$$\text{and } b_n - b_{n-1} = \frac{1}{n} - \frac{4}{3}\log\left(\frac{n+\frac{1}{2}}{n-\frac{1}{2}}\right) + \frac{1}{6}\log\left(\frac{n+1}{n-1}\right)$$
$$= \frac{1}{n} - \frac{4}{3}\log\left(\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right) + \frac{1}{6}\log\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)$$
$$= \frac{1}{n} - \frac{8}{3}\left(\frac{1}{2n} + \frac{1}{3(2n)^3} + \frac{1}{5(2n)^5} + O\left(\frac{1}{n^7}\right)\right)$$
$$+ \frac{1}{3}\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5}\right) + O\left(\frac{1}{n^7}\right) = \frac{1}{20n^5} + O\left(\frac{1}{n^7}\right). \qquad (5)$$

Hence not only the initial terms (3) but also the consecutive term differences (4), (5) would seem to indicate that the new increasing monotone sequence $\{b_n\}$ converges to γ more rapidly than the original decreasing monotone sequence $\{a_n\}$.

Note finally that, while $a_{20} > 0.6$, we have $b_{20} = 0.57721559\ldots$ which compares well with $\gamma = 0.5772156649\ldots$ gleaned from the Internet.

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109.41 Elementary proofs of some trigonometric inequalities

In this Note, we present some results concerning inequalities using both geometric and trigonometric methods.

Geometrical proofs

In order to prove the inequality

$$\frac{\tan x + \tan y}{2} \geqslant \tan\left(\frac{x+y}{2}\right),\ x \in \left[0, \tfrac{1}{2}\pi\right),\ y \in \left[0, \tfrac{1}{2}\pi\right), \qquad (1)$$

we are using Figure 1, where the semicircle is a part of the circle of radius $OA = 1$, and the segment EA is tangent to a given circle at the point A.

Let $x = \angle AOB$, $y = \angle AOE$. According to Figure 1, if C_1 is the midpoint of arc $\overset{\frown}{B_1E_1}$, then $\overset{\frown}{B_1C_1} = \overset{\frown}{C_1E_1} = \frac{1}{2}\overset{\frown}{B_1E_1} = \frac{1}{2}(y - x)$.

The point D is the midpoint of the segment BE, OC represents an angle-bisector and OD a median in the triangle BOE. Using the result that a median is at least as long as the corresponding angle-bisector, [1], then $OD \geqslant OC$ and after that $AD \geqslant AC$, equivalent to

$$AD = AB + BD = AB + \frac{AE - AB}{2} = \frac{AB + AE}{2} = \frac{\tan x + \tan y}{2},$$
$$AC = \tan\angle AOC_1 = \tan\left(x + \frac{y-x}{2}\right) = \tan\left(\frac{x+y}{2}\right).$$

The inequality (1) is validated.

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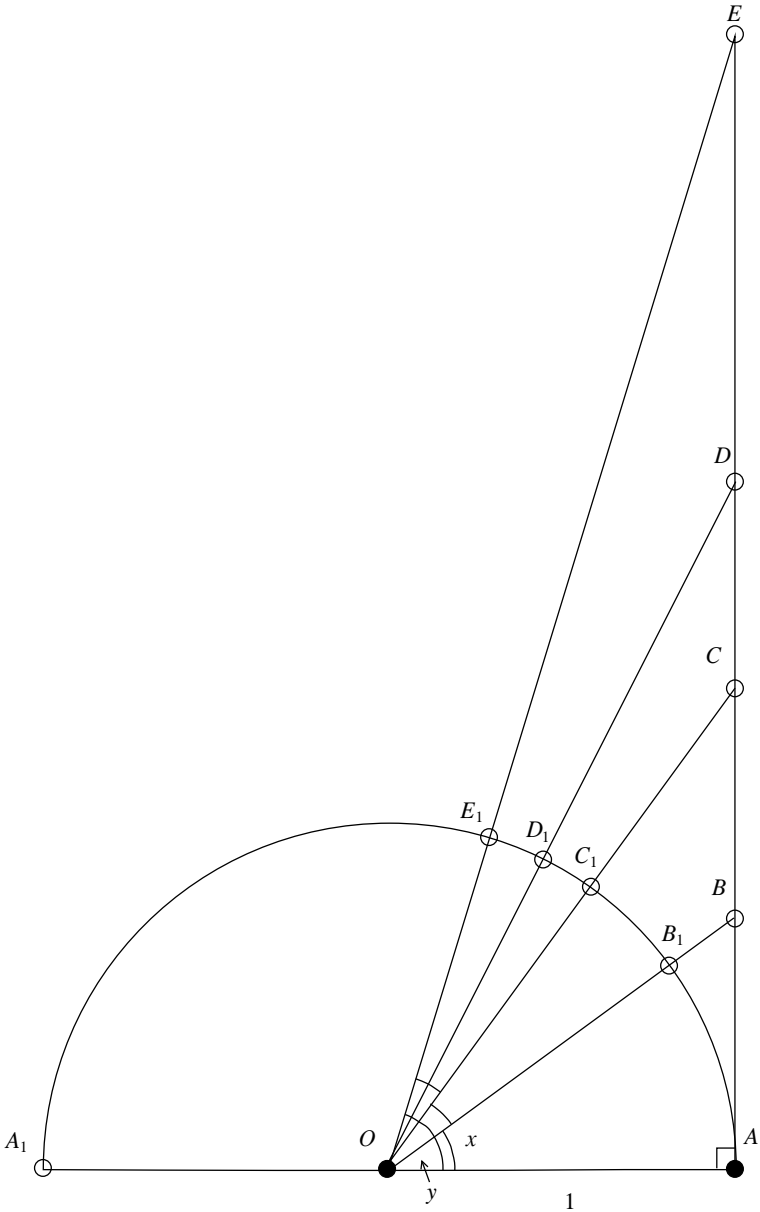


FIGURE 1

In the next step we prove the following inequality

$$\frac{\sin x + \sin y}{2} \leqslant \sin\left(\frac{x+y}{2}\right). \tag{2}$$

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In Figure 2, the point C is the middle of the circular arc $\overset{\frown}{BD}$, $\angle AOB = x$, $\angle AOD = y$. Let $BD \cap OC = \{E\}$ and the points B_1, C_1, E_1, D_1 are projections of the points B, C, E, D on the segment OA. Since the point E is the midpoint of the segment BD, then $EE_1 = \frac{1}{2}(BB_1 + DD_1) \leqslant CC_1$ is equivalent to

$$\frac{\sin x + \sin y}{2} \leqslant \sin\left(x + \frac{y - x}{2}\right) = \sin\left(\frac{x + y}{2}\right),$$

and $OE_1 = \frac{1}{2}(OD_1 + OB_1) \leqslant OC_1$ leads to $\frac{1}{2}(\cos x + \cos y) \leqslant \cos\frac{1}{2}(x + y)$. This also follows by replacing x, y by $\frac{1}{2}\pi - x$, $\frac{1}{2}\pi - y$ in the sine inequality.

Trigonometric proofs

$$\begin{aligned}\frac{\tan x + \tan y}{2} &= \frac{1}{2}\left(\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}\right) = \frac{\sin x \cos y + \sin y \cos x}{2 \cos x \cos y} \\ &= \frac{\sin(x + y)}{\cos(x - y) + \cos(x + y)} \geqslant \frac{\sin(x + y)}{1 + \cos(x + y)} \\ &= \frac{2 \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)}{2\cos^2\left(\frac{x+y}{2}\right)} = \frac{\sin\left(\frac{x+y}{2}\right)}{\cos\left(\frac{x+y}{2}\right)} = \tan\left(\frac{x+y}{2}\right).\end{aligned}$$

The inequality (1) is proved. For inequality (2):

$$\frac{\sin x + \sin y}{2} = \sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right) \leqslant \sin\left(\frac{x + y}{2}\right).$$

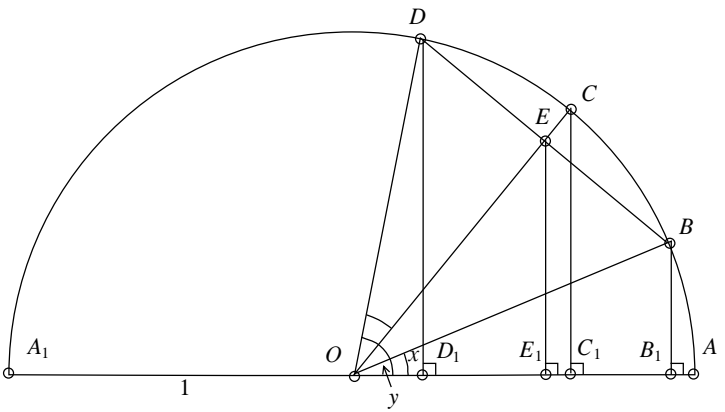


FIGURE 2

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MARIAN DINCA
Strada Aleea Buhusi, nr. 1,Bloc 11,Sc. 1, et. 8,ap.57, sector 3, Bucuresti, Romania

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