Equality holds if, and only if, $\frac{a_i^p}{b_i^q} = \frac{a_j^p}{b_j^q}$ for all i and j.

Our proof is similar to the traditional proof for Hölder's inequality based on Young's inequality [1, p. 15] or weighted AM-GM inequality [1, p. 19]. Recall that Young's inequality states:

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

for all positive a and b.

Proof: Using Young's inequality,

$$\frac{w_k^{1/p} a_k}{\left(\sum_{k=1}^n w_k a_k^p\right)^{1/p}} \frac{w_k^{1/q} b_k}{\left(\sum_{k=1}^n w_k b_k^q\right)^{1/q}} \leqslant \frac{1}{p} \frac{w_k a_k^p}{\sum_{k=1}^n w_k a_k^p} + \frac{1}{q} \frac{w_k b_k^q}{\sum_{k=1}^n w_k b_k^q}. \tag{3}$$

By summing over k from 1 to n on both sides of (3), we obtain

$$\frac{\sum_{k=1}^n w_k a_k b_k}{\left(\sum_{k=1}^n w_k a_k^p\right)^{1/p} \left(\sum_{k=1}^n w_k b_k^q\right)^{1/q}} \leqslant \frac{1}{p} + \frac{1}{q} = 1, \tag{4}$$

which gives us (2).

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109.28 A trick with the gamma function

The gamma function is probably the most common of the special functions and has many applications in physics [1]. For the definition and basic properties of this function, readers may consult [2]. In some cases, it can also be used to rewrite singular integrals in terms of a series expansion which is more tractable for obtaining an asymptotic estimate of the singular behaviour by elementary means. Consider a function $f(x)$ defined by an integral as follows:

$$f(x) = \int_0^1 dt \frac{\sqrt{t}}{(1 - xt^2)^\beta},$$

which depends on real $x < 1$ and real $\beta > 1$. Note that $f(0) = \frac{2}{3}$ for all β. Some readers may recognise this as a type of hypergeometric function.

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Begin by observing that

$$(1 - z)^{-\beta} = \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(k + \beta) z^k}{k!}. \tag{1}$$

for $|z| < 1$. By comparison with (1), we then have

$$f(x) = \int_0^1 dt \sqrt{t} \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(k + \beta)}{k!} x^k t^{2k} = \frac{2}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(k + \beta)}{(4k + 3) k!} x^k.$$

The radius of convergence of this series can be found using the ratio test:

$$\frac{u_k}{u_{k+1}} = \frac{\Gamma(k + \beta)}{(4k + 3) k!} \frac{(k + 1)! (4k + 4)}{\Gamma(k + \beta + 1)}$$

$$= (k + 1) \frac{4k + 4}{4k + 3} \frac{\Gamma(k + \beta)}{(k + \beta) \Gamma(k + \beta)} = \frac{(k + 1)(4k + 4)}{(k + \beta)(4k + 3)}.$$

This ratio tends to 1 as k goes to infinity, so the radius of convergence is 1. Note that u_k denotes the k-th term of the series, and not the coefficient. To be clear, the limit as k tends to infinity of $\frac{u_{k+1}}{u_k}$ is $\frac{x}{R}$, whose modulus must not exceed 1. The above calculation shows that $R = 1$, hence $|x| < 1$.

It turns out that in some cases other properties of a power series apart from the radius of convergence can be deduced from the behaviour of ratios of terms in the series. If there is a singularity on the circle of convergence, the terms will depend on it and so will the speed of convergence. It can be shown that if the singularity is at $x = a$, one has the following asymptotic relation as $k \to \infty$:

$$\frac{u_k}{u_{k-1}} \sim \frac{1}{a} \left(1 - \frac{1 + \alpha}{k}\right) + O\left(k^{-2}\right), \tag{2}$$

where α is the order of the singularity. For the terms in our series, we have

$$\frac{u_k}{u_{k-1}} \sim \frac{\Gamma(k + \beta)}{(4k - 1) k!} \frac{(k - 1)! (4k + 2)}{\Gamma(k + \beta - 1)} = 1 + \frac{\beta - 2}{k} + O\left(k^{-2}\right).$$

By comparison with (2), we see that $f(x)$ has a singularity of order $\alpha = 1 - \beta$, which implies that for x close to 1, one has

$$f(x) \sim A(1 - x)^{1-\beta},$$

for some constant A.

In order for this expression to provide a useful asymptotic estimate of the singular behaviour of $f(x)$, it is necessary to evaluate or estimate the value of A. This can be done in elementary fashion. Making a substitution $s = xt^2$, we have

$$f(x) = \frac{x^{-3/4}}{2} \int_0^x \frac{s^{-1/4}}{(1 - s)^{\beta}} ds.$$

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If we change sign by writing $x = 1 - \varepsilon$ and relabel, we have

$$f(x) = \frac{(1-\varepsilon)^{-3/4}}{2}\int_{\varepsilon}^{1} t^{-\beta}(1-t)^{-1/4}\,dt.$$

We have an expansion

$$(1-t)^{-1/4} = 1 + \frac{t}{4} + \frac{5t^2}{32} + \ldots. \tag{3}$$

When substituted in, this leads to an asymptotic estimate for $f(x)$ which will contain multiple terms which diverge in the limit as x goes to 1. The number of these terms and their explicit expressions will depend on the value which is assumed for β, but the leading term will be the same for all real $\beta > 1$. Since we only need the leading term to find A, we only need β between 1 and 2 and so we way rewrite $f(x)$ as

$$f(x) \sim \frac{(1-\varepsilon)^{-3/4}}{2}\left(\int_{\varepsilon}^{1} t^{-\beta}\left((1-t)^{-1/4} - 1\right)dt + \int_{\varepsilon}^{1} t^{-\beta}dt\right).$$

By comparing the first integral when integrated between ε and 1 and between 0 and 1, it can be seen that this integral is regular at $\varepsilon = 0$. We then have

$$f(x) \sim \frac{(1-\varepsilon)^{-3/4}}{2}\left(\int_{0}^{1} t^{-\beta}\left((1-t)^{-1/4} - 1\right)dt + o(1) + \int_{\varepsilon}^{1} t^{-\beta}dt\right).$$

This evaluates to

$$f(x) \sim \frac{(1-x)^{1-\beta}}{2(\beta-1)} + O(1),$$

hence $A = (2\beta - 2)^{-1}$. We will end with a question for possible investigation by readers: does $f(x) \to 0$ algebraically as $x \to \infty$?

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