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REPRESENTING CONJUNCTIVE DEDUCTIONS BY DISJUNCTIVE DEDUCTIONS

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Abstract. A skeleton of the category with finite coproducts $\mathcal{D}$ freely generated by a single object has a subcategory isomorphic to a skeleton of the category with finite products $\mathcal{C}$ freely generated by a countable set of objects. As a consequence, we obtain that $\mathcal{D}$ has a subcategory equivalent with $\mathcal{C}$. From a proof-theoretical point of view, this means that up to some identifications of formulae the deductions of pure conjunctive logic with a countable set of propositional letters can be represented by deductions in pure disjunctive logic with just one propositional letter. By taking opposite categories, one can replace coproduct by product, i.e., disjunction by conjunction, and the other way round, to obtain the dual results.

§1. Introduction. In general proof theory, as conceived by Prawitz (1971), one addresses the question "What is a proof?" or "What is a deduction?"—a deduction being a hypothetical proof, i.e., a proof with hypotheses—by dealing with questions related to normal forms for deductions, and in particular with the question of identity criteria for deductions. This theory deals with the structure of deductions, which in one way can be shown with the help of the typed lambda calculus in the Curry–Howard correspondence, and not with their strength measured by ordinals, which is what one finds in proof theory that arose out of Hilbert's programme.

Much of general proof theory is the field of categorial (or categorical) proof theory. Fundamental notions of category theory like the notion of adjoint functor, and very important structures like cartesian closed categories, came to be of central concern for logic in that field. Through results of categorial proof theory called coherence results, which provide a model theory for equality of deductions, logic finds new ties with geometry, topology, and algebra (see the books Došen (1999), Došen & Petrić (2004), and Došen & Petrić (2007), the more recent introductory survey Došen (2014), and references therein).

In general proof theory, and in particular in categorial proof theory, one deals with an algebra of deductions, and for that, one concentrates on the operations of this algebra, which come with the rules of inference. As an equational theory, the algebra of deductions involves the question of identity criteria for deductions, the central question of general proof theory. (This question may be found, at least implicitly, in Hilbert's 24th problem; see Došen (2015).)

In categorial proof theory, one usually studies a freely generated category of a certain kind equationally presented. This freely generated category is constructed out of syntactical material, as in universal algebra one constructs a freely generated algebra of a certain kind equationally presented by factoring through an equivalence relation on terms. In categories we have partial algebras—the arrow terms out of which the equivalence classes are built have types, their sources and targets—but there is no significant mathematical difference

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in the construction when compared with what one has in universal algebra without types (see Došen & Petrić, 2004, chap. 2, in particular in sec. 2.3). The objects of this freely constructed categories are propositions, i.e., formulae, and the arrows, i.e., the equivalence classes of arrow terms, are deductions, i.e., equivalence classes of particular derivations, whose sources are premises and whose targets are conclusions. (Our dealing only with deductions with not more than one premise will not limit the mathematical importance of the results we consider, because once conjunction becomes available, a finite number of premises can always be replaced by their conjunction, and zero premises can be replaced by the propositional constant $\top$.) For deductions we have the partial operation of composition and identity deductions (this is essential for them; see Došen (2016)). The categories in question are interesting if they are not flatly preorders, i.e., if not all arrows with the same source and the same target are equal. Otherwise, the proof theory is trivial: any deductions with the same premises and the same conclusions become equal.

We deal with pure disjunctive logic with the alphabet including just the symbol $\vee$ for the connective of disjunction, the propositional constant $\bot$, and a single propositional letter. The proof theory of this fragment, according to Prawitz's equivalence of derivations in its beta-eta version (see Prawitz, 1971) and Lawvere's and Lambek's ideas about identity of deductions (see Lawvere, 1969 and Lambek, 1972), is to be identified with the category with finite, possibly empty, coproducts freely generated by a single object. In this proof theory we have only hypothetical proofs, i.e., we have no proofs without hypotheses. We also consider pure conjunctive logic based on the alphabet including just the symbol $\wedge$ for the connective of conjunction, the propositional constant $\top$, and a countably infinite set of propositional letters. The proof theory of that fragment is to be identified with the category with finite, possibly empty, products freely generated by a countable set of objects.

What one thinks of immediately when one has to connect conjunction with disjunction is to put everything upside down, i.e., dualize. The question whether without dualizing disjunction can imitate conjunction, or the other way round, is however different, and the answer to this question is not straightforward. The goal of this paper is to give an answer to it in the context of categorial proof theory. Our result that, up to some identifications of formulae, the deductions of pure conjunctive logic with a countable set of propositional letters can be represented by deductions in pure disjunctive logic with just one propositional letter, without dualizing, is rather unexpected.

To achieve this result, the propositional letters of conjunctive logic are coded by disjunctive formulae with a single propositional letter. So propositional variables are not translated by propositional variables.

We will show that there is an embedding, i.e., faithful functor, which is one-one on objects from a category equivalent to the latter category to a category equivalent to the former category. The main ingredient of our proof is the faithfulness of a structure-preserving functor from a free cartesian category to the category *Set*. Čubrić proved in Dj. Čubrić (1998) that there exists a faithful structure-preserving functor from a free cartesian closed category to the category *Set*. Our result may be taken as related to that previous result, but it cannot be inferred from it.

In our proofs we rely on coherence results and results involving the representation theory of algebras called Brauerian algebras we have obtained before, for which we give references at appropriate places in the text. The results about representations of Brauerian algebras are however generalized. All these results enabled us to get new results about faithful embeddings, mentioned in the preceding paragraph, in a simple manner. We think that without proceeding in a manner such as ours, by relying on our previous coherence and representation results, what is achieved in this paper would be quite difficult to reach.

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§2. The category Set_ω and its coproducts. Let ω be the set of finite ordinals $0, 1, \ldots, n, \ldots$, i.e., $\emptyset, \{0\}, \ldots, \{0, \ldots, n-1\}, \ldots$, and let Set_ω be the full subcategory of *Set* whose set of objects is ω. This category is a strict monoidal category with finite coproducts freely generated by the single object 1. This can be demonstrated as follows.

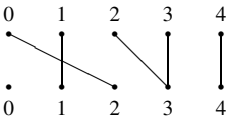
The empty coproduct in Set_ω is the object 0 and the binary coproduct on objects is given by addition. The binary coproduct on arrows is given by putting "side by side," i.e., the coproduct of $f : n \to m$ and $f' : n' \to m'$ is given by the function $g : n + n' \to m + m'$ such that

$$g(i) = \begin{cases} f(i) & \text{if} \quad 0 \leq i \leq n-1, \\ m + f'(i-n) & \text{if} \quad n \leq i \leq n + n' - 1. \end{cases}$$

For example, if $f : 2 \vdash 3$ and $f' : 3 \vdash 2$ are given, respectively, by the following two pictures (representing functions going downwards, with sources, i.e., domains, above, and targets, i.e., codomains, below)



then $f + f' : 5 \to 5$ is given by

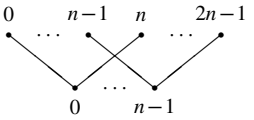


It is clear that $(Set_\omega, +, 0)$ is a strict monoidal category (see Mac Lane, 1971, sec. VII.1). This category is skeletal in the sense of Mac Lane (1971, sec. IV.4), i.e., any two objects isomorphic in it are identical. So this category is a skeleton of itself, where a *skeleton* of a category $\mathcal{K}$ is any full subcategory $\mathcal{A}$ of $\mathcal{K}$ such that each object of $\mathcal{K}$ is isomorphic in $\mathcal{K}$ to exactly one object of $\mathcal{A}$. (The category $(Set_\omega, +, 0)$ is a PROP in the sense of Mac Lane (1965, chap. V).)

The unique arrow $\check{\kappa} : 0 \to n$ is the empty function. The first injection $\check{k}^1 : n \to n + m$ and the second injection $\check{k}^2 : m \to n + m$ are as expected:



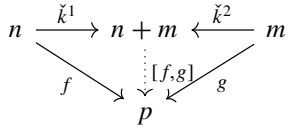
The codiagonal arrow $\check{\omega} : n + n \to n$ is given by



Note that, for complete precision, we would have to label $\check{k}^1$ and $\check{k}^2$ above by n and m; analogously we would have to label $\check{\kappa}$ and $\check{\omega}$ by n. We will however omit these labels, since they can be reconstructed from the contexts.

For arrows $f : n \to p$ and $g : m \to p$, the unique arrow $[f, g] : n + m \to p$, which makes the following diagram commute

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is the composition $\check{w} \circ (f + g)$.

Let Cat_+ be the subcategory of Cat whose objects are the categories with finite, strict monoidal coproducts and whose arrows are functors preserving this structure "on the nose." We have just shown that Set_ω is an object of Cat_+. Let $\mathcal{D}$ (where D comes for disjunction) be a category with finite, strict monoidal, coproducts freely generated by the set $\{p\}$ of objects. This category is the image of $\{p\}$ under the left adjoint of the forgetful functor from Cat_+ to Set that maps a category to the set of its objects. One can construct the category $\mathcal{D}$ out of syntactic material, but we will not go into this construction here (see Došen & Petrić, 2004, chap. 2).

Since $\mathcal{D}$ is freely generated by $\{p\}$, there is a unique arrow $F : \mathcal{D} \to Set_\omega$ of Cat_+, which extends the function from $\{p\}$ to ω mapping p to 1. The following proposition stems from Kelly (1972) (see p. 129, where a related dual result is announced), Mints (1980, theorem 2.2), Troelstra & Schwichtenberg (1996, theorem 8.2.3, p. 207), Petrić (2002, sec. 7), and Došen & Petrić (2001).

PROPOSITION 2.1. *The functor F is an isomorphism.*

Proof. It is trivial to check that F is one-one and onto on objects. The faithfulness of F is a result just dual to Cartesian Coherence of Došen & Petrić (2004, sec. 9.2). It remains to show that F is full, and since F is one-one on objects, it is enough to verify that an arbitrary function $f : n \to m$ can be built in terms of identities, composition, empty functions, injections, and brackets [,]. We proceed by induction on $n \geq 0$.

If $n = 0$, then $f = \check{\kappa}$, the empty function. If $n = m = 1$, then f is the identity. Suppose $n = 1$ and $m > 1$. If $f(0) = 0$, then $f = \check{k}^1$, and if $f(0) = m-1$, then $f = \check{k}^2$. If $f(0) = i$ for $0 < i < m-1$, then $f = \check{k}^2 \circ \check{k}^1$, for $\check{k}^1 : 1 \to m-i$ and $\check{k}^2 : m-i \to m$.

If $n = i + 1$, for $i \geq 1$, then $f = [f_1, f_2]$ for $f_1 : i \to m$ and $f_2 : 1 \to m$, and we just apply the induction hypothesis to f_1 and f_2. □

§3. Products in Set_ω. Besides the structure given by finite coproducts, the category Set_ω has the structure given by finite products. The empty product in Set_ω is the object 1 and the binary product on objects is given by multiplication. As in the case of coproducts, we will omit the labels of all the arrows since they can be reconstructed from the contexts.

Let $\iota : n \times m \to n \cdot m$ be the bijection defined by $\iota(i, j) = i \cdot m + j$. The inverse of this function is defined by $\iota^{-1}(i) = (\lfloor i/m \rfloor, i \bmod m)$, where $\lfloor i/m \rfloor$ is the quotient and $i \bmod m$ is the remainder for the division of i by m. In other words, $\iota^{-1}(i)$ is the element at the i-th place (starting from 0) in the lexicographically ordered set $n \times m$. For example, if $n = 2$ and $m = 3$, then $\iota^{-1}(1) = (0, 1)$ and $\iota^{-1}(5) = (1, 2)$.

The product $f_1 \cdot f_2 : n_1 \cdot n_2 \to m_1 \cdot m_2$ of arrows $f_1 : n_1 \to m_1$ and $f_2 : n_2 \to m_2$ is defined as $\iota \circ (f_1 \times f_2) \circ \iota^{-1}$, where $\times$ is the standard product in Set, i.e.,

$$(f_1 \times f_2)(i_1, i_2) = (f_1(i_1), f_2(i_2)).$$

For example, if $f_1 : 2 \to 2$ and $f_2 : 3 \to 2$ are given, respectively, by



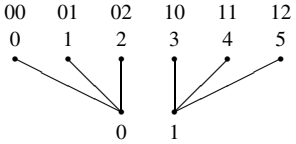
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then $f_1 \cdot f_2 : 6 \to 4$ and $f_2 \cdot f_1 : 6 \to 4$ are given, respectively, by

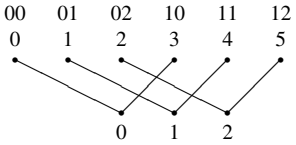


It is easy to verify that the structure on Set_ω given by $\cdot$ and 1 is again strict monoidal. Moreover, since Set_ω is skeletal, the product commutes on objects; we always have $n \cdot m = m \cdot n$ but this does not mean, as it is shown above, that we always have $f_1 \cdot f_2 = f_2 \cdot f_1$.

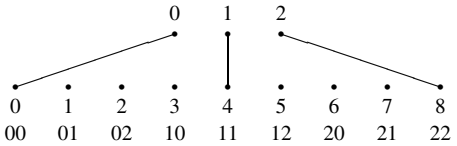
For every $n \in \omega$ there is a unique function $\hat{\kappa} : n \to 1$. The first projection $\hat{k}^1 : n \cdot m \to n$ is defined as $\pi_1 \circ \iota^{-1}$, where $\pi_1 : n \times m \to n$ is the ordinary first projection, and analogously for $\hat{k}^2 : n \cdot m \to m$. For example, the first projection $\hat{k}^1 : 6 \to 2$ is given by



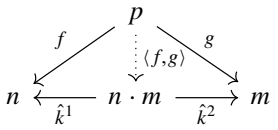
while the second projection $\hat{k}^2 : 6 \to 3$ is given by



The diagonal arrow $\hat{w} : n \to n \cdot n$ is defined as $\iota \circ \Delta$, where $\Delta : n \to n \times n$ is the ordinary diagonal map. For example, $\hat{w} : 3 \to 9$ is given by



For arrows $f : p \to n$ and $g : p \to m$, the unique arrow $\langle f, g \rangle : p \to n \cdot m$, which makes the following diagram commute



is the composition $(f \cdot g) \circ \hat{w}$.

§4. Mapping $\mathcal{C}$ into Set_ω. Let $Cat_\times$ be the subcategory of Cat whose objects are the categories with finite, strict monoidal products and whose arrows are functors preserving this structure "on the nose." We have just shown that Set_ω is an object of $Cat_\times$. Let $\mathcal{C}$ (where C stands for conjunction) be a category with finite strict monoidal products freely generated by a countable set $P = \{p_1, p_2, \ldots\}$ of objects. This category is the image of

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P under the left adjoint for the forgetful functor from $Cat_\times$ to Set that maps a category to the set of its objects. The category $\mathcal{C}$ can as $\mathcal{D}$ be constructed out of syntactic material, but we will not go into this construction here (see Došen & Petrić, 2004, chap. 2). We assume only that the objects of $\mathcal{C}$ are the finite sequences of elements of P.

In the proof below we rely on prime numbers, because when natural numbers greater than 0 are generated with multiplication and 1, the prime numbers are free generators. Disjunctive logic with a single propositional variable corresponds to generating the natural numbers with addition and 0 out of the free generator 1, while conjunctive logic with countably many propositional variables corresponds to building them with multiplication and 1 with the prime numbers as free generators. Using the prime numbers enables us to obtain that the functor mentioned in Proposition 4.11 is one-one on objects.

Since $\mathcal{C}$ is freely generated by P, there is a unique arrow $H : \mathcal{C} \to Set_\omega$ of $Cat_\times$, which extends the function from P to ω mapping p_n to the n-th prime number $\mathbf{p}_n$. Our goal is to prove the following proposition.

PROPOSITION 4.1. *The functor H is faithful.*

For this, we rely on Cartesian Coherence of Došen & Petrić (2004, sec. 9.2) and Proposition 5 of Došen & Petrić (2003b, sec. 5). We start with some auxiliary notions.

Since Set_ω is an object of Cat_+, its opposite category Set_ω^{op}, with $+$ as the product and 0 as the terminal object, is an object of $Cat_\times$. Hence, there is a unique arrow $G : \mathcal{C} \to Set_\omega^{op}$ of $Cat_\times$, which extends the function from P to ω mapping every p_n to 1.

The following lemma follows from Cartesian Coherence of Došen & Petrić (2004, sec. 9.2).

LEMMA 4.2. *The functor G is faithful.*

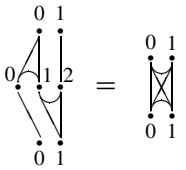
Let *Gen* be a category whose objects are again the finite ordinals. An arrow of *Gen* from n to m is an equivalence relation defined on $n + m$, called a *split equivalence*. (This is any equivalence relation on the ordinal $n + m$, and it is called *split*, because its domain $n + m$ is divided into the source n and the target m; for more on these matters see Došen & Petrić (2013).) The identity arrow from n to n is the split equivalence with n equivalence classes of the form $\{i, i + n\}$. (We follow here the presentation of *Gen* in Došen & Petrić (2003b), rather than that in Došen & Petrić (2013), which yields an isomorphic category.)

Composition of arrows is defined, roughly speaking, as the transitive closure of the union of the two relations composed, where we omit the ordered pairs one of whose members is in the middle (see Došen & Petrić (2003b, sec. 2), Došen & Petrić (2003a, sec. 2), and Došen & Petrić (2013, sec. 2) for a detailed definition). For example, the split equivalences R_1 and R_2 given, respectively, by

R_1 

R_2 

are composed so as to yield the split equivalence $R_2 \circ R_1$ given by the picture on the right-hand side of the equation sign:

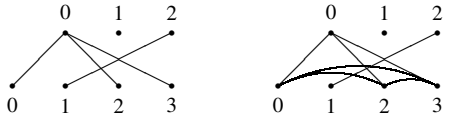
$R_2 \circ R_1$ 

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Consider the function, which maps the arrow $f^{op} : n \to m$ of Set_{ω}^{op} to the split equivalence (an arrow of *Gen*) between n and m with n equivalence classes, one for each $i \in n$, of the form

$$\{i\} \cup \{j + n \mid j \in m \text{ and } f(j) = i\}.$$

(For every value of the function f we put in the same class together with this value all the arguments with that value; besides that we have singleton equivalence classes for elements of the codomain of f that are not in the image of f.) For example, $f^{op} : 3 \to 4$ given by the picture on the left-hand side is mapped to the split equivalence given by the picture on the right-hand side:



It is not difficult to check that this is the arrow function of a faithful functor $J : Set_{\omega}^{op} \to Gen$, which is identity on objects (see Došen & Petrić, 2013, end of sec. 2).

Let $R : n \to m$ be an arrow of *Gen*, and $\overrightarrow{ab}$ the sequence $a_0 \dots a_{n-1} b_0 \dots b_{m-1}$ of not necessarily distinct finite ordinals greater than or equal to 2. We define a relation (not a split equivalence) $F_{\overrightarrow{ab}}(R)$ between the ordinal $a_0 \cdot \dots \cdot a_{n-1}$ (which is 1 when $n = 0$) and the ordinal $b_0 \cdot \dots \cdot b_{m-1}$ in the following way.

For $\overrightarrow{d}$ being the sequence $d_0 \dots d_{k-1}$, with $k \geq 0$, let $\imath_{\overrightarrow{d}} : d_0 \times \dots \times d_{k-1} \to d_0 \cdot \dots \cdot d_{k-1}$ be a function defined by

$$\imath_{\overrightarrow{d}}(i_0, \dots, i_{k-1}) = i_0 \cdot d_1 \cdot \dots \cdot d_{k-1} + \dots + i_{k-2} \cdot d_{n-1} + i_{k-1}.$$

Its inverse $\imath^{-1}_{\overrightarrow{d}} : d_0 \cdot \dots \cdot d_{k-1} \to d_0 \times \dots \times d_{k-1}$ is defined by

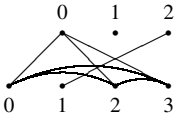
$$\imath^{-1}_{\overrightarrow{d}}(i) = (\lfloor i/(d_1 \cdot \dots \cdot d_{k-1}) \rfloor, \dots, \lfloor i/d_{k-1} \rfloor \bmod d_{k-2}, i \bmod d_{k-1}).$$

Note that $\imath$ and $\imath^{-1}$ defined at the beginning of §3 are just $\imath_{\overrightarrow{d}}$ and $\imath^{-1}_{\overrightarrow{d}}$ for $\overrightarrow{d}$ being the two-element sequence nm. The bijection $\imath_{\overrightarrow{d}}$ assigns to an element of $d_0 \times \dots \times d_{k-1}$ its position in the lexicographical order.

Let $F_{\overrightarrow{ab}}(R)$ be defined as the set of all ordered pairs (i, j) in $(a_0 \cdot \dots \cdot a_{n-1}) \times (b_0 \cdot \dots \cdot b_{m-1})$ such that for π_x the x-th projection, π_y the y-th projection, and $\overrightarrow{c}$ the $n+m$-tuple obtained by concatenating the n-tuple $\imath^{-1}_{\overrightarrow{d}}(i)$ with the m-tuple $\imath^{-1}_{\overrightarrow{b}}(j)$ we have

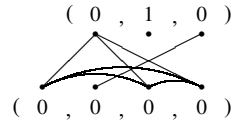
$$(\forall x, y \in n+m)\ ((x, y) \in R \Rightarrow \pi_x(\overrightarrow{c}) = \pi_y(\overrightarrow{c})).$$

Roughly speaking, the idea is to connect by $F_{\overrightarrow{ab}}(R)$ an element of $a_0 \times \dots \times a_{n-1}$ with an element of $b_0 \times \dots \times b_{m-1}$ when this pair "matches" R. For example, if $R : 3 \to 4$ is given by



and $a_0 = 3$, $a_1 = a_2 = b_0 = \dots = b_3 = 2$, then 2 from $3 \cdot 2^2$, which corresponds to the triple $(0, 1, 0)$, is connected to 0 from 2^4, which corresponds to the quadruple $(0, 0, 0, 0)$, and the pair $((0, 1, 0), (0, 0, 0, 0))$ matches R, since we have the picture

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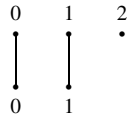


which is as the picture given for R with every element of an equivalence class of R replaced by the same number. In the same manner, we conclude that 0, 1, 3, 4, 5, 6, and 7 from $3 \cdot 2^2$ are connected, respectively, to 0, 4, 4, 11, 15, 11, and 15 from 2^4, and that there are no other pairs corresponding to the elements of $F_{\overrightarrow{ab}}(R)$. Hence, in this case $F_{\overrightarrow{ab}}(R)$ is not a function.

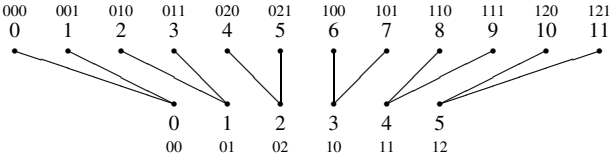
For $R : n \to m$ an arrow of *Gen* we say that a sequence of finite ordinals $a_0 \dots a_{n-1}$ $b_0 \dots b_{m-1}$ is *appropriate* for R when $(i, j) \in R$ and $i, j < n$ implies $a_i = a_j$, $(i, j) \in R$ and $i < n$ and $j \geq n$ implies $a_i = a_j$, and $(i, j) \in R$ and $i, j \geq n$ implies $b_i = b_j$. The following lemma, with a straightforward proof, provides sufficient conditions for $F_{\overrightarrow{ab}}(R)$ to be a function.

LEMMA 4.3. *For $R = J(f^{op})$ and $\overrightarrow{ab}$ a sequence appropriate for R, we have that $F_{\overrightarrow{ab}}(R)$ is a function.*

EXAMPLE 4.4. Let $f^{op} : 2 + 1 \to 2$ be the first projection in Set_ω^{op} given by



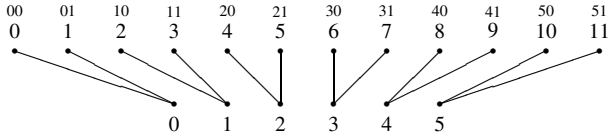
For $R = J(f^{op})$ and an appropriate sequence $\overrightarrow{ab} = 2\ 3\ 2\ \ 2\ 3$ we have that $F_{\overrightarrow{ab}}(R)$ is given by



which for the isomorphism

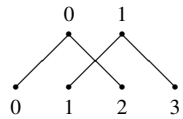
$$\iota^{-1} : (2 \cdot 3) \cdot 2 \to (2 \cdot 3) \times 2$$

is equal to the composition $\pi_1 \circ \iota^{-1}$ given by



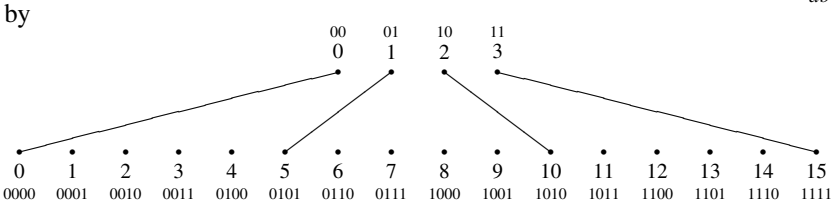
Hence, this is the first projection from $(2 \cdot 3) \cdot 2$ to $2 \cdot 3$ in Set_ω.

EXAMPLE 4.5. Let $f^{op} : 2 \to 2 + 2$ be the diagonal arrow in Set_ω^{op} given by



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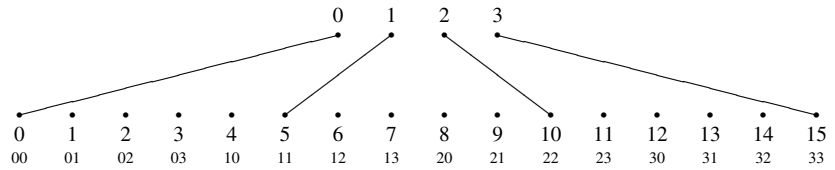
For $R = J(f^{op})$ and an appropriate sequence $\overrightarrow{ab} = 2\ 2\ \ 2\ 2\ 2\ 2$ we have that $F_{\overrightarrow{ab}}(R)$ is given by



which for the isomorphism

$$\iota\colon (2 \cdot 2) \times (2 \cdot 2) \to (2 \cdot 2) \cdot (2 \cdot 2)$$

is equal to the composition $\iota \circ \Delta$ given by



Hence, this is the diagonal arrow from $2 \cdot 2$ to $(2 \cdot 2) \cdot (2 \cdot 2)$ in Set_ω.

By reasoning as in Examples 4.4 and 4.5, we can prove the following lemma.

LEMMA 4.6. *The composition $F_{\overrightarrow{ab}} \circ J$ from Set_ω^{op} to Set_ω, for appropriate $\overrightarrow{ab}$, maps projections to projections and diagonal arrows to diagonal arrows.*

If $a_0 = \cdots = b_{m-1} = p \geq 2$, then $F_{\overrightarrow{ab}}(R)$ is denoted by $F_p(R)$, and it coincides with $F_p(R)$ defined in Došen & Petrić (2003b, sec. 5). In the following lemma we have F_p with $p = 2$.

LEMMA 4.7. *If $F_{\overrightarrow{ab}}(R) = F_{\overrightarrow{ab}}(S)$, then $F_2(R) = F_2(S)$.*

Proof. Let 2^k in the index of ι^{-1} denote the sequence of k occurrences of 2. For $R, S\colon n \to m$, for every $(i, j) \in 2^n \times 2^m$, we have that $(i, j) \in F_2(R)$

iff $(\forall x, y \in n+m)\ ((x, y) \in R \Rightarrow \pi_x(\iota_{2^n}^{-1}(i)\, \iota_{2^m}^{-1}(j)) = \pi_y(\iota_{2^n}^{-1}(i)\, \iota_{2^m}^{-1}(j)))$,

iff $(\iota_{\overrightarrow{a}}(\iota_{2^n}^{-1}(i)), \iota_{\overrightarrow{b}}(\iota_{2^m}^{-1}(j))) \in F_{\overrightarrow{ab}}(R)$,

since $\iota_{2^n}^{-1}(i) = \iota_{\overrightarrow{a}}^{-1}(\iota_{\overrightarrow{a}}(\iota_{2^n}^{-1}(i)))$ and $\iota_{2^m}^{-1}(j) = \iota_{\overrightarrow{b}}^{-1}(\iota_{\overrightarrow{b}}(\iota_{2^m}^{-1}(j)))$.

We conclude the same for R replaced by S. □

Then, as a corollary of Proposition 5 of Došen & Petrić (2003b, sec. 5), we obtain the following lemma.

LEMMA 4.8. *If $F_2(R) = F_2(S)$, then $R = S$.*

We prove the following lemma along the lines of Proposition 4 of Došen & Petrić (2003b, sec. 5).

LEMMA 4.9. *The function $F_{\overrightarrow{aa}}$ maps the identity to the identity, and for the arrows $R\colon n \to m$ and $S\colon m \to p$ of Gen and appropriate sequences $\overrightarrow{ab}$ and $\overrightarrow{bc}$, we have that*

$$F_{\overrightarrow{bc}}(S) \circ F_{\overrightarrow{ab}}(R) = F_{\overrightarrow{ac}}(S \circ R).$$

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Let Gen_P be the category whose objects are all the finite sequences of elements of P (as in the category $\mathcal{C}$) and whose arrows are all the triples of the form $(R, p_{i_0} \dots p_{i_{n-1}}, p_{j_0} \dots p_{j_{m-1}})$, where $R : n \to m$ is an arrow of Gen while $p_{i_0} \dots p_{i_{n-1}}$ and $p_{j_0} \dots p_{j_{m-1}}$ are objects of the category $\mathcal{C}$. From the faithfulness of the composition $J \circ G : \mathcal{C} \to Gen$, it follows that the functor $(J \circ G)_P : \mathcal{C} \to Gen_P$, which is identity on objects, and which maps the arrow $f : p_{i_0} \dots p_{i_{n-1}} \to p_{j_0} \dots p_{j_{m-1}}$ of $\mathcal{C}$ to $(JGf, p_{i_0} \dots p_{i_{n-1}}, p_{j_0} \dots p_{j_{m-1}})$ is also faithful.

Let Rel_ω be the category whose arrows are binary relations between finite ordinals, and let F be the functor from Gen to Rel_ω defined on objects by

$$F(p_{i_0} \dots p_{i_{n-1}}) = \mathbf{p}_{i_0} \cdot \dots \cdot \mathbf{p}_{i_{n-1}},$$

and on arrows, for $\overrightarrow{ab}$ the sequence $\mathbf{p}_{i_0} \dots \mathbf{p}_{j_{m-1}}$, by

$$F(R, p_{i_0} \dots p_{i_{n-1}}, p_{j_0} \dots p_{j_{m-1}}) = F_{\overrightarrow{ab}}(R).$$

From Lemmata 4.7–4.9, it follows that F is a faithful functor. Hence, the functor $F \circ (J \circ G)_P : \mathcal{C} \to Rel_\omega$ is also faithful. By Lemma 4.3, we may restrict this functor to a functor from $\mathcal{C}$ to Set_ω, which we also call $F \circ (J \circ G)_P$, for which we have the following.

LEMMA 4.10. $F \circ (J \circ G)_P = H$.

Proof. The functor $F \circ (J \circ G)_P$ maps a generator p_i to the prime number $\mathbf{p}_i$ and it preserves finite products. By Lemma 4.6, it preserves also the rest of the finite product structure. It remains to apply the uniqueness of H with these properties. □

Proposition 4.1, which asserts that H is faithful, follows from Lemma 4.10 and the faithfulness of $F \circ (J \circ G)_P$.

Let $sk(\mathcal{C})$ be a skeleton of $\mathcal{C}$, and let I be the inclusion functor from $sk(\mathcal{C})$ to $\mathcal{C}$. We may consider $sk(\mathcal{C})$ to be the full subcategory of $\mathcal{C}$ on objects of the form $p_{i_1} \dots p_{i_n}$ with $i_1 \leq \dots \leq i_n$.

PROPOSITION 4.11. *The composition $H \circ I : sk(\mathcal{C}) \to H\mathcal{C}$ is an isomorphism.*

Proof. From Proposition 4.1 we have that this composition is faithful. Since p_i is mapped by $H \circ I$ to the i-th prime number, and this functor preserves products, it is one-one on objects. □

The category $sk(\mathcal{C})$ is also a skeleton of the category with finite products freely generated by P. It is free in the sense that every function from P to the set of objets of a category from $Cat_\times$ in which product is commutative on objects extends to a unique arrow of $Cat_\times$ from $sk(\mathcal{C})$ to this category.

Our categories $\mathcal{D}$ and $sk(\mathcal{C})$ are not exactly the categories mentioned in the second paragraph of the introduction. However, they are equivalent to these categories. In logical terminology, this is just as if we worked with equivalence classes of formulae instead of usually defined formulae. In the case of disjunctive formulae, they are identified up to associativity (we get commutativity for free because we have a single letter), and in the case of conjunctive formulae, they are identified up to associativity and commutativity.

By taking opposite categories, one can replace coproducts with products, disjunction with conjunction and vice versa to obtain the dual results. Hence, there is nothing asymmetric that gives priority to disjunction over conjunction in these matters.

§5. Connection with the exponential and contravariant power-set functors. In this concluding section, we consider matters that connect our representation of conjunctive

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deductions by disjunctive deductions with a particular, well-behaved and rather familiar case of the Brauerian representation of Došen & Petrić (2003b). We consider first the representation of an equivalence relation R by a set of functions $\mathcal{F}^{=}(R)$, which we dealt with in Došen & Petrić (2003b, sec. 4), and which engenders the representation by $F_p(R)$, which we dealt with in Došen & Petrić (2003b, sec. 5), and which is closely related to the representation by $F^{\rightarrow}_{ab}(R)$ of this paper (see §4 above). The set of functions $\mathcal{F}^{=}(R)$ can be replaced by a relation between functions $\mathcal{F}_{X_1,X_2}(R)$, with X being the disjoint union of X_1 and X_2. For every equivalence relation $R \subseteq X^2$ there is a function $\Phi\colon X_2 \to X_1$ such that $\mathcal{F}_{X_1,X_2}(R)$ is equal to the function p^{Φ} that maps a function $f_1 : X_1 \to p$ to the function $f_1 \circ \Phi : X_2 \to p$. Finally, we consider how our representation of conjunctive deductions by disjunctive deductions is related to the exponential functor p^{-} from *Set* to Set^{op}, which on arrows is defined as p^{Φ}. The exponential functor 2^{-} is naturally isomorphic to the contravariant power-set functor.

For an arbitrary equivalence relation $R \subseteq X^2$ and an arbitrary set p such that for $p_0 \neq p_1$ we have $p_0, p_1 \in p$, let $\mathcal{F}^{=}(R)$ be the set of all functions $f\colon X \to p$ such that

$$(*) \qquad (\forall x, y \in X)(xRy \Rightarrow f(x) = f(y)).$$

It is shown in Došen & Petrić (2003b, sec. 4, Corollary) that for $R_1, R_2 \subseteq X^2$ equivalence relations we have $R_1 = R_2$ iff $\mathcal{F}^{=}(R_1) = \mathcal{F}^{=}(R_2)$.

For an equivalence relation $R \subseteq X^2$, consider the partition of X induced by R. Let X_1 be a set of representatives of these equivalence classes, one for each class (for the existence of this set one relies on the Axiom of Choice when X is infinite), and let X_2 be the complement of X_1 with respect to X. (The sets X_1 and X_2 can both be empty and X_1 can be nonempty with X_2 empty, but X_1 cannot be empty with X_2 nonempty.) So $X = X_1 + X_2$, where $+$ is disjoint union (coproduct in *Set*).

Let $\Phi\colon X_2 \to X_1$ be the function that maps every element of X_2 to the representative of its equivalence class. Assuming the sets X_1 and X_2 are given, for every function $f\colon X \to p$ there is a unique pair of functions $(f_1 : X_1 \to p, f_2 : X_2 \to p)$ such that $f = [f_1, f_2]$. We can easily verify that $(*)$ above is equivalent with

$$(**) \qquad (\forall x_1 \in X_1)(\forall x_2 \in X_2)(x_1 R x_2 \Rightarrow f_1(x_1) = f_2(x_2)).$$

We also have that $x_1 R x_2$ iff $\Phi(x_2) = x_1$. From that we infer that $(**)$ is equivalent with $f_1 \circ \Phi = f_2$.

Let $\mathcal{F}_{X_1,X_2}(R)$ be the set of all pairs $(f_1 : X_1 \to p, f_2 : X_2 \to p)$ such that $[f_1, f_2] \in \mathcal{F}^{=}(R)$. For $\Phi : X_2 \to X_1$ as above, let $p^{\Phi} : p^{X_1} \to p^{X_2}$ be the function that maps a function $f_1 : X_1 \to p$ to the function $f_1 \circ \Phi : X_2 \to p$. We have that

$$\begin{aligned}(f_1, f_2) \in \mathcal{F}_{X_1,X_2}(R) &\text{ iff } [f_1, f_2] \in \mathcal{F}^{=}(R),\\ &\text{iff } f_1 \circ \Phi = f_2,\\ &\text{iff } p^{\Phi}(f_1) = f_2.\end{aligned}$$

So $\mathcal{F}_{X_1,X_2}(R)$ and p^{Φ} are the same function.

For every cartesian closed category $\mathcal{K}$ (see Mac Lane, 1971, sec. IV.10 and Lambek & Scott, 1986, sec. I.3) and every object C of $\mathcal{K}$ there is an exponential functor C^{-} from $\mathcal{K}$ to $\mathcal{K}^{op}$, which assigns to an object A of $\mathcal{K}$ the exponential object C^A of $\mathcal{K}$, and to an arrow $f : A \to B$ of $\mathcal{K}$ the canonical arrow $C^f : C^B \to C^A$ produced by the cartesian closed structure of $\mathcal{K}$. (In the notation of Lambek & Scott (1986, sec. I.1) the arrow C^f is $\varepsilon_{C,B} \circ (\mathbf{1}_{C^B} \times f))^{*}$.) The category *Set* is cartesian closed, and in it we have the following exponential functor p^{-} from *Set* to Set^{op}, for p an arbitrary set. The set p^A is the set of all

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functions $h : A \to p$. For $f : A \to B$, the function $p^f : p^B \to p^A$ is defined by taking that for $g : B \to p$ we have

$$p^f(g) = g \circ f : A \to p.$$

The function p^Φ above is a particular case of p^f.

Consider next the contravariant power-set functor of *Set* (see Mac Lane, 1971, sec. II.2). This is the functor $\overline{P}$ from *Set* to Set^{op} such that $\overline{P}A$ is the power set of the set A, and for $f : A \to B$ the function $\overline{P}f : \overline{P}B \to \overline{P}A$ is the inverse-image function under f, which means that for $Y \in \overline{P}B$, i.e., $Y \subseteq B$, we have

$$(\overline{P}f)(Y) = \{a \in A \mid f(a) \in Y\}.$$

It is an easy exercise to verify that the functors 2^- and $\overline{P}$ are naturally isomorphic.

The image-function under f and the inverse-image function under f make a covariant Galois connection, i.e., a trivial adjunction (see Došen, 1998 or Došen, 1999, sec. 2.4.4).

The functors p^- and $\overline{P}$ can be restricted to the category *Finset*, the full subcategory of *Set* whose objects are finite sets. The category Set_ω is a skeleton of *Finset*. The functor p^- maps coproduct into product, and this fact is related to the arithmetical equation $p^{n+m} = p^n \cdot p^m$.

This paper gave a representation of product through coproduct in Set_ω. This worked because product is tied to functions, and functions in Set_ω in general are representable through coproduct. This is not specific for functions tied to product. Any other functions would be representable through coproduct in Set_ω. For example, functions tied to exponentiation, which logically corresponds to implication. Since, as we noted at the end of the preceding section, coproduct can be represented through product in Set_ω, anything representable through coproduct can also be represented through product in Set_ω.

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BIBLIOGRAPHY

Čubrić, Dj. (1998). Embedding of a free cartesian closed category into the category of sets. *Journal of Pure and Applied Algebra*, **126**, 121–147.

Došen, K. (1998). Functions redefined. *The American Mathematical Monthly*, **105**, 631–635.

Došen, K. (1999). *Cut Elimination in Categories*. Dordrecht: Kluwer.

Došen, K. (2014). Algebras of deductions in category theory. In Jokanović, D., editor. *Third Mathematical Conference of the Republic of Srpska, Proceedings, Trebinje 2013, Zbornik radova*, Vol. I. Trebinje: Univerzitet u Istočnom Sarajevu, Fakultet za proizvodnju i menadžment, pp. 11–18 (available at: http://www.mi.sanu.ac.rs/∼kosta/publications.htm).

Došen, K. (2015). General proof theory. In Schroeder-Heister, P., Heinzmann, G., Hodges, W., and Bour, P. E., editors. *Logic, Methodology and Philosophy of Science - Proceedings of the 14th International Congress* (*Nancy*): *Logic and Science Facing the New Technologies*. Introduction to the Symposium on General Proof Theory. London: College Publications, pp. 149–151 (preprint available at: http://www.mi.sanu.ac.rs/∼kosta/publications.htm).

Došen, K. (2016). On the paths of categories: An introduction to deduction. In Piecha, T. and Schroeder-Heister, P., editors. *Advances in Proof-Theoretic Semantics*. Cham: Springer, pp. 65–77 (available at: http://www.mi.sanu.ac.rs/∼kosta/publications.htm).

https://doi.org/10.1017/S175502031600037X Published online by Cambridge University Press

Došen, K. & Petrić, Z. (2001). The maximality of cartesian categories. *Mathematical Logic Quarterly*, **47**, 137–144.
Došen, K. & Petrić, Z. (2003a). A Brauerian representation of split preorders. *Mathematical Logic Quarterly*, **49**, 579–586 (version with misprints corrected available at arXiv).
Došen, K. & Petrić, Z. (2003b). Generality of proofs and its Brauerian representation. *The Journal of Symbolic Logic*, **68**, 740–750.
Došen, K. & Petrić, Z. (2004). *Proof-Theoretical Coherence*. London: KCL Publications (College Publications) (revised version of 2007 available at: http://www.mi.sanu.ac.rs/∼kosta/coh.pdf).
Došen, K. & Petrić, Z. (2007). *Proof-Net Categories*. Monza: Polimetrica (preprint of 2005 available at: http://www.mi.sanu.ac.rs/ kosta/pn.pdf).
Došen, K. & Petrić, Z. (2013). Syntax for split preorders. *Annals of Pure and Applied Logic*, **164**, 443–481.
Kelly, G. M. (1972). An abstract approach to coherence. In Mac Lane, S., editor. *Coherence in Categories*. Lecture Notes in Mathematics, Vol. 281. Berlin: Springer, pp. 106–147.
Lambek, J. (1972). Deductive systems and categories III: Cartesian closed categories, intuitionist propositional calculus, and combinatory logic. In Lawvere, F. W., editor. *Toposes, Algebraic Geometry and Logic*. Lecture Notes in Mathematics, Vol. 274. Berlin: Springer, pp. 57–82.
Lambek, J. & Scott, P. J. (1986). *Introduction to Higher-Order Categorical Logic*. Cambridge: Cambridge University Press.
Lawvere, F. W. (1969). Adjointness in foundations. *Dialectica*, **23**, 281–296.
Mac Lane, S. (1965). Categorical algebra. *Bulletin of the American Mathematical Society*, **71**, 40–106.
Mac Lane, S. (1971). *Categories for the Working Mathematician*. Berlin: Springer (expanded second edition, 1998).
Mints, G. E. (1980). Category theory and proof theory. *Aktual'nye voprosy logiki i metodologii nauki*. Kiev: Naukova Dumka, pp. 252–278 (in Russian) (English translation, with permuted title, in: G. E. Mints, *Selected Papers in Proof Theory*, Bibliopolis, Naples, 1992).
Petrić, Z. (2002). Coherence in substructural categories. *Studia Logica*, **70**, 271–296.
Prawitz, D. (1971). Ideas and results in proof theory. In Fenstad, J. E., editor. *Proceedings of the Second Scandinavian Logic Symposium*. Amsterdam: North-Holland, pp. 235–307.
Troelstra, A. S. & Schwichtenberg, H. (1996). *Basic Proof Theory*. Cambridge: Cambridge University Press (second edition, 2000).

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