|\epsilon_0(\cdot) - \epsilon_p(\omega)\right\|_{\mathbb{L}^{\infty}(D_j)} \sim 1$, gives

$$|T_0(x)| \lesssim \frac{1}{|x|} \sum_{j=1}^{\aleph} \|E\|_{\mathbb{L}^2(D_j)} \ |D_j|^{\frac{1}{2}} \lesssim \frac{1}{|x|} \ \|E\|_{\mathbb{L}^2(D)} \ |D|^{\frac{1}{2}} \overset{(3.21)}{=} \mathcal{O}\left(\frac{a^{3-h-s}}{|x|}\right).$$

Hence,

$$E^s(x) - V^s(x) = \frac{e^{i\,k\,|x|}}{|x|} \left[-\omega^2 \mu \sum_{j=1}^{\aleph} \int_{D_j} G_k^{\infty}(\hat{x},y) \cdot E(y) \left(\epsilon_0(y) - \epsilon_p(\omega)\right) dy + \mathcal{O}\left(\frac{a^{3-h-s}}{|x|}\right)\right].$$

This implies,

$$E^{\infty}(\hat{x}) - V^{\infty}(\hat{x}) = -\omega^2 \mu \sum_{j=1}^{\aleph} \int_{D_j} G_k^{\infty}(\hat{x},y) \cdot E(y) \left(\epsilon_0(y) - \epsilon_p(\omega)\right) dy.$$

Furthermore, by repeating the same arguments used to derive (3.40) and letting $|x| \gg 1$ in (3.40), we obtain

$$\begin{aligned} & E^{\infty}(\hat{x}) - V^{\infty}(\hat{x}) \\ & = -\mu\, a^3\, \omega^2 \sum_{j=1}^{\aleph} \frac{\epsilon_0(z_j) \left(\epsilon_0(z_j) - \Lambda_{n_0 j}(\omega)\right)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0 j}(\omega)} \left\langle V(z_j,\theta,q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle G_k^{\infty}(\hat{x},z_j) \cdot \int_B e_{n_0}^{(3)}(y)\, dy \\ & \quad + \mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right), \end{aligned}$$

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which by using (3.38) further indicates,

$$\begin{aligned} &E^\infty(\hat{x}) - V^\infty(\hat{x}) \\ &\quad = -\mu\, a^3 \sum_{j=1}^{\aleph} \frac{\omega_{P_\ell,n_0,j}^2\, \epsilon_0(z_j)\, \big(\epsilon_0(z_j) - \Lambda_{n_0,j}(\omega)\big)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} \left\langle V(z_j,\theta,q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle G_k^\infty(\hat{x}, z_j) \cdot \int_B e_{n_0}^{(3)}(y)\, dy \\ &\qquad + \mathcal{O}\Big(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\Big), \end{aligned}$$

with $\hat{x} \in \mathbb{S}^2$ and $G_k^\infty(\cdot, z)$ being the far field associated to the Green's kernel solution to (1.10). Besides, we have

$$\begin{aligned} \frac{\epsilon_0(z_j)\, \big(\epsilon_0(z_j) - \Lambda_{n_0,j}(\omega)\big)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} &= \frac{\epsilon_0(z_j)\, \big(\epsilon_0(z_j) - \Lambda_{n_0,j}\big(\omega_{P_\ell,n_0,j}\big)\big)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} + \frac{\epsilon_0(z_j)\, \big(\Lambda_{n_0,j}\big(\omega_{P_\ell,n_0,j}\big) - \Lambda_{n_0,j}(\omega)\big)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} \\ &= \frac{\epsilon_0(z_j)\, \big(\epsilon_0(z_j) - \Lambda_{n_0,j}\big(\omega_{P_\ell,n_0,j}\big)\big)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} + \mathcal{O}\,(1). \end{aligned}$$

Then,

$$\begin{aligned} &E^\infty(\hat{x}) - V^\infty(\hat{x}) \\ &\quad = -\mu\, a^3 \sum_{j=1}^{\aleph} \frac{\omega_{P_\ell,n_0,j}^2\, \epsilon_0(z_j)\, \big(\epsilon_0(z_j) - \Lambda_{n_0,j}\big(\omega_{P_\ell,n_0,j}\big)\big)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} \left\langle V(z_j,\theta,q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle G_k^\infty(\hat{x}, z_j) \cdot \int_B e_{n_0}^{(3)}(y)\, dy \\ &\qquad + \mathcal{O}\Big(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\Big). \end{aligned} \tag{3.43}$$

We will delay the proof of Theorem 1.1 until we announce and comment on a technical result named *Mixed Electromagnetic Reciprocity Relation*. Because the kernel $G_k^\infty(\cdot,\cdot)$ is unknown on the right-hand side of (3.43), since the kernel $G_k(\cdot,\cdot)$ is unknown, we need to substitute it with a known (measured) term. To achieve this, we propose setting the following result.

Proposition 3.1. *The following Mixed Electromagnetic Reciprocity Relation holds,*

$$\big(G_k^\infty\big)^\top(\hat{x}, z) \cdot (\hat{x} \times q) = -\frac{1}{4\,\pi}\, V(z, -\hat{x}, q), \quad z \in \mathbb{R}^3 \quad \textit{and} \quad \hat{x}, q \in \mathbb{S}^2. \tag{3.44}$$

Proof. See Subsection A.1. To avoid confusion, let us recall that the electromagnetic reciprocity relation refers to the relationship between two existing electromagnetic wave fields in terms of their incident, propagation and polarization directions. As per the proposition above, there is a correlation between the total field $V(\cdot,\cdot,\cdot)$ and the far field corresponding to the Green's kernel $G_k(\cdot,\cdot)$, solution of (1.10) for the heterogeneous medium, and it is named the *Mixed Electromagnetic Reciprocity Relation*. It's worth to emphasize that for the case of a homogeneous medium a reciprocity relation has been proved in [46, Section 5.1], and for the particular case of an exterior domain with specific boundary conditions, the mixed reciprocity relation has already been proved in [49, Theorem 5]. Here, we extend this mixed reciprocity relation to our transmission problem. □

By taking the inner product on each side of the equation given by (3.43) with the vector $(\hat{x} \times q)$, we obtain

$$\begin{aligned} &\langle (E^\infty - V^\infty)(\hat{x}), (\hat{x} \times q) \rangle \\ &\quad = -\mu\, a^3 \sum_{j=1}^{\aleph} \frac{\omega_{P_\ell,n_0,j}^2\, \epsilon_0(z_j)\, \big(\epsilon_0(z_j) - \Lambda_{n_0,j}\big(\omega_{P_\ell,n_0,j}\big)\big)}{\lambda_{n_0}^{(3)}(B)\, \Lambda_{n_0,j}(\omega)} \left\langle V(z_j,\theta,q), \int_B e_{n_0}^{(3)}(y)\, dy \right\rangle \\ &\qquad \cdot \left\langle G_k^\infty(\hat{x}, z_j) \cdot \int_B e_{n_0}^{(3)}(y)\, dy, (\hat{x} \times q) \right\rangle + \mathcal{O}\Big(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\Big), \end{aligned}$$

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or, equivalently

$$\begin{aligned}&\left\langle (E^{\infty}-V^{\infty})\,(\hat{x}),(\hat{x}\times q)\right\rangle\\&\quad=-\mu\,a^{3}\sum_{j=1}^{\aleph}\frac{\omega^{2}_{P_{\ell},n_0,j}\,\epsilon_0(z_j)\left(\epsilon_0(z_j)-\Lambda_{n_0,j}\left(\omega_{P_{\ell},n_0,j}\right)\right)}{\lambda^{(3)}_{n_0}(B)\,\Lambda_{n_0,j}(\omega)}\left\langle V(z_j,\theta,q),\int_B e^{(3)}_{n_0}(y)\,dy\right\rangle\\&\qquad\cdot\left\langle\int_B e^{(3)}_{n_0}(y)\,dy,\left(G^{\infty}_{k}\right)^{\top}(\hat{x},z_j)\cdot(\hat{x}\times q)\right\rangle+\mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right).\end{aligned}$$

Thanks to the Mixed Electromagnetic Reciprocity Relation in Proposition 3.1, formula (3.44), the above equation becomes

$$\begin{aligned}&\langle(E^{\infty}-V^{\infty})(\hat{x}),(\hat{x}\times q)\rangle\\&\quad=\frac{\mu\,a^{3}}{4\,\pi}\sum_{j=1}^{\aleph}\frac{\omega^{2}_{P_{\ell},n_0,j}\,\epsilon_0(z_j)\left(\epsilon_0(z_j)-\Lambda_{n_0,j}\left(\omega_{P_{\ell},n_0,j}\right)\right)}{\lambda^{(3)}_{n_0}(B)\,\Lambda_{n_0,j}(\omega)}\left\langle V(z_j,\theta,q),\int_B e^{(3)}_{n_0}(y)\,dy\right\rangle\\&\qquad\cdot\left\langle V(z_j,-\hat{x},q),\int_B e^{(3)}_{n_0}(y)\,dy\right\rangle+\mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right),\end{aligned}$$

which proves (1.13). In the particular case for the back-scattered direction, i.e. $\hat{x}=-\theta$, we deduce

$$\begin{aligned}&\langle(E^{\infty}-V^{\infty})\,(-\theta),(\theta\times q)\rangle\\&\quad=-\frac{\mu\,a^{3}}{4\,\pi}\sum_{j=1}^{\aleph}\frac{\omega^{2}_{P_{\ell},n_0,j}\,\epsilon_0(z_j)\left(\epsilon_0(z_j)-\Lambda_{n_0,j}\left(\omega_{P_{\ell},n_0,j}\right)\right)}{\lambda^{(3)}_{n_0}(B)\,\Lambda_{n_0,j}(\omega)}\left(\left\langle V(z_j,\theta,q),\int_B e^{(3)}_{n_0}(y)\,dy\right\rangle\right)^{2}\\&\qquad+\mathcal{O}\left(a^{\min\left((3-s),\left(6-3t-2h-\frac{3s}{2}\right)\right)}\right).\end{aligned}$$

This ends the proof Theorem 1.1.

4. Conclusion

In the recent years, there has been a growing interest, in the engineering community, in using contrast agents for different imaging modalities as ultrasound, optic tomography, elastography and different hybridized modalities. Such contrast agents are usually given by micro-scaled bubbles, nano-scaled particles or micro-mechanical inclusions. In this work, we described and analysed a quantitative optical imaging modality using plasmonic nanoparticles as contrast agents. We provided with a novel method to reconstruct the permittivity distribution, of an object to image, from the remotely measured electromagnetic fields.

The key principle is that these nanoparticles enjoy resonant effects and hence have the potential in enhancing the applied incident fields, while excited at certain particular frequencies called plasmonic resonances. We characterized these resonances, at the subwavelenght regime, as related to the eigenvalues of the Magnetization operator and then derived the generated electromagnetic fields in a heterogeneous medium containing multiple of such plasmonic nanoparticles. Based on these expansions, we designed an imaging functional, built-up from the remotely measured electric fields, that depends solely on the used incident frequencies. We showed that such a functional reaches its maximum values, in terms of the used incident frenquencies, only on the related plasmonic resonances. This allowed us to recover these resonances from which we extract the values of the electric permittivity distribution.

The proposed idea looks to be original and promising as it is fundamentally based on the resonant character of the contrast agents. As next steps, we need to confirm this approach computationally by providing numerical tests. In addition, we need to improve it in a number of directions. For example, we plan to study this approach while injecting the nanoparticles all-at-once and not one-by-one as it is proposed currently. To tackle this, we propose first to consider the case when these contrast agents are distributed regularly following a given density of distribution. In this case, we expect deterministic

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estimates. The ultimate step is to handle the case where such contrast agents are sent randomly and then derive probabilistic estimates.

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Appendix

This section will be divided into four parts. The first subsection is devoted to demonstrate the Mixed Electromagnetic Reciprocity Relation that was announced in Proposition 3.1. In the second subsection, we prove an a priori estimate related to the electric field that is used in Section 3. In the third subsection, we give sense of the formula (A.5). The last subsection is dedicated to justifying formula (A.12), which is related to the Green's kernel $G_k(\cdot,\cdot)$ and used in the proof of the Mixed Electromagnetic Reciprocity Relation.

A.1. Proof Proposition 3.1 (Mixed Electromagnetic Reciprocity Relation)

We start by recalling, from (1.10), that the Green's kernel associated to heterogeneous medium is the solution to

$$\nabla \times \nabla \times (G_k(\cdot,z)) - k^2(\cdot)\, G_k(\cdot,z) = \delta\,(\cdot,z)\; I_3, \quad \text{in} \quad \mathbb{R}^3, \tag{A.1}$$

where $k^2(\cdot) = k^2$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. The corresponding Green's kernel to the homogeneous medium is given by

$$\nabla \times \nabla \times (\Pi_k(\cdot,z)) - k^2\, \Pi_k(\cdot,z) = \delta\,(\cdot,z)\; I_3, \quad \text{in} \quad \mathbb{R}^3. \tag{A.2}$$

It is known that

$$\Pi_k(x,z) = \frac{1}{k^2} \underset{x}{\nabla} \underset{x}{\nabla} \Phi_k(x,z) + \Phi_k(x,z)\, I_3, \quad x \neq z, \tag{A.3}$$

where $\Phi_k(\cdot,\cdot)$ is the fundamental solution to the Helmholtz equation given by (1.8). Subtracting (A.2) from (A.1), it yields

$$\nabla \times \nabla \times ((G_k - \Pi_k)(\cdot,z)) - k^2\,(G_k - \Pi_k)(\cdot,z) = (k^2(\cdot) - k^2)\, G_k(\cdot,z), \quad \text{in} \quad \mathbb{R}^3, \tag{A.4}$$

and its solution is given by

$$(G_k - \Pi_k)(x,z) = \int_\Omega \Pi_k(x,y) \cdot (k^2(y) - k^2)\, G_k(y,z)\, dy, \quad x \in \mathbb{R}^3, \tag{A.5}$$

which will be provided later, see Subsection A.3. Ω represents the support of the right-hand side of (A.4). Hence, by taking its transpose and using the fact that $\Pi_k^\top(\cdot,\cdot) = \Pi_k(\cdot,\cdot)$, we obtain

$$\big(G_k^\top - \Pi_k\big)(x,z) = \int_\Omega G_k^\top(y,z) \cdot (k^2(y) - k^2)\, \Pi_k(x,y)\, dy, \quad x \in \mathbb{R}^3. \tag{A.6}$$

The corresponding far-field to (A.6) will be given by

$$\big(G_k^\top - \Pi_k\big)^\infty(\hat{x},z) = \int_\Omega G_k^\top(y,z) \cdot (k^2(y) - k^2)\, \Pi_k^\infty(\hat{x},y)\, dy, \quad x \in \mathbb{R}^3. \tag{A.7}$$

Moreover, from (A.3), we know that

$$\Pi_k^\infty(\hat{x},y) = \frac{e^{-ik\hat{x}\cdot y}}{4\pi}\,(I_3 - \hat{x} \otimes \hat{x}). \tag{A.8}$$

Then, by plugging the above expression into (A.7), we obtain

$$\big(G_k^\top - \Pi_k\big)^\infty(\hat{x},z) = \hat{x} \times \left(\int_\Omega \frac{e^{-ik\hat{x}\cdot y}}{4\pi}\,(k^2(y) - k^2)\, G_k^\top(y,z)\, dy \times \hat{x} \right).$$

Besides, by multiplying each side with the vector $(\hat{x} \times q)$, with the relation $\big(G_k^\top\big)^\infty = \big(G_k^\infty\big)^\top$, we obtain

$$\begin{aligned} \big(G_k^\infty\big)^\top(\hat{x},z) \cdot (\hat{x} \times q) &= \Pi_k^\infty(\hat{x},z) \cdot (\hat{x} \times q) + \int_\Omega \frac{e^{-ik\hat{x}\cdot y}}{4\pi}\,(k^2(y) - k^2)\, G_k^\top(y,z) \cdot (\hat{x} \times q)\, dy \\ &\overset{(4.8)}{=} \frac{e^{-ik\hat{x}\cdot z}}{4\pi}\,(\hat{x} \times q) + \int_\Omega \frac{e^{-ik\hat{x}\cdot y}}{4\pi}\,(k^2(y) - k^2)\, G_k^\top(y,z) \cdot (\hat{x} \times q)\, dy. \end{aligned} \tag{A.9}$$

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Now, from (1.6), we recall that

$$V^{Inc}(z,d,q)=(d\times q)\,e^{i\,k\,z\cdot d},\quad \text{for}\quad d,q\in\mathbb{S}^2,$$

we deduce, from (A.9), the following equation

$$\left(G_k^{\infty}\right)^{\top}(\hat{x},z)\cdot(\hat{x}\times q)=-\frac{1}{4\,\pi}\,V^{Inc}(z,-\hat{x},q)-\frac{1}{4\,\pi}\int_{\Omega}(k^2(y)-k^2)\,G_k^{\top}(y,z)\cdot V^{Inc}(y,-\hat{x},q)\,dy. \quad \text{(A.10)}$$

The total field equation should be recalled to analyse the second term on the R.H.S of the above equation. We have

$$\underset{z}{\nabla}\times\underset{z}{\nabla}\times(V(z,\hat{x},q))-k^2(z)\,V(z,\hat{x},q)=0,\quad z\in\mathbb{R}^3\quad\text{and}\quad\hat{x},q\in\mathbb{S}^2,$$

and, by knowing that $V(\cdot,\cdot,\cdot)=V^s(\cdot,\cdot,\cdot)+V^{Inc}(\cdot,\cdot,\cdot)$ with

$$\underset{z}{\nabla}\times\underset{z}{\nabla}\times(V^{Inc}(z,\hat{x},q))-k^2\,V^{Inc}(z,\hat{x},q)=0,\quad z\in\mathbb{R}^3,$$

we deduce

$$\underset{z}{\nabla}\times\underset{z}{\nabla}\times(V^{s}(z,\hat{x},q))-k^2(z)\,V^{s}(z,\hat{x},q)=(k^2(z)-k^2)\,V^{Inc}(z,\hat{x},q).$$

The solution to the above equation will be given by

$$V^s(z,\hat{x},q)=\int_{\Omega}G_k(z,y)\cdot(k^2(y)-k^2)\,V^{Inc}(y,\hat{x},q)\,dy. \quad \text{(A.11)}$$

Then, as we have

$$G_k(z,y)=G_k^{\top}(y,z),\quad y\neq z, \quad \text{(A.12)}$$

see Subsection A.4, we rewrite (A.11) as

$$V^s(z,\hat{x},q)=\int_{\Omega}G_k^{\top}(y,z)\cdot(k^2(y)-k^2)\,V^{Inc}(y,\hat{x},q)\,dy. \quad \text{(A.13)}$$

By gathering (A.10) and (A.13), we end up with the following equation

$$\left(G_k^{\infty}\right)^{\top}(\hat{x},z)\cdot(\hat{x}\times q)=-\frac{1}{4\,\pi}\,V^{Inc}(z,-\hat{x},q)-\frac{1}{4\,\pi}\,V^{s}(z,-\hat{x},q)=-\frac{1}{4\,\pi}\,V(z,-\hat{x},q).$$

This proves (3.44) and justifies the proof of Proposition 3.1.

A.2. A-priori estimate

Proposition A.1. *Under the condition*

$$3-3\,t\,-s-h>0,$$

we have,

$$\|u_1\|_{\mathbb{L}^2(D)}\lesssim a^{-h}\,\|u_0\|_{\mathbb{L}^2(D)}. \quad \text{(A.14)}$$

where the parameter h is such that $0<h<1$, *the parameter s fulfills* $\aleph\sim a^{-s}$ *and the parameter t is such that* $d\sim a^t$.

Proof. We assume that $D=\underset{j=1}{\overset{\aleph}{\cup}}D_j$ with $\aleph\sim a^{-s}$. To investigate Proposition A.1, we recall, from (3.6), that the total field $u_1(\cdot)$ is the solution to

$$u_1(x)+\omega^2\,\mu\int_D G_k(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy=u_0(x),\quad x\in\mathbb{R}^3.$$

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Now, we let $x \in D_m$ and we rewrite the above equation as

$$u_1(x) + \omega^2\,\mu \int_{D_m} G_k(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy + \omega^2\,\mu \sum_{\substack{j=1\\ j\neq m}}^{\aleph} \int_{D_j} G_k(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy = u_0(x), \quad x\in D_m. \tag{A.15}$$

In addition, thanks to [25, Theorem 2.1], we know that

$$G_k(x,y) = \frac{1}{\omega^2\,\mu\,\epsilon_0(y)}\,\nabla_y\,\nabla_y \Phi_0(x,y) + \Gamma_2(x,y), \qquad x\neq y. \tag{A.16}$$

Then, by plugging (A.16) into (A.15), we obtain

$$\begin{aligned} &u_1(x) - \nabla M_{D_m}(u_1(\cdot)\,\tau(\cdot))(x) + \omega^2\,\mu \sum_{\substack{j=1\\ j\neq m}}^{\aleph} \int_{D_j} G_k(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy \\ &= u_0(x) - \omega^2\,\mu \int_{D_m} \Gamma_2(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy, \qquad x\in D_m, \end{aligned} \tag{A.17}$$

where

$$\tau(y) := \frac{(\epsilon_0(y)-\epsilon_p(\omega))}{\epsilon_0(y)},$$

and $\nabla M_{D_m}(\cdot)$ is the Magnetization operator defined by (1.9). Furthermore, using Taylor expansion, the equation (A.17) becomes,

$$\begin{aligned} &\big[I - \tau(z_m)\,\nabla M_{D_m}\big](u_1)(x) \\ &= u_0(x) - \omega^2\,\mu \sum_{\substack{j=1\\ j\neq m}}^{\aleph} (\epsilon_0(z_j)-\epsilon_p(\omega))\,G_k(z_m,z_j)\cdot \int_{D_j} u_1(y)\;dy \\ &\quad - \omega^2\,\mu \sum_{\substack{j=1\\ j\neq m}}^{\aleph} (\epsilon_0(z_j)-\epsilon_p(\omega)) \int_0^1 \nabla_x G_k(z_m + t(x-z_m), z_j)\cdot(x-z_m)\,dt \cdot \int_{D_j} u_1(y)\;dy \\ &\quad - \omega^2\,\mu \sum_{\substack{j=1\\ j\neq m}}^{\aleph} (\epsilon_0(z_j)-\epsilon_p(\omega)) \int_{D_j}\int_0^1 \nabla_y G_k(z_m, z_j + t(y-z_j))\cdot(y-z_j)\,dt\cdot u_1(y)\;dy \\ &\quad - \omega^2\,\mu \sum_{\substack{j=1\\ j\neq m}}^{\aleph} \int_{D_j} G_k(x,y)\cdot u_1(y) \int_0^1 \nabla\epsilon_0(z_m + t(y-z_m))\cdot(y-z_m)\,dt\,dy \\ &\quad + \nabla M_{D_m}\left(u_1(\cdot)\int_0^1 \nabla\tau(z_m + t(\cdot - z_m))\cdot(\cdot - z_m)\,dt\right)(x) \\ &\quad - \omega^2\,\mu \int_{D_m} \Gamma_2(x,y)\cdot u_1(y)\,(\epsilon_0(y)-\epsilon_p(\omega))\,dy, \qquad x\in D_m. \end{aligned}$$

Besides, by scaling the above equation from the domain D_m to the domain B, i.e. we let $x = z_m + a\,\eta$ and $y = z_j + a\,\xi$ with $\eta,\ \xi \in B \subset B(0,1)$, we obtain

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$$
\begin{aligned}
&\big[I-\tau(z_m)\,\nabla M_B\big](\tilde{u}_{1,m})(\eta)\\
&\quad=\tilde{u}_{0,m}(\eta)-\omega^2\,\mu\,a^3\sum_{\substack{j=1\\ j\neq m}}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\,G_k(z_m,z_j)\cdot\int_B\tilde{u}_{1,j}(\xi)\,d\xi\\
&\qquad-\omega^2\,\mu\,a^4\sum_{\substack{j=1\\ j\neq m}}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\int_0^1\nabla_\eta G_k(z_m+t\,a\eta,z_j)\cdot\eta\,dt\cdot\int_B\tilde{u}_{1,j}(\xi)\,d\xi\\
&\qquad-\omega^2\,\mu\,a^4\sum_{\substack{j=1\\ j\neq m}}^{\aleph}(\epsilon_0(z_j)-\epsilon_p(\omega))\int_B\int_0^1\nabla_\xi G_k(z_m,\,z_j+t\,a\,\xi)\cdot\xi\,dt\cdot\tilde{u}_{1,j}(\xi)\,d\xi\\
&\qquad-\omega^2\,\mu\,a^4\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\int_B G_k(z_m+a\,\eta,z_j+a\,\xi)\cdot\tilde{u}_{1,j}(\xi)\int_0^1\nabla\epsilon_0(z_m+t\,a\,\xi)\cdot\xi\,dt\,d\xi\\
&\qquad+a\,\nabla M_B\left(\tilde{u}_{1,m}(\cdot)\int_0^1\nabla\tau(z_m+t\,a\,\cdot)\cdot\,dt\right)(\eta)\\
&\qquad-\omega^2\,\mu\,a^3\int_B\Gamma_2(z_m+a\eta,z_m+a\xi)\cdot\tilde{u}_{1,m}(\xi)\,(\epsilon_0(z_m+a\,\xi)-\epsilon_p(\omega))\,d\xi,\qquad\eta\in B,
\end{aligned}
$$

where $\tilde{u}_{k,j}(\eta):=u_k(z_j+a\,\eta)$, for $k=0\,,\ 1$ and $1\leq j\leq\aleph$. For the above equation, we successively take the inverse of the operator $[I-\tau(z_m)\,\nabla M_B]$, and then the $\|\cdot\|_{\mathbb{L}^2(B)}$-norm on the both sides, by using the fact that $\|\nabla M_B\|_{\mathcal{L}\left(\mathbb{L}^2(B);\mathbb{L}^2(B)\right)}=1$ and

$$
\left\|[I-\tau(z_m)\,\nabla M_B]^{-1}\right\|_{\mathcal{L}\left(\mathbb{L}^2(B);\mathbb{L}^2(B)\right)}=\mathcal{O}(a^{-h}),
$$

see for instance [25, Subsection 4.1], to obtain

$$
\begin{aligned}
\|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)}\lesssim a^{-h}\Bigg[&\|\tilde{u}_{0,m}\|_{\mathbb{L}^2(B)}+a^3\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\big|G_k(z_m,z_j)\big|\ \left\|\tilde{u}_{1,j}\right\|_{\mathbb{L}^2(B)}\\
&+a^4\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\big|\nabla G_k(z_m,z_j)\big|\ \left\|\tilde{u}_{1,j}\right\|_{\mathbb{L}^2(B)}+a\,\|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)}\\
&+a^3\left\|\int_B\Gamma_2(z_m+a\cdot,z_m+a\xi)\cdot\tilde{u}_{1,m}(\xi)\,(\epsilon_0(z_m+a\,\xi)-\epsilon_p(\omega))\,d\xi\right\|_{\mathbb{L}^2(B)}\Bigg].
\end{aligned}
$$

In addition, knowing that $h<1$ and

$$
\Gamma_2(z_m+a\cdot,z_m+a\xi)\sim\Phi_0^2(z_m+a\cdot,z_m+a\xi)=a^{-2}\,\frac{1}{|\cdot-\xi|^2},
$$

from singularity analysis point of view, see for instance [25, Theorem 2.1], we deduce that

$$
\begin{aligned}
\|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)}\lesssim a^{-h}\Bigg[&\|\tilde{u}_{0,m}\|_{\mathbb{L}^2(B)}+a^3\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\big|G_k(z_m,z_j)\big|\ \left\|\tilde{u}_{1,j}\right\|_{\mathbb{L}^2(B)}\\
&+a^4\sum_{\substack{j=1\\ j\neq m}}^{\aleph}\big|\nabla G_k(z_m,z_j)\big|\ \left\|\tilde{u}_{1,j}\right\|_{\mathbb{L}^2(B)}+a\,\|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)}\Bigg].
\end{aligned}
$$

Moreover, using the fact that

$$
\big|G_k(z_m,z_j)\big|\sim\frac{1}{d_{mj}^3}\quad\text{and}\quad\big|\nabla G_k(z_m,z_j)\big|\sim\frac{1}{d_{mj}^4},
$$

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we deduce that

$$\|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)} \lesssim a^{-h} \left[\|\tilde{u}_{0,m}\|_{\mathbb{L}^2(B)} + \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \left(\frac{a^3}{d_{mj}^3} + \frac{a^4}{d_{mj}^4} \right) \|\tilde{u}_{1,j}\|_{\mathbb{L}^2(B)} \right].$$

Besides, we have

$$\left(\frac{a^3}{d_{mj}^3} + \frac{a^4}{d_{mj}^4} \right) = \frac{a^3}{d_{mj}^3} \left(1 + \frac{a}{d_{mj}} \right) \leq \frac{a^3}{d_{mj}^3} \left(1 + \frac{a}{d} \right) = \frac{a^3}{d_{mj}^3} \left(1 + a^{1-t} \right) = \mathcal{O}\left(\frac{a^3}{d_{mj}^3} \right),$$

where the last equality is due to the fact that $t < 1$. Then,

$$\|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)} \lesssim a^{-h} \left[\|\tilde{u}_{0,m}\|_{\mathbb{L}^2(B)} + a^3 \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \frac{1}{d_{mj}^3} \|\tilde{u}_{1,j}\|_{\mathbb{L}^2(B)} \right].$$

By taking the square and summing up with respect to the index m on the both sides, we end up with the following estimation

$$\sum_{m=1}^{\aleph} \|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)}^2 \lesssim a^{-2h} \left[\sum_{m=1}^{\aleph} \|\tilde{u}_{0,m}\|_{\mathbb{L}^2(B)}^2 + a^6 \sum_{m=1}^{\aleph} \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \frac{1}{d_{mj}^6} \sum_{j=1}^{\aleph} \|\tilde{u}_{1,j}\|_{\mathbb{L}^2(B)}^2 \right].$$

Since

$$\sum_{m=1}^{\aleph} \sum_{\substack{j=1 \\ j\neq m}}^{\aleph} \frac{1}{d_{mj}^6} \leq d^{-6}\, \aleph^2 = \mathcal{O}(a^{-6t-2s}).$$

Then,

$$\sum_{m=1}^{\aleph} \|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)}^2 \lesssim a^{-2h} \sum_{m=1}^{\aleph} \|\tilde{u}_{0,m}\|_{\mathbb{L}^2(B)}^2 + a^{6-6t-2s-2h} \sum_{j=1}^{\aleph} \|\tilde{u}_{1,j}\|_{\mathbb{L}^2(B)}^2,$$

and, under the condition

$$3 - 3\,t - s - h > 0,$$

we deduce that

$$\sum_{m=1}^{\aleph} \|\tilde{u}_{1,m}\|_{\mathbb{L}^2(B)}^2 \lesssim a^{-2h} \sum_{m=1}^{\aleph} \|\tilde{u}_{0,m}\|_{\mathbb{L}^2(B)}^2$$

$$\|\tilde{u}_1\|_{\mathbb{L}^2(B)} \lesssim a^{-h} \|\tilde{u}_0\|_{\mathbb{L}^2(B)}.$$

This implies, by scaling back to the domain D,

$$\|u_1\|_{\mathbb{L}^2(D)} \lesssim a^{-h} \|u_0\|_{\mathbb{L}^2(D)}.$$

This proves (A.14) and completes the proof of Proposition A.1. □

A.3. Comprehending the formula (A.5)

In order to finish the proof of Proposition 3.1, it is necessary to comprehend (A.5). Recall (A.5) that

$$(G_k - \Pi_k)(x, z) = \int_\Omega \Pi_k(x, y) \cdot (k^2(y) - k^2)\, G_k(y, z)\, dy.$$

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To give sense to the R.H.S of the above equation, we use the explicit expression of the Green dyadic function $\Pi_k(\cdot,\cdot)$ given by (A.3), to obtain

$$\int_\Omega \Pi_k(x,y)\cdot(k^2(y)-k^2)\,G_k(y,z)\,dy = -\frac{1}{k^2}\nabla_x\int_\Omega \nabla_y\,\Phi_k(x,y)\cdot(k^2(y)-k^2)\,G_k(y,z)\,dy + \int_\Omega \Phi_k(x,y)\,(k^2(y)-k^2)\,G_k(y,z)\,dy,$$

which, by using the definition of the Magnetization operator and the Newtonian operator, see (1.7), can be rewritten as

$$\int_\Omega \Pi_k(x,y)\cdot(k^2(y)-k^2)\,G_k(y,z)\,dy = -\frac{1}{k^2}\nabla_x M_\Omega^k\left((k^2(\cdot)-k^2)\,G_k(\cdot,z)\right)(x) + N_\Omega^k\left((k^2(\cdot)-k^2)\,G_k(\cdot,z)\right)(x). \tag{A.18}$$

Now, for an arbitrary fixed point z, write

$$G_k(\cdot,z) = \Pi_k(\cdot,z) + (G_k-\Pi_k)(\cdot,z), \tag{A.19}$$

in Ω with $(G_k-\Pi_k)(\cdot,z)$ being a regular function, see for instance [25, Theorem 2.1]. More precisely, based on the proof of [25, Theorem 2.1], we know that

$$(G_k-\Pi_k)(\cdot,z)\in\mathbb{L}^{\frac{3}{2}-\delta}(\Omega),\quad \text{with } \delta\to 0^+. \tag{A.20}$$

Then, using (A.19), the equation (A.18) takes the following form,

$$\begin{aligned}\int_\Omega \Pi_k(x,y)\cdot(k^2(y)-k^2)\,G_k(y,z)\,dy = &-\frac{1}{k^2}\nabla_x M_\Omega^k\big((k^2(\cdot)-k^2)\,\Pi_k(\cdot,z)\big)(x)\\ &+N_\Omega^k((k^2(\cdot)-k^2)\,\Pi_k(\cdot,z))(x)\\ &-\frac{1}{k^2}\nabla_x M_\Omega^k((k^2(\cdot)-k^2)\,(G_k-\Pi_k)(\cdot,z))(x)\\ &+N_\Omega^k((k^2(\cdot)-k^2)\,(G_k-\Pi_k)(\cdot,z))(x).\end{aligned} \tag{A.21}$$

The third term and the fourth term on the R.H.S are well defined, as we know that $(G_k-\Pi_k)(\cdot,z)$ is a regular function and both the Magnetization operator and the Newtonian operator are bounded from $\mathbb{L}^p(\mathbb{R}^3)$ to $\mathbb{L}^p(\mathbb{R}^3)$, with $p>1$. Consequently, based on (A.20), we deduce that

$$\begin{aligned}T_0(x,z) := &-\frac{1}{k^2}\nabla_x M_\Omega^k((k^2(\cdot)-k^2)\,(G_k-\Pi_k)(\cdot,z))(x)\\ &+N_\Omega^k((k^2(\cdot)-k^2)\,(G_k-\Pi_k)(\cdot,z))(x)\ \in\ \mathbb{L}^{\frac{3}{2}-\delta}(\Omega).\end{aligned} \tag{A.22}$$

Hence, to give sense of the L.H.S of (A.21), it is enough to study the first term and the second term on the R.H.S accordingly. To accomplish this, we set

$$\begin{aligned}T_1(x,z) &:= \nabla_x M_\Omega^k((k^2(\cdot)-k^2)\,\Pi_k(\cdot,z))(x)\\ &\overset{(4.3)}{=} \frac{1}{k^2}\nabla_x M_\Omega^k\big((k^2(\cdot)-k^2)\,\nabla\nabla\Phi_k(\cdot,z)\big)(x)+\nabla_x M_\Omega^k\big((k^2(\cdot)-k^2)\,\Phi_k(\cdot,z)I_3\big)(x).\end{aligned}$$

Observe that the second term on the R.H.S is well defined as $\Phi_k(\cdot,z)I_3\in\mathbb{L}^{3-\delta}(\Omega)$ and $\nabla M_\Omega^k(\cdot)$ is bounded from $\mathbb{L}^p(\mathbb{R}^3)$ to $\mathbb{L}^p(\mathbb{R}^3)$, with $p>1$. Then,

$$T_{1,2}(\cdot,z) := \nabla_x M_\Omega^k\big((k^2(\cdot)-k^2)\,\Phi_k(\cdot,z)I_3\big)\in\mathbb{L}^{3-\delta}(\Omega), \tag{A.23}$$

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which implies,

$$\begin{aligned} T_1(x,z) &= \frac{1}{k^2}\nabla_x M_\Omega^k\big((k^2(\cdot)-k^2)\,\nabla\nabla\Phi_k(\cdot,z)\big)(x) + T_{1,2}(x,z) \\ &= -\frac{1}{k^2}\nabla_x M_\Omega^k\Big((k^2(\cdot)-k^2)\,\nabla_z\nabla\Phi_k(\cdot,z)\Big)(x) + T_{1,2}(x,z) \\ &= -\frac{1}{k^2}\nabla_z\Big(\nabla_x M_\Omega^k\big((k^2(\cdot)-k^2)\,\nabla\Phi_k(\cdot,z)\big)(x)\Big) + T_{1,2}(x,z). \end{aligned}$$

Next, we need to analyse the regularity of the term

$$T_{1,1}(x,z) := -\frac{1}{k^2}\nabla_z\Big(\nabla_x M_\Omega^k\left((k^2(\cdot)-k^2)\,\nabla\Phi_k(\cdot,z)\right)(x)\Big). \tag{A.24}$$

In order to avoid lengthy computation, we only examine the reduced formula given by

$$\begin{aligned} T_{1,1,r}(x,z) &= \nabla_z\Big(\nabla_x M_\Omega^k\left(\nabla\Phi_k(\cdot,z)\right)(x)\Big) \\ &\overset{(1.9)}{=} \nabla_z\nabla_x\int_\Omega \nabla_y\Phi_k(x,y)\cdot\nabla_y\Phi_k(y,z)\,dy \\ &= \nabla_x\Big(\nabla_z\Big(\nabla_z\cdot\Big(\nabla_x\cdot\big(N_\Omega^k\left(\Phi_k(\cdot,z)\,I_3\right)(x)\big)\Big)\Big)\Big), \end{aligned} \tag{A.25}$$

where $N^k(\cdot)$ is the Newtonian operator defined by (1.7). By using the Helmholtz decomposition (1.11), we have

$$\Phi_k(\cdot,z)\,I_3 = \sum_{j=1}^{3}\sum_{n\in\mathbb{N}}\left\langle\Phi_k(\cdot,z)\,I_3, e_n^{(j)}\right\rangle_{\mathbb{L}^2(\Omega)}\otimes e_n^{(j)}(\cdot) \overset{(1.7)}{=} \sum_{j=1}^{3}\sum_{n\in\mathbb{N}} N_\Omega^k\big(e_n^{(j)}\big)(z)\otimes e_n^{(j)}(\cdot).$$

Hence,

$$N_\Omega^k\left(\Phi_k(\cdot,z)\,I_3\right)(x) = \sum_{j=1}^{3}\sum_{n\in\mathbb{N}} N_\Omega^k\big(e_n^{(j)}\big)(z)\otimes N_\Omega^k\big(e_n^{(j)}\big)(x). \tag{A.26}$$

Then, by plugging (A.26) into (A.25), we deduce

$$\begin{aligned} T_{1,1,r}(x,z) &= \sum_{j=1}^{3}\sum_{n\in\mathbb{N}}\nabla_x\Big(\nabla_z\Big(\nabla_z\cdot\Big(\nabla_x\cdot\big(N_\Omega^k\big(e_n^{(j)}\big)(z)\otimes N_\Omega^k\big(e_n^{(j)}\big)(x)\big)\Big)\Big)\Big) \\ &= \sum_{j=1}^{3}\sum_{n\in\mathbb{N}}\nabla_x\Big(\nabla_z\Big(\nabla_z\cdot\big(N_\Omega^k\big(e_n^{(j)}\big)(z)\big)\nabla_x\cdot\big(N_\Omega^k\big(e_n^{(j)}\big)(x)\big)\Big)\Big), \end{aligned}$$

and, using the fact that for an arbitrary vector field F, there holds

$$\nabla M_\Omega^k(F) = -\nabla\nabla\cdot N_\Omega^k(F),$$

we derive that

$$T_{1,1,r}(x,z) = \sum_{j=1}^{3}\sum_{n\in\mathbb{N}}\nabla M_\Omega^k\big(e_n^{(j)}\big)(z)\otimes\nabla M_\Omega^k\big(e_n^{(j)}\big)(x).$$

Furthermore, knowing that ∇M_Ω^k restricted to $\mathbb{H}_0(\mathrm{div}\,=0)$ is a vanishing operator, we obtain

$$T_{1,1,r}(x,z) = \sum_{j=2}^{3}\sum_{n\in\mathbb{N}}\nabla M_\Omega^k\big(e_n^{(j)}\big)(z)\otimes\nabla M_\Omega^k\big(e_n^{(j)}\big)(x).$$

By recalling that $e_n^{(2,3)}(\cdot)\in\mathbb{L}^2(\Omega)$ and $\nabla M_\Omega^k(\cdot)$ is a bounded operator from $\mathbb{L}^2(\Omega)$ to $\mathbb{L}^2(\Omega)$, we deduce that, for z fixed in Ω,

$$T_{1,1,r}(\cdot,z)\in\mathbb{L}^2(\Omega).$$

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Hence, for (A.24),

$$T_{1,1}(\cdot, z) \in \mathbb{L}^2(\Omega). \tag{A.27}$$

Then, by gathering (A.23) and (A.27), we deduce that

$$T_1(\cdot, z) \in \mathbb{L}^2(\Omega). \tag{A.28}$$

In a similar way, for z fixed in Ω, we set

$$\begin{aligned} T_2(x, z) &:= N_\Omega^k((k^2(\cdot) - k^2)\, \Pi_k(\cdot, z))(x) \\ &\overset{(4.3)}{=} \frac{1}{k^2} N_\Omega^k((k^2(\cdot) - k^2)\, \nabla\nabla\Phi_k(\cdot, z))(x) + N_\Omega^k((k^2(\cdot) - k^2)\, \Phi_k(\cdot, z)\, I_3)(x). \end{aligned}$$

Since the second term on the R.H.S fulfills $\Phi_k(\cdot, z) I_3 \in \mathbb{L}^{3-\delta}(\Omega)$, we deduce that

$$N_\Omega^k\left((k^2(\cdot) - k^2)\, \Phi_k(\cdot, z)\, I_3\right)(x) \in \mathbb{W}^{2,3-\delta}(\Omega) \subset \mathbb{L}^{3-\delta}(\Omega).$$

Hence, in the sequel, to write short formulas, we denote it by $T_{2,2}(x, z)$. Then,

$$\begin{aligned} T_2(x, z) &= \frac{1}{k^2} N_\Omega^k\left((k^2(\cdot) - k^2)\, \nabla\nabla\Phi_k(\cdot, z)\right)(x) + T_{2,2}(x, z) \\ &= \frac{1}{k^2} \underset{x}{\nabla} \underset{x}{\nabla} N_\Omega^k\left((k^2(\cdot) - k^2)\, \Phi_k(\cdot, z)\right)(x) + T_{2,2}(x, z). \end{aligned}$$

Now, using the Calderon–Zygmund inequality, see [33, Theorem 9.9], we deduce that

$$\underset{x}{\nabla} \underset{x}{\nabla} N_\Omega^k\left((k^2(\cdot) - k^2)\, \Phi_k(\cdot, z)\right)(x) \in \mathbb{L}^{3-\delta}(\Omega).$$

This implies,

$$T_2(\cdot, z) \in \mathbb{L}^{3-\delta}(\Omega). \tag{A.29}$$

Gathering (A.22), (A.28) and (A.29) allows us to deduce

$$\int_\Omega \Pi_k(\cdot, y) \cdot (k^2(y) - k^2)\, G_k(y, z)\, dy \in \mathbb{L}^{\frac{3}{2}-\delta}(\Omega),$$

and proves the well-posed character of (A.5).

A.4. Justification of A.12

Let E and F be two compactly supported smooth vector fields. We consider the solutions to the two following problems:

$$\nabla \times \nabla \times \left(V^E(x)\right) - k^2(x)\, V^E(x) = E(x), \quad \text{in} \quad \mathbb{R}^3, \tag{A.30}$$

$$\nabla \times \nabla \times \left(V^F(x)\right) - k^2(x)\, V^F(x) = F(x), \quad \text{in} \quad \mathbb{R}^3, \tag{A.31}$$

where V^E and V^F satisfsfy the Silver–Müller radiation conditions. Based on (1.10), the solution $V^E(\cdot)$ can be given by

$$V^E(x) := (G_k \star E)(x), \quad x \in \mathbb{R}^3,$$

where the convolution operator should be understood in the following sense,

$$V^E(x) = \int_{\mathbb{R}^3} G_k(x, y) \cdot E(y)\, dy. \tag{A.32}$$

The formula above can be comprehended as explained in Section 3, formula (3.2). In the same manner, we have

$$V^F(x) := (G_k \star F)(x) = \int_{\mathbb{R}^3} G_k(x, y) \cdot F(y)\, dy, \quad x \in \mathbb{R}^3. \tag{A.33}$$

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Since the vector fields E and F are compactly supported, then we can replace the integral set by a ball of centre zero and radius R, i.e. $B(0,R)$, with R large enough such that $\Omega \subset B(0,R)$. In $\mathbb{L}^2(B(0,R))$, by taking the inner product for (A.31) with V^E and for (A.30) with V^F, we can obtain that

$$\langle E, V^F\rangle_{\mathbb{L}^2(B(0,R))} - \langle V^E, F\rangle_{\mathbb{L}^2(B(0,R))} = \int_{B(0,R)} \nabla_x \cdot [(\nabla \times (V^E)) \times V^F + V^E \times (\nabla \times (V^F))]\, dx.$$

Moreover, for the same reason that E and F are of compact support, we can replace the inner product $\langle \cdot, \cdot\rangle_{\mathbb{L}^2(B(0,R))}$, appearing on the L.H.S, by $\langle \cdot, \cdot\rangle_{\mathbb{L}^2(\mathbb{R}^3)}$. Then,

$$\langle E, V^F\rangle_{\mathbb{L}^2(\mathbb{R}^3)} - \langle V^E, F\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = \int_{B(0,R)} \nabla_x \cdot \left[\left(\nabla \times (V^E)\right) \times V^F + V^E \times \left(\nabla \times (V^F)\right)\right]\, dx.$$

In addition, since the L.H.S is independent on the parameter R, by taking the limit as $R \to \infty$, we obtain

$$\langle E, V^F\rangle_{\mathbb{L}^2(\mathbb{R}^3)} - \langle V^E, F\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = J, \tag{A.34}$$

where

$$J := \lim_{R\to+\infty} \int_{B(0,R)} \nabla_x \cdot [(\nabla \times (V^E)) \times V^F + V^E \times (\nabla \times (V^F))]\, dx.$$

Now, we show that J is vanishing. By using the divergence theorem to rewrite J as,

$$\begin{aligned} J &= \lim_{R\to+\infty} \int_{\partial B(0,R)} (\nabla \times (V^E) \cdot (V^F \times \nu) - \nabla \times (V^F) \cdot (V^E \times \nu))\, d\sigma \\ &= \lim_{R\to+\infty} \int_{\partial B(0,R)} (\nabla \times (V^E) \cdot (V^F \times \nu - \nabla \times (V^F)) + \nabla \times (V^E) \cdot \nabla \times (V^F))\, d\sigma \\ &\quad - \lim_{R\to+\infty} \int_{\partial B(0,R)} (\nabla \times (V^F) \cdot (V^E \times \nu - \nabla \times (V^E)) + \nabla \times (V^F) \cdot \nabla \times (V^E))\, d\sigma, \end{aligned}$$

which can be simplified as

$$\begin{aligned} J &= \lim_{R\to+\infty} \int_{\partial B(0,R)} \nabla \times (V^E) \cdot (V^F \times \nu - \nabla \times (V^F))\, d\sigma \\ &\quad - \lim_{R\to+\infty} \int_{\partial B(0,R)} \nabla \times (V^F) \cdot (V^E \times \nu - \nabla \times (V^E))\, d\sigma. \end{aligned} \tag{A.35}$$

We set and estimate the second term on the R.H.S. as

$$\begin{aligned} J_2 &:= \lim_{R\to+\infty} \int_{\partial B(0,R)} \nabla \times V^F \cdot (V^E \times \nu - \nabla \times V^E)\, d\sigma \\ |J_2| &\le \lim_{R\to+\infty} \|\nabla \times V^F\|_{\mathbb{L}^2(\partial B(0,R))}\, \|V^E \times \nu - \nabla \times V^E\|_{\mathbb{L}^2(\partial B(0,R))}. \end{aligned} \tag{A.36}$$

Since $V^E(\cdot)$ is a radiating solution to the Maxwell equation, the following Silver–Müller radiation condition holds,

$$\lim_{|x|\to+\infty} |x|\, (V^E \times \nu - \nabla \times V^E)(x) = 0, \tag{A.37}$$

see [15, Definition 6.6]. The formula (A.37), for $|x| \gg 1$, implies

$$|(V^E \times \nu - \nabla \times V^E)(x)| \lesssim \frac{1}{|x|^{1+\alpha}}, \quad \alpha \in \mathbb{R}^+, \tag{A.38}$$

hence, for $R \gg 1$, we deduce

$$\begin{aligned} \|V^E \times \nu - \nabla \times V^E\|^2_{\mathbb{L}^2(\partial B(0,R))} &:= \int_{\partial B(0,R)} |V^E \times \nu - \nabla \times V^E|^2(x)\, d\sigma(x) \\ &\overset{(4.38)}{\lesssim} \int_{\partial B(0,R)} \frac{1}{|x|^{2(1+\alpha)}}\, d\sigma(x) = \frac{4\pi}{|R|^{2\alpha}}. \end{aligned}$$

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Using the above estimation, the inequality (A.36) becomes,

$$|J_2| \lesssim \lim_{R\to+\infty} \|\nabla \times V^F\|_{\mathbb{L}^2(\partial B(0,R))} \frac{1}{R^\alpha}. \tag{A.39}$$

Now, we are in a position to estimate $\left\|\nabla \times V^F\right\|_{\mathbb{L}^2(\partial B(0,R))}$. By taking the Curl operator on the both sides of (A.33), there holds

$$\begin{aligned}\nabla_x \times (V^F(x)) &= \nabla_x \times \int_\Omega G_k(x,y)\cdot F(y)\,dy \\ &= \nabla_x \times \int_\Omega \Pi_k(x,y)\cdot F(y)\,dy + \nabla_x \times \int_\Omega (G_k-\Pi_k)(x,y)\cdot F(y)\,dy.\end{aligned}$$

Now, by using the $\Pi_k(\cdot,\cdot)$'s expression given by (A.3), the definition of both the Magnetization and the Newtonian operators given by (1.7) as well as the expression of $(G_k-\Pi_k)(\cdot,\cdot)$ given by (A.5), we can rewrite the above equation as

$$\nabla \times (V^F) = \nabla \times \left(N^k_\Omega(F)\right) + \nabla \times \left(N^k_\Omega((k^2(\cdot)-k^2)\,V^F(\cdot))\right).$$

Now, using the fact that

$$\nabla \times N^k_\Omega(E) = N^k_\Omega(\nabla \times E) - SL^k_{\partial\Omega}(\nu \times E),$$

where $SL^k_{\partial\Omega}$ is the single-layer operator defined by

$$SL^k_{\partial\Omega}(f)(x) := \int_{\partial\Omega} \phi_k(x,y) f(y)\,d\sigma(y), \quad x \in \mathbb{R}^3,$$

we deduce

$$\begin{aligned}\nabla \times (V^F) = N^k_\Omega(\nabla \times F) - SL^k_{\partial\Omega}(\nu \times F) + N^k_\Omega(\nabla \times ((k^2(\cdot)-k^2)V^F(\cdot))) \\ - SL^k_{\partial\Omega}((k^2(\cdot)-k^2)\nu \times V^F(\cdot)).\end{aligned}$$

Then, by taking $\|\cdot\|_{\mathbb{L}^2(\partial B(0,R))}$-norm on the both sides of the above equation, we obtain

$$\begin{aligned}\|\nabla \times V^F\|^2_{\mathbb{L}^2(\partial B(0,R))} &\lesssim \|N^k_\Omega(\nabla \times F)\|^2_{\mathbb{L}^2(\partial B(0,R))} + \|SL^k_{\partial\Omega}(\nu \times F)\|^2_{\mathbb{L}^2(\partial B(0,R))} \\ &\quad + \|N^k_\Omega(\nabla \times ((k^2(\cdot)-k^2)\,V^F(\cdot)))\|^2_{\mathbb{L}^2(\partial B(0,R))} \\ &\quad + \|SL^k_{\partial\Omega}((k^2(\cdot)-k^2)\nu \times V^F(\cdot))\|^2_{\mathbb{L}^2(\partial B(0,R))} \\ &\lesssim \int_{\partial B(0,R)}\int_\Omega |\Phi_0(x,y)|^2\,dy\,d\sigma(x)\ \rho_1 + \int_{\partial B(0,R)}\int_{\partial\Omega} |\Phi_0(x,y)|^2\,d\sigma(y)\,d\sigma(x)\ \rho_2,\end{aligned}$$

where

$$\begin{aligned}\rho_1 &:= \|\nabla \times F\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla \times ((k^2(\cdot)-k^2)V^F(\cdot))\|^2_{\mathbb{L}^2(\Omega)} \\ \rho_2 &:= \|\nu \times F\|^2_{\mathbb{L}^2(\partial\Omega)} + \|(k^2(\cdot)-k^2)\nu \times V^F\|^2_{\mathbb{L}^2(\partial\Omega)}\end{aligned}$$

As we have assumed the vector field $F(\cdot)$ to be smooth, hence $V^F(\cdot)$ will also be smooth and the following relation holds

$$\max(\rho_1\,;\rho_2) = \mathcal{O}(1).$$

This implies that

$$\|\nabla \times V^F\|^2_{\mathbb{L}^2(\partial B(0,R))} \lesssim \int_{\partial B(0,R)}\int_\Omega |\Phi_0(x,y)|^2\,dy\,d\sigma(x) + \int_{\partial B(0,R)}\int_{\partial\Omega} |\Phi_0(x,y)|^2\,d\sigma(y)\,d\sigma(x).$$

Since $|x| \gg 1$, we obtain

$$\|\nabla \times V^F\|^2_{\mathbb{L}^2(\partial B(0,R))} \lesssim \int_{\partial B(0,R)} \frac{1}{|x|^2}\,d\sigma(x) = 4\pi.$$

Therefore, in (A.39), we deduce that

$$|J_2| \lesssim \lim_{R\to+\infty} \frac{1}{R^\alpha} = 0.$$

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Similar arguments allow us to justify that

$$J_1 := \lim_{R\to+\infty} \int_{\partial B(0,R)} Curl(V^E)\cdot(V^F\times \nu - Curl(V^F))\,d\sigma = 0.$$

Then, as $J = J_1 - J_2$, see (A.35), we deduce that

$$J = 0,$$

from (A.34), which further indicates that

$$\langle E, V^F\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = \langle V^E, F\rangle_{\mathbb{L}^2(\mathbb{R}^3)}.$$

By using (A.32) and (A.33), we can know that

$$\begin{aligned}
\int_{\mathbb{R}^3} E(x)\cdot\int_{\mathbb{R}^3} G_k(x,y)\cdot F(y)\,dy\,dx &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_k(x,y)\cdot E(y)\,dy\cdot F(x)\,dx\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_k(x,y)\cdot E(y)\cdot F(x)\,dy\,dx\\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} E(y)\cdot G_k^{\top}(x,y)\cdot F(x)\,dy\,dx\\
&= \int_{\mathbb{R}^3} E(y)\cdot\int_{\mathbb{R}^3} G_k^{\top}(x,y)\cdot F(x)\,dx\,dy\\
&= \int_{\mathbb{R}^3} E(x)\cdot\int_{\mathbb{R}^3} G_k^{\top}(y,x)\cdot F(y)\,dy\,dx.
\end{aligned}$$

Since E and F are two arbitrary vector fields, we deduce that

$$G_k(x,y) = G_k^{\top}(y,x), \quad x\neq y.$$

This justifies (A.12).

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