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KRONECKER CLASSES AND CLIQUES IN DERANGEMENT GRAPHS

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Abstract

Given a permutation group G, the derangement graph of G is defined with vertex set G, where two elements x and y are adjacent if and only if xy^{-1} is a derangement. We establish that if G is transitive with degree exceeding 30, then the derangement graph of G contains a complete subgraph with four vertices. In the process, we determine all transitive groups whose derangement graph does not contain a complete subgraph on four vertices. As a consequence, if G is a normal subgroup of A such that $|A : G| = 3$ and U is a subgroup of G satisfying $G = \bigcup_{a\in A} U^a$, then $|G : U| \leq 10$. This provides support for a conjecture by Neumann and Praeger concerning Kronecker classes.

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1. Introduction

In this paper, we study an analog of the Erdős–Ko–Rado theorem [7] in the setting of permutation groups. Let G be a finite permutation group acting on a finite set Ω. A subset $\mathcal{F} \subseteq G$ is called ***intersecting*** if, for all $g, h \in \mathcal{F}$, the permutation gh^{-1} fixes at least one point in Ω.

Each point stabilizer G_ω is an example of an intersecting set and, more generally, so is any of its cosets. For the symmetric group $G = \mathrm{Sym}(\Omega)$, it was shown independently by Cameron and Ku [5], and Larose and Malvenuto [12], that these examples are optimal: any intersecting subset of $\mathrm{Sym}(\Omega)$ has size at most $(|\Omega| - 1)!$, with equality if and only if it is a coset of a point stabilizer.

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However, in general permutation groups, such neat characterizations do not always hold. There are groups in which the largest intersecting sets are not cosets of point stabilizers and the point stabilizers themselves may fail to be of maximum size among intersecting subsets. In other settings, multiple distinct types of maximal intersecting sets can exist. For instance, in the 2-transitive action of $\mathrm{PGL}_d(q)$ on the projective space $\mathrm{PG}_{d-1}(q)$, maximal intersecting sets are cosets of either point stabilizers or hyperplane stabilizers [19].

To help quantify how closely a group G follows an Erdős–Ko–Rado-type behavior, Li, Song, and Pantangi introduced in [13] the notion of ***intersection density***. Given $\omega \in \Omega$, let G_ω be a point stabilizer in G of maximum cardinality. The intersection density of an intersecting set $\mathcal{F} \subseteq G$ is defined by $\rho(\mathcal{F}) = |\mathcal{F}|/|G_\omega|$, and the ***intersection density*** of the group G is given by

$$\rho(G) = \max\{\rho(\mathcal{F}) \mid \mathcal{F} \subseteq G \text{ intersecting}\}.$$

A useful tool in this context is the ***derangement graph*** Γ_G of G, whose vertex set is G, with an edge between x and y whenever xy^{-1} is a derangement (that is, an element with no fixed points). Equivalently, Γ_G is the Cayley graph of G with connection set equal to the set of derangements. We note that some authors require a Cayley graph to be defined on a generating set, whereas the set of derangements may generate a proper subgroup. Intersecting subsets of G correspond precisely to independent sets in Γ_G. We write $\omega(\Gamma_G)$ and $\alpha(\Gamma_G)$ for the clique number and independence number of Γ_G, respectively. Then, the standard clique–coclique inequality [9, Theorem 2.1.1] implies

$$\alpha(\Gamma_G) \cdot \omega(\Gamma_G) \leq |G|,$$

which leads to a bound on the intersection density:

$$\rho(G) \leq \frac{|\Omega|}{\omega(\Gamma_G)}. \tag{1-1}$$

In [14], Meagher *et al.* proved that for any transitive group G with $|\Omega| \geq 3$, the derangement graph Γ_G contains a triangle. As a consequence, from (1-1), they showed that $\rho(G) \leq |\Omega|/3$. Their work also raised the following broader question [14, Question 6.1].

Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every transitive group G of degree n, if the derangement graph Γ_G contains no k-clique, then $n \leq f(k)$?

The main result of [8] provides a positive answer to [14, Question 6.1] for arbitrary values of k, albeit under the stronger assumption that the permutation group G is ***innately transitive***, meaning that G possesses a minimal normal subgroup that is transitive. For small values of k, partial answers are known: when $k = 2$, Jordan's classical result implies $n \leq 1$ and, as noted above, when $k = 3$, the result of [14] yields $n \leq 2$. The present paper continues this line of investigation by settling the case $k = 4$.

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THEOREM 1.1. *Let G be a finite transitive group on Ω. Then, the derangement graph of G does not contain a clique of size at least 4 if and only if one of the following holds:*

(1) $|\Omega| \leq 3$;
(2) $|\Omega| = 6$ *and* $G \cong \mathrm{Alt}(4)$;
(3) $|\Omega| = 18$, $|G| = 324$, *and* G *is* G_{18};
(4) $|\Omega| = 30$ *and* G *is* G_{30} *or* G_{30}^*.

In cases (2)–(4), the derangement graph of G contains cliques of size 3, whereas, in case (1), the derangement graph of G contains cliques of size $|\Omega|$.

In Section 2, we describe the three exceptional groups G_{18}, G_{30}, and G_{30}^*, whose Magma IDs are *TransitiveGroup*(18, 142), *TransitiveGroup*(30, 126), and *TransitiveGroup*(30, 233), respectively.

From (1-1), Theorem 1.1 shows that most transitive groups have intersection density at most $|\Omega|/4$.

This result also contributes to a broader set of problems concerning subgroup coverings and their applications to field extensions. In particular, there is a connection to the Kronecker classes, explored in [11] and summarized in [3, Section 1.7]. We recall the following definition.

DEFINITION 1.2. Let A be a finite group with a normal subgroup H. A subgroup $U \leq H$ is said to be an A-***covering subgroup*** of H if $H = \bigcup_{a \in A} U^a$.

A central open problem in this area is the following conjecture by Neumann and Praeger [16, Conjecture 4.3], also recorded as [10, 11.71] in the Kourovka Notebook.

CONJECTURE 1.3 (Neumann–Praeger). *There is a function $g : \mathbb{N} \to \mathbb{N}$ such that, whenever U is an A-covering subgroup of H with $|A : H| = n$, we have $|H : U| \leq g(n)$.*

The conjecture is Jordan's theorem when $n = 1$ and was proved in [18] for $n = 2$. Using Theorem 1.1, we establish it for $n = 3$.

PROOF OF CONJECTURE 1.3 FOR $n = 3$. Let Ω be the set of right cosets of U in A and consider the natural action of A on Ω. Since U is a point stabilizer and $H = \bigcup_{a \in A} U^a$, it follows that H contains no derangements.

Let C be a clique in the derangement graph of A acting on Ω. For any distinct $x, y \in C$, the element xy^{-1} is a derangement and thus belongs to a distinct right coset of H in A. Therefore, $|C| \leq |A : H| = 3$. By Theorem 1.1, this implies that $|A : U| \leq 30$ and, hence, $|H : U| \leq 10$. □

The proof above is a particular case of an argument given in [8, page 934]. The authors have shown the following theorem.

THEOREM 1.4 (See [8, page 934]). *If* [14, Question 6.1] *has a positive answer, then the Neumann–Praeger conjecture* [10, 11.71] *is true, and the function g can be chosen to be $g(n) = f(n)/n$.*

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It is unclear whether the Praeger–Neumann conjecture [10, 11.71] is actually equivalent to a positive answer to [14, Question 6.1].

The groups of degree 30 in Theorem 1.1(4) in fact show that the function g in Conjecture 1.3 can be chosen so that $g(3) = 10$. In Section 2, we collect some information on the groups appearing in Theorem 1.1(2)–(4). This information is useful to provide a complete classification of the subgroup chains $U < G < A$ satisfying the hypotheses of Conjecture 1.3 with $n = 3$.

2. A description of the groups in Theorem 1.1(2)–(4)

We start by describing G_{18}. Let v_1, v_2, v_3 be the canonical basis for the three-dimensional vector space $\mathbb{F}_3^3$ over the field $\mathbb{F}_3$ with 3 elements. Let $V = \langle v_1, v_2, v_3 \rangle$, so V is an elementary abelian 3-group of order 27.

Now, let $x, y, u \in \mathrm{GL}_3(3)$ be defined as follows:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and let $K = \langle x, y, u \rangle$. It is easy to verify that $K \cong \mathrm{Alt}(4)$. Finally, let $H = V \rtimes \langle x, y \rangle$ and $A = V \rtimes K$. In particular, $|A| = 27 \cdot 12 = 324$, $H \trianglelefteq A$, and $|A : H| = 3$. Now, let $U = \langle v_1, v_2 - v_3 \rangle \rtimes \langle x \rangle$. Clearly, $|U| = 9 \cdot 2 = 18$. It is not hard to verify that $H = \bigcup_{a \in A} U^a$ and, hence, U is an A-covering subgroup of H.

Now, let G be the permutation group induced by the action of A on the set Ω of right cosets of U in A. Clearly, G has degree $324/18 = 18$ and, in fact, G is permutation isomorphic to G_{18}. The lattice of subgroups of A containing U is the chain: $U < V \rtimes \langle x \rangle < H < A$.

The group G_{30} can be described in a similar manner. Let v_1, v_2 be the canonical basis for the two-dimensional vector space $\mathbb{F}_5^2$ over the field $\mathbb{F}_5$ with 5 elements and let $V = \langle v_1, v_2 \rangle$.

Now, let $x, y, u \in \mathrm{GL}_2(5)$ be defined as follows:

$$x = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix},$$

and let $K = \langle x, y, u \rangle$. It is easy to verify that $\langle x, y \rangle \cong Q_8$, where Q_8 is the quaternion group of order 8, and that u has order 3 and normalizes $\langle x, y \rangle$. Finally, let $H = V \rtimes \langle x, y \rangle$ and $A = V \rtimes K$. In particular, $|A| = 25 \cdot 24 = 600$, $H \trianglelefteq A$, and $|A : H| = 3$. Now, let $U = \langle v_2 \rangle \rtimes \langle x \rangle$. Clearly, $|U| = 5 \cdot 4 = 20$. As above, it is not hard to verify that U is an A-covering subgroup of H.

The permutation group induced by the action of A on the set of right cosets of U has degree $600/20 = 30$ and is permutation isomorphic to G_{30}. The lattice of subgroups of A containing U is the chain: $U < V \rtimes \langle x \rangle < H < A$.

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The group G_{30}^* can be easily described, as it contains the group G_{30} as a subgroup of index 2. Let v_1, v_2, x, y, u be as in the previous paragraph, let $z \in \mathrm{GL}_2(5)$ be defined as follows:

$$z = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and let $K = \langle x, y, z, u \rangle$. It is easy to verify that $\langle x, y, z \rangle \cong Q_8 \circ C_4$ has order 16 and that $K \cong (Q_8 \rtimes C_3) \circ C_4$ has order 48. Finally, let $H = V \rtimes \langle x, y, z \rangle$ and $A = V \rtimes K$. In particular, $|A| = 25 \cdot 48 = 1200$, $H \trianglelefteq A$, and $|A : H| = 3$. Now, let $U = \langle v_2 \rangle \rtimes \langle x, z \rangle$. Clearly, $|U| = 5 \cdot 8 = 40$. Moreover, it can be verified that U is an A-covering subgroup of H.

The permutation group induced by the action of A on the set of right cosets of U has degree $1200/40 = 30$ and is permutation isomorphic to G_{30}^*. The lattice of subgroups of A containing U is the chain: $U < V \rtimes \langle x \rangle < H < A$.

3. Preliminaries: basic definitions and basic results

This section is divided into three parts. In the first part, we present basic definitions and recall standard results from permutation group theory. The second part is devoted to deeper auxiliary results that support the main arguments of this paper. Finally, in the third part, we introduce and employ a new concept—hyper-transitive (a, b)-hypergraphs—particularly suited for formulating statements that are used inductively in the proof of Theorem 1.1.

3.1. Basic definitions and a basic result. For future reference, we provide a summary of key definitions.

Let G be a group and let Σ be a system of imprimitivity for the action of G on Ω and let $\Delta \in \Sigma$. We let $G_{\{\Delta\}} = \{g \in G \mid \Delta^g = \Delta\}$ be the setwise stabilizer of Δ in G and we let $G_{(\Delta)} = \{g \in G \mid \delta^g = \delta \text{ for all } \delta \in \Delta\}$ be the pointwise stabilizer of Δ in G. Moreover, we let $G_{\{\Delta\}}^{\Delta}$ denote the permutation group induced by $G_{\{\Delta\}}$ on Δ.

LEMMA 3.1. *Let A and B be two commuting regular subgroups of* $\mathrm{Sym}(\Delta)$. *Then, either there exist $a \in A \setminus \{1\}$ and $b \in B \setminus \{1\}$ such that ab is a derangement, or $|\Delta| \leq 2$.*

PROOF. Fix $\delta_0 \in \Delta$ and let $\varepsilon \colon \Delta \to A$ be the map defined by $\delta\varepsilon = a$ when $\delta_0^a = \delta$. Observe that ε is a well-defined bijection, because A is a regular subgroup of $\mathrm{Sym}(\Delta)$. Now, consider the function $\iota_\varepsilon \colon \mathrm{Sym}(\Delta) \to \mathrm{Sym}(A)$ defined by $\sigma\iota_\varepsilon = \varepsilon^{-1} \circ \sigma \circ \varepsilon$. Observe that the image of $A \leq \mathrm{Sym}(\Delta)$ under ι_ε is the right regular representation of A. As $B \leq \mathrm{Sym}(\Delta)$ is a regular subgroup commuting with A, the image of B under ι_ε is a regular subgroup of $\mathrm{Sym}(A)$ commuting with the right regular representation of A. Thus, by [4, 1.5, page 27], the image of B under ι_ε is the left regular representation of A.

Summing up, replacing Δ, A, and B if necessary, we may suppose that Δ is a group X, and that A and B are the right and left regular representations of X, respectively.

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Assume $|X| = |\Delta| \neq 1$. Let $x, y \in X \setminus \{1\}$ and consider $\rho_x\lambda_y$. If $\rho_x\lambda_y$ is not a derangement on X, then there exists $\omega \in X$ with

$$\omega = \omega^{\rho_x\lambda_y} = y^{-1}\omega x,$$

that is, $\omega x\omega^{-1} = y$. In particular, x and y belong to the same X-conjugacy class. If X has only one conjugacy class of nonidentity elements, then $|X| = 2$. □

3.2. The O'Nan–Scott theorem and auxiliary results for the proof of Theorem 1.1. Let G be a group and let N be the socle of G. The O'Nan–Scott theorem describes in detail the embedding of N in G and collects some useful information about the action of N. In [17], eight types of primitive groups are defined (depending on the group- and action-structure of the socle), namely HA (Affine), HS (Holomorph of simple), HC (Holomorph of compound), AS (Almost Simple), PA (Product Action), SD (Simple Diagonal), CD (Compound Diagonal), and TW (Twisted Wreath), and it is shown that every primitive group belongs to exactly one of these types.

We do not give a full description of these types here and instead refer to [17] for details. Here, we just state some basic facts. Let G be primitive with domain Ω and socle N.

If G is of type HA, then N is an elementary abelian p-group, for some prime number p, and it is a minimal normal subgroup. A similar structure holds when G is of TW type; indeed, N is a minimal normal subgroup acting regularly on Ω, but there, the socle is nonabelian.

When G is of type HS or HC, G has exactly two minimal normal subgroups, say M_1 and M_2; these are pairwise isomorphic and nonabelian. Moreover, $N = M_1 \times M_2$, and M_1, M_2 act regularly on Ω. The stabilizer of a point in N embeds diagonally in N. When G is of type HS, M_1, M_2 are nonabelian simple; when G is of type HC, M_1, M_2 are the direct products of more than one nonabelian simple group.

When G is of type SD or CD, N is a minimal normal subgroup of G and N is nonabelian. The stabilizer of a point in N embeds diagonally in N. When G is of type SD, the stabilizer of a point in N is a nonabelian simple group; when G is of type CD, the stabilizer of a point in N is the direct product of more than one nonabelian simple group.

When G is of type AS, N is a nonabelian simple group, but here there is no description of the action of N on Ω. However, there is a strong algebraic description of the whole of G, because G, as an abstract group, is contained in the automorphism group of N.

When G is of type PA, N is a minimal normal subgroup of G and is nonabelian. Here, G preserves a nontrivial Cartesian decomposition of Ω, and the action on a Cartesian factor is primitive of type AS.

Before applying the O'Nan–Scott theorem, we first recall a theorem of Saxl.

THEOREM 3.2 (Saxl; see [18, Proposition 2]). *Let T be a nonabelian simple group and let M be a subgroup of T. If*

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$$T = \bigcup_{\varphi \in \mathrm{Aut}(T)} M^{\varphi},$$

then $T = M$.

We have the following observation.

LEMMA 3.3. *Let G be a transitive permutation group on Ω and let Σ be a system of imprimitivity for the action of G on Ω. If $C \subseteq G$ is a clique for the derangement graph $\Gamma_{G,\Sigma}$ of G in its action on Σ, then C is also a clique for the derangement graph $\Gamma_{G,\Omega}$ of G in its action on Ω.*

PROOF. Let $C \subseteq G$ be clique for the derangement graph of G in its action on Σ. Now, given two distinct elements g and h of C, we have that gh^{-1} fixes no element of Σ. Since Σ is a system of imprimitivity for Ω, gh^{-1} cannot fix any element of Ω. Therefore, C is a clique for the derangement graph of G in its action on Ω. □

For improving the flow of the argument in the proof of Proposition 3.5, we prove the following lemma.

LEMMA 3.4. *Let G be a transitive group on Ω, let Σ be a system of imprimitivity for the action of G on Ω, and let $\Delta \in \Sigma$. Suppose:*

- *$G_{\{\Delta\}}$ acts primitively on Δ;*
- *$G_{(\Sigma)} \neq 1$; and*
- *$G_{(\Delta)} = 1$.*

Let M be a minimal normal subgroup of G contained in $G_{(\Sigma)}$. Then, either:

(1) *M is a minimal normal subgroup of $G_{\{\Delta\}}$ for every $\Delta \in \Sigma$; or*
(2) *M is the socle of $G_{\{\Delta\}}$ and is the direct product of two isomorphic minimal normal subgroups $M_{1,\Delta}, M_{2,\Delta}$ acting regularly on Δ for every $\Delta \in \Sigma$.*

PROOF. Assume first that M is abelian. Since $G_{\{\Delta\}}$ is primitive on Δ and $M \trianglelefteq G_{\{\Delta\}}$, we deduce that M is transitive on Δ. As $G_{(\Delta)} = 1$, $G_{\{\Delta\}}$ is faithful on Δ and, hence, we deduce that M is regular on Δ. In particular, the primitive group $G_{\{\Delta\}}$ has a normal abelian regular subgroup. From the O'Nan–Scott theorem, the primitive groups having a normal abelian regular subgroup are of affine type; thus, M is actually the socle of $G_{\{\Delta\}}$ and it is the unique minimal normal subgroup of $G_{\{\Delta\}}$. Since this argument does not depend on Δ, we get that M is the socle of $G_{\{\Delta'\}}$ and is the unique minimal normal subgroup of $G_{\{\Delta'\}}$ for every $\Delta' \in \Sigma$. In particular, part (1) holds.

Assume now that M is nonabelian, and write $M = T_1 \times \cdots \times T_\ell$ for some positive integer ℓ and for some pairwise isomorphic nonabelian simple groups $T_1, \ldots, T_\ell$. Let M_1 be a minimal normal subgroup of $G_{\{\Delta\}}$ with $M_1 \leq M$. Therefore, relabeling the indexed set $\{1, \ldots, \ell\}$ if necessary, we may suppose that $M_1 = T_1 \times \cdots \times T_\kappa$ for some $1 \leq \kappa \leq \ell$. If $\kappa = \ell$, then M is a minimal normal subgroup of $G_{\{\Delta\}}$. As $M \trianglelefteq G$ and G is transitive on Σ, we deduce that M is a minimal normal subgroup of $G_{\{\Delta'\}}$ for every $\Delta' \in \Sigma$. Therefore, part (1) holds.

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Suppose $\kappa < \ell$. Let M_2 be a minimal normal subgroup of $G_{\{\Delta\}}$ with $T_{\kappa+1} \leq M_2 \leq M$. From [6, Theorem 4.3B], a finite primitive group has at most two minimal normal subgroups and, in the case that it does have two minimal normal subgroups, these are isomorphic. Hence, we deduce that $M_1 \cong M_2$, $\ell = 2\kappa$, and $M = M_1 \times M_2$. Since M_1 and M_2 are transitive on Δ and centralize each other, by [4, 1.5, page 27], M_1 and M_2 both act regularly on Δ. In particular, M is the socle of $G_{\{\Delta\}}$. As above, since $M \trianglelefteq G$ and G is transitive on Σ, M is the socle of $G_{\{\Delta'\}}$ for every $\Delta' \in \Sigma$. Thus, part (2) holds. □

PROPOSITION 3.5. *Let G be a transitive group on Ω, let Σ be a system of imprimitivity for the action of G on Ω, and let $\Delta \in \Sigma$. Suppose:*

- *$G_{\{\Delta\}}$ acts primitively on Δ;*
- *$G_{(\Sigma)} \neq 1$; and*
- *$G_{(\Delta)} = 1$.*

Then, there exists $g \in G_{(\Sigma)}$ acting as a derangement on Ω.

PROOF. Since $G_{(\Delta)} = 1$ and since $G_{(\Sigma)} \leq G_{\{\Delta\}}$, the group $G_{(\Sigma)}$ acts faithfully on Δ.

We use the O'Nan–Scott theorem. Let $\Delta' \in \Sigma$ and let $g \in G$ with $\Delta' = \Delta^g$. As $G_{\{\Delta\}}$ and $G_{\{\Delta'\}}$ are G-conjugate, the O'Nan–Scott type of the group $G_{\{\Delta\}}$ acting on Δ is identical to that of $G_{\{\Delta'\}}$ acting on Δ'.

Let S be a minimal normal subgroup of G contained in $G_{(\Sigma)}$. We use Lemma 3.4.

If $G^{\Delta}_{\{\Delta\}}$ has type HA or TW, then S is the socle of $G_{\{\Delta'\}}$ and hence it acts regularly on Δ', for every $\Delta' \in \Sigma$. Therefore, every non-identity element of S is a derangement on Ω.

If $G^{\Delta}_{\{\Delta\}}$ has type HS or HC, then $S = S_{1,\Delta'} \times S_{2,\Delta'}$, where $S_{1,\Delta'}$ and $S_{2,\Delta'}$ are the two minimal normal subgroups of $G_{\{\Delta'\}}$ generating the socle of $G_{\{\Delta'\}}$ for all $\Delta' \in \Sigma$. From the structure of groups of type HS and HC, every simple direct factor of S acts semiregularly on Δ' for each $\Delta' \in \Sigma$. Therefore, every nonidentity element of S lying in a simple direct factor of S is a derangement on Ω.

Suppose $G^{\Delta}_{\{\Delta\}}$ has type SD or CD. Then,

$$S = T_1 \times \cdots \times T_\ell,$$

where $T_1, \ldots, T_\ell$ are pairwise isomorphic nonabelian simple groups. Observe that, from the structure of groups of type SD and CD, for each $i \in \{1, \ldots, \ell\}$, the group T_i acts semiregularly on the domain Δ' for each $\Delta' \in \Sigma$. In particular, from this, it follows that, for every i, the nonidentity elements of T_i are derangements on Ω.

Suppose that $G^{\Delta}_{\{\Delta\}}$ is of type AS or PA. In particular, S is the unique minimal normal subgroup of $G_{\{\Delta'\}}$ and is the socle of $G_{\{\Delta'\}}$ for all $\Delta' \in \Sigma$. Suppose that S has no derangement in its action on Ω. Let $\omega_0 \in \Delta$. Then,

$$S = \bigcup_{\omega \in \Omega} S_\omega = \bigcup_{g \in G} S^g_{\omega_0}. \tag{3-1}$$

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For every $g \in G$, the action of g by conjugation on S induces an automorphism of S. Therefore, from (3-1), we have

$$S = \bigcup_{\varphi \in \mathrm{Aut}(S)} S_{\omega_0}^{\varphi}. \tag{3-2}$$

From the definition of primitive groups of AS and PA type, the $G_{\{\Delta\}}$-set Δ admits a Cartesian decomposition and, hence, $\Delta = \Lambda^{\ell}$ for some finite set Λ with $|\Lambda| \geq 5$ and for some $\ell \geq 1$ (the value $\ell = 1$ corresponds to the case where $G_{\{\Delta\}}$ is of AS type). Moreover,

$$G_{\{\Delta\}} \leq H \operatorname{wr} \mathrm{Sym}(\ell),$$

where $H \leq \mathrm{Sym}(\Lambda)$, H is almost simple and acts primitively on Λ, and $G_{\{\Delta\}}$ is endowed with the product action. We identify the elements of Δ with the elements of Λ^{ℓ} and we identify $G_{\{\Delta\}}$ with its image in the embedding in $H \operatorname{wr} \mathrm{Sym}(\ell)$. Here, the socle S of $G_{\{\Delta\}}$ is T^{ℓ}, where T is the socle of H. As $\Delta = \Lambda^{\ell}$, we have $\omega_0 = (\lambda_1, \ldots, \lambda_{\ell})$ for some $\lambda_1, \ldots, \lambda_{\ell} \in \Lambda$. Since $G_{\{\Delta\}}$ is transitive on $\Delta = \Lambda^{\ell}$, there exists $g \in G_{\{\Delta\}}$ with $\omega_0^g = (\lambda, \ldots, \lambda)$ for some $\lambda \in \Lambda$. Thus, replacing ω_0 with ω_0^g if necessary, we may suppose that $\omega_0 = (\lambda, \ldots, \lambda)$ for some $\lambda \in \Lambda$. Therefore,

$$S_{\omega_0} = T_{\lambda} \times \cdots \times T_{\lambda} = T_{\lambda}^{\ell}.$$

Since $\mathrm{Aut}(S) = \mathrm{Aut}(T^{\ell}) = \mathrm{Aut}(T) \operatorname{wr} \mathrm{Sym}(\ell)$, from (3-2), we deduce

$$T^{\ell} = S = \bigcup_{\varphi \in \mathrm{Aut}(S)} S_{\omega_0}^{\varphi} = \bigcup_{\varphi \in \mathrm{Aut}(T^{\ell})} (T_{\lambda}^{\ell})^{\varphi} = \bigcup_{\varphi \in \mathrm{Aut}(T) \operatorname{wr} \mathrm{Sym}(\ell)} (T_{\lambda}^{\ell})^{\varphi}. \tag{3-3}$$

Since T_{λ}^{ℓ} is normalized by $\mathrm{Sym}(\ell)$, that is,

$$(T_{\lambda}^{\ell})^{\varphi} = T_{\lambda}^{\ell} \quad \text{for all } \varphi \in \mathrm{Sym}(\ell),$$

from (3-3), we deduce

$$T^{\ell} = \bigcup_{\varphi_1, \ldots, \varphi_{\ell} \in \mathrm{Aut}(T)} T_{\lambda}^{\varphi_1} \times \cdots \times T_{\lambda}^{\varphi_{\ell}}. \tag{3-4}$$

In particular, by considering only the first direct factor of T^{ℓ}, from (3-4), we get

$$T = \bigcup_{\varphi \in \mathrm{Aut}(T)} T_{\lambda}^{\varphi}.$$

However, this contradicts Theorem 3.2. □

3.3. (a, b)-Hypergraphs. A finite ***hypergraph*** is an ordered pair $\Gamma = (V, E)$, where V is a finite set and E is a collection of subsets of V. It is customary to say that Γ is a-uniform if every element in E has cardinality a.

Let a and b be positive integers with $a \geq 2$. An (a, b)-***hypergraph*** is an ordered pair $\Gamma = (V, E)$, where V is a finite set and where the elements e of E are partitions of

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ab-subsets of V into b parts each having cardinality a. For instance, when $a = 2$ and $b = 3$, a typical element of E is of the form

$$\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}\}.$$

In the special case where $b = 1$, our definition returns the notion of an a-uniform hypergraph, except for having an extra set of braces around the elements; that is, when $b = 1$, a typical element of E in an (a, b)-hypergraph is of the form $\{\{v_1, \ldots, v_a\}\}$. This connection motivates our terminology. In general, an (a, b)-hypergraph is not a hypergraph.

An ***automorphism*** φ of the (a, b)-hypergraph Γ is a permutation of V preserving E, that is, $e^\varphi \in E$ for every $e \in E$. Let G be a group acting on Γ by automorphisms (in particular, we are not insisting that G acts faithfully on V). We say that Γ is ***G-vertex transitive*** if G acts transitively on V and, analogously, we say that Γ is ***G-edge transitive*** if G acts transitively on E.

In Proposition 3.6, we are interested only in certain (a, b)-hypergraphs. We say that G is ***hyper-transitive*** for Γ if:

- Γ is G-vertex and G-edge-transitive; and
- for every edge $e = \{p_1, \ldots, p_b\} \in E$, the edge stabilizer G_e is transitive on the b parts of e, and either:
  - G_e is also transitive on $p_1 \cup \cdots \cup p_b$, that is, G_e is transitive on the underlying set of vertices appearing in e; or
  - $a = 2$.

Given a positive integer c, a ***proper coloring*** of the (a, b)-hypergraph Γ with c colors is a function $\eta : V \to \{1, \ldots, c\}$ such that there exists no $e = \{p_1, \ldots, p_b\} \in E$, where p_i is ***monochromatic*** for every i, that is, all of its elements have the same image via η. The ***chromatic number*** $\chi(\Gamma)$ of Γ is the minimum c such that Γ admits a proper coloring with c colors. In the special case where $b = 1$ and $a = 2$, our definition of chromatic number coincides with the classic definition.

Proposition 3.6 complements Proposition 3.5. In the proof of Proposition 3.6 and throughout the rest of the paper, we use a slight abuse of notation, which we now explain. Let ℓ be a positive integer and let X be a group. A ***diagonal subgroup*** of the Cartesian product X^ℓ is a subgroup of the form

$$\{(x^{\varphi_1}, x^{\varphi_2}, \ldots, x^{\varphi_\ell}) \mid x \in X\}$$

for some $\varphi_1, \ldots, \varphi_\ell \in \mathrm{Aut}(X)$. We denote such a subgroup by $\mathrm{Diag}(X^\ell)$; this is where our abuse of notation lies, since the subgroup depends on the choice of $\varphi_1, \ldots, \varphi_\ell$, and different choices of automorphisms give rise to different subgroups.

PROPOSITION 3.6. *Let G be a transitive group on Ω, let Σ be a system of imprimitivity for the action of G on Ω, and let $\Delta \in \Sigma$. Suppose:*

- *$G_{\{\Delta\}}$ acts primitively on Δ;*
- *$G_{(\Sigma)} \neq 1$;*

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- $G_{(\Delta)} \neq 1$*; and*
- *the primitive permutation group* $G^{\Delta}_{\{\Delta\}}$ *has nonabelian socle.*

Let S *be the socle of* $G_{(\Sigma)}$ *and let* $\omega \in \Delta$*. Then, either* $G_{(\Sigma)}$ *contains a derangement on* Ω *or* $S = N_1 \times \cdots \times N_s$ *such that:*

(1) N_i *is a minimal normal subgroup of* $G_{(\Sigma)}$ *for each* $i \in \{1, \ldots, s\}$ *and* $s \geq 5$*;*
(2) $N_i \cong T^{\kappa}$ *for some nonabelian simple group* T *and some* $\kappa \geq 1$*, and if* $s = 5$*, then* $N_i \cong T \cong \mathrm{Alt}(5)$*;*
(3) G *acts transitively by conjugation on* $\{N_1, \ldots, N_s\}$*;*
(4) *there exists an* (a, b)*-hypergraph* Γ *such that:*
   (a) Γ *has vertex set* $\{N_1, \ldots, N_s\}$*;*
   (b) $\chi(\Gamma) \geq t + 1$*, where* t *is the number of* $\mathrm{Aut}(T^{\kappa})$*-conjugacy classes in* T^{κ}*;*
   (c) $G/G_{(\Sigma)}$ *is hyper-transitive for* Γ*;*
   (d) $G_{\{\Delta\}}$ *fixes some edge* $e = \{p_1, \ldots, p_b\}$ *of* Γ*;*
(5) *for* $i \in \{1, \ldots, b\}$*, let* $N_{p_i} = \prod_{N \in p_i} N$ *and let* $\mathrm{Diag}(N_{p_i})$ *be a diagonal subgroup of the direct product* N_{p_i}*, we have*

$$S \cap G_{\omega} = \prod_{i=1}^{b} \mathrm{Diag}(N_{p_i}) \times \prod_{N \notin p_1 \cup \cdots \cup p_b} N.$$

PROOF. Let W be a maximal subgroup of $G_{(\Sigma)}$ with $G_{\omega} \cap G_{(\Sigma)} \leq W$. Assume that $G_{(\Sigma)}$ has no derangement on Ω. As $G_{(\Sigma)}$ has no derangements on Ω, $G_{(\Sigma)} \trianglelefteq G$, and $G_{\omega} \cap G_{(\Sigma)} \leq W$, we have $G_{(\Sigma)} = \bigcup_{a \in G} W^g$. Therefore, W is a G-covering subgroup of $G_{(\Sigma)}$. Let $K = \bigcap_{a \in G} W^a$.

Assume for a contradiction that $K \neq 1$. Since $G_{\{\Delta\}}$ is primitive on Δ and since $K \trianglelefteq G_{\{\Delta\}}$, we deduce that either K is transitive on Δ or $K \leq G_{(\Delta)}$. The latter possibility is excluded by the fact that $1 \neq K \trianglelefteq G$. Therefore, K is transitive on Δ and, hence, $G_{\{\Delta\}} = KG_{\omega}$. Intersecting both sides with $G_{(\Sigma)}$ and using the modular law, we get

$$G_{(\Sigma)} = K(G_{(\Sigma)} \cap G_{\omega}) \leq KW \leq W \leq G_{(\Sigma)},$$

which contradicts the fact that $W \neq G_{(\Sigma)}$. This shows that $\bigcap_{a \in G} W^a = K = 1$ and, hence, the hypotheses of [15, Proposition 3.1] are all satisfied, that is, W is a G-covering subgroup of $G_{(\Sigma)}$ that is maximal in $G_{(\Sigma)}$ and is core-free in G. Observe that the socle S of $G_{(\Sigma)}$ cannot be abelian because we are assuming the primitive permutation group $G^{\Delta}_{\{\Delta\}}$ has nonabelian socle. Now, parts (1)–(3) follow immediately from [15, Proposition 3.1 (i) and (ii)].

Let $T_1, \ldots, T_{\kappa s}$ be the simple direct factor of S. We label these subgroups so that $N_y = T_{(y-1)\kappa+1} \times \cdots \times T_{y\kappa}$. Consider the subsets $p = \{y \in \{1, \ldots, s\} \mid N_y \not\leq G_{(\Delta)}\}$ and $P = \{x \in \{1, \ldots, \kappa s\} \mid T_x \not\leq G_{(\Delta)}\}$. Thus,

$$P = \{(y-1)\kappa + v \mid y \in p, v \in \{1, \ldots, \kappa\}\}.$$

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If we let $P_y = \{(y-1)\kappa + v \mid v \in \{1, \dots, \kappa\}\}$, then $\{P_y \mid y \in p\}$ is a partition of P and $N_y = \prod_{x \in P_y} T_x$.

Since by part (1) N_y is a minimal normal subgroup of $G_{(\Sigma)}$ for each $y \in \{1, \dots, s\}$, we deduce $S \cap G_{(\Delta)} = S_{(\Delta)} = \prod_{y \notin p} N_y$ and

$$S = \prod_{y \in p} N_y \times S_{(\Delta)},$$

where $\prod_{y \in p} N_y$ acts faithfully on Δ. Moreover, as $S_\omega \geq S_{(\Delta)}$, we may write $S_\omega = Z \times S_{(\Delta)}$ for some subgroup Z of $\prod_{y \in p} N_y$. From [15, Proposition 3.1 (iii)], there exist two distinct indices $i, j \in \{1, \dots, s\}$ with

$$S \cap W = D \times \prod_{\ell \neq i,j} N_\ell, \tag{3-5}$$

where D is a diagonal subgroup of $N_i \times N_j$. Relabeling the indexed set $\{1, \dots, s\}$ if necessary, we suppose $i = 1$ and $j = 2$. Clearly, $S \cap W \geq S_\omega$. We now use the O'Nan–Scott theorem to determine the type of $G^\Delta_{\{\Delta\}}$ and to obtain some preliminary information for the proof of parts (4) and (5).

This action cannot be of type AS or PA, because in these cases, the structure of S_ω is incompatible with the description of $S \cap W$ in (3-5) and with the inclusion $S_\omega \leq S \cap W$. It is also clear that it cannot be of type HA, since S is nonabelian. If it were of type TW, then $\prod_{x \in P} T_x$ would act regularly on Δ and, hence, $Z = 1$ and $S_\omega = S_{(\Delta)}$. In particular, the elements of S_ω can be viewed as κs-tuples of elements of T whose entries are 1 in the coordinates corresponding to the simple direct factors indexed by the elements of P. Now, since S has no derangements in its action on Ω, we would have

$$S = \bigcup_{g \in G} S_\omega^g = \bigcup_{g \in G} S_{(\Delta)}^g.$$

However, this leads to a contradiction: if we take $g_i \in T_i \setminus \{1\}$ for every $i \in \{1, \dots, \kappa s\}$, then $g = (g_i)_i \in S$, but $g \notin \bigcup_{g \in G} S_{(\Delta)}^g$, because no coordinate of g is 1. It follows that the action of $G_{\{\Delta\}}$ on Δ is of type HS, SD, HC, or CD.

Assume $G^\Delta_{\{\Delta\}}$ is of type HS. Then, from the description of the primitive groups of SD type,

$$|T| = |\Delta| = |S : S_\omega| \geq |S : S \cap W| = |N_1 \times N_2 : \mathrm{Diag}(N_1 \times N_2)| = |T|^\kappa.$$

Hence, $\kappa = 1$ and $N_x = T_x$ for every $x \in \{1, \dots, s\}$. Moreover, $S \cap W = S_\omega$ and $D = Z = \mathrm{Diag}(T_1 \times T_2)$. In this case, for the proof of parts (4) and (5), we set $e = \{\{T_1, T_2\}\}$, $a = 2$, and $b = 1$.

Assume that $G^\Delta_{\{\Delta\}}$ is of type SD. Thus,

$$Z = \mathrm{Diag}\Big(\prod_{x \in P} T_x\Big) \cong T.$$

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From the description of the primitive groups of SD type, we deduce that $G_{\{\Delta\}}$ acts primitively by conjugation on $\{T_x \mid x \in P\}$. Since $G_{(\Sigma)} \trianglelefteq G_{\{\Delta\}}$, it follows that either $G_{(\Sigma)}$ acts trivially by conjugation on $\{T_x \mid x \in P\}$ or $G_{(\Sigma)}$ acts transitively by conjugation on $\{T_x \mid x \in P\}$. We now prove that the latter case does not occur. Indeed, if $G_{(\Sigma)}$ acted transitively by conjugation on $\{T_x \mid x \in P\}$, then $\prod_{x\in P} T_x$ would be a minimal normal subgroup of $G_{(\Sigma)}$ contained in S and, hence, $N_y = \prod_{x\in P} T_x$ for some $y \in \{1, \dots, s\}$, because $N_1, \dots, N_s$ are the unique minimal normal subgroups of $G_{(\Sigma)}$ contained in S by part (1). Consequently,

$$|T|^\kappa = |N_y| = \left|\prod_{x\in P} T_x\right| = |T|^{|P|} = |T|^{\kappa|p|}$$

and $|p| = 1$. Thus, $p = \{1\}$, $P = P_1 = \{1, \dots, \kappa\}$, and

$$S_\omega = Z \times \prod_{y\neq 1} N_y = \mathrm{Diag}(N_1) \times \prod_{y\neq 1} N_y.$$

However, this is incompatible with the inclusion $S_\omega \leq S \cap W$ and with the description of $S \cap W$ in (3-5). Therefore, this contradiction shows that $G_{(\Sigma)}$ acts trivially by conjugation on $\{T_x \mid x \in P\}$. In particular, $\kappa = 1$ and $N_x = T_x$ for every $x \in \{1, \dots, s\}$. In this case, for the proof of parts (4) and (5), we set

$$e = \{\{T_x \mid x \in p\}\}, \quad a = |p|, \quad b = 1.$$

Assume $G^\Delta_{\{\Delta\}}$ is of type HC. In this case, $G^\Delta_{\{\Delta\}}$ is a subgroup of the holomorph $T^{\kappa b} \rtimes \mathrm{Aut}(T^{\kappa b})$, where $|P| = 2\kappa b$. Therefore, Z is isomorphic to $\mathrm{Diag}(T^{\kappa b} \times T^{\kappa b}) \cong T^{\kappa b}$ and is the direct product of κb subgroups of the form $\mathrm{Diag}(T \times T)$. We now give a more precise description. As $G^\Delta_{\{\Delta\}}$ is of type HC, the group $G_{\{\Delta\}}$ has two orbits of the same cardinality κb (respectively b) in its action by conjugation on $\{T_x \mid x \in P\}$ (respectively $\{N_y \mid y \in p\}$). Therefore, $p = p' \cup p''$ admits a partition into two sets of cardinality b such that $S' = \prod_{y\in p'} N_y$ and $S'' = \prod_{y\in p''} N_y$ are the two minimal normal subgroups of $G_{\{\Delta\}}$ contained in $\prod_{y\in p} N_y$. Moreover, $S_\omega = \mathrm{Diag}(S' \times S'') \times S_{(\Delta)}$. In particular, there exists a bijection $F : \bigcup_{y\in p'} P_y \to \bigcup_{y\in p''} P_y$ such that

$$S_\omega = \prod_{x\in\bigcup_{y\in p'} P_y} \mathrm{Diag}(T_x \times T_{F(x)}) \times S_{(\Delta)}.$$

As $S_\omega \trianglelefteq G_{(\Sigma)} \cap G_\omega$ and as $G_{(\Sigma)} \cap G_\omega$ acts transitively by conjugation on the simple direct factors of N_y for each y, we deduce that the bijection F induces a bijection $f : p' \to p''$ such that

$$S_\omega = \prod_{y\in p'} \mathrm{Diag}(N_y \times N_{f(y)}) \times S_{(\Delta)}.$$

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Thus, the set p indexing the $G_{(\Sigma)}$-minimal normal subgroups of $\prod_{y\in p} N_y$ admits a partition $\{p_1, \dots, p_b\}$ into b parts of cardinality 2. Namely, $\{\{y, f(y)\} \mid y \in p'\}$. If we set $p_i = \{v_i, w_i\}$ for each i, then $p' = \{v_1, v_2, \dots, v_b\}$, $p'' = \{w_1, w_2, \dots, w_b\}$, and

$$Z = \mathrm{Diag}(S' \times S'') = \prod_{y\in p'} \mathrm{Diag}(N_y \times N_{f(y)}) = \mathrm{Diag}(N_{v_1} \times N_{w_1}) \times \cdots \times \mathrm{Diag}(N_{v_b} \times N_{w_b}).$$

In this case, for the proof of parts (4) and (5), we set

$$e = \{\{N_y \mid y \in p_i\} \mid i \in \{1, \dots, b\}\}, \quad a = 2.$$

Assume $G^{\Delta}_{\{\Delta\}}$ is of type CD. Hence, $G_{\{\Delta\}}/G_{(\Delta)}$ is embedded in

$$Y \text{ wr } \mathrm{Sym}(B),$$

with the primitive product action, where Y is a primitive group of SD type and $B \geq 2$. In particular, Z is a direct product of B diagonal subgroups pairwise isomorphic to T of the direct product $\prod_{x\in P} T_x$ of $|P| = \kappa|p|$ copies of T. The group $G_{\{\Delta\}}$ acts transitively by conjugation on $\{T_x \mid x \in P\}$ and on $\{N_y \mid y \in p\}$. As S_ω is the direct product of B diagonal subgroups isomorphic to T, the set P admits a uniform partition $\{P'_1, \dots, P'_B\}$ such that

$$S_\omega = \mathrm{Diag}\Big(\prod_{x\in P'_1} T_x\Big) \times \cdots \times \mathrm{Diag}\Big(\prod_{x\in P'_B} T_x\Big) \times S_{(\Delta)}. \tag{3-6}$$

Observe that, as Y has type SD, the normalizer of $\prod_{x\in P'_i} T_x$ acts primitively by conjugation on $\{T_x \mid x \in P'_i\}$. The set P admits also a second uniform partition (with $|P|/\kappa = |p|$ parts of cardinality κ) given by the orbits of the action of $G_{(\Sigma)}$ by conjugation on $\{T_x \mid x \in P\}$. Indeed, this second uniform partition of P is $\{P_y \mid y \in p\}$.

We claim that $|P_y \cap P'_i| \leq 1$ for every $y \in p$ and for every $i \in \{1, \dots, B\}$. Assume $P_y \cap P'_i \neq \emptyset$. Since the partitions $\{P'_1, \dots, P'_B\}$ and $\{P_y \mid y \in p\}$ label two systems of imprimitivity for the action of $G_{\{\Delta\}}$ on $\{T_x \mid x \in P\}$, since the meet of two systems of imprimivity is a system of imprimivity, and since the normalizer of $\prod_{x\in P'_i} T_x$ acts primitively by conjugation on $\{T_x \mid x \in P'_i\}$, we deduce that either $\{P_y \mid y \in p\}$ is a coarser partition than $\{P'_1, \dots, P'_B\}$ or the meet of $\{P'_1, \dots, P'_B\}$ and $\{P_y \mid y \in p\}$ is the trivial partition. In the first case, arguing as in the SD case, we obtain a contradiction using (3-5) and the inclusion $S_\omega \leq S \cap W$. Therefore, the meet of $\{P'_1, \dots, P'_B\}$ and $\{P_y \mid y \in p\}$ is the trivial partition, that is, $|P_y \cap P'_i| \leq 1$ for every $y \in p$ and for every $i \in \{1, \dots, B\}$. From this, (3-6), and arguing as in the SD case, it follows

$$S_\omega = \prod_{i=1}^{b} \mathrm{Diag}(N_{p_i}) \times S_{(\Delta)}$$

for some partition $\{p_1, \dots, p_b\}$ of p, with $b = B/\kappa$. In this case, we set

$$e = \{\{N_y \mid y \in p_i\} \mid i \in \{1, \dots, b\}\}, \quad a = |P|/B.$$

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We are now ready to prove parts (4) and (5). We consider the function θ having domain Σ and codomain the set of all subsets of $\{N_1, \dots, N_s\}$. For each $\Delta' \in \Sigma$, we let $\theta(\Delta') = \{N_i \mid N_i \not\leq G_{(\Delta')}\}$. For instance, $\theta(\Delta) = \{N_y \mid y \in p\}$. Since G acts transitively on Σ, the sets $\theta(\Delta')$ all have the same cardinality, say $k = |p|$. Moreover, from the O'Nan–Scott type of $G^{\Delta}_{\{\Delta\}}$ that we have highlighted in the previous paragraph, we deduce that $\theta(\Delta')$ is endowed with a natural partition into b sets of cardinality k/b. Let $V = \{N_1, \dots, N_s\}$, let $a = k/b$, and let E be the image of θ. In particular, $\Gamma = (V, E)$ is an (a, b)-hypergraph and part (a) is satisfied. Moreover, $G_{\{\Delta\}}$ fixes the edge $\theta(\Delta)$ of Γ by construction and, hence, part (d) is satisfied. Since the action of G by conjugation on V is transitive and since G is transitive on Σ, we deduce that Γ is G-vertex and G-edge transitive. By construction, the group $G_{\{\Delta\}}$ acts transitively on the b parts of $\theta(\Delta)$. Finally, when Y has O'Nan–Scott type SD or CD, $G_{\{\Delta\}}$ is also transitive on $p_1 \cup \cdots \cup p_b$ and, when Y has O'Nan–Scott type HS or HC, we have $a = 2$. We deduce that part (c) is satisfied.

Let t be the number of $\mathrm{Aut}(T^\kappa)$-conjugacy classes in T^κ and let $x_1, \dots, x_t$ be a set of representatives. Let c be the chromatic number of Γ. Suppose $c \leq t$. In particular, there exists a coloring η of V where we can use as 'colors' the elements in $\{x_1, \dots, x_t\}$. Using the coloring η, consider

$$g = (\eta(N_1), \eta(N_2), \dots, \eta(N_s)) \in N_1 \times \cdots \times N_s = S,$$

in other words, we are considering the element of S where in the ith coordinate, we put the element having color $\eta(N_i)$. Since we are dealing with the case that $G_{(\Sigma)}$ has no derangements, we deduce that g fixes some point of Ω, because $S \leq G_{(\Sigma)}$. Without loss of generality, g fixes ω and, hence, $g \in S \cap G_\omega$. However, from part (5), we see that the coordinates of g belonging to the same part of the edge $\theta(\Delta)$ must be conjugate under some automorphism of T^κ, that is, all the parts of the edge $\theta(\Delta)$ are monochromatic, which contradicts the fact that η is a coloring of Γ. This contradiction has arisen from assuming $c \leq t$ and, hence, part (b) follows. □

REMARK 3.7. Here, we give a natural example of a transitive permutation group satisfying the assumptions in Proposition 3.6. Consider the nonabelian simple group $T = \mathrm{Alt}(5)$, let $G = T \operatorname{wr} \mathrm{Sym}(5)$, and let $N = T^5$ be the base group of G. Let $H = \{(t_1, t_2, t_3, t_4, t_5) \in N \mid t_1 = t_2\} \cong T^4$ and let Ω be the set of right cosets of H in G. Thus, $|\Omega| = |\mathrm{Sym}(5)|[N : H] = 120 \cdot 60$ and G can be viewed as a permutation group on Ω.

Let Σ be the system of imprimitivity on Ω consisting of the orbits of N. The kernel of the action of G on Σ is $G_{(\Sigma)} = \bigcap_{\Delta \in \Sigma} G_{\{\Delta\}} = N \neq 1$. For each $\Delta \in \Sigma$, we have $G_{\{\Delta\}} = N$, since the quotient group G/N acts regularly on Σ. Moreover, as H is a maximal subgroup of N, we deduce that $G_{\{\Delta\}}$ acts primitively on Δ. Finally, as the core of H in N is isomorphic to T^3, we deduce $G_{(\Delta)} \cong T^3 \neq 1$.

Let $(t_1, t_2, t_3, t_4, t_5) \in N = G_{(\Sigma)}$. As T has only four $\mathrm{Aut}(T)$-conjugacy classes, there exist two distinct indices $i, j \in \{1, \dots, 5\}$ with $t_i^{\mathrm{Aut}(T)} = t_j^{\mathrm{Aut}(T)}$. Since Sym(5) acts 2-transitively on the five simple direct factors of N, there exists $\sigma \in G$ with

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$(t_1, t_2, t_3, t_4, t_5)^\sigma \in H$. This shows that every element of $G_{(\Sigma)}$ fixes some point of Ω and, hence, no element of $G_{(\Sigma)}$ acts as a derangement on Ω.

In this example, the auxiliary (a, b)-hypergraph arising in Proposition 3.6(4) has parameters $a = 2$ and $b = 1$, and is the complete graph on five vertices.

4. The action of G on a maximal system of imprimitivity Σ_1

In this section, we start the proof of Theorem 1.1 and we fix the notation that is maintained for the rest of the paper.

In the proof of Theorem 1.1, we argue by induction on $|G|$. Specifically, by substituting G with a proper transitive subgroup if necessary, we may assume that

$$G \text{ is minimally transitive,} \tag{4-1}$$

meaning that every proper subgroup of G acts intransitively on Ω.

Let G be a transitive group having domain Ω and suppose that the derangement graph $\Gamma_{G,\Omega}$ of G in its action on Ω has no cliques of size 4. If $\Gamma_{G,\Omega}$ has no triangles, then the result follows from [14], indeed, part (1) is satisfied with $|\Omega| \leq 2$. Therefore, we may suppose that $\Gamma_{G,\Omega}$ has at least one triangle. In particular, $|\Omega| > 2$.

LEMMA 4.1. *The group G possesses a system of imprimitivity Σ_1 with $|\Sigma_1| = 3$. The permutation group induced by G on Σ_1 is $\mathrm{Alt}(\Sigma_1)$ and $G_{(\Sigma_1)}$ has no derangements.*

PROOF. Let Σ_1 be a system of imprimitivity for the action of G on Ω of minimal cardinality. Consider the natural homomorphism

$$^- : G \to \mathrm{Sym}(\Sigma_1)$$

arising from the action of G on Σ_1.

Since the derangement graph of G on Ω lacks cliques of size 4, by Lemma 3.4, the derangement graph $\Gamma_{\bar{G},\Sigma_1}$ also lacks such cliques.

Now, suppose the derangement graph $\Gamma_{\bar{G},\Sigma_1}$ has no triangles. As $|\Sigma_1| > 1$, according to [14], we have $|\bar{G}| = |\Sigma_1| = 2$. Consequently, $[G : G_{(\Sigma_1)}] = 2$ and $\bar{G} \cong \mathrm{Sym}(2)$. Notice that each element in $G \setminus G_{(\Sigma_1)}$ acts as a derangement on Σ_1 and, thus, on Ω as well.

Consider $g \in G \setminus G_{(\Sigma_1)}$. If $G_{(\Sigma_1)}$ contains a derangement k in its action on Ω, then gk and gkg^{-1} are both derangements since $gk \in G \setminus G_{(\Sigma_1)}$ and gkg^{-1} is conjugate to k. Consequently,

$$C = \{1, g, k, gk\}$$

forms a clique of size four in $\Gamma_{G,\Omega}$, which contradicts the assumption that $\Gamma_{G,\Omega}$ has no 4-cliques. Hence, $G_{(\Sigma_1)}$, in its action on Ω, contains no derangements. Consequently, the derangement graph $\Gamma_{G,\Omega}$ is bipartite with bipartition

$$G_{(\Sigma_1)}, G \setminus G_{(\Sigma_1)}.$$

However, this contradicts the fact that $\Gamma_{G,\Omega}$ admits a triangle.

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This contradiction demonstrates that the derangement graph $\Gamma_{\bar{G},\Sigma_1}$ of $\bar{G}$ has a triangle but no 4-clique. Since $\bar{G}$ is primitive, by [8, Theorem 1.2], we conclude that $|\Sigma_1| = 3$ and $\mathrm{Alt}(\Sigma_1) \leq \bar{G} \leq \mathrm{Sym}(\Sigma_1)$.

Now, let $\tilde{G}$ be the preimage under $\bar{}$ of $\mathrm{Alt}(\Sigma_1)$. We have $[G : \tilde{G}] = 1$ when $\bar{G} = \mathrm{Alt}(\Sigma_1)$ and $[G : \tilde{G}] = 2$ when $\bar{G} = \mathrm{Sym}(\Sigma_1)$. We claim that $\tilde{G}$ is transitive on Ω. This is clear when $G = \tilde{G}$. Suppose then $|G : \tilde{G}| = 2$. If $G_{\omega_1} \leq \tilde{G}$, then from the bijection between systems of imprimitivity of G on Ω and the interval lattice of G/G_{ω_1}, we deduce that G has a system of imprimitivity of size 2, because $|G : \tilde{G}| = 2$ and $G_{\omega_1} \leq \tilde{G}$. However, this contradicts the fact that we have shown that the minimal cardinality of a system of imprimitivity for G is 3. Therefore, $G_{\omega_1} \not\leq \tilde{G}$. Since $|G : \tilde{G}| = 2$, we have $G = G_{\omega_1}\tilde{G}$. We deduce by the Frattini argument that $\tilde{G}$ is transitive on Ω.

By (4-1), we conclude that $G = \tilde{G}$, which implies $\bar{G} = \mathrm{Alt}(\Sigma_1)$.

Suppose $G_{(\Sigma_1)}$ contains a derangement k in its action on Ω. Now, let $c \in G$ such that $\bar{c}$ acts as a cycle of length 3 on Σ_1, that is, $\bar{G} = \mathrm{Alt}(\Sigma_1) = \langle \bar{c} \rangle$. Following the previous argument, we observe that $C = \{1, c, c^2, k, ck, c^2k\}$ forms a clique of size six in $\Gamma_{G,\Omega}$, which contradicts the assumption that $\Gamma_{G,\Omega}$ has no 4-cliques. Therefore, $G_{(\Sigma_1)}$ in its action on Ω contains no derangement. □

5. The action of G on a refinement Σ_2 of Σ_1

NOTATION 5.1. We establish some necessary notation for the subsequent steps of the proof of Theorem 1.1. From Lemma 4.1, we know that $G_{(\Sigma_1)}$ has no derangements in its action on Ω, and thus,

$$G_{(\Sigma_1)} = \bigcup_{\omega \in \Omega} G_\omega. \tag{5-1}$$

Let $\Sigma_1 = \{\Delta_1, \Delta_2, \Delta_3\}$ and choose $\omega_i \in \Delta_i$ for all $i \in \{1, 2, 3\}$.

Since G is imprimitive on Ω with a system of imprimitivity Σ_1, and G induces $\mathrm{Alt}(\Sigma_1)$ on Σ_1 by Lemma 4.1, we embed

$$G \leq H \,\mathrm{wr}\, \mathrm{Alt}(3),$$

where $H \leq \mathrm{Sym}(\Delta)$, H is transitive, and Δ is a finite set with $|\Delta| = |\Delta_i|$. Under this identification, we may write

$$\Delta_1 = \Delta \times \{1\},\ \Delta_2 = \Delta \times \{2\},\ \Delta_3 = \Delta \times \{3\},$$

and regard G as endowed with its imprimitive action on $\Omega = \Delta \times \{1, 2, 3\}$. Specifically,

$$G_{(\Sigma_1)} \leq H \times H \times H.$$

For each $i \in \{1, 2, 3\}$, let $\pi_i : G_{(\Sigma_1)} \to H$ be the projection onto the ith direct factor. Without loss of generality (by potentially replacing H with the permutation group induced by $G_{(\Sigma_1)}$ on its action on Δ_i), we may assume that

$$H = \pi_1(G_{(\Sigma_1)}) = \pi_2(G_{(\Sigma_1)}) = \pi_3(G_{(\Sigma_1)}).$$

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This allows a concrete representation of the elements of G; indeed, each element of G takes the form $(h_1, h_2, h_3)(1\,2\,3)^i$ for some $h_1, h_2, h_3 \in H$ and $i \in \{0, 1, 2\}$.

Let $c \in G \setminus G_{(\Sigma_1)}$ with $\Delta_1^c = \Delta_2$. Then, $c = (h_1, h_2, h_3)(1\,2\,3)$ for some $h_1, h_2, h_3 \in H$ and $G = G_{(\Sigma_1)}\langle c\rangle$. Let $h = h_1h_2h_3$. Note that

$$\begin{aligned}(1, h_1^{-1}, h_2^{-1}h_1^{-1})^{-1}c(1, h_1^{-1}, h_2^{-1}h_1^{-1}) &= (1, h_1, h_1h_2)(h_1, h_2, h_3)(1\,2\,3)(1, h_1^{-1}, h_2^{-1}h_1)\\ &= (h_1, h_1h_2, h_1h_2h_3)(h_1^{-1}, h_2^{-1}h_1^{-1}, 1)(1\,2\,3)\\ &= (1, 1, h)(1\,2\,3).\end{aligned}$$

This shows that, by replacing G with its conjugate $G^{(1,h_1^{-1},h_2^{-1}h_1^{-1})}$ in H wr Alt(3), we may assume that

$$c = (1, 1, h)(1\,2\,3).$$

Of course, this substitution does not affect the fact that $\Gamma_{G,\Omega}$ has no 4-clique.

Notice that if G is primitive on Ω, then $\Sigma_1 = \Omega$, which implies that part (1) of Theorem 1.1 holds with $|\Omega| = 3$. Hence, for the remainder of the proof of Theorem 1.1, we assume that G is imprimitive on Ω, meaning $|\Omega| > 3$.

LEMMA 5.2. *The group G admits a system of imprimitivity Σ_2 that is a refinement of Σ_1, such that the permutation group induced by G on Σ_2 is permutation isomorphic to the action of* Alt(4) *on the six 2-subsets of* $\{1, 2, 3, 4\}$.

PROOF. From Notation 5.1, we have $\bar{G} = \mathrm{Alt}(\Sigma_1)$ and, hence, $G_{\{\Delta_i\}} = G_{(\Sigma_1)}$, because $\mathrm{Alt}(\Sigma_1)$ acts regularly on Σ_1. As $G_{\{\Delta_i\}}$ is transitive on Δ_i, we deduce that

$$G_{(\Sigma_1)} \text{ is transitive on } \Delta_i \quad \text{for each } i \in \{1, 2, 3\}.$$

Since $|\Omega| > |\Sigma_1| = 3$, G admits a system of imprimitivity $\Sigma_2 \neq \Sigma_1$ that is a refinement of Σ_1. Among all such systems of imprimitivity, choose one so that $|\Sigma_2|$ is as small as possible. Just as in the proof of Lemma 4.1, we may replace G with the permutation group induced by G on Σ_2 and, hence, we may suppose that $\Sigma_2 = \{\{\omega\} \mid \omega \in \Omega\}$. Therefore, we need to show that the action of G on Ω itself is permutation isomorphic to the action of Alt(4) on the six 2-subsets of $\{1, 2, 3, 4\}$.

The minimality of $|\Sigma_2|$ implies that $G_{\{\Delta_1\}} = G_{(\Sigma_1)}$ is primitive in its action on Δ_1. If $G_{(\Delta_1)} = 1$, then the hypotheses of Proposition 3.5 are satisfied and, hence, $G_{(\Sigma_1)}$ contains a derangement in its action on Ω, which contradicts (5-1) (or the conclusion of Lemma 4.1). This contradiction implies that $G_{(\Delta_1)} \neq 1$, that is, $G_{\{\Delta_1\}}$ does not act faithfully on Δ_1. Although we are in a position to apply Proposition 3.6, we choose not to do so here. Instead, we argue using an *ad hoc* method, as we believe this approach could be applied more generally. However, at present, we do not know how to achieve this broader generalization.

Since $G_{(\Sigma_1)}$ is primitive on Δ_i, G_{ω_i} is maximal in $G_{(\Sigma_1)}$. We claim that

$$G_{(\Delta_1\cup\Delta_2)} = G_{(\Delta_1\cup\Delta_3)} = G_{(\Delta_2\cup\Delta_3)} = 1. \tag{5-2}$$

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Observe that, since G acts transitively by conjugation on $\Sigma_1 = \{\Delta_1, \Delta_2, \Delta_3\}$, the groups $G_{(\Delta_1\cup\Delta_2)}$, $G_{(\Delta_1\cup\Delta_3)}$, and $G_{(\Delta_2\cup\Delta_3)}$ are G-conjugate. Hence, if one of them is trivial, all must be trivial. We argue by contradiction and we suppose that $G_{(\Delta_1\cup\Delta_2)} \neq 1$. Observe that

$$G_{(\Delta_1)} \cap G_{(\Delta_2)} \cap G_{(\Delta_3)} = G_{(\Delta_1\cup\Delta_2\cup\Delta_3)} = G_{(\Omega)} = 1.$$

Since $G_{(\Delta_1\cup\Delta_2)} \trianglelefteq G_{(\Sigma_1)}$ and since G_{ω_3} is a maximal subgroup of $G_{(\Sigma_1)}$, we deduce that either $G_{(\Sigma_1)} = G_{\omega_3}G_{(\Delta_1\cup\Delta_2)}$ or $G_{(\Delta_1\cup\Delta_2)} \leq G_{\omega_3}$. If $G_{(\Delta_1\cup\Delta_2)} \leq G_{\omega_3}$, then, using the fact that $G_{(\Delta_1\cup\Delta_2)} \trianglelefteq G_{(\Sigma_1)}$ and that $G_{(\Sigma_1)}$ is transitive on Δ_3, we get

$$G_{(\Delta_1\cup\Delta_2)} \leq \bigcap_{k\in G_{(\Sigma_1)}} G_{\omega_3}^k = \bigcap_{\omega\in\Delta_3} G_\omega = G_{(\Delta_3)}.$$

Thus, $G_{(\Delta_1\cup\Delta_2)} = G_{(\Delta_1\cup\Delta_2\cup\Delta_3)} = 1$, which contradicts the fact that we are assuming $G_{(\Delta_1\cup\Delta_2)} \neq 1$. Therefore,

$$G_{(\Sigma_1)} = G_{\omega_3}G_{(\Delta_1\cup\Delta_2)}.$$

Since $G_{(\Sigma_1)}$ is transitive on Δ_3, the Frattini argument implies that $G_{(\Delta_1\cup\Delta_2)}$ is transitive on Δ_3. In particular, from Jordan's theorem, there exists $k_3 \in G_{(\Delta_1\cup\Delta_2)}$ such that k_3 is a derangement on Δ_3 and fixes each point in $\Delta_1 \cup \Delta_2$. With a similar argument, we obtain $k_1 \in G_{(\Delta_2\cup\Delta_3)}$ acting as a derangement on Δ_1 and $k_2 \in G_{(\Delta_1\cup\Delta_3)}$ acting as a derangement on Δ_2. The support of the permutation k_i is exactly Δ_i and, hence, $k = k_1k_2k_3 \in G_{(\Sigma_1)}$ is a permutation having support exactly $\Delta_1 \cup \Delta_2 \cup \Delta_3 = \Omega$. Therefore, k is a derangement of Ω belonging to $G_{(\Sigma_1)}$, which contradicts (5-1) (or the conclusion of Lemma 4.1). This contradiction has established the veracity of (5-2).

Now, we use the embedding $G \leq H$ wr Alt(3); see Notation 5.1. Let

$$A := \pi_2(G_{(\Delta_1)}) \leq \mathrm{Sym}(\Delta),$$
$$B := \pi_3(G_{(\Delta_1)}) \leq \mathrm{Sym}(\Delta).$$

Thus,

$$G_{(\Delta_1)} \leq 1 \times A \times B. \tag{5-3}$$

Using the maximality of G_{ω_2} and G_{ω_3} in $G_{(\Sigma_1)}$, we get $G_{(\Sigma_1)} = G_{\omega_2}G_{(\Delta_1)} = G_{\omega_3}G_{(\Delta_1)}$. Thus, from the Frattini argument, we deduce that $G_{(\Delta_1)}$ is transitive on Δ_2 and on Δ_3, that is, A and B are transitive subgroups of Sym(Δ).

Conjugating $G_{(\Delta_1)}$ by $c = (1, 1, h)(1\,2\,3)$ (see Notation 5.1) we deduce

$$\begin{aligned} G_{(\Delta_2)} = G_{(\Delta_1)}^c &\leq (1 \times A \times B)^c = (1 \times A \times B)^{(1,1,h)(1\,2\,3)} \\ &= (1 \times A \times B^h)^{(1\,2\,3)} = B^h \times 1 \times A. \end{aligned}$$

In fact, this computation shows that

$$\pi_1(G_{(\Delta_2)}) = B^h \quad \text{and} \quad \pi_3(G_{(\Delta_2)}) = A. \tag{5-4}$$

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As $G_{(\Delta_1)}, G_{(\Delta_2)} \trianglelefteq G_{(\Sigma_1)}$, we have

$$[G_{(\Delta_1)}, G_{(\Delta_2)}] \leq G_{(\Delta_1)} \cap G_{(\Delta_2)} = G_{(\Delta_1 \cup \Delta_2)} = 1,$$

where in the last equality, we have used (5-2). Hence, $G_{(\Delta_1)}$ commutes with $G_{(\Delta_2)}$. From (5-3) and (5-4), we deduce that $[A, B] = 1$. As A and B are both transitive subgroups of $\mathrm{Sym}(\Delta)$, from [4, page 27, 1.5], we get

$$B = \mathbf{C}_{\mathrm{Sym}(\Delta)}(A) \quad \text{and} \quad A = \mathbf{C}_{\mathrm{Sym}(\Delta)}(B).$$

We have $A = B$ if and only if A is abelian and, in this case, A acts regularly on Δ. If $A \neq B$, then A and B both act regularly on Δ, and we may identify the action of A on Δ via its right regular representation and the action of B on Δ via the left regular representation of A itself.

Conjugating by c again, we deduce

$$G_{(\Delta_3)} = G_{(\Delta_2)}^c \leq (B^h \times 1 \times A)^c = A^h \times B^h \times 1.$$

As above, this computation shows that

$$\pi_1(G_{(\Delta_3)}) = A^h \quad \text{and} \quad \pi_2(G_{(\Delta_2)}) = B^h. \tag{5-5}$$

Arguing as above, we see that $G_{(\Delta_3)}$ commutes with $G_{(\Delta_1)}$ and with $G_{(\Delta_2)}$. From (5-3) and (5-5), we deduce that $[A, B^h] = 1$. As A and B^h are both transitive subgroups of $\mathrm{Sym}(\Delta)$, from [4, page 27, 1.5], we get

$$B^h = \mathbf{C}_{\mathrm{Sym}(\Delta)}(A) = B.$$

Thus, h normalizes B. As $A = \mathbf{C}_{\mathrm{Sym}(\Delta)}(B)$, we get

$$A^h = \mathbf{C}_{\mathrm{Sym}(\Delta)}(B^h) = \mathbf{C}_{\mathrm{Sym}(\Delta)}(B) = A.$$

Thus, h normalizes A. Now, from (5-3), (5-4), and (5-5), we have

$$\begin{aligned} G_{(\Delta_1)} &\leq 1 \times A \times B, \\ G_{(\Delta_2)} &\leq B \times 1 \times A, \\ G_{(\Delta_3)} &\leq A \times B \times 1, \end{aligned} \tag{5-6}$$

where $B = \mathbf{C}_{\mathrm{Sym}(\Delta)}(A)$ and A is a regular subgroup of $\mathrm{Sym}(\Delta)$.

Now, we use (5-2) combined with (5-6). Indeed, from (5-2) and from the first line in (5-6), we deduce that, for every $a \in A$, there exists a unique $b \in B$ with $(1, a, b) \in G_{(\Delta_1)}$. In particular, the mapping $\varphi : A \to B$, characterized by the property that $(1, a, a^\varphi) \in G_{(\Delta_1)}$ for every $a \in A$, defines a group isomorphism from A to B. Similarly, (5-2) and the remaining two lines of (5-6) imply that there exist two more group isomorphisms $\psi, \eta : A \to B$ with

$$\begin{aligned} G_{(\Delta_1)} &= \{(1, a, a^\varphi) \mid a \in A\}, \\ G_{(\Delta_2)} &= \{(a^\psi, 1, a) \mid a \in A\}, \\ G_{(\Delta_3)} &= \{(a, a^\eta, 1) \mid a \in A\}. \end{aligned}$$

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We are now ready to conclude the proof of this result. Since A and B are commuting regular subgroups of $\mathrm{Sym}(\Delta)$, we may apply Lemma 3.1. Assume there exist $a \in A \setminus \{1\}$ and $b \in B \setminus \{1\}$ such that ab is a derangement (on Δ). Then,

$$\underbrace{(a^{\psi}, 1, a)}_{\in G_{(\Delta_2)}}\underbrace{(1, b^{\varphi^{-1}}, b)}_{\in G_{(\Delta_1)}} = (a^{\psi}, b^{\varphi^{-1}}, ab) \in G_{(\Sigma_1)}.$$

The first two coordinates of this element lie in regular subgroups, and are different from the identity since φ and ψ are group isomorphisms. Therefore, each coordinate acts as a derangement on its corresponding block and we deduce that $(a^{\psi}, b^{\varphi^{-1}}, ab)$ is a derangement on Ω, which contradicts (5-1). This contradiction shows that only the other case of Lemma 3.1 can occur, namely $|\Delta| = 2$ and $|\Omega| = |\Sigma_1||\Delta| = 6$.

This has shown that G is a subgroup of $\mathrm{Sym}(2)$ wr $\mathrm{Alt}(3)$ in its natural imprimitive action of degree 6. It can be checked, either by hand or with the aid of a computer, that $\mathrm{Sym}(2)$ wr $\mathrm{Alt}(3)$ has three conjugacy classes of transitive subgroups, namely: $\mathrm{Sym}(2)$ wr $\mathrm{Alt}(3)$; $\mathrm{Alt}(4)$ in its action on the six 2-subsets of $\{1, 2, 3, 4\}$; and the cyclic group C_6 of order 6. Since G is minimally transitive, we have $G \neq \mathrm{Sym}(2)$ wr $\mathrm{Alt}(3)$. Moreover, since the derangement graph of C_6 has cliques of size 6, we also have $G \neq C_6$. Thus, G is permutation-isomorphic to $\mathrm{Alt}(4)$ in its action on the six 2-subsets of $\{1, 2, 3, 4\}$. □

6. The action of G on a refinement Σ_3 of Σ_2

NOTATION 6.1. Building upon Notation 5.1 and using Lemma 5.2, we introduce additional notation crucial for the subsequent steps of the proof.

We let Σ_2 be a system of imprimitivity of G on Ω having cardinality 6, which is a refinement of Σ_1. We let $\Sigma_2 = \{\Delta_{1,1}, \Delta_{1,2}, \Delta_{2,1}, \Delta_{2,2}, \Delta_{3,1}, \Delta_{3,2}\}$, where $\Delta_i = \Delta_{i,1} \cup \Delta_{i,2}$ for all $i \in \{1, 2, 3\}$. Moreover, we choose $\omega_{i,j} \in \Delta_{i,j}$ for all $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$.

Using the system of imprimitivity Σ_2, we obtain an embedding

$$G \leq K \,\mathrm{wr}\, \mathrm{Alt}(4),$$

where $K \leq \mathrm{Sym}(\Delta)$, Δ is a finite set with $|\Delta| = |\Delta_{i,j}|$, and K is the permutation group $G_{\{\Delta_{i,j}\}}^{\Delta_{i,j}}$. Under this identification, we may write

$$\begin{aligned}\Delta_{1,1} &= \Delta \times \{1\}, \Delta_{1,2} = \Delta \times \{2\},\\ \Delta_{2,1} &= \Delta \times \{3\}, \Delta_{2,2} = \Delta \times \{4\},\\ \Delta_{3,1} &= \Delta \times \{5\}, \Delta_{3,2} = \Delta \times \{6\},\end{aligned}$$

and we may think about G as endowed with its imprimitive action on $\Omega = \Delta \times \{1, 2, 3, 4, 5, 6\}$. In particular,

$$G_{(\Sigma_2)} \leq K \times K \times K \times K \times K \times K.$$

For each $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, let $\pi_{i,j} : G_{(\Sigma_2)} \to K$ be the projection obtained by restricting a permutation of $G_{(\Sigma_2)}$ to $\Delta_{i,j}$. Since the union of the G-conjugates of $G_{\omega_{1,1}}$

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is $G_{(\Sigma_1)}$ by (5-1), we have $G_{\omega_{1,1}} \not\leq G_{(\Sigma_2)}$. As $[G_{\{\Delta_{1,1}\}} : G_{(\Sigma_2)}] = 2$ and $G_{\omega_{1,1}} \leq G_{\{\Delta_{1,1}\}}$, we deduce

$$G_{\{\Delta_{1,1}\}} = G_{\omega_{1,1}} G_{(\Sigma_2)}.$$

Now, the Frattini argument implies that $G_{(\Sigma_2)}$ is transitive on $\Delta_{1,1}$. As $G_{(\Sigma_2)} \trianglelefteq G$, we get that $G_{(\Sigma_2)}$ is transitive on $\Delta_{i,j}$ for every $i \in \{1,2,3\}$ and $j \in \{1,2\}$.

We let

$$L = \pi_{i,j}(G_{(\Sigma_2)}),$$

and observe that L does not depend on i, j and, from the paragraph above, L is a transitive subgroup of K with $[K : L] \leq 2$.

We claim that

$$G_{\{\Delta_i\}} = G_{\{\Delta_{i,j}\}} G_\omega \quad \text{for all } i \in \{1,2,3\}, j \in \{1,2\}, \omega \in \Omega \setminus \Delta_i.$$

Indeed, as $[G_{\{\Delta_i\}} : G_{\{\Delta_{i,j}\}}] = 2$, we have either $G_{\{\Delta_i\}} = G_{\{\Delta_{i,j}\}} G_\omega$ or $G_\omega \leq G_{\{\Delta_{i,j}\}}$. In the latter case, if we let $\Delta_{i',j'}$ be the block of Σ_2 containing ω, we have $G_\omega \leq G_{\{\Delta_{i,j}\}} \cap G_{\{\Delta_{i',j'}\}} = G_{(\Sigma_2)}$. However, this contradicts the fact that $G_{(\Sigma_2)}$ is transitive on each block of Σ_2.

The embedding of G in K wr Alt(4) allows a concrete representation of the elements of G; indeed, each element of G is of the form $(k_1, k_2, k_3, k_4, k_5, k_6)\sigma$ for some $k_1, k_2, k_3, k_4, k_5, k_6 \in K$ and

$$\sigma \in \langle (1\,3\,5)(2\,4\,6), (1\,2)(3\,4) \rangle \cong \mathrm{Alt}(4).$$

In particular, the element c defined in Notation 5.1 can now be written as

$$c = (1, 1, 1, 1, k_1, k_2)(1\,3\,5)(2\,4\,6).$$

If $G_{(\Sigma_1)}$ is primitive on Δ_i, then $\Sigma_2 = \Omega$. Now, to establish part (2) of Theorem 1.1, it is necessary to omit the assumption in (4-1). This is deduced through computation: the derangement graph of every proper overgroup of Alt(4) in its degree 6 action contains a clique of size 4. Therefore, in the rest of the proof of Theorem 1.1, we may assume that $G_{(\Sigma_1)}$ is imprimitive on Δ_i, that is, $|\Omega| > 6 = |\Sigma_2|$.

Lemma 5.2 has already shown that $|\Omega|$ is a multiple of 6.

LEMMA 6.2. *Let X be the group* Alt(4) *in its transitive degree* 6 *action, and let Γ be an (a, b)-hypergraph such that X is hyper-transitive for Γ and the stabilizer of a point in X fixes some edge of Γ. Then, $\chi(\Gamma) \leq 4$.*

PROOF. Let $\Gamma = (V, E)$ be an (a, b)-hypergraph that is hyper-transitive for X and suppose that the stabilizer of a point in X fixes some edge of Γ. Since X acts transitively on both V and E, it follows that these sets are transitive X-sets.

If $|V| \leq 4$, then it is clear that $\chi(\Gamma) \leq 4$ because we may assign to each element of V a different color. Suppose instead that $|V| > 4$. In particular, the action of X is either permutation isomorphic to the action of Alt(4) on the 2-subsets of $\{1, \ldots, 4\}$ or to the regular action of Alt(4) on itself by right multiplication.

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Assume the first case holds. Then,

$$X = \langle (1\,6)(4\,5),\ (1\,6)(2\,3),\ (1\,2\,4)(3\,5\,6)\rangle.$$

Let $e = \{p_1, \ldots, p_b\} \in E$. By hypothesis, up to replacing e by a suitable conjugate, we may assume $X_e \geq \langle (1\,6)(4\,5)\rangle$. Thus, we have the following possibilities:

- $X_e = X$;
- $X_e = \langle (1\,6)(4\,5),\ (1\,6)(2\,3)\rangle$; or
- $X_e = \langle (1\,6)(4\,5)\rangle$.

If $X_e = X$, then $E = \{e\}$ because X is transitive on E, and $e = \{\{1, 2, 3, 4, 5, 6\}\}$ or $e = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, as X_e must be transitive on the b parts of e. In this case, $\chi(\Gamma) = 2$. A 2-coloring is given by assigning $\chi(x) = \textit{red}$ when $x \neq 6$ and $\chi(x) = \textit{blue}$ when $x = 6$.

If $X_e = \langle (1\,6)(4\,5),\ (1\,6)(2\,3)\rangle$, then

$$E = \{\{1, 6\}\},\ \{\{4, 5\}\},\ \{\{2, 3\}\}.$$

A 2-coloring is given by assigning $\chi(x) = \textit{red}$ for all $x \in \{1, 3, 5\}$ and $\chi(x) = \textit{blue}$ for all $x \in \{2, 4, 6\}$, so $\chi(\Gamma) = 2$.

Finally, if $X_e = \langle (1\,6)(4\,5)\rangle$, then, up to relabeling the points, we may assume $e = \{\{1, 4\}, \{5, 6\}\}$ and

$$E = \{\{\{1, 3\}, \{2, 6\}\}, \{\{2, 5\}, \{3, 4\}\}, \{\{1, 5\}, \{4, 6\}\}, \{\{1, 2\}, \{3, 6\}\},\ \{\{1, 4\}, \{5, 6\}\}, \{\{2, 4\}, \{3, 5\}\}\}.$$

In this case, $\chi(\Gamma) \leq 3$, since a proper 3-coloring is given by assigning $\chi(x) = \textit{red}$ for $x \in \{1, 3\}$, $\chi(x) = \textit{blue}$ for $x \in \{2, 4\}$, and $\chi(x) = \textit{black}$ for $x \in \{5, 6\}$.

The case where the action of X is permutation isomorphic to the regular action of Alt(4) on itself by right multiplication can be treated in a similar manner. □

LEMMA 6.3. *The group G admits a system of imprimitivity Σ_3 that is a refinement of Σ_2, such that the permutation group induced by G on Σ_3 is permutation equivalent to the group* (18, 142)*,* (30, 126)*, or* (30, 233) *in the library of transitive groups in* `Magma`.

Before proceeding with the proof of Lemma 6.3, we observe that the group G_{30} is a subgroup of G_{30}^* and, moreover, that G_{30} is minimally transitive—that is, it is transitive but has no proper transitive subgroups. This observation is relevant for (4-1).

PROOF. Since $|\Omega| > |\Sigma_2| = 6$, G admits a system of imprimitivity $\Sigma_3 \neq \Sigma_2$ that is a refinement of Σ_2. Among all such systems of imprimitivity, choose one so that $|\Sigma_3|$ is as small as possible. Just as in the proof of Lemma 4.1 and of Lemma 5.2, we may replace G with the permutation group induced by G on Σ_3 and, hence, we may suppose that $\Sigma_3 = \{\{\omega\} \mid \omega \in \Omega\}$.

If $G_{(\Sigma_2)} = 1$, then G acts faithfully on Σ_2 and, hence, $G \cong \mathrm{Alt}(4)$, because the permutation group induced by G on Σ_2 is Alt(4) in its degree-six action. Since

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$12 = |G| \geq |\Omega| > |\Sigma_2| = 6$, we deduce $|\Omega| = 12$ and G acts regularly on Ω, which is clearly a contradiction. Therefore,

$$G_{(\Sigma_2)} \neq 1. \tag{6-1}$$

The minimality of $|\Sigma_3|$ implies that $G_{\{\Delta_{1,1}\}}$ is primitive in its action on $\Delta_{1,1}$. If $G_{(\Delta_{1,1})} = 1$, then the hypotheses of Proposition 3.5 are satisfied and, hence, $G_{(\Sigma_2)}$ contains a derangement in its action on Ω, which contradicts (5-1). This contradiction implies that

$$G_{(\Delta_{i,j})} \neq 1 \quad \text{for all } i \in \{1,2,3\}, j \in \{1,2\}. \tag{6-2}$$

Suppose that the primitive permutation group $G^{\Delta_{i,j}}_{\{\Delta_{i,j}\}}$ has nonabelian socle. Summing up:

- $G_{\{\Delta_{i,j}\}}$ acts primitively on $\Delta_{i,j}$, by Notation 6.1;
- $G_{(\Sigma_2)} \neq 1$, by (6-1);
- $G_{(\Delta_{i,j})} \neq 1$, by (6-2); and
- $G^{\Delta_{i,j}}_{\{\Delta_{i,j}\}}$ has nonabelian socle, by the current hypothesis.

Therefore, we are now in a position to apply Proposition 3.6 (with $\Sigma = \Sigma_2$ and $\Delta = \Delta_{i,j}$) and the notation therein. Since, by (5-1), $G_{(\Sigma_2)}$ has no derangements, we deduce that parts (1)–(5) of Proposition 3.6 are satisfied. In particular, by parts (c) and (d) of Proposition 3.6, there exists an (a,b)-hypergraph Γ that is hyper-transitive for $G/_{(\Sigma_2)} \cong \mathrm{Alt}(4)$ and with the property that $G_{\{\Delta_{i,j}\}}/G_{(\Sigma_2)} \cong C_2$ fixes some edge of Γ. In particular, the stabilizer of a point in Alt(4) (in its action of degree 6) fixes some edge of Γ. From Lemma 6.2, $\chi(\Gamma) \leq 4$.

Observe that, by Burnside's $p^\alpha q^\beta$ theorem, a nonabelian simple group has order divisible by at least three distinct primes and, hence, the nonabelian simple group T appearing in the statement of Proposition 3.6 has at least 4 $\mathrm{Aut}(T)$-conjugacy classes. From Proposition 3.6 part (b) and the previous paragraph, we immediately obtain a contradiction.

Assume that $G^{\Delta}_{\{\Delta\}}$ has abelian socle. Let V be a minimal normal subgroup of G with $V \leq G_{(\Sigma_2)}$. Then, V is an elementary abelian p-group for some prime number p. From (5-1), we have

$$V = \bigcup_{\omega \in \Omega} V_\omega.$$

Since V_ω is normalized by V and by G_ω, we get $\mathbf{N}_G(V_\omega) \geq G_\omega V = G_{\{\Delta\}}$ and, hence, the union above consists of at most $|\Sigma_2|$ elements. Therefore,

$$p^a = |\Delta| = |V : V_\omega| < |\Sigma_2| = 6.$$

Thus, $|\Omega| = p^a|\Sigma_2| \leq 30$. The rest of the result follows from a computer computation using the database of transitive groups of degree at most 30. □

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7. Final step

The objective of this section is to establish Theorem 1.1.

LEMMA 7.1. *Let G be the group* (18, 142) *or* (30, 126) *in the library of transitive groups in* `Magma`*, and let Γ be an (a, b)-hypergraph Γ such that G is hyper-transitive for Γ and such that the stabilizer of a point in G fixes some edge of Γ. Then, $\chi(\Gamma) \leq 4$.*

PROOF. This follows from a computer computation; see also the proof of Lemma 6.2. Given G as above, we constructed all (a, b)-hypergraphs Γ such that G is hyper-transitive for Γ and such that the stabilizer of a point in G fixes some edge of Γ. We give some details of the computation. Let $\Gamma = (V, E)$ be hyper-transitive for G, where V is the set of vertices and E is the set of edges of Γ. Since Γ is hyper-transitive for G, it follows that V and E are transitive G-sets. In particular, this significantly narrows down the number of possibilities for V and for E. Indeed, V can be any transitive G-set. However, once V is fixed, edge transitivity implies that the set of edges E is uniquely determined by e^G, where $e = \{p_1, \ldots, p_b\}$ is a fixed edge of Γ, which, by definition, is a partition of an ab-subset of V into b parts of cardinality a.

This partition e is in turn very limited by the assumption that the stabilizer of a point in G fixes e. Moreover, the choice of e is further restricted because, by definition, the edge stabilizer G_e is transitive on the b parts of e, and either G_e is also transitive on the underlying set of e or $a = 2$. All of these conditions impose severe constraints on Γ.

Then, for each (a, b)-hypergraph $\Gamma = (V, E)$, we randomly constructed 10^6 maps $\eta : V \to \{1, 2, 3, 4\}$ and checked whether each map was a coloring of Γ. In all cases, we found such a map. Incidentally, these (a, b)-hypergraphs may have a smaller chromatic number. The choice of 10^6 bears no particular meaning; it was simply large enough to ensure that the computer could find a suitable coloring for each Γ by random search within a reasonable amount of time. □

THEOREM 7.2 (See [1, Theorems 1 and 2]). *Let p be an odd prime number and $a \geq 1$. If $p + 1 < b < 3(p + 1)/2$ or if $b = p + 1$ and $a \geq 2$, then every transitive group of degree $p^a \cdot b$ contains a derangement of p-power order.*

LEMMA 7.3. *Let X be a finite group and let N be a normal subgroup of X such that X/N is solvable. Then, either N is contained in the Frattini subgroup of X or $X = NM$ for some maximal subgroup M of X. Therefore, in either case, there exists a solvable subgroup Y of X such that $X = YN$.*

PROOF. Let $\Phi(X)$ denote the Frattini subgroup of X. If $N \not\leq \Phi(X)$, then there exists a maximal subgroup M with $N \not\leq M$. The maximality of M implies $X = NM$.

For the final statement, we argue by induction on $|X|$. If $|X| = 1$, the result is trivial. If $N \leq \Phi(X)$, then X is solvable because X/N is solvable and $\Phi(X)$ is nilpotent. Hence, we may take $Y = X$ in this case.

If $N \not\leq \Phi(X)$, then as noted above, $X = NM$ for some maximal subgroup M of X. Since $|M| < |X|$ and $M/(M \cap N) \cong X/N$ is solvable, by the inductive hypothesis, there

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exists a solvable subgroup Y of M such that $M = Y(M \cap N)$. It follows that $X = NM = Y(M \cap N)N = YN$, as required. □

PROOF THEOREM 1.1. We use the notation established throughout Sections 4, 5, and 6.

Let Σ_4 be a system of imprimitivity for G that is a refinement of Σ_3. Just as in the proof of Lemmas 4.1, 5.2, and 6.3, we may replace G with the permutation group induced by G on Σ_4 and, hence, we may suppose that $\Sigma_4 = \{\{\omega\} | \omega \in \Omega\}$.

Suppose first $G_{(\Sigma_3)} = 1$. Then G, as an abstract group, is isomorphic to the permutation group induced by G on Σ_3 and, hence, it is isomorphic to the transitive group (18, 142) or (30, 126) in the library of transitive groups in `Magma` [2]. We have checked with a computer that these two permutation groups of degree 18 and 30 do not admit a transitive action on a set $|\Omega| > |\Sigma_3|$ such that the derangement graph has no 4-cliques. Therefore, $G_{(\Sigma_3)} \neq 1$. Let $\Delta \in \Sigma_3$.

Suppose $G_{(\Delta)} = 1$. Then, by Proposition 3.5, $G_{(\Sigma_3)}$ admits a derangement for its action on Ω, which contradicts (5-1). Therefore, $G_{(\Delta)} \neq 1$.

Assume that $G^{\Delta}_{\{\Delta\}}$ has nonabelian socle. We now apply Proposition 3.6 and the notation therein. Since by (5-1), $G_{(\Sigma_3)}$ has no derangements, we deduce that parts (1)–(5) of Proposition 3.6 are satisfied. Observe that, by Burnside's $p^\alpha q^\beta$ theorem, a nonabelian simple group has order divisible by at least three distinct primes and, hence, the nonabelian simple group T appearing in the statement of Proposition 3.6 has at least $4\,\mathrm{Aut}(T)$-conjugacy classes. From Proposition 3.6(b) and the previous paragraph, we immediately obtain a contradiction.

Assume that $G^{\Delta}_{\{\Delta\}}$ has abelian socle. Let V be a minimal normal subgroup of G with $V \leq G_{(\Sigma_3)}$. Then, V is an elementary abelian p-group for some prime number p. From (5-1), we have $V = \bigcup_{\omega\in\Omega} V_\omega$. Since V_ω is normalized by V and by G_ω, we get $\mathbf{N}_G(V_\omega) \geq G_\omega V = G_{\{\Delta\}}$ and, hence, the union above consists of at most $|\Sigma_3|$ elements. Therefore,

$$p^a = |\Delta| = |V : V_\omega| < |\Sigma_3|.$$

Observe that, since $|G : G_{(\Sigma_1)}| = 3$ and $G_{(\Sigma_1)} \trianglelefteq G$, if $p \neq 3$, then every p-element of G is contained in $G_{(\Sigma_1)}$. Hence, G has no derangements of p-power order with $p \neq 3$ by (5-1). Therefore, using Theorem 7.2:

- $|\Sigma_3| = 18$ and $|\Delta| \in \{2, 3, 4, 5, 7, 8, 9, 11, 16, 17\}$; or
- $|\Sigma_3| = 30$ and $|\Delta| \in \{2, 3, 4, 7, 8, 9, 11, 13, 16, 17, 19, 27, 29\}$.

We claim that G is solvable. Observe that $G/G_{(\Sigma_3)}$ is solvable and, hence, by Lemma 7.3, there exists $Y \leq G$ with Y solvable and $G = G_{(\Sigma_3)}Y$. In particular Y is transitive on Σ_3 and, hence, VY is transitive on Ω. The minimality of G implies $G = YV$ and, hence, G is solvable.

Using the computer algebra system `Magma`, we have dealt with the cases:

- $|\Sigma_3| = 18$ and $|\Delta| \in \{2, 3, 4, 5\}$; and
- $|\Sigma_3| = 30$ and $|\Delta| \in \{2, 3, 4\}$.

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In all of these cases, no counterexample arises. The main tool used in these computations is the built-in function `MaximalSubgroups` in `Magma`, with *ad hoc* arguments for some of the cases. Indeed, for each solvable primitive group X of affine type of degree p^d, we have constructed $X \mathrm{wr}\, G/G_{(\Sigma_3)}$ and working inductively on the subgroup lattice, we have constructed the relevant groups. Observe that, by the minimality of G and by the fact that G is solvable, we may suppose that X is a $\{2, 3, p\}$-group when $|\Sigma_3| = 18$ and a $\{2, 3, 5, p\}$-group when $|\Sigma_3| = 30$. Here, by relevant, we mean a transitive subgroup Y of $X \mathrm{wr}\, G/G_{(\Sigma_3)}$ projecting surjectively to $G/G_{(\Sigma_3)}$ and with the stabilizer of $\Delta \in \Delta_3$ in Y projecting surjectively to X.

For the rest of the argument, we exclude the cases $|\Sigma_3| = 18$ and $|\Delta| \in \{2, 3, 4, 5\}$, and $|\Sigma_3| = 30$ and $|\Delta| \in \{2, 3, 4\}$. We claim that

$$G_{(\Delta_1)} \leq G_{(\Sigma_3)}. \tag{7-1}$$

When $|\Sigma_3| = 30$, it can be verified with a computer that in G_{30}, the identity is the unique element fixing the ten elements in Σ_3 contained in Δ_1. In G_{18}, the pointwise stabilizer of the six elements of Σ_3 contained in Δ_1 has order 3 and fixes none of the remaining 12 elements of Σ_3. This implies that, if $G_{(\Delta_1)} \not\leq G_{(\Sigma_3)}$, then there exists $g \in G_{(\Delta_1)}$ fixing pointwise Δ_1 and fixing setwise no element $\Delta \in \Sigma_3$ with $\Delta \subseteq \Delta_2 \cup \Delta_3$. Now, let $\Lambda_1, \ldots, \Lambda_6$ be the remaining six elements of Σ_3 contained in Δ_1 and let $\omega_i \in \Lambda_i$ for each i. If

$$V_{\omega_1} \cup V_{\omega_2} \cup V_{\omega_3} \cup V_{\omega_4} \cup V_{\omega_5} \cup V_{\omega_6} \subsetneq V,$$

then there exists $z \in V \setminus (V_{\omega_1} \cup \cdots \cup V_{\omega_6})$. Clearly, gz is a derangement on Ω, because gz fixes no block of Σ_3 outside Δ_1 and acts as the derangement z on Δ_1. However, this contradicts (5-1). Therefore, $V_{\omega_1} \cup V_{\omega_2} \cup V_{\omega_3} \cup V_{\omega_4} \cup V_{\omega_5} \cup V_{\omega_6} = V$ and, hence, $6 > |V : V_{\omega_i}| = p^d$. However, we are excluding these cases here. This has established (7-1).

We claim that

$$G_{(\Delta_{1,1})} \leq G_{(\Sigma_3)} \quad \text{and} \quad |\Delta| \in \{11, 16\} \quad \text{when } |\Sigma_3| = 30.$$

The argument here is similar to that in the paragraph above. In G_{30}, the pointwise stabilizer of the five elements of Σ_3 contained in $\Delta_{1,1}$ has order 5 and fixes none of the remaining 25 elements of Σ_3. This implies that if $G_{(\Delta_{1,1})} \not\leq G_{(\Sigma_3)}$, then there exists $g \in G_{(\Delta_{1,1})}$ fixing pointwise $\Delta_{1,1}$ and fixing setwise no element $\Delta \in \Sigma_3$ with $\Delta \subseteq \Delta_{1,2} \cup \Delta_2 \cup \Delta_3$. Now, let $\Delta_{1,1,1}, \ldots, \Delta_{1,1,5}$ be the remaining five elements of Σ_3 contained in $\Delta_{1,1}$ and let $\omega_i \in \Delta_{1,1,i}$ for each i. If

$$V_{\omega_1} \cup V_{\omega_2} \cup V_{\omega_3} \cup V_{\omega_4} \cup V_{\omega_5} \subsetneq V,$$

then there exists $z \in V \setminus (V_{\omega_1} \cup \cdots \cup V_{\omega_5})$. Clearly, gz is a derangement on Ω, because it fixes no block of Σ_3 outside $\Delta_{1,1}$ and it acts as the derangement z on $\Delta_{1,1}$. However, this contradicts (5-1). Therefore, $V_{\omega_1} \cup V_{\omega_2} \cup V_{\omega_3} \cup V_{\omega_4} \cup V_{\omega_5} = V$ and, hence, $5 > |V : V_{\omega_i}| = p^d$. However, we are excluding these cases here. This has established

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$G_{(\Delta_{1,1})} \leq G_{(\Sigma_3)}$. Now, the group $G_{\{\Delta_{1,1}\}}/G_{(\Delta_{1,1})}$ is embedded in $\mathrm{AGL}_d(p)\,\mathrm{wr}\,L$, where L is the permutation group induced by G_{30} on the five elements of Σ_3 contained in $\Delta_{1,1}$. It can be verified that L is a Frobenius group of order 20 and that in G_{30}, the stabilizer of a block of cardinality 5 has order 100. As $G_{(\Delta_{1,1})} \leq G_{(\Sigma_3)}$, $|G_{\{\Delta_{1,1}\}} : G_{(\Sigma_3)}| = 100$ divides the order of $\mathrm{AGL}_d(p)\,\mathrm{wr}\,L$ and, hence, $100/20 = 5$ divides $|\mathrm{GL}_d(p)|$. This implies $|\Delta| \in \{11, 16\}$.

A similar argument holds when $|\Sigma_3| = 18$. We claim that

$$|G_{(\Delta_{1,1})} : G_{(\Delta_{1,1})} \cap G_{(\Sigma_3)}| \leq 3 \quad \text{and} \quad |\Delta| \in \{9, 16\} \quad \text{when } |\Sigma_3| = 18.$$

In G_{18}, the pointwise stabilizer of the three elements of Σ_3 contained in $\Delta_{1,1}$ has order 9. Let $\Delta_{1,1,1}$, $\Delta_{1,1,2}$, and $\Delta_{1,1,3}$ be the three elements of Σ_3 with $\Delta_{1,1} = \Delta_{1,1,1} \cup \Delta_{1,1,2} \cup \Delta_{1,1,3}$. In particular,

$$|G_{\{\Delta_{1,1,1}\}} \cap G_{\{\Delta_{1,1,2}\}} \cap G_{\{\Delta_{1,1,3}\}} : G_{(\Sigma_3)}| = 9.$$

Assume now

$$G_{\{\Delta_{1,1,1}\}} \cap G_{\{\Delta_{1,1,2}\}} \cap G_{\{\Delta_{1,1,3}\}} = G_{(\Delta_{1,1})} G_{(\Sigma_3)}.$$

This implies that there exists $g \in G_{(\Delta_{1,1})}$ fixing pointwise $\Delta_{1,1}$ and fixing setwise no element $\Delta \in \Sigma_3$ with $\Delta \subseteq \Delta_{1,2} \cup \Delta_2 \cup \Delta_3$. Now, let $\omega_i \in \Delta_{1,1,i}$ for each $i \in \{1, 2, 3\}$. If $V_{\omega_1} \cup V_{\omega_2} \cup V_{\omega_3} \subsetneq V$, then there exists $z \in V \setminus (V_{\omega_1} \cup V_{\omega_2} \cup V_{\omega_3})$. Clearly, gz is a derangement on Ω, which contradicts (5-1). Therefore, $V_{\omega_1} \cup V_{\omega_2} \cup V_{\omega_3} = V$ and, hence, $3 > |V : V_{\omega_i}| = p^d$. However, we are excluding these cases here. This has established $|G_{(\Delta_{1,1})} : G_{(\Delta_{1,1})} \cap G_{(\Sigma_3)}| \leq 3$. Now, the group $G_{\{\Delta_{1,1}\}}/G_{(\Delta_1,1)}$ is embedded in $Z\,\mathrm{wr}\,L$, where L is the permutation group induced by G_{18} on the three elements of Σ_3 contained in $\Delta_{1,1}$ and Z is a solvable primitive subgroup of $\mathrm{AGL}_d(p)$. It can be verified that L is the symmetric group of degree 3 and that in G_{18}, the stabilizer of a block of cardinality 3 has order 54. As $|G_{(\Delta_{1,1})} : G_{(\Delta_{1,1})} \cap G_{(\Sigma_3)}| \leq 3$, $|G_{\{\Delta_{1,1}\}} : G_{(\Sigma_3)} G_{(\Delta_{1,1})}|$ is either 18 or 54 (depending on whether $|G_{(\Delta_{1,1})} : G_{(\Delta_{1,1})} \cap G_{(\Sigma_3)}|$ equals 3 or 1). In particular, 18 divides the order of $\mathrm{AGL}_d(p)\,\mathrm{wr}\,L$ and, hence, $18/6 = 3$ divides $|\mathrm{GL}_d(p)|$. This implies $|\Delta| \in \{9, 16\}$.

Summing up, the remaining cases are:

- $|\Sigma_3| = 18$ and $|\Delta| \in \{7, 9, 16\}$; and
- $|\Sigma_3| = 30$ and $|\Delta| \in \{11, 16\}$.

The rest of the proof is computational with *ad hoc* arguments for each of the cases. □

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