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Textile systems and one-sided resolving automorphisms and endomorphisms of the shift

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Abstract. Two results on textile systems are obtained. Using these we prove that for any automorphism φ of any topologically-transitive subshift of finite type, if φ is expansive and φ or φ^{-1} has memory zero or anticipation zero, then φ is topologically conjugate to a subshift of finite type. Moreover, this is generalized to a result on chain recurrent onto endomorphisms of topologically-transitive subshifts of finite type. Using textile systems and textile subsystems, we develop a structure theory concerning expansiveness with the pseudo orbit tracing property on directionally essentially weakly one-sided resolving automorphisms and endomorphisms of subshifts.

1. *Introduction*

For a subshift (X, σ) over an alphabet A and $k \geq 1$, let $L_k(X)$ denote the set of all words $a_j \ldots a_{j+k-1}$ that appear on some point $(a_j)_{j\in\mathbb{Z}} \in X$ with $a_j \in A$. Let (X, σ) and (X', σ') be subshifts. Let $m, n \geq 0$. A mapping $\phi : X \to X'$ is called a *block map of* (m, n) *type* or a *block map* if there exists a *local rule* $f : L_{m+n+1}(X) \to L_1(X')$ such that

$$\phi((a_j)_{j\in\mathbb{Z}}) = (a'_j)_{j\in\mathbb{Z}} \quad \text{with } a'_j = f(a_{j-m} \ldots a_{j+n}) \text{ for all } j \in \mathbb{Z}.$$

A block map of (m, n) type is said to have *memory* m and *anticipation* n.

For subshifts (X, σ) and (X', σ'), a mapping $\phi : X \to X'$ is a homomorphism of (X, σ) into (X', σ') (i.e. a continuous map with $\phi\sigma = \sigma'\phi$) if and only if ϕ is a block map (Curtis–Hedlund–Lyndon Theorem [**14**]). Hence we use the same terminology as above for homomorphisms between subshifts.

The following problem remains open: for any expansive automorphism φ of a topologically-transitive subshift of finite type (SFT) (X, σ), is the dynamical system (X, φ) conjugate to an SFT? This was asked in [**20**] for automorphisms of general SFTs, but Doris Fiebig [**13**] gave examples of an automorphism φ of a non-transitive SFT (X, σ) such that (X, φ) is not conjugate to an SFT. (Her examples are included in Boyle [**6**] as an appendix.) Recently Boyle [**6**] proved that if the dynamical system defined by an expansive automorphism of a non-wandering SFT is conjugate to an almost Markov sofic

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system, then it must be conjugate to an SFT. However, from Fiebig's result until Boyle's result, no progress on the problem was made. This contrasts with the complete solution of the similar problems for positively-expansive endomorphisms of one-sided and two-sided SFTs and expansive automorphisms of one-sided SFTs and remarkable further progress on these and related subjects [**4, 7, 8, 12, 13, 16, 20–23, 25**] (for a quick survey, see [**24**]). The open problem above is extended to the following problem: for any automorphism φ of a transitive SFT (X, σ), determine the dynamics of $\varphi^i\sigma^j$ for $i, j \in \mathbb{Z}$. The starting point of the research on this problem was given by Boyle and Krieger [**9**]: they proved that if φ is an automorphism of an SFT (X, σ), then $(X, \varphi\sigma^j)$ is conjugate to an SFT for all j greater than $\max\{k, n\}$ if φ has memory k and φ^{-1} has anticipation n. This result was improved to the following in [**20**]: for any automorphism φ of a topological Markov shift (X, σ), $\varphi\sigma^j$ is an 'LR' automorphism for some $j \in \mathbb{Z}$; if φ is LR, then φ^i is LR for all $i \geq 0$ and $\varphi\sigma$ is LR and expansive; and if φ is LR and expansive, then (X, φ) is conjugate to an SFT (which is explicitly given by using the 'LR' property). It turned out that an automorphism φ of an SFT is essentially LR (ELR) if and only if φ essentially has the property that φ has memory zero and φ^{-1} has anticipation zero [**21**]. Here the term 'essentially' means 'up to topological conjugacy between endomorphisms of dynamical systems'.

In this paper we prove that for any automorphism φ of any transitive SFT (X, σ), if φ is expansive and φ or φ^{-1} has memory zero or anticipation zero, then (X, φ) is topologically conjugate to an SFT (Theorem 5.2).

This is generalized to a result on chain recurrent onto endomorphisms of transitive SFTs. Using textile systems we define the notions of 'p-L', 'p-R', 'q-L' and 'q-R' for onto endomorphisms of topological Markov shifts and say that an onto endomorphism of a topological Markov shift is 'one-sided resolving' if it has at least one of these four properties. An onto endomorphism of an SFT is essentially p-L (respectively, p-R) if and only if it essentially has memory (respectively, anticipation) zero; an automorphism of an SFT is essentially q-L (respectively, q-R) if and only if its inverse essentially has memory (respectively, anticipation) zero. We obtain the result that if φ is a chain recurrent onto endomorphism of a transitive SFT (X, σ) with at least one of the four properties of (i) being essentially p-L and left σ-expansive, (ii) being essentially p-R and right σ-expansive, (iii) being essentially q-L and left σ-expansive, and (iv) being essentially q-R and right σ-expansive, then φ has the pseudo orbit tracing property (POTP) (Theorem 6.3). This, combined with a result of Boyle and Kitchens [**8**] unifies and generalizes the known affirmative solutions for the versions of the problem mentioned above. (For the definitions of left and right σ-expansiveness, see the beginning of §6.)

Before proving the above results, we prove two main results on textile systems: Propositions 3.10 and 4.3. The former is proved by using a method developed in [**23**] and a fundamental result of Boyle in [**6**]; the latter is proved by using the method of [**23**] with a new ingredient, the chain recurrence condition, which in symbolic dynamics was explained by Boyle [**6**]. Our result stated first is proved in two ways: one uses Proposition 3.10 and the other uses Proposition 4.3, and the generalized result stated next is proved by using Proposition 4.3. Proposition 3.10 gives necessary and sufficient conditions for the dynamical system defined by an expansive automorphism of a mixing SFT to be

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conjugate to an SFT, but we can only make use of an extremely special case of it in this paper.

We introduce the notions of 'weakly p-L', 'weakly p-R', 'weakly q-L' and 'weakly q-R' for onto endomorphisms of subshifts, using 'textile subsystems'. An onto endomorphism of a subshift is essentially weakly p-L (respectively, p-R) if and only if it essentially has memory (respectively, anticipation) zero; an automorphism of a subshift is essentially weakly q-L (respectively, q-R) if and only if its inverse essentially has memory (respectively, anticipation) zero. In §7, we develop the basic properties of these essentially 'weakly one-sided resolving' endomorphisms of subshifts.

Let X be a zero-dimensional compact metric space, $S(X)$ the monoid of surjective continuous maps of X and $H(X)$ the group of homeomorphisms in $S(X)$. Let τ be an expansive element of $H(X)$. Let J be a commutative submonoid of $S(X)$. Let $PL_J(\tau)$ denote the set of all elements of J that are a directionally essentially weakly p-L endomorphism of (X, τ). (For any property P of endomorphisms of dynamical systems, we say that an endomorphism φ of (X, τ) is 'directionally' P to mean that for some $r, s \geq 1$ the endomorphism φ^r of (X, τ^s) is P.) Similarly, we define $QR_J(\tau)$. Let $PL_J^\circ(\tau)$ and $QR_J^\circ(\tau)$ be the set of all right τ-expansive elements of $PL_J(\tau)$ and the set of all left τ-expansive elements of $QR_J(\tau)$, respectively. If $\tau \in J$, then we can imagine that $PL_J(\tau)$ is a kind of 'cone' with $\{\tau^n \mid n = 0, 1, \ldots\}$ as an 'inner half-line of points' and with the 'interior' $PL_J^\circ(\tau)$, and can do similarly for $QR_J(\tau)$. We define $C_J(\tau)$ to be the set of all directionally essentially weakly LR endomorphisms of τ, and define $C_J^\circ(\tau)$ to be the set of all expansive elements of $C_J(\tau)$. (Here the term 'expansive' is used in the sense of 'c-expansive' in [**2**]; see §2.) If $\tau \in J$, then we can also imagine that $C_J(\tau)$ is a kind of 'cone' with $\{\tau^n \mid n = 0, 1, \ldots\}$ as an 'inner half-line of points' and with the 'interior' $C_J^\circ(\tau)$. We prove that $C_J^\circ(\tau) = PL_J^\circ(\tau) \cap QR_J^\circ(\tau)$ (Proposition 8.1). For the case when τ has POTP, we prove that if $J \subset H(X)$ or τ is transitive, then every element of $PL_J(\tau)$ (respectively, $QR_J(\tau)$) is an essentially p-L (respectively, q-R) endomorphism of (X, τ) and every element of $PL_J(\tau) \cap QR_J(\tau)$ is an essentially LR endomorphism of (X, τ), so that $PL_J(\tau) \cap QR_J(\tau) = C_J(\tau)$ (Theorem 8.8). We also prove that $QL_J^\circ(\tau) \cap QR_J^\circ(\tau)$ is the set of all positively expansive onto endomorphisms of (X, τ) in J (Proposition 8.3), where $QL_J^\circ(\tau)$ denotes the set of all right τ-expansive elements of $QL_J(\tau)$.

Let $D_J(\tau) = PL_J(\tau) \cup QR_J(\tau)$. Let τ have POTP. Then $D_J(\tau) \cup D_J(\tau^{-1})$ is the set of all essentially one-sided resolving endomorphisms of (X, τ) in J. Our main result above (Theorem 6.3) implies that if τ is transitive, then every expansive element of J with periodic points dense and without POTP, if any, cannot exist in $D_J(\tau) \cup D_J(\tau^{-1})$. For an onto endomorphism φ of a transitive SFT (X, σ) and $J = \{\varphi^i \sigma^j \mid i \geq 0, j \in \mathbb{Z}\}$, if $D_J(\sigma) \cap D_J(\sigma^{-1}) \neq \{i_X\}$, then $J = D_J(\sigma) \cup D_J(\sigma^{-1})$ and hence all expansive elements of J have POTP (Proposition 8.9). Typical examples satisfying the hypothesis are an automorphism φ of a transitive SFT (X, σ) with both φ and φ^{-1} having memory 0, and a positively expansive endomorphism φ of a transitive SFT (X, σ) (Corollary 8.10).

In §9, we develop the properties of the 'cones' $PL_K(\tau)$, $QR_K(\tau)$ and $C_K(\tau)$, where K is a commutative subgroup of $H(X)$. (A part of the results on $C_K(\tau)$ were stated in [**21**] without proof.) For $\varphi, \tau \in H(X)$ let us write $\varphi \rightharpoonup \tau$ (respectively, $\varphi \rightharpoondown \tau$, $\varphi \to \tau$) to mean that τ is expansive and φ is a directionally essentially weakly p-L (respectively,

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q-R, LR) automorphism of (X, τ). The properties of the relations $\rightharpoonup$, $\rightharpoondown$ and $\to$ obtained are summarized in Propositions 9.4 and 9.5. We show in particular that the restriction of $\to$ on the set of expansive elements of K, say $E(K)$, is an equivalence relation on $E(K)$, so that for any element τ of $E(K)$ and any $\tau' \in C_K^\circ(\tau)$, $C_K(\tau) = C_K(\tau')$. Hence we call $C_K(\tau)$ a cone in K. If $\tau \in E(K)$ has POTP, then $C_K(\tau)$ is the set of all essentially LR automorphisms of (X, τ) (as follows from the above), and every element of $C_K(\tau)$ has POTP (Proposition 9.9). We call such a cone $C_K(\tau)$ an ELR (essentially LR) cone in K. The interiors of cones and those of ELR cones are closely related with the zero-dimensional cases of the 'expansive components and Markov components of 1-frames' for a $\mathbb{Z}^d$-action in the theory of expansive subdynamics due to Boyle and Lind [**11**].

Moreover, we show the following: the equivalence relation $\to$ on $E(K)$ is compatible with each of the relations $\rightharpoonup$ and $\rightharpoondown$ restricted on $E(K)$ (Proposition 9.7), so that these induce the quotient relations $\rightharpoonup$ and $\rightharpoondown$ on the set of all interiors C°'s of all cones C's in K; and if C and C' are two of these with $C^\circ \rightharpoonup C'^\circ$ or $C^\circ \rightharpoondown C'^\circ$ and either one of them is an ELR cone containing an expansive, topologically-mixing element, then so is the other (Corollary 9.12).

2. *Preliminaries*

Let $\varphi : X \to X$ be an onto continuous map of a compact metric space. A bisequence $(x_i)_{i\in\mathbb{Z}}$ of points x_i of X is called an *orbit* of φ if $\varphi(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$. For $\delta > 0$, it is called a δ-*pseudo orbit* of φ if $d_X(\varphi(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. For $\epsilon > 0$, an orbit $(x_i)_{i\in\mathbb{Z}}$ of φ is said to ϵ-*trace* a δ-pseudo orbit $(x'_i)_{i\in\mathbb{Z}}$ of φ if $d_X(x_i, x'_i) < \epsilon$ for all $i \in \mathbb{Z}$. We say that φ has *the pseudo orbit tracing property*, or *POTP* for short, if for any $\epsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo orbit of φ is ϵ-traced by some orbit of φ. We say that φ is *chain recurrent* if for any $x \in X$ and any $\delta > 0$ there exists a periodic δ-pseudo orbit of φ passing through x. We say that φ is *chain transitive* if for any $x, y \in X$ and any $\delta > 0$ there exists a periodic δ-pseudo orbit of φ passing through x and y. We say that φ is *expansive* if there exists $\delta > 0$ such that for any orbits $(x_i)_{i\in\mathbb{Z}}$ and $(y_i)_{i\in\mathbb{Z}}$ of φ, if $d_X(x_i, y_i) < \delta$ for all $i \in \mathbb{Z}$, then $(x_i)_{i\in\mathbb{Z}} = (y_i)_{i\in\mathbb{Z}}$. (This notion appears in Aoki and Hiraide [**2,** p. 57] with the term 'c-expansive'.)

A *commuting system* (X, τ, φ) means an ordered pair of commuting continuous maps $\tau : X \to X$ and $\varphi : X \to X$ of a compact metric space. If (X, τ, φ) is a commuting system, then φ is an *endomorphism* of the dynamical system (X, τ) (and hence τ is an endomorphism of the dynamical system (X, φ)). When φ is a homeomorphism in this definition, then φ is an *automorphism* of (X, τ).

A commuting system (X, τ, φ) (respectively, an endomorphism φ of a dynamical system (X, τ)) is said to be *topologically conjugate* to another (X', τ', φ') (respectively, another φ' of another (X', τ')) if there exists a *a topological conjugacy* $\theta : (X, \tau, \varphi) \to (X', \tau', \varphi')$, i.e. a homeomorphism $\theta : X \to X'$ which gives topological conjugacies $\theta : (X, \tau) \to (X, \tau')$ and $\theta : (X, \varphi) \to (X, \varphi')$ between the dynamical systems at the same time.

Let A be an alphabet (i.e. a finite non-empty set of symbols). Let $A^{\mathbb{Z}}$ be endowed with the metric d such that for $x = (a_j)_{j\in\mathbb{Z}}$ and $y = (b_j)_{j\in\mathbb{Z}}$ with $a_j, b_j \in A$, $d(x, y) = 0$ if $x = y$, and otherwise $d(x, y) = 1/(1 + k)$, where $k = \min\{|j| \mid a_j \neq b_j\}$. The metric d is

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compatible with the product topology of the discrete topology on A. Let $\sigma_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be defined by $\sigma_A((a_j)_{j\in\mathbb{Z}}) = (a_{j+1})_{j\in\mathbb{Z}}$. The dynamical system $(A^{\mathbb{Z}}, \sigma_A)$ is called the *full shift* over A. For a closed subset X of $A^{\mathbb{Z}}$ with $\sigma(X) = X$, the dynamical system (X, σ_X) is called a *subshift* over A, where σ_X is the restriction of σ_A on X.

Let $A^{\mathbb{N}} = \{(a_j)_{j\in\mathbb{N}} \mid a_j \in A\}$ be endowed with a metric compatible with the product topology of the discrete topology on A. Let $\tilde{\sigma}_A : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be defined by $\tilde{\sigma}_A((a_j)_{j\in\mathbb{N}}) = (a_{j+1})_{j\in\mathbb{N}}$. The dynamical system $(A^{\mathbb{N}}, \tilde{\sigma}_A)$ is called the *one-sided full shift* over A. For a subshift (X, σ) over A, let $\tilde{X} = \{(a_j)_{j\in\mathbb{N}} \mid \exists (a_j)_{j\in\mathbb{Z}} \in X\}$. Then, with the onto continuous map $\tilde{\sigma} = \tilde{\sigma}_A|\tilde{X}$, we have a dynamical system $(\tilde{X}, \tilde{\sigma})$ which is called a *one-sided subshift* over A and is said to *be induced by* (X, σ). From $(\tilde{X}, \tilde{\sigma})$ we can uniquely recover (X, σ), so that (X, σ) is said to *be induced by* $(\tilde{X}, \tilde{\sigma})$. Let $s_X : X \to \tilde{X}$ be the continuous map which maps $(a_j)_{j\in\mathbb{Z}}$ to $(a_j)_{j\in\mathbb{N}}$. If a homomorphism ϕ of a subshift (X, σ) into another (X', σ') has memory zero, then it uniquely *induces* a homomorphism $\tilde{\phi}$ between the induced one-sided subshifts $(\tilde{X}, \tilde{\sigma})$ and $(\tilde{X}', \tilde{\sigma}')$ such that $s_{X'}\phi = \tilde{\phi}s_X$, and conversely each homomorphism $\tilde{\phi}$ of a one-sided subshift $(\tilde{X}, \tilde{\sigma})$ into another $(\tilde{X}', \tilde{\sigma}')$ uniquely *induces* a homomorphism ϕ between the induced subshifts (X, σ) and (X', σ') such that ϕ has memory zero and $s_{X'}\phi = \tilde{\phi}s_X$.

A subshift (X, σ) over an alphabet A is called a *subshift of finite type*, or an *SFT* for short, if there is a finite set F of words (blocks) over A such that X is the set of all points $(a_j)_{j\in\mathbb{Z}}$ in $A^{\mathbb{Z}}$ with $a_j \ldots a_{j'} \notin F$ for all $j \le j'$. A *one-sided SFT* is a one-sided subshift induced by an SFT.

It is well known (Hedlund [**14**], Reddy [**26**]) that for a zero-dimensional, compact metric space X, if $\tau : X \to X$ is an expansive homeomorphism, then (X, τ) is conjugate to a subshift, and if $\tau : X \to X$ is a positively expansive onto continuous map, then (X, τ) is conjugate to a one-sided subshift. It is also well known (Walters [**27**]) that a subshift is of finite type if and only if it has POTP.

The reader is referred to [**15**] or [**17**] for a comprehensive introduction to symbolic dynamics, and to [**2**] for more information on topological dynamics.

Let P be a property of endomorphisms of dynamical systems. We say that an endomorphism φ of a dynamical system (X, τ) *essentially has* P or *is essentially* P if there exists an endomorphism φ_0 of a dynamical system (X_0, τ_0) such that φ_0 has P or is P and (X, τ, φ) is topologically conjugate to (X_0, τ_0, φ_0).

Finally we make the following remark. Let P be a property of endomorphisms of SFTs such that (i) if an endomorphism φ of an SFT (X, σ) has P, then the endomorphism φ^r of (X, σ^s) essentially has P for all $r, s \ge 1$, and such that (ii) if an endomorphism φ of an SFT which is the disjoint union of finitely many SFTs (X_j, σ_j) with $\varphi(X_j) = X_j$ has P, then the restriction of φ on each (X_j, σ_j) has P. Consider the claim that if an onto endomorphism φ of a topologically-transitive (or more generally, non-wandering) SFT (X, σ) essentially has P, then φ has POTP. To prove this, it suffices to prove it under the additional condition that σ is topologically mixing. To see this, let φ be an onto endomorphism of a topologically-transitive (or non-wandering) SFT (X, σ). Then for some $s \ge 1$, (X, σ^s) is conjugate to a disjoint union of finitely many mixing SFTs. Since φ is an onto endomorphism of (X, σ^s), up to the conjugacy, it permutes these mixing SFTs and hence φ^r fixes each of them for some $r \ge 1$. If the restriction of φ^r on each of them has

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POTP, then so does φ^r itself, so that φ has POTP. (An analogous argument is found in Boyle [**6,** the proof of Theorem 5.5].)

3. *Textile systems*

Let G be a graph. Here a *graph* means a directed graph which may have multiple arcs and loops. Let A_G and V_G denote the arc-set and the vertex-set, respectively, of G. Let $i_G : A_G \to V_G$ and $t_G : A_G \to V_G$ be such that for each arc $a \in A_G$, $i_G(a)$ and $t_G(a)$ are its initial and terminal vertices. Hence the graph G is represented by $V_G \xleftarrow{i_G} A_G \xrightarrow{t_G} V_G$. Let X_G be the set of all points $(a_j)_{j\in\mathbb{Z}}$ in $A_G^{\mathbb{Z}}$ such that $t_G(a_j) = i_G(a_{j+1})$ for all $j \in \mathbb{Z}$. Then we have a subshift (X_G, σ_G), which is called the *topological Markov shift* defined by G. Throughout this paper we assume that any graph defines a non-empty topological Markov shift.

An SFT can be defined as a subshift which is topologically conjugate to a topological Markov shift. A *sofic system* is a subshift which is the image of a topological Markov shift under a block map.

For graphs Γ and G, a *graph-homomorphism* h of Γ into G, written by $h : \Gamma \to G$, is a pair (h_A, h_V) of mappings $h_A : A_\Gamma \to A_G$ (*arc-map*) and $h_V : V_\Gamma \to V_G$ (*vertex-map*) such that the following diagram is commutative.

$$\begin{array}{ccccc} V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ {\scriptstyle h_V}\downarrow & & {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle h_V} \\ V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \end{array}$$

We call a block map of (0, 0) type between subshift spaces a *1-block map*. For a graph-homomorphism $h : \Gamma \to G$, we define $\phi_h : X_\Gamma \to X_G$ to be the 1-block map with the local rule $h_A : A_\Gamma \to A_G$.

A *textile system* T *over* a graph G is an ordered pair of graph-homomorphisms $p : \Gamma \to G$ and $q : \Gamma \to G$. (NB The additional condition of the one–one correspondence between $\alpha \in A_\Gamma$ and the quadruple $(i_\Gamma(\alpha), t_\Gamma(\alpha), p_A(\alpha), q_A(\alpha))$ imposed on the original definition in [**20**] is removed. In earlier theorems on textile systems, this condition automatically follows from the hypotheses of the theorems, and hence its removal has no impact on the statements of the theorems. The removal makes the definition simpler and makes it easier to define the composition of textile systems.) We write

$$T = (p, q : \Gamma \to G).$$

We have the following commutative diagram.

$$\begin{array}{ccccc} V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \\ \uparrow{\scriptstyle p_V} & & {\scriptstyle p_A}\uparrow & & \uparrow{\scriptstyle p_V} \\ V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ {\scriptstyle q_V}\downarrow & & {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_V} \\ V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \end{array}$$

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If we observe this diagram vertically, then we have the ordered pair of graph-homomorphisms

$$\begin{array}{ccc} V_G & \xleftarrow{i_G} & A_G \\ \uparrow{\scriptstyle p_V} & & {\scriptstyle p_A}\uparrow \\ V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma \\ \downarrow{\scriptstyle q_V} & & {\scriptstyle q_A}\downarrow \\ V_G & \xleftarrow{i_G} & A_G \end{array} \quad \text{and} \quad \begin{array}{ccc} A_G & \xrightarrow{t_G} & V_G \\ {\scriptstyle p_A}\uparrow & & \uparrow{\scriptstyle p_V} \\ A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_V} \\ A_G & \xrightarrow{t_G} & V_G \end{array}.$$

This defines another textile system

$$T^* = (p^*, q^* : \Gamma^* \to G^*)$$

called the *dual* of T, where $i_{\Gamma^*} = p_A$, $t_{\Gamma^*} = q_A$, $i_{G^*} = p_V$ and $t_{G^*} = q_V$.

Let $T = (p, q : \Gamma \to G)$ be a textile system with $T^* = (p^*, q^* : \Gamma^* \to G^*)$. Every point in X_G is called a *G-thread* or simply a *thread*, and every point in X_Γ is called a *Γ-thread*. Let $\xi = \phi_p$ and let $\eta = \phi_q$. For $n \geq 1$, a two-dimensional configuration $(\alpha_{i,j})_{1 \leq i \leq n, j \in \mathbb{Z}}$ of arcs $\alpha_{i,j} \in A_\Gamma$ is called an *obi of width n woven by T* if $(\alpha_{i,j})_{j \in \mathbb{Z}} \in X_\Gamma$ for all $1 \leq i \leq n$, $\eta((\alpha_{i,j})_{j \in \mathbb{Z}}) = \xi((\alpha_{i+1,j})_{j \in \mathbb{Z}})$ for all $1 \leq i \leq n-1$ and in addition

$$(\alpha_{i,j})_{1 \leq i \leq n} \in L_n(X_{\Gamma^*}) \quad \text{for all } \ j \in \mathbb{Z}. \tag{3.1}$$

Our definition of an obi is its original definition in [**20**] to which the condition (3.1) is added for some technical reason. We note that if $l = \#A_G + 1$ and $(\alpha_{i,j})_{1 \leq i \leq k+2l, j \in \mathbb{Z}}$ with $k \geq 1$ is an obi without condition (3.1) imposed on, then its sub-obi $(\alpha_{i,j})_{l+1 \leq i \leq k+l, j \in \mathbb{Z}}$ is an obi satisfying condition (3.1). Hence, it is easily seen that all notions defined for T below (including in particular 'upwardly finitely saturated' and 'downwardly finitely saturated') are independent of the addition of condition (3.1).

Let $n \geq 1$. Let $W_{T,n}$ denote the set of all obis of width n woven by T. If we regard each (vertical) path $(\alpha_i)_{1 \leq i \leq n} \in L_n(X_{\Gamma^*})$ as an arc going from vertex $(i_\Gamma(\alpha_i))_{1 \leq i \leq n} = (p_A^*(\alpha_i))_{1 \leq i \leq n}$ to vertex $(t_\Gamma(\alpha_i))_{1 \leq i \leq n} = (q_A^*(\alpha_i))_{1 \leq i \leq n}$, then we have a graph with arc-set $L_n(X_{\Gamma^*})$, which is denoted by $\Delta_{T,n}$. Since the space of the topological Markov shift defined by the graph $\Delta_{T,n}$ is equal to $W_{T,n}$ (indeed, each point satisfies condition (3.1)), the topological Markov shift is denoted by $(W_{T,n}, \varsigma_{T,n})$ and called the *obi shift of width n of T*.

Let $X_{T,0,0}$ be the set of all G-threads $(a_j)_{j \in \mathbb{Z}}$ with $a_j \in A_G$ such that for all $j \in \mathbb{Z}$, a_j appears on some point of X_{Γ^*} as a vertex of Γ^*. For $m, n \geq 0$, let $Z_{T,m,n}$ denote the set of all $(m+1)$st Γ-threads $(\alpha_{m+1,j})_{j \in \mathbb{Z}}$ of some obi $(\alpha_{i,j})_{1 \leq i \leq m+n+1, j \in \mathbb{Z}}$ of width $m+n+1$ with $\alpha_{i,j} \in A_\Gamma$; let $X_{T,m,n+1} = \xi(Z_{T,m,n})$ and let $X_{T,m+1,n} = \eta(Z_{T,m,n})$. Then we have subshifts $(X_{T,m,n}, \sigma_{T,m,n})$ and $(Z_{T,m,n}, \varsigma_{T,m,n})$ for $m, n \geq 0$. These are sofic systems, as $(X_{T,0,0}, \sigma_{T,0,0})$ is an SFT and the others are the images of the obi shifts of T under 1-block maps. Let $\xi_{T,m,n} : Z_{T,m,n} \to X_{T,m,n+1}$ and $\eta_{T,m,n} : Z_{T,m,n} \to X_{T,m+1,n}$ be defined as the restrictions of ξ and η, respectively. Then $\xi_{T,m,n}$ and $\eta_{T,m,n}$ are onto. It is clear that if

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$0 \le m \le m'$ and $0 \le n \le n'$, then $X_{T,m,n} \supset X_{T,m',n'}$, and $Z_{T,m,n} \supset Z_{T,m',n'}$. Define

$$X_{T,m,\infty} = \bigcap_{n=0}^{\infty} X_{T,m,n} \quad \text{and} \quad Z_{T,m,\infty} = \bigcap_{n=0}^{\infty} Z_{T,m,n} \quad \text{for } m \ge 0,$$

$$X_{T,\infty,n} = \bigcap_{m=0}^{\infty} X_{T,m,n} \quad \text{and} \quad Z_{T,\infty,n} = \bigcap_{m=0}^{\infty} Z_{T,m,n} \quad \text{for } n \ge 0.$$

Define

$$X_T = X_{T,0,\infty} \cap X_{T,\infty,0} \quad \text{and} \quad Z_T = Z_{T,0,\infty} \cap Z_{T,\infty,0}.$$

A two-dimensional configuration $(\alpha_{ij})_{i,j\in\mathbb{Z}}$, $\alpha_{ij} \in A_\Gamma$, is called a *textile* woven by T if $(\alpha_{ij})_{j\in\mathbb{Z}} \in X_\Gamma$ and $\eta((\alpha_{i-1,j})_{j\in\mathbb{Z}}) = \xi((\alpha_{ij})_{j\in\mathbb{Z}})$ for all $i \in \mathbb{Z}$. Then X_T is the set of all G-threads that appear on some textile woven by T, and Z_T is the set of all Γ-threads that appear on some textile woven by T. We have subshifts (X_T, σ_T) and (Z_T, ς_T). We call (X_T, σ_T) the *woof shift* of T. The woof shift of the dual T^* of T is called the *warp shift* of T. Let $\xi_T : Z_T \to X_T$ and $\eta_T : Z_T \to X_T$ be the restrictions of ξ and η, respectively. We say that T is *one-sided 1-1* if ξ_T is one-to-one. If T is one-sided 1-1, then we define an endomorphism φ_T of the woof shift (X_T, σ_T) by $\varphi_T = \eta_T \xi_T^{-1}$. We say that T is *1-1* if both ξ_T and η_T are one-to-one. If T is 1-1, then φ_T is an automorphism of the woof shift (X_T, σ_T). We say that T is *finitely one-sided 1-1* if there exist $m, n \ge 0$ such that $\xi_{T,m,n}$ is one-to-one. We say that T is *finitely 1-1* if there exist $m, n \ge 0$ such that both $\xi_{T,m,n}$ and $\eta_{T,m,n}$ are one-to-one. We say that T is *finitely bounded-1* if there exist $m, n \ge 0$ such that both $\xi_{T,m,n}$ and $\eta_{T,m,n}$ are bounded-to-one.

We say that T is *non-degenerate* if $\xi = \phi_p$ and $\eta = \phi_q$ are onto. Clearly, T is non-degenerate if and only if $X_G = X_T$. We say that T is *finitely saturated* if there exist $m, n \ge 0$ with $X_T = X_{T,m,n}$. We say that T is *upwardly finitely saturated* if there exists $m \ge 0$ such that $X_{T,m,0} = X_{T,\infty,0}$. We say that T is *downwardly finitely saturated* if there exists $n \ge 0$ such that $X_{T,0.n} = X_{T,0,\infty}$.

Remark 3.1. Let T be a textile system. Then T is finitely saturated if and only if T is both upwardly finitely saturated and downwardly finitely saturated.

Proof. Assume that $T = (p, q : \Gamma \to G)$ is finitely saturated. If $X_{T,0,0} = X_T$, then $X_{T,0,0} = X_{T,0,\infty} = X_{T,\infty,0}$. Assume that $X_T = X_{T,m,n}$ with $m, n \ge 0$ and $m + n \ge 1$. Let $w = (\alpha_{i,j})_{1\le i\le m+n, j\in\mathbb{Z}} \in W_{T,m+n}$. Then the $(m+1)$-st thread $(p_A(\alpha_{m+1,j}))_{j\in\mathbb{Z}}$ of w is in X_T, so that there exists a textile $(\beta_{i,j})_{i,j\in\mathbb{Z}}$ woven by T such that $(p_A(\beta_{0,j}))_{j\in\mathbb{Z}} = (p_A(\alpha_{m+1,j}))_{j\in\mathbb{Z}}$, which is in $X_{T,0,\infty} \cap X_{T,\infty,0}$. Hence the first (uppermost) thread $(p_A(\alpha_{1,j}))_{j\in\mathbb{Z}}$ of w is in $X_{T,0,\infty}$ and the last (lowermost) one $(q_A(\alpha_{m+n,j}))_{j\in\mathbb{Z}}$ is in $X_{T,\infty,0}$. This implies that $X_{T,0,m+n} \subset X_{T,0,\infty}$ and $X_{T,m+n,0} \subset X_{T,\infty,0}$, so that $X_{T,0,m+n} = X_{T,0,\infty}$ and $X_{T,m+n,0} = X_{T,\infty,0}$.

Conversely, assume that $X_{T,m,0} = X_{T,\infty,0}$ and $X_{T,0,n} = X_{T,0,\infty}$ with $m, n \ge 0$. Then $X_{T,m,n} = X_{T,m,0} \cap X_{T,0,n} = X_{T,\infty,0} \cap X_{T,0,\infty} = X_T$. □

We recall the most fundamental property of textile systems.

THEOREM 3.2. [**20,** Theorem 2.5] *Let T be a one-sided 1-1 textile system. Then φ_T is expansive if and only if T^* is 1-1. If both T and T^* are 1-1, then the commuting systems $(X_T, \sigma_T, \varphi_T)$ and $(X_{T^*}, \varphi_{T^*}, \sigma_{T^*})$ are topologically conjugate.*

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Remark 3.3. Let T be a textile system. Suppose the dual $T^* = (p^*, q^* : \Gamma^* \to G^*)$ is one-sided 1-1 and non-degenerate. Let $n \geq 0$. Let $\delta = 1/(n+1)$. Then the obis of width $2n+1$ woven by T are the same as the δ-pseudo orbits of φ_{T^*} in the sense of the following. For any δ-pseudo orbit $((b_{i,j})_{i\in\mathbb{Z}})_{j\in\mathbb{Z}}$ of φ_{T^*} with $(b_{i,j})_{i\in\mathbb{Z}} \in X_{T^*}$, $b_{i,j} \in A_{G^*}$, there exists an obi $(\alpha_{i,j})_{-n\leq i\leq n, j\in\mathbb{Z}}$ of width $2n+1$ woven by T such that $(p_A^*(\alpha_{i,j}))_{-n\leq i\leq n} = (b_{i,j})_{-n\leq i\leq n}$, and conversely any obi $(\alpha_{i,j})_{-n\leq i\leq n, j\in\mathbb{Z}}$ of width $2n+1$ woven by T is extendable to a configuration $(\alpha_{i,j})_{i,j\in\mathbb{Z}}$ such that $((p_A^*(\alpha_{i,j}))_{i\in\mathbb{Z}})_{j\in\mathbb{Z}}$ is a δ-pseudo orbit of φ_{T^*}.

Proof. Since T^* is non-degenerate, $X_{T^*} = X_{G^*}$ and hence $Z_{T^*} = X_{\Gamma^*}$. Therefore, for $(b_i)_{i\in\mathbb{Z}}, (b'_i)_{i\in\mathbb{Z}} \in X_{T^*}$, $\varphi_{T^*}((b_i)_{i\in\mathbb{Z}}) = (b'_i)_{i\in\mathbb{Z}}$ if and only if there exists $(\alpha_i)_{i\in\mathbb{Z}} \in X_{\Gamma^*}$ with $(p_A^*(\alpha_i))_{i\in\mathbb{Z}} = (b_i)_{i\in\mathbb{Z}}$ and $(q_A^*(\alpha_i))_{i\in\mathbb{Z}} = (b'_i)_{i\in\mathbb{Z}}$. Hence, by noting the definition of an obi, in particular the condition (3.1), the remark is easily proved. □

PROPOSITION 3.4. *Let T be a textile system. Suppose the dual T^* is one-sided 1-1 and non-degenerate. Then φ_{T^*} has POTP if and only if T is finitely saturated.*

Proof. Let $T = (p, q : \Gamma \to G)$ and let $T^* = (p^*, q^* : \Gamma^* \to G^*)$.

Assume that T is finitely saturated. Then there exists $n \geq 0$ such that $X_{T,n,n} = X_T$. Let $\epsilon > 0$. Let k be a positive integer with $1/(k+1) < \epsilon$. Let $\delta = 1/(n+k+1)$. Let $((b_{i,j})_{i\in\mathbb{Z}})_{j\in\mathbb{Z}}$ be any δ-pseudo orbit of φ_{T^*} with $(b_{i,j})_{i\in\mathbb{Z}} \in X_{G^*}$, $b_{i,j} \in A_{G^*}$. Then there exists an obi $(\alpha_{i,j})_{-k-n\leq i\leq k+n, j\in\mathbb{Z}}$ of width $2(n+k)+1$ woven by T such that $(p_A^*(\alpha_{i,j}))_{-k-n\leq i\leq k+n} = (b_{i,j})_{-k-n\leq i\leq k+n}$ for $j \in \mathbb{Z}$. Since $(p_A(\alpha_{i,j}))_{j\in\mathbb{Z}} \in X_{T,n,n}$ for $i = -k, \ldots, k+1$, we have $(p_A(\alpha_{i,j}))_{j\in\mathbb{Z}} \in X_T$ for $i = -k, \ldots, k+1$. This implies that there exists a textile $(\beta_{i,j})_{i,j\in\mathbb{Z}}$ woven by T such that $(\beta_{i,j})_{-k\leq i\leq k, j\in\mathbb{Z}} = (\alpha_{i,j})_{-k\leq i\leq k, j\in\mathbb{Z}}$. We see that $((p_A^*(\beta_{i,j}))_{i\in\mathbb{Z}})_{j\in\mathbb{Z}}$ ϵ-traces $((b_{i,j})_{i\in\mathbb{Z}})_{j\in\mathbb{Z}}$.

Conversely, assume that φ_{T^*} has POTP. Let $k \geq 0$ be such that $\xi_{T^*}^{-1}$ is a block map of (k,k) type. Then there exists $\delta > 0$ such that every δ-pseudo orbit of φ_{T^*} is ϵ-traced with $\epsilon = 1/(2k+1)$. Let n be a positive integer with $1/(n+1) < \delta$. To see that $X_{T,n,n+1} = X_T$, let $(\alpha_{i,j})_{-n\leq i\leq n, j\in\mathbb{Z}}$ be an obi of width $2n+1$ woven by T. Then it is extendable to a configuration $(\alpha_{i,j})_{i,j\in\mathbb{Z}}$ such that $((p_A^*(\alpha_{i,j}))_{i\in\mathbb{Z}})_{j\in\mathbb{Z}}$ is a δ-pseudo orbit of φ_{T^*}. Since this is ϵ-traced by φ_{T^*}, there exists a textile $(\beta_{i,j})_{i,j\in\mathbb{Z}}$ woven by T such that $p_A^*(\beta_{i,j}))_{-k\leq i\leq k} = ((p_A^*(\alpha_{i,j}))_{-k\leq i\leq k}$ for all $j \in \mathbb{Z}$. Since $\xi_{T^*}^{-1}$ is a block map of (k,k) type, $(\alpha_{0,j})_{j\in\mathbb{Z}} = (\beta_{0,j})_{j\in\mathbb{Z}}$. Thus $(p_A(\alpha_{0,j}))_{j\in\mathbb{Z}} \in X_T$. □

LEMMA 3.5. *Let Z be the space of a topologically-transitive sofic system and let ϕ_1 and ϕ_2 be bounded-to-one block-maps with Z as their common domain. Then $\phi_1(Z) \subset \phi_2(Z)$ implies $\phi_1(Z) = \phi_2(Z)$.*

Proof. This follows from [**23,** Lemma 2.1]. □

PROPOSITION 3.6. *Let T be a finitely bounded-1 textile system such that the obi shift of width n of T is topologically transitive for infinitely many n. Then the following are equivalent.*

(1) *T is finitely saturated.*
(2) *T is upwardly finitely saturated.*
(3) *T is downwardly finitely saturated.*

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Proof. By Remark 3.1, (1) implies each of (2) and (3).

Assume (2). Then there exists $k \geq 0$ with $X_{T,k,0} = X_{T,\infty,0}$ and hence we have

$$X_{T,k,l} = X_{T,k',l} \quad \text{for all } k' \geq k \text{ and } l \geq 0, \tag{3.2}$$

because $X_{T,k,l} = X_{T,k,0} \cap X_{T,0,l}$ and $X_{T,k',l} = X_{T,k',0} \cap X_{T,0,l}$ with $X_{T,\infty,0} \subset X_{T,k',0} \subset X_{T,k,0} = X_{T,\infty,0}$. Since T is finitely bounded-1, there exist non-negative integers m and n such that $\xi_{T,m,n} : Z_{T,m,n} \to X_{T,m,n+1}$ and $\eta_{T,m,n} : Z_{T,m,n} \to X_{T,m+1,n}$ are bounded-to-one. For any $m' \geq m$, $\xi_{T,m',n}$ and $\eta_{T,m',n}$ are bounded-to-one because $Z_{T,m',n} \subset Z_{T,m,n}$ and $\xi_{T,m',n}$ and $\eta_{T,m',n}$ are the restrictions of $\xi_{T,m,n}$ and $\eta_{T,m,n}$, respectively, on $Z_{T,m',n}$. Hence, by replacing m by sufficiently large one if necessary, we may assume that $m \geq k$ and the obi shift of width $m + n + 1$ of T is topologically transitive. Using (3.2) we have

$$\begin{aligned} \xi_{T,m,n}(Z_{T,m,n}) &= X_{T,m,n+1} = X_{T,m+1,n+1} \\ &\subset X_{T,m+1,n} = \eta_{T,m,n}(Z_{T,m,n}). \end{aligned}$$

Since $(Z_{T,m,n}, \varsigma_{T,m,n})$ is the image of the topologically-transitive topological Markov shift $(W_{T,m+n+1}, \varsigma_{T,m+n+1})$ under a block map, it is a topologically-transitive sofic system. Therefore, since both $\xi_{T,m,n}$ and $\eta_{T,m,n}$ are bounded-to-one 1-block maps, it follows from Lemma 3.5 that $X_{T,m+1,n} = X_{T,m,n+1}$. Hence, using equation (3.2) we have

$$X_{T,m,n} = X_{T,m,n+1}. \tag{3.3}$$

To prove by induction on l that $X_{T,m,n} = X_{T,m,n+l}$ for all $l \geq 1$, assume that $X_{T,m,n} = X_{T,m,n+l}$. Let $x \in X_{T,m,n}$. Then since $x \in X_{T,m,n+l}$ by the induction hypothesis, there exists an obi $w = (\alpha_{i,j})_{1 \leq i \leq m+n+l,\, j \in \mathbb{Z}} \in W_{T,m+n+l}$ such that the $(m+1)$st thread of w is x. Since the $(m+l+1)$st thread, say x', of w is in $X_{T,m+l,n} \subset X_{T,m,n}$, it follows from equation (3.3) that x' is in $X_{T,m,n+1} \subset X_{T,0,n+1}$. Hence there exists $w' \in W_{T,n+1}$ such that the first thread of w' is x'. Sewing w' to the subobi $(\alpha_{i,j})_{1 \leq i \leq m+l,\, j \in \mathbb{Z}}$ of w in the bottom, we obtain an obi in $W_{T,m+n+l+1}$ in which x is the $(m+1)$st thread. Thus $x \in X_{T,m,n+l+1}$. Therefore $X_{T,m,n} \subset X_{T,m,n+l+1}$, so that $X_{T,m,n} = X_{T,m,n+l+1}$.

We have proved that $X_{T,m,n} = X_{T,m,n+l}$ for all $l \geq 1$. Hence by equation (3.2), $X_{T,m,n} = X_{T,m',n+l}$ for all $m' \geq m$ and $l \geq 1$, so that $X_{T,m,n} = X_T$ and (1) is proved.

The proof that (3) implies (1) is similar. □

PROPOSITION 3.7. *Let T be a textile system. Then T is one-sided 1-1 if and only if T is finitely one-sided 1-1, and T is 1-1 if and only if T is finitely 1-1.*

Proof. The proof is similar to [**23,** the proof of Proposition 3.1]. □

Boyle [**6**] proved the following theorem, a special case of which has an important interpretation in the theory of textile systems.

THEOREM 3.8. (Boyle [**6**]) *If φ is an expansive automorphism of a mixing sofic system, then φ^k is chain transitive for all $k \geq 1$.*

COROLLARY 3.9. *If T is a 1-1 textile system with T^* 1-1 and non-degenerate and σ_{T^*} topologically mixing, then for all $n \geq 1$ the obi shift of width n of T is topologically mixing.*

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Proof. By Theorem 3.2, φ_{T^*} is an expansive automorphism of the mixing SFT (X_{T^*}, σ_{T^*}). It therefore follows from Boyle's theorem that $\varphi_{T^*}^k$ is chain transitive for all $k \geq 1$. Hence it easily follows from Remark 3.3 that for all $n \geq 1$ the obi shift of width n of T is topologically mixing. □

The following is one of the two main propositions of this paper.

PROPOSITION 3.10. *Let T be a 1-1 textile system such that T^* is 1-1 and non-degenerate with σ_{T^*} topologically mixing. Then the following are equivalent.*

(1) *(X_T, σ_T) is an SFT.*
(2) *T is upwardly finitely saturated.*
(3) *T is downwardly finitely saturated.*

Proof. By the corollary to Boyle's theorem above, for all $n \geq 1$ the obi shift of width n of T is topologically transitive. Since T is 1-1, it is finitely 1-1 by Proposition 3.7. Therefore, by Proposition 3.6, each of (2) and (3) is equivalent to the condition that T is finitely saturated, which is equivalent to (1) by Proposition 3.4, Theorem 3.2 and the result of Walters [**27**] mentioned in §2 or directly by [**20,** Theorem 2.7]. □

Proposition 3.10 gives necessary and sufficient conditions for the dynamical system defined by an expansive automorphism of a mixing topological Markov shift to be conjugate to an SFT, but we can only make use of an extremely special case of it in this paper (see §5).

4. *One-sided resolving textile systems*

LEMMA 4.1. *Let $n \geq 1$. Let T be a textile system such that:*

(i) *$X_{T,0,n} \subset X_{T,\infty,0}$;*
(ii) *$\xi_{T,0,n} : Z_{T,0,n} \to X_{T,0,n+1}$ and $\eta_{T,0,n} : Z_{T,0,n} \to X_{T,1,n}$ are bounded-to-one; and*
(iii) *the obi shift $(W_{T,n}, \varsigma_{T,n})$ of width n is topologically transitive.*

Then

$$X_{T,0,n} = X_T.$$

Proof. The idea of the proof comes from [**23,** the proof of Proposition 3.2].

Let $T = (p, q : \Gamma \to G)$. We define two graph-homomorphisms $\hat{p} : \hat{\Gamma} \to G$ and $\hat{q} : \hat{\Gamma} \to G$ as follows: $\hat{\Gamma} = \Delta_{T,n+1}$, and for $(\alpha_i)_{0 \leq i \leq n} \in A_{\Delta_{T,n+1}}$, which is a (vertical) path in $L_{n+1}(X_{\Gamma^*})$ with $\alpha_i \in A_{\Gamma^*} = A_\Gamma$,

$$\hat{p}_A((\alpha_i)_{0 \leq i \leq n}) = p_A(\alpha_0), \quad \text{and} \quad \hat{q}_A((\alpha_i)_{0 \leq i \leq n}) = q_A(\alpha_0).$$

Then $X_{\hat{\Gamma}} = W_{T,n+1}$, $\phi_{\hat{p}}(X_{\hat{\Gamma}}) = X_{T,0,n+1}$ and $\phi_{\hat{q}}(X_{\hat{\Gamma}}) = X_{T,1,n}$. (Recall that for a graph-homomorphism h, the 1-block map ϕ_h is defined in §3.)

Let $\hat{h}$ be the graph-homomorphism of $\hat{\Gamma} = \Delta_{T,n+1}$ into $\Delta_{T,n}$ such that $\hat{h}_A$ maps each $(\alpha_i)_{0 \leq i \leq n} \in A_{\Delta_{T,n+1}}$ with $\alpha_i \in A_\Gamma$ to $(\alpha_i)_{1 \leq i \leq n} \in A_{\Delta_{T,n}}$. By (i), $X_{T,0,n} \subset X_{T,1,0}$, and hence for each obi $(\alpha_{i,j})_{1 \leq i \leq n, j \in \mathbb{Z}} \in W_{T,n}$ with $\alpha_{i,j} \in A_\Gamma$, there exists $(\alpha_{0,j})_{j \in \mathbb{Z}}$ with $\alpha_{0,j} \in A_\Gamma$ such that $(\alpha_{i,j})_{0 \leq i \leq n, j \in \mathbb{Z}}$ is an obi in $W_{T,n+1}$. This implies that $\phi_{\hat{h}}$ is onto. Since $\Delta_{T,n}$ is irreducible by (iii), there exists an irreducible component (i.e. a maximal irreducible subgraph), say $\check{\Gamma}$, of $\hat{\Gamma}$ such that if $\check{h} : \check{\Gamma} \to \Delta_{T,n}$ is the graph-homomorphism

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given as the restriction of $\hat{h}$, then $\phi_{\check{h}}$ is onto, by [**19,** Lemma 6]. The fact that $\phi_{\check{h}}$ is onto implies that $X_{T,0,n} \subset \phi_{\hat{q}}(X_{\check{\Gamma}})$. Therefore, since $\phi_{\hat{q}}(X_{\check{\Gamma}}) \subset \phi_{\hat{q}}(X_{\hat{\Gamma}}) = X_{T,1,n} \subset X_{T,0,n}$, we have

$$\phi_{\hat{q}}(X_{\check{\Gamma}}) = X_{T,0,n}.$$

Let π be the 1-block map of $X_{\hat{\Gamma}} = W_{T,n+1}$ onto $Z_{T,0,n}$ which maps each obi of width $n+1$ to its first (i.e. uppermost) Γ-thread. Then $\pi(X_{\check{\Gamma}})$ is the space of a transitive sofic system because $\check{\Gamma}$ is irreducible. By (ii), $\xi_{T,0,n}$ and $\eta_{T,0,n}$ are bounded-to-one 1-block maps. Moreover, we have

$$\begin{aligned}\xi_{T,0,n}(\pi(X_{\check{\Gamma}})) = \phi_{\hat{p}}(X_{\check{\Gamma}}) \subset \phi_{\hat{p}}(X_{\hat{\Gamma}}) = X_{T,0,n+1} \\ \subset X_{T,0,n} = \phi_{\hat{q}}(X_{\check{\Gamma}}) = \eta_{T,0,n}(\pi(X_{\check{\Gamma}})).\end{aligned}$$

Therefore, applying Lemma 3.5 to the restrictions of $\xi_{T,0,n}$ and $\eta_{T,0,n}$ on $\pi(X_{\check{\Gamma}})$, we see that $X_{T,0,n} = X_{T,0,n+1}$, which implies that $X_{T,0,n} = X_{T,0,\infty}$. Combining this with (i), we have $X_{T,0,n} = X_T$. □

A graph-homomorphism $h : \Gamma \to G$ is said to be *right resolving* if for each pair (u, a) of $u \in V_\Gamma$ and $a \in A_G$ with $i_G(a) = h_V(u)$, there exists unique $\alpha \in A_\Gamma$ with $i_\Gamma(\alpha) = u$ and $h_A(\alpha) = a$. It is said to be *left resolving* if for each pair (u, a) of $u \in V_\Gamma$ and $a \in A_G$ with $t_G(a) = h_V(u)$, there exists unique $\alpha \in A_\Gamma$ with $t_\Gamma(\alpha) = u$ and $h_A(\alpha) = a$.

LEMMA 4.2. *Let T be a textile system with $T^* = (p^*, q^* : \Gamma^* \to G^*)$ such that:*
(i) *p^* or q^* is left resolving;*
(ii) *T^* is one-sided 1-1 and non-degenerate;*
(iii) *φ_{T^*} is chain recurrent; and*
(iv) *σ_{T^*} is topologically mixing.*
If there exists $n \geq 1$ such that $\xi_{T,0,n} : Z_{T,0,n} \to X_{T,0,n+1}$ is one-to-one, then the obi shift $(W_{T,m}, \varsigma_{T,m})$ of width m is topologically mixing for all $m \geq 1$.

Proof. The idea of the proof comes from [**23,** the proof of Proposition 3.3].

Suppose $\xi_{T,0,n}$ is one-to-one with $n \geq 1$. We note that if $m \geq n$, then $\xi_{T,0,m}$ is one-to-one. We also note that for $1 \leq m \leq n$, if $(W_{T,n}, \varsigma_{T,n})$ is topologically mixing, then so is $(W_{T,m}, \varsigma_{T,m})$. (Let $1 \leq m \leq n$. Since T^* is non-degenerate, every path (in G^*) in $L_n(X_{G^*})$ appears on some obi in $W_{T,n} = X_{\Delta_{T,n}}$ as a vertex in $\Delta_{T,n}$, so that every vertex in $\Delta_{T,m} = L_m(X_{G^*})$ appears on the uppermost subobi of width m of some obi in $W_{T,n}$ as the initial subpath of length m of a path in $L_n(X_{G^*})$. Hence if $\Delta_{T,n}$ is irreducible and aperiodic, then so is $\Delta_{T,m}$.) Therefore, it suffices to show that $(W_{T,n}, \varsigma_{T,n})$ is topologically mixing, additionally assuming that n is sufficiently large for a σ_{G^*}-periodic point of period n to exist in X_{G^*}.

First we observe that for any integer $r \geq 1$, $\Delta_{T,r}$ is the disjoint union of irreducible graphs. Let s be an integer with $2s + 1 \geq r$. Since T^* is non-degenerate and φ_{T^*} is chain recurrent, each point of $X_{G^*} = X_{T^*}$ is on some periodic δ-pseudo orbit of φ_{T^*} with $\delta = 1/(s+1)$. Hence every vertex in $V_{\Delta_{T,2s+1}} = L_{2s+1}(X_{G^*})$ is on some periodic point of the obi shift $(W_{T,2s+1}, \varsigma_{T,2s+1})$, so that every vertex in $V_{\Delta_{T,r}} = L_r(X_{G^*})$ is on some periodic point of the obi shift $(W_{T,r}, \varsigma_{T,r})$. Thus $\Delta_{T,r}$ is the disjoint union of irreducible graphs.

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Let $T = (p, q : \Gamma \to G)$. Let $\bar{T} = (\bar{p}, \bar{q} : \bar{\Gamma} \to G)$ be the textile system defined as follows: $\bar{\Gamma} = \Delta_{T,n}$, and for $(\alpha_i)_{1\le i\le n} \in A_{\Delta_{T,n}} = L_n(X_{\Gamma^*})$ with $\alpha_i \in A_{\Gamma^*} = A_\Gamma$,

$$\bar{p}_A((\alpha_i)_{1\le i\le n}) = p_A(\alpha_1) \quad \text{and} \quad \bar{q}_A((\alpha_i)_{1\le i\le n}) = q_A(\alpha_n).$$

Then $Z_{\bar{T}} = W_{T,n}$. Since $\xi_{T,0,n} : Z_{T,0,n} \to X_{T,0,n+1}$ is one-to-one, it follows that $\xi_{\bar{T},0,1} : Z_{\bar{T},0,1} \to X_{\bar{T},0,2}$ is one-to-one. Since q is right or left resolving by (i), $\bar{q}$ is right or left resolving.

Let $\breve{u}$ be a path in $L_n(X_{G^*})$ which appears on some σ_{G^*}-periodic point of period n of X_{G^*}. Let $\breve{v} = i_{G^*}(\breve{u})$. Then $\breve{u}$ is a circuit in G^* going from $\breve{v}$ to $\breve{v}$. Let $\bar{G}^*$ be the graph over which $\bar{T}^*$ is defined. Then the periodic sequence of period 1 on which $\breve{u}$ is the unique symbol is a $\sigma_{\bar{G}^*}$-fixed point of $X_{\bar{G}^*}$. Hence, since T^* is non-degenerate, there exists a textile $(\bar{\alpha}_{i,j})_{i,j\in\mathbb{Z}}$ woven by $\bar{T}$ such that $i_{\bar{\Gamma}}(\bar{\alpha}_{i,0}) = \breve{u}$ for all $i \in \mathbb{Z}$.

Let $\breve{G}$ be the irreducible component of G which contains $\breve{v}$. Let $\breve{\Gamma}$ be the irreducible component of $\bar{\Gamma} = \Delta_{T,n}$ which contains the vertex $\breve{u}$ in $\bar{\Gamma}$. If $\bar{p}(\breve{\Gamma})$ and $\bar{q}(\breve{\Gamma})$ denote the subgraphs of G with arc-sets $\bar{p}_A(A_{\breve{\Gamma}})$ and $\bar{q}_A(A_{\breve{\Gamma}})$, respectively, then they are irreducible. Therefore, since $\bar{p}_V(\breve{u}) = \breve{v}$ and $\bar{q}_V(\breve{u}) = \breve{v}$, $\bar{p}(\breve{\Gamma})$ and $\bar{q}(\breve{\Gamma})$ are subgraphs of $\breve{G}$. Hence we can define a textile system $\breve{T} = (\breve{p}, \breve{q} : \breve{\Gamma} \to \breve{G})$ such that $\breve{p}$ and $\breve{q}$ are the restrictions of $\bar{p}$ and $\bar{q}$, respectively, on $\breve{\Gamma}$. $\breve{T}$ weaves a non-empty set of textiles. Indeed, $(\bar{\alpha}_{i,j})_{i,j\in\mathbb{Z}}$ is woven by $\breve{T}$, because $(\bar{\alpha}_{i,j})_{j\in\mathbb{Z}} \in X_{\breve{\Gamma}}$ for all $i \in \mathbb{Z}$, since $\breve{\Gamma}$ is an irreducible component of $\bar{\Gamma} = \Delta_{T,n}$ which is the disjoint union of irreducible graphs and hence $\breve{\Gamma}$ is closed in $\bar{\Gamma}$ in the sense that every arc in $\bar{\Gamma}$ starting from a vertex in $\breve{\Gamma}$ ends in a vertex in $\breve{\Gamma}$ and every arc in $\bar{\Gamma}$ ending in a vertex in $\breve{\Gamma}$ starts from a vertex in $\breve{\Gamma}$.

Since $\bar{q}$ is right or left resolving, $\breve{q}$ is right or left resolving. It therefore follows that $X_{\breve{T},0,1} \subset X_{\breve{T},\infty,0}$.

Since $\xi_{\bar{T},0,1} : Z_{\bar{T},0,1} \to X_{\bar{T},0,2}$ is one-to-one, $\xi_{\breve{T},0,1} : Z_{\breve{T},0,1} \to X_{\breve{T},0,2}$ is one-to-one. Since $\breve{q}$ is right or left resolving, $\eta_{\breve{T},0,1} : Z_{\breve{T},0,1} \to X_{\breve{T},1,1}$ is bounded-to-one.

Since $\breve{\Gamma}$ is irreducible, $(W_{\breve{T},1}, \varsigma_{\breve{T},1})$ is topologically transitive.

Therefore, applying Lemma 4.1 to $\breve{T}$ we have $X_{\breve{T},0,1} = X_{\breve{T}}$. This implies that if $(\alpha_{i,j})_{1\le i\le n, j\in\mathbb{Z}}$ is an obi of width n woven by T such that $(\alpha_{i,j})_{1\le i\le n, j\in\mathbb{Z}} \in X_{\breve{\Gamma}}$, then there exists a textile $(\beta_{i,j})_{i,j\in\mathbb{Z}}$ woven by T such that

$$\xi((\beta_{1,j})_{j\in\mathbb{Z}}) = \xi((\alpha_{1,j})_{j\in\mathbb{Z}}) \quad \text{and} \quad (\beta_{tn+i,j})_{1\le i\le n, j\in\mathbb{Z}} \in X_{\breve{\Gamma}} \text{ for all } t \in \mathbb{Z}.$$

Assume that $\bar{\Gamma}$ is not irreducible. Then there exists $\bar{u} \in V_{\bar{\Gamma}} - V_{\breve{\Gamma}}$. Since $\breve{u}$, $\bar{u} \in L_n(X_{T^*})$ and (X_{T^*}, σ_{T^*}) is topologically mixing, there exists a textile $(\alpha_{i,j})_{i,j\in\mathbb{Z}}$ woven by T such that

$$(i_\Gamma(\alpha_{i,0}))_{1\le i\le n} = \breve{u} \quad \text{and} \quad (i_\Gamma(\alpha_{tn+i,0}))_{1\le i\le n} = \bar{u} \text{ for some } t \ge 1.$$

Since $\breve{\Gamma}$ is closed in $\bar{\Gamma}$, $(\alpha_{i,j})_{1\le i\le n, j\in\mathbb{Z}} \in X_{\breve{\Gamma}}$. Therefore, by the above, there exists a textile $(\beta_{i,j})_{i,j\in\mathbb{Z}}$ woven by T such that

$$\xi((\beta_{1,j})_{j\in\mathbb{Z}}) = \xi((\alpha_{1,j})_{j\in\mathbb{Z}}) \quad \text{and} \quad (\beta_{tn+i,j})_{1\le i\le n, j\in\mathbb{Z}} \in X_{\breve{\Gamma}} \text{ for all } t \in \mathbb{Z}.$$

Clearly the half-textiles $(\alpha_{i,j})_{i\in\mathbb{N}, j\in\mathbb{Z}}$ and $(\beta_{i,j})_{i\in\mathbb{N}, j\in\mathbb{Z}}$ are different, but they have the same uppermost thread, which contradicts that $\xi_{T,0,n}$ is one-to-one.

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Thus we have proved that $\bar{\Gamma}$ is irreducible. Therefore $(W_{T,n}, \varsigma_{T,n}) = (X_{\bar{\Gamma}}, \sigma_{\bar{\Gamma}})$ is topologically transitive.

To show that $(W_{T,n}, \varsigma_{T,n})$ is topologically mixing, let $k \geq 1$ and define the textile system $T^{(k)} = (p^{(k)}, q^{(k)} : \Gamma^{(k)} \to G^{(k)})$ as follows: $G^{(k)}$ is the graph such that $A_{G^{(k)}} = L_k(X_G)$, $V_{G^{(k)}} = V_G$, and $i_{G^{(k)}}$ and $t_{G^{(k)}}$ map each $a_1 \dots a_k \in A_{G^{(k)}}$ with $a_j \in A_G$ to $i_G(a_1)$ and $t_G(a_k)$, respectively; similarly $\Gamma^{(k)}$ is defined from Γ; $p_A^{(k)}$ and $q_A^{(k)}$ map each $\alpha_1 \dots \alpha_k \in A_{\Gamma^{(k)}} = L_k(X_\Gamma)$ with $\alpha_j \in A_\Gamma$ to $p_A(\alpha_1) \dots p_A(\alpha_k)$ and $q_A(\alpha_1) \dots q_A(\alpha_k)$, respectively. Then the conditions (i), (ii), (iii) and (iv) of the lemma are satisfied for $(T^{(k)})^*$ instead of T^*, and $\xi_{T^{(k)},0,n} : Z_{T^{(k)},0,n} \to X_{T^{(k)},0,n+1}$ is one-to-one. Thus by the above we conclude that $(W_{T^{(k)},n}, \varsigma_{T^{(k)},n})$ is transitive. Hence $\varsigma_{T,n}^k$ is transitive, because $(W_{T^{(k)},n}, \varsigma_{T^{(k)},n})$ is conjugate to $(W_{T,n}, \varsigma_{T,n}^k)$.

Therefore $(W_{T,n}, \varsigma_{T,n})$ is an SFT such that $\varsigma_{T,n}^k$ is transitive for all $k \geq 1$, so that $\varsigma_{T,n}$ is topologically mixing. □

The following is the other main proposition of this paper.

PROPOSITION 4.3. *Let T be a one-sided 1-1 textile system with $T^* = (p^*, q^* : \Gamma^* \to G^*)$ such that:*

(i) *p^* or q^* is left resolving;*
(ii) *T^* is one-sided 1-1 and non-degenerate;*
(iii) *φ_{T^*} is chain recurrent; and*
(iv) *σ_{T^*} is topologically mixing.*

Then φ_{T^*} has POTP and is topologically mixing.

Proof. Let $T = (p, q : \Gamma \to G)$. Then q is left or right resolving by (i). Therefore it follows that $X_{T,0,1} \subset X_{T,\infty,0}$.

Since T is one-sided 1-1, T is finitely one-sided 1-1 by Proposition 3.7. Since T is finitely one-sided 1-1 and $X_{T,0,1} \subset X_{T,\infty,0}$, there exists $n \geq 1$ such that $\xi_{T,0,n} : Z_{T,0,n} \to X_{T,0,n+1}$ is one-to-one. Therefore it follows from Lemma 4.2 that the obi shift $(W_{T,m}, \varsigma_{T,m})$ is topologically mixing for all $m \geq 1$.

Since q is right or left resolving, $\eta_{T,0,n} : Z_{T,0,n} \to X_{T,1,n}$ is bounded-to-one. Since $X_{T,0,1} \subset X_{T,\infty,0}$, we have $X_{T,0,n} \subset X_{T,\infty,0}$. Therefore, it follows from Lemma 4.1 that $X_{T,0,n} = X_T$, so that T is finitely saturated. Thus φ_{T^*} has POTP by Proposition 3.4.

It remains to be shown that φ_{T^*} is topologically mixing.

Let $x, y \in X_{T^*}$. Let $\epsilon > 0$. It suffices to show that there exists $l \geq 1$ such that if $k \geq l$, then $\varphi_{T^*}^k(B(x, \epsilon)) \cap B(y, \epsilon) \neq \emptyset$, where $B(x, \epsilon) = \{x' \in X_{T^*} \mid d_{X_{T^*}}(x', x) < \epsilon\}$. There exist points $(\alpha_i)_{i\in\mathbb{Z}}$ and $(\beta_i)_{i\in\mathbb{Z}}$ of Z_{T^*} with $\alpha_i, \beta_i \in A_\Gamma$ such that $(i_\Gamma(\alpha_i))_{i\in\mathbb{Z}} = x$ and $(t_\Gamma(\beta_i))_{i\in\mathbb{Z}} = y$. Since φ_{T^*} has POTP, there exists $\delta > 0$ such that any δ-pseudo orbit of φ_{T^*} is ϵ-traced by some orbit of φ_{T^*}. Let s be a positive integer with $1/(s+1) < \delta$. Since $\Delta_{T,2s+1}$ is irreducible and aperiodic (because $(W_{T,2s+1}, \varsigma_{T,2s+1})$ is topologically mixing), there exists $l > 2$ such that for all $m \geq l - 2$ there exists a path γ_m of length m in $\Delta_{T,2s+1}$ going from vertex $(t_\Gamma(\alpha_i))_{-s\leq i\leq s}$ to vertex $(i_\Gamma(\beta_i))_{-s\leq i\leq s}$. Let $k \geq l$. There exists an obi $w = (\alpha_{i,j})_{-s\leq i\leq s, j\in\mathbb{Z}}$ of width $2s+1$ woven by T with $\alpha_{i,j} \in A_\Gamma$ such that $(\alpha_{i,0})_{-s\leq i\leq s} = (\alpha_i)_{-s\leq i\leq s}$, $(\alpha_{i,j})_{-s\leq i\leq s, 1\leq j\leq k-2} = \gamma_{k-2}$ and $(\alpha_{i,k-1})_{-s\leq i\leq s} = (\beta_i)_{-s\leq i\leq s}$. Considering a configuration to which w is extended as in Remark 3.3, we see that there exists a δ-pseudo

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orbit $(x'_j)_{j\in\mathbb{Z}}$ of φ_{T^*} such that $x'_0 = x$ and $x'_k = y$. There exists an orbit $(x_j)_{j\in\mathbb{Z}}$ of φ_{T^*} which ϵ-traces $(x'_j)_{j\in\mathbb{Z}}$. Therefore we have $d_{X_{T^*}}(x, x_0) < \epsilon$ and $d_{X_{T^*}}(y, x_k) < \epsilon$, so that $\varphi^k_{T^*}(B(x, \epsilon)) \cap B(y, \epsilon)$ contains x_k and hence is non-empty as required. □

5. *One-sided resolving automorphisms*

A textile system $T = (p, q : \Gamma \to G)$ is said to be p-L if p is left resolving, p-R if p is right resolving, q-L if q is left resolving, and q-R if q is right resolving. An onto endomorphism φ of a topological Markov shift (X, σ) is called a *p-L endomorphism* if there exists a one-sided 1-1, non-degenerate, p-L textile system with $(X, \sigma, \varphi) = (X_T, \sigma_T, \varphi_T)$. We similarly define p-R, q-L, and q-R *endomorphisms* of topological Markov shifts. An onto endomorphism of a topological Markov shift is said to be *one-sided resolving* if it is p-L, p-R, q-L or q-R.

For a subshift (X, σ) and an integer $s \geq 1$, we define the *sth power system* of (X, σ) to be the subshift $(X^{(s)}, \sigma^{(s)})$ over the alphabet $L_s(X)$ with $X^{(s)} = \{(a_{js} \dots a_{(j+1)s-1})_{j\in\mathbb{Z}} \mid (a_j)_{j\in\mathbb{Z}} \in X,\ a_j \in L_1(X)\}$. For an endomorphism φ of a subshift (X, σ) and $s \geq 1$, let $(\varphi)^{(s)}$ denote the endomorphism of $(X^{(s)}, \sigma^{(s)})$ such that if φ maps $(a_j)_{j\in\mathbb{Z}}$ to $(b_j)_{j\in\mathbb{Z}}$ with $a_j, b_j \in L_1(X)$, then $(\varphi)^{(s)}$ maps $(a_{js} \dots a_{(j+1)s-1})_{j\in\mathbb{Z}}$ to $(b_{js} \dots b_{(j+1)s-1})_{j\in\mathbb{Z}}$. Clearly (X, σ^s, φ) and $(X^{(s)}, \sigma^{(s)}, \varphi^{(s)})$ are conjugate.

If both φ_1 and φ_2 are p-L (respectively, p-R, q-L, q-R) endomorphisms of a topological Markov shift, then so is $\varphi_2\varphi_1$; if φ is a p-L (respectively, p-R, q-L, q-R) endomorphism of a topological Markov shift (X, σ), then $((\varphi)^{(s)})^r$ is a p-L (respectively, p-R, q-L, q-R) endomorphism of the topological Markov shift $(X^{(s)}, \sigma^{(s)})$ for all $r, s \geq 1$ (see [**20,** pp. 55, 56]). Hence if φ is an essentially p-L (respectively, p-R, q-L, q-R) endomorphism of an SFT (X, σ), then φ^r is an essentially p-L (respectively, p-R, q-L, q-R) endomorphism of (X, σ^s) for all $r, s \geq 1$.

Here we recall the definitions of the following 'resolving textile systems' and 'resolving endomorphisms of topological Markov shifts' introduced and studied in [**20**]. A textile system is said to be *LR* if it is p-L and q-R, *RL* if it is p-R and q-L, *LL* if it is p-L and q-L, *RR* if it is p-R and q-R and *q-biresolving* if it is q-L and q-R. An onto endomorphism φ of a topological Markov shift (X, σ) is called an *LR endomorphism* if there exists a one-sided 1-1, non-degenerate, LR textile system T with $(X_T, \sigma_T, \varphi_T) = (X, \sigma, \varphi)$. The definitions of *RL*, *LL*, *RR* and *q-biresolving endomorphisms* of topological Markov shifts are similarly given.

For a subshift (X, σ) and $n \geq 1$, we define the *higher block system of order n* of (X, σ) to be the subshift $(X^{[n]}, \sigma^{[n]})$ over the alphabet $L_n(X)$ with $X^{[n]} = \{(a_j \dots a_{j+n-1})_{j\in\mathbb{Z}} \mid (a_j)_{j\in\mathbb{Z}} \in X, a_j \in L_1(X)\}$. For an endomorphism φ of a subshift (X, σ) and $n \geq 1$, let $\varphi^{[n]}$ denote the endomorphism of $(X^{[n]}, \sigma^{[n]})$ such that if φ maps $(a_j)_{j\in\mathbb{Z}}$ to $(b_j)_{j\in\mathbb{Z}}$ with $a_j, b_j \in L_1(X)$, then $\varphi^{[n]}$ maps $(a_j \dots a_{j+n-1})_{j\in\mathbb{Z}}$ to $(b_j \dots b_{j+n-1})_{j\in\mathbb{Z}}$. Clearly (X, σ, φ) and $(X^{[n]}, \sigma^{[n]}, \varphi^{[n]})$ are conjugate.

We define the *higher block graph $G^{[n]}$ of order n* of G to be the graph such that for $n \geq 2$, $A_{G^{[n]}} = L_n(X_G)$, $V_{G^{[n]}} = L_{n-1}(X_G)$ and $i_{G^{[n]}}$ and $t_{G^{[n]}}$ send $a_1 \dots a_n \in A_{G^{[n]}}$, $a_j \in A_G$ to $a_1 \dots a_{n-1}$ and $a_2 \dots a_n$, respectively, and $G^{[1]}$ is the minimal subgraph G' of G with $X_{G'} = X_G$.

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PROPOSITION 5.1. *Let φ be an onto endomorphism of a topological Markov shift (X_G, σ_G).*

(1) *φ has memory (respectively, anticipation) 0 if and only if φ is p-L (respectively, p-R).*

(2) [**21**] *If φ is one-to-one and has memory 0 and φ^{-1} has anticipation 0, then $\varphi^{[n]}$ is LR for some $n \geq 1$.*

Proof. (1) If φ is of $(0, l)$ type with a local rule $f : L_{l+1}(X_G) \to A_G$, then let $T = (p, q : \Gamma \to G)$ be the textile system such that $\Gamma = G^{[l+1]}$, $p_A(a_1 \dots a_{l+1}) = a_1$ for $a_1 \dots a_{l+1} \in L_{l+1}(X_G)$, $a_j \in A_G$, and $q_A = f$. Then T is one-sided 1-1 and non-degenerate with p left resolving and $(X_T, \sigma_T, \varphi_T) = (X_G, \sigma_G, \varphi)$.

Conversely, if $T = (p, q : \Gamma \to G)$ is a one-sided 1-1 non-degenerate textile system with p left resolving and $\varphi_T = \varphi$, then there exist $m, n \geq 1$ and a local rule $g : L_{m+n+1}(X_G) \to A_\Gamma$ such that if $\phi_p^{-1}((a_j)_{\in\mathbb{Z}}) = (\alpha_j)_{\in\mathbb{Z}}$, $a_j \in A_G$, $\alpha_j \in A_\Gamma$, then $\alpha_j = g(a_{j-m} \dots a_{j+n})$ for all $j \in \mathbb{Z}$. Since p is left resolving, for any $\alpha_{-m} \dots \alpha_n \in L_{m+n+1}(X_\Gamma)$, $\alpha_{-m} \dots \alpha_{-1} \in L_m(X_\Gamma)$ is determined by $p_A(\alpha_{-m}) \dots p_A(\alpha_{-1})$ and $t_\Gamma(\alpha_{-1}) = i_\Gamma(\alpha_0) = i_\Gamma(g(p_A(\alpha_{-m}) \dots p_A(\alpha_n)))$. Hence we can define a local rule $g' : L_{m+n+1}(X_G) \to A_\Gamma$ by $g'(p_A(\alpha_{-m}) \dots p_A(\alpha_n)) = \alpha_{-m}$ with $\alpha_{-m} \dots \alpha_n \in L_{m+n+1}(X_\Gamma)$. Then for $(\alpha_j)_{\in\mathbb{Z}} \in X_\Gamma$, $q_A(\alpha_{-m+j}) = q_A g'(p_A(\alpha_{-m+j}) \dots p_A(\alpha_{j+n}))$ for all $j \in \mathbb{Z}$, so that φ_T is of $(0, m + n + 1)$ type.

By symmetry the second version follows.

(2) This is a consequence of [**21,** Propositions 6.4 and 6.5]. Below is a direct proof. A similar proof (attributed to Kitchens) to prove a somewhat different result is found in [**3**].

There exists $n \geq 1$ such that φ is of $(0, n)$ type and φ^{-1} is of $(n, 0)$ type. Let Z' be the set of all elements of the form $(a_j/b_j)_{j\in\mathbb{Z}}$, where $(a_j)_{j\in\mathbb{Z}} \in X_G$ and $\varphi((a_j)_{j\in\mathbb{Z}}) = (b_j)_{j\in\mathbb{Z}}$ with $a_j, b_j \in A_G$. Then Z' is the space of an SFT. Let $T' = (p', q' : \Gamma' \to G')$ be the textile system defined as follows: $G' = G^{[n+1]}$; $A_{\Gamma'} = L_{n+1}(Z')$, $V_{\Gamma'} = L_n(Z')$, and for each arc $(a_1/b_1) \dots (a_{n+1}/b_{n+1}) \in A_{\Gamma'}$, $i_{\Gamma'}$, $t_{\Gamma'}$, p'_A and q'_A map it to $(a_1/b_1) \dots (a_n/b_n)$, $(a_2/b_2) \dots (a_{n+1}/b_{n+1})$, $a_1 \dots a_{n+1}$ and $b_1 \dots b_{n+1}$, respectively. Then noting that $(X_{\Gamma'}, \sigma_{\Gamma'}) = (Z'^{[n+1]}, \sigma'^{[n+1]})$, we see that T' is 1-1, non-degenerate and LR with $(X_{T'}, \sigma_{T'}, \varphi_{T'}) = (X_G^{[n+1]}, \sigma_G^{[n+1]}, \varphi^{[n+1]})$. □

We can now show that every expansive, essentially one-sided resolving automorphism of a transitive SFT is conjugate to an SFT.

THEOREM 5.2. *If an expansive automorphism φ of a topologically-transitive SFT (X, σ) essentially has the property that φ or φ^{-1} has memory zero or anticipation zero, then (X, φ) is topologically conjugate to an SFT.*

Proof. It suffices to prove the theorem only for φ which has memory zero, under the additional condition that σ is topologically mixing (by the final remark of §2). There exists $n \geq 1$ such that $(X^{[n]}, \sigma^{[n]})$ is a topological Markov shift. Since $\varphi^{[n]}$ has memory zero, it follows from Proposition 5.1 that there exists a 1-1, non-degenerate, p-L textile system T with $(X_T, \sigma_T, \varphi_T)$ conjugate to (X, σ, φ). Since σ is topologically mixing, so is σ_T. Since φ is expansive, so is φ_T and hence it follows from Theorem 3.2 that T^* is 1-1 and (X_T, φ_T) is conjugate to (X_{T^*}, σ_{T^*}).

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We give two proofs for proving that (X_T, φ_T) is conjugate to an SFT.

The first proof: since T is p-L, T^* is q-R (i.e. if $T^* = (p^*, q^* : \Gamma^* \to G^*)$, then q^* is right resolving). It therefore follows that T^* is upwardly finitely saturated. Hence by (the dual form of) Proposition 3.10, (X_{T^*}, σ_{T^*}) is an SFT.

The second proof: since φ_T is an automorphism of a mixing topological Markov shift, φ_T has periodic points dense, so that it is chain recurrent. (This argument is found in Boyle's paper [**6**].) Hence by (the dual form of) Proposition 4.3, φ_T has POTP. Since it is expansive, (X_T, φ_T) is conjugate to an SFT. □

Remark 5.3. Let φ be an endomorphism of a subshift (X, σ).

(1) If φ has memory (respectively, anticipation) zero, then for all $i \geq 0$, φ^i has memory (respectively, anticipation) zero.

(2) If φ is of (k, l) type with $k, l \geq 0$, then $\varphi\sigma^s$ is of $(\max\{k - s, 0\}, \max\{l + s, 0\})$ type for $s \in \mathbb{Z}$.

(3) If φ is an automorphism of (X, σ) and φ^{-1} is of (m, n) type with $m, n \geq 0$, then $(\varphi\sigma^s)^{-1}$ is of $(\max\{m + s, 0\}, \max\{n - s, 0\})$ type for $s \in \mathbb{Z}$.

COROLLARY 5.4. *Let φ be an automorphism of a topologically-transitive SFT (X, σ) such that each of φ and φ^{-1} essentially has memory 0. Then for all $i, j \in \mathbb{Z}$, $(X, \varphi^i\sigma^j)$ is topologically conjugate to an SFT whenever $\varphi^i\sigma^j$ is expansive.*

Proof. By Remark 5.3 we see that for all $i, j \geq 0$, $\varphi^i\sigma^j$ essentially has memory zero and so does $\varphi^{-i}\sigma^j$. Hence the result follows from Theorem 5.2. □

Example 5.5. Kōji Akatsuka [**1**] developed a computer program which generates all local rules $f : \{0, 1\}^6 \to \{0, 1\}$ with $f(0^6) = 0$ that give automorphisms of the full shift (X, σ) over the alphabet $\{0, 1\}$. For each rule f it additionally computes the non-negative integers p_f, q_f such that f is 'p_f right definite' and 'q_f left definite' (see [**18,** Definition 3]) from which we can immediately know $m_f, n_f \geq 0$ such that if φ_f is an automorphism of (0, 5) type with local rule f, then φ_f^{-1} is of (m_f, n_f) type (see [**18,** Theorem 7 and Corollary 2]). One example from his results is as following. Let f be the local rule such that the numbers $a_1 2^5 + a_2 2^4 + \cdots + a_5 2 + a_6$ with $f(a_1 a_2 \ldots a_6) = 0$ are 0, 1, 2, 3, 8, 10, 13, 15, 16, 17, 18, 19, 24, 26, 27, 29, 32, 34, 35, 40, 42, 43, 45, 48, 49, 50, 51, 53, 56, 58, 59 and 61. Then $(m_f, n_f) = (9, 3)$ ($p_f = 9$ and $q_f = 8$). Therefore, by Remark 5.3 and Proposition 5.1, we know the following.

The automorphisms $\varphi_f^i\sigma^j$ with $j \geq 0$ and $i \geq -j/9$ are p-L and their inverses $\varphi_f^i\sigma^j$ with $j \leq 0$ and $i \leq -j/9$ are q-L. The automorphisms $\varphi_f^i\sigma^j$ with $j \geq 0$ and $j/3 \geq i \geq -j/5$ are q-R and their inverses $\varphi_f^i\sigma^j$ with $j \leq 0$ and $j/3 \leq i \leq -j/5$ are p-R. These are one-sided resolving automorphisms of (X, σ), but we cannot say that they are all of the essentially one-sided resolving automorphisms of the form $\varphi_f^i\sigma^j$ of (X, σ); we cannot even say that they are all of the one-sided resolving automorphisms of the form $\varphi_f^i\sigma^j$ of (X, σ).

The automorphisms $\varphi_f^i\sigma^j$ with $j \geq 0$ and $j/3 \geq i \geq -j/9$ (respectively, with $j \leq 0$ and $j/3 \leq i \leq -j/9$) are essentially LR (respectively, essentially RL) automorphisms of (X, σ), but we cannot say that they are all of the essentially LR (respectively, essentially RL) automorphisms of the form $\varphi_f^i\sigma^j$ of (X, σ).

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Example 5.6. Here the reader is referred to Examples 1–3 in [**20,** §10].

(1) The automorphism $\varphi^2\sigma_M^{-1}$ of the mixing topological Markov shift (X, σ) in the Example 1 is not an essentially one-sided resolving automorphism of (X_M, σ_M), as will be seen in Example 8.12. As was shown in the Example 1, $(X_M, \varphi^2\sigma_M^{-1})$ is conjugate to an SFT.

(2) Let φ be the non-expansive automorphism of the full 2-shift (X, σ) found in [**14,** §20] and considered in the Example 2. Both φ and φ^{-1} are of (1, 2) type. Therefore the automorphisms $\varphi^k\sigma^n$ with $n \geq 0$ and $|k| \leq n$ are p-L, and hence their inverses $\varphi^k\sigma^n$ with $n \leq 0$ and $|k| \leq -n$ are q-L. The automorphisms $\varphi^k\sigma^n$ with $n \geq 0$ and $|k| \leq n/2$ are essentially LR, and hence their inverses $\varphi^k\sigma^n$ with $n \leq 0$ and $|k| \leq -n/2$ are essentially RL. In fact, $\varphi^{[2]}(\sigma^{[2]})^2$ and $(\varphi^{[2]})^{-1}(\sigma^{[2]})^2$ are LR automorphisms of $(X^{[2]}, \sigma^{[2]})$. The automorphisms $\varphi^k\sigma^n$ with $|n|/2 \leq |k| \leq |n|$ are non-expansive. If $|k| > |n|$, then $\varphi^k\sigma^n$ is not an essentially one-sided resolving automorphism of (X, σ), as will be seen in Example 8.12.

Let φ' be the *symmetry* of φ, i.e. φ' is of (2, 1) type with the local rule $f' : \{0, 1\}^4 \to \{0, 1\}$ such that $f'(a_1a_2a_3a_4) = f(a_4a_3a_2a_1)$ for $a_j \in \{0, 1\}$, where f is the local rule for φ. By the above it follows that the automorphisms $\varphi'^k\sigma^n$ with $n \geq 0$ and $|k| \leq n$ are q-R and the automorphisms $\varphi'^k\sigma^n$ with $n \geq 0$ and $|k| \leq n/2$ are essentially LR.

Let us consider the direct product $(X_0, \sigma_0) = (X \times X, \sigma \times \sigma)$ and the direct product $\varphi_0 = \varphi'\sigma \times \varphi\sigma^{-1}$. Then φ_0 is an automorphism of (X_0, σ_0). From the above, we know the following: the automorphisms $\varphi_0^k\sigma_0^n$ with $-n/3 \leq k \leq n/2$ are p-L, the automorphisms $\varphi_0^k\sigma_0^n$ with $-n/2 \leq k \leq n/3$ are q-R, and the automorphisms $\varphi_0^k\sigma_0^n$ with $-n/3 \leq k \leq n/3$ are essentially LR. The automorphisms $\varphi_0^k\sigma_0^n$ with $|n|/3 \leq |k| \leq |n|/2$ are non-expansive. If $|k| > |n|/2$, then $\varphi^k\sigma^n$ is not an essentially one-sided resolving automorphism of (X_0, σ_0). (Consider the automorphism $\varphi_0^{[2]}$ of $(X_0^{[2]}, \sigma_0^{[2]})$ and use the obvious fact that the direct product $\varphi_1 \times \varphi_2$ of a p-L (respectively, q-R) endomorphism φ_1 of a topological Markov shift (X_1, σ_1) and a p-L (respectively, q-R) endomorphism φ_2 of another (X_2, σ_2) is a p-L (respectively, q-R) endomorphism of the direct product $(X_1 \times X_2, \sigma_1 \times \sigma_2)$.)

(3) Let ψ be the automorphism of the full shift (X_B, σ_B) dually induced by a bipermutive block map of (r, s) type over an alphabet A $(\#A \geq 2)$ with $r, s \geq 1$ and $B = A^{r+s}$ in the Example 3. Then ψ is LL (and hence ψ and ψ^{-1} have memory 0). $\psi^k\sigma_B^n$ is non-expansive for $(n, k) \in \mathbb{Z}^2$ only on three lines: $n = 0$, $k = rn$ and $k = -sn$. For all $n \geq 0$, $\psi^k\sigma_B^n$ is a p-L automorphism of (X, σ) and if $-sn \leq k \leq rn$, then it is an LR automorphism of (X, σ); for all $n \leq 0$, $\psi^k\sigma_B^n$ is a q-L automorphism of (X, σ) and if $rn \leq k \leq -sn$, then it is an RL automorphism of (X, σ).

6. *One-sided resolving endomorphisms*

Let (X, τ, φ) be a commuting system, where τ is a homeomorphism. For any two φ-orbits $(x_i)_{i\in\mathbb{Z}}$ and $(y_i)_{i\in\mathbb{Z}}$, let

$$\Delta((x_i)_{i\in\mathbb{Z}}, (y_i)_{i\in\mathbb{Z}}) = \sup\{d_X(x_i, y_i) \mid i \in \mathbb{Z}\}.$$

We say that φ is *left τ-expansive* if there exists $\delta > 0$ such that for any two φ-orbits $(x_i)_{i\in\mathbb{Z}}$ and $(y_i)_{i\in\mathbb{Z}}$, if $\Delta((\tau^{-j}(x_i))_{i\in\mathbb{Z}}, (\tau^{-j}(y_i))_{i\in\mathbb{Z}}) < \delta$ for all $j \geq 0$, then $(x_i)_{i\in\mathbb{Z}} = (y_i)_{i\in\mathbb{Z}}$.

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We say that φ is *right τ-expansive* if φ is left τ^{-1}-expansive. It is clear that if φ is expansive, then φ is left τ-expansive and right τ-expansive.

We can easily see the following proposition.

PROPOSITION 6.1. *Let (X, τ, φ) and (X', τ', φ') be topologically-conjugate commuting systems such that τ and τ' are homeomorphisms. Then φ is left τ-expansive if and only if φ' is left τ'-expansive; φ is right τ-expansive if and only if φ' is right τ'-expansive.*

The proof of the following proposition is easy and hence omitted.

PROPOSITION 6.2. *Let T be a one-sided 1-1 textile system. Then φ_T is left σ_T-expansive if and only if ξ_{T^*} is one-to-one (T^* is one-sided 1-1); φ_T is right σ_T-expansive if and only if η_{T^*} is one-to-one.*

The following theorem generalizes Theorem 5.2.

THEOREM 6.3. *Suppose that φ is a chain recurrent onto endomorphism of a topologically-transitive SFT (X, σ) and has at least one of the following properties.*

(i) *φ is essentially p-L (or essentially has memory zero) and left σ-expansive.*
(ii) *φ is essentially p-R (or essentially has anticipation zero) and right σ-expansive.*
(iii) *φ is essentially q-L and left σ-expansive.*
(iv) *φ is essentially q-R and right σ-expansive.*

Then φ has POTP. If σ is topologically mixing in addition, then φ is topologically mixing.

Proof. By the final remark of §2, it suffices to prove the theorem under the additional condition that σ is topologically mixing. Assume that φ satisfies (i) or (iii). There exists a one-sided 1-1, non-degenerate, p-L or q-L textile system T with $(X_T, \sigma_T, \varphi_T)$ conjugate to (X, σ, φ). Since φ is left σ-expansive, T^* is one-sided 1-1 by Propositions 6.1 and 6.2. Since φ is chain recurrent, so is φ_T. Since σ is topologically mixing, so is σ_T. Hence it follows from Proposition 4.3 that φ_T has POTP and is topologically mixing. Thus we have the conclusions of the theorem.

By symmetry, we have the same conclusions when φ satisfies (ii) or (iv). □

COROLLARY 6.4. *A chain recurrent, expansive onto endomorphism of a topologically-transitive one-sided SFT has POTP.*

Proof. Let $\tilde{\varphi}$ be a chain recurrent, expansive onto endomorphism of a transitive one-sided SFT $(\tilde{X}, \tilde{\sigma})$. Then the subshift (X, σ) induced by $(\tilde{X}, \tilde{\sigma})$ is a transitive SFT and the endomorphism φ of (X, σ) induced by $\tilde{\varphi}$ has memory zero. It is easy to see that φ is chain recurrent, left σ-expansive and onto. Therefore by Theorem 6.3, φ has POTP. Hence, using the fact that φ has memory zero and a standard compactness argument, we easily see that $\tilde{\varphi}$ has POTP. (For any $\delta > 0$, there exists $\tilde{\delta} > 0$ such that every $\tilde{\delta}$-pseudo orbit $((b_{i,j})_{j\in\mathbb{N}})_{i\in\mathbb{Z}}$ of $\tilde{\varphi}$ with $b_{i,j} \in L_1(X)$ is extendable to a δ-pseudo orbit $((b_{ij})_{j\in\mathbb{Z}})_{i\in\mathbb{Z}}$ of φ with $b_{i,j} \in L_1(X)$. We also use the uniform continuity of $s_X : X \to \tilde{X}$.) □

It is an open problem whether an onto endomorphism φ of a mixing SFT has φ-periodic points dense [**5, 8, 10**]. If the answer is affirmative, then the chain recurrence condition on φ in Theorem 6.3 and that on $\tilde{\varphi}$ in Corollary 6.4 can be removed. However Boyle and

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Kitchens [**8**] gave the following result by which, in particular for φ satisfying (iii) and (iv), Theorem 6.3 holds without assuming the chain recurrence of φ.

A homomorphism $\phi : (X, \tau) \to (X', \tau')$ between dynamical systems is said to be *right closing* (respectively, *left closing*) if it never collapses distinct backwardly (respectively, forwardly) τ-asymptotic points.

THEOREM 6.5. (Boyle and Kitchens [**8**]) *If φ is a right or left closing endomorphism of a topologically-mixing SFT (X, σ), then the points which are jointly σ-periodic and φ-periodic are dense in X.*

COROLLARY 6.6. *Suppose that φ is an onto endomorphism of a topologically-transitive SFT (X, σ) and has at least one of the following properties.*

(i) *φ is right or left closing, essentially p-L and left σ-expansive.*

(ii) *φ is right or left closing, essentially p-R and right σ-expansive.*

(iii) *φ is essentially q-L and left σ-expansive.*

(iv) *φ is essentially q-R and right σ-expansive.*

Then φ has POTP. If σ is additionally topologically mixing, then φ is topologically mixing.

Proof. We note the following: if φ is essentially q-R or essentially q-L, then φ is right closing or left closing; if φ has periodic points dense, then φ is chain recurrent. Thus the corollary is proved by Theorems 6.3 and 6.5 in the case σ is topologically mixing. Hence it is proved by the final remark of §2. □

COROLLARY 6.7. *Suppose $\tilde{\varphi}$ is an expansive endomorphism of a topologically-transitive one-sided SFT $(\tilde{X}, \tilde{\sigma})$ such that its induced endomorphism φ of the induced shift (X, σ) is right or left closing. Then $\tilde{\varphi}$ has POTP.*

Proof. If $\tilde{\sigma}$ is topologically mixing, then so is σ, so that φ is chain recurrent by Theorem 6.5. Thus the corollary is proved by the proof of Corollary 6.4, in case $\tilde{\sigma}$ is topologically mixing. Hence it is proved for a reason similar to that in the final remark of §2. □

This corollary is a generalization of the main theorem of [**23**] that an expansive automorphism of a mixing one-sided SFT is conjugate to an SFT.

Blanchard and Tisseur [**5**] proved that if an onto endomorphism φ of a topologically-transitive SFT (X, σ) has a point of equicontinuity, then φ has periodic points dense. One can see that such φ is neither left σ-expansive nor right σ-expansive if X has infinitely many points.

7. *Weakly one-sided resolving endomorphisms of subshifts*

Throughout the remainder of this paper we require the notion of 'textile subsystem', which was introduced in [**21,** §6] with the benefit of the criticism [**11,** Remark 8.11] to the textile system machinery [**20**].

Let $T = (p, q : \Gamma \to G)$ be a textile system. Then the set U_T of all textiles woven by T is a closed subset of $A_\Gamma^{\mathbb{Z}^2}$ equipped with the product topology of the discrete topology on A_Γ and is *shift invariant* (i.e. if $(\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U_T$ with $\alpha_{i,j} \in A_\Gamma$, then $(\alpha_{i+k,j+l})_{i,j\in\mathbb{Z}} \in U_T$ for all $k, l \in \mathbb{Z}$). We define a *textile subsystem* of T to be a closed, shift-invariant subset of U_T. Let U be a textile subsystem of T. The dual U^* of U is defined by

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$U^* = \{(\alpha_{j,i})_{i,j\in\mathbb{Z}} \mid (\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U, \alpha_{i,j} \in A_\Gamma\}$. Clearly U^* is a textile subsystem of T^*. The *woof shift* (X_U, σ_U) of U is defined by $X_U = \{\xi_T((\alpha_{0,j})_{j\in\mathbb{Z}}) \mid (\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U\}$ and $\sigma_U = \sigma_T|X_U$. We also define another subshift (Z_U, ς_U) by $Z_U = \{(\alpha_{0,j})_{j\in\mathbb{Z}} \mid (\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U\}$ and $\varsigma_U = \varsigma_T|X_U$. We define onto maps $\xi_U : Z_U \to X_U$ and $\eta_U : Z_U \to X_U$ restricting ξ_T and η_T, respectively. If U is *one-sided 1-1*, i.e. ξ_U is one-to-one, then we have an onto endomorphism φ_U of (X_U, σ_U) by $\varphi_U = \eta_U\xi_U^{-1}$. We say that U is *1-1* if both ξ_U and η_U are one-to-one. For $n \geq 1$, we define a textile subsystem $U^{[n]}$, which is called the *higher block system of order n* of U, by $U^{[n]} = \{(\alpha_{i,j} \ldots \alpha_{i,j+n-1})_{i,j\in\mathbb{Z}} \mid (\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U\}$. Clearly $U^{[n]}$ is a textile subsystem of the *higher block system* $T^{[n]}$ *of order n* of T which is defined by

$$T^{[n]} = (p^{[n]}, q^{[n]} : \Gamma^{[n]} \to G^{[n]}),$$

where $p^{[n]}$ and $q^{[n]}$ map each $\alpha_1 \ldots \alpha_n \in A_{\Gamma^{[n]}} = L_n(\Gamma)$ with $\alpha_j \in A_\Gamma$ to $p_A(\alpha_1) \ldots p_A(\alpha_n)$ and $q_A(\alpha_1) \ldots q_A(\alpha_n)$, respectively.

Let T be a textile system. If U is a textile subsystem of T, then for $n \geq 1$, $Z_{((U^*)^{[n]})^*}$ is the subshift space consisting of all obis of width n (woven by T) that appear on some textile (woven by T) in U, and the subshift spaces $Z_{((U^*)^{[n]})^*}$, $n = 1, 2, \ldots$, define U (because the union of the sets $\mathcal{F}_n$ of 'forbidden blocks' for $Z_{((U^*)^{[n]})^*}$ gives the set of 'forbidden patterns' for U). Therefore if U is one-sided 1-1, then Z_U defines U. For every subshift space $Z \subset Z_T$ with $\xi_T(Z) = \eta_T(Z)$ and $\xi_T|Z$ one-to-one, there exists a unique one-sided 1-1 textile subsystem U of T with $Z_U = Z$.

Remark 7.1. [**21**] Let φ be an onto endomorphism of a subshift (X, σ). Then there exists a one-sided 1-1 textile subsystem U with $(X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi)$.

Proof. Assume that φ is a block map of (k, l) type with $k, l \geq 0$ and let $f : L_{k+l+1}(X) \to L_1(X)$ be the local rule which gives φ. Let $T_{f,k,l} = (p, q : \Gamma \to G)$ be the textile system defined as follows: G is the one-vertex graph with $A_G = L_1(X)$; if $k = l = 0$, then $\Gamma = G$ with $p_A(a) = a$ and $q_A(a) = f(a)$ for $a \in A_\Gamma = L_1(X)$; if $k + l \geq 1$, then $A_\Gamma = L_{k+l+1}(X)$, $V_\Gamma = L_{k+l}(X)$, and for $w = a_1 \ldots a_{k+l+1} \in A_\Gamma$ with $a_j \in L_1(X)$, $i_\Gamma(w) = a_1 \ldots a_{k+l}$, $t_\Gamma(w) = a_2 \ldots a_{k+l+1}$, $p_A(w) = a_{k+1}$ and $q_A(w) = f(w)$. Then $T_{f,k,l}$ is one-sided 1-1. Since $X^{[k+l+1]} \subset Z_{T_{f,k,l}}$ and $\xi_{T_{f,k,l}}(X^{[k+l+1]}) = X = \eta_{T_{f,k,l}}(X^{[k+l+1]})$, there exists a unique one-sided 1-1 textile subsystem $U_{f,k,l}$ of $T_{f,k,l}$ with $Z_{U_{f,k,l}} = X^{[k+l+1]}$. Clearly $U = U_{f,k,l}$ has the properties required. □

PROPOSITION 7.2. *Let U be a one-sided 1-1 textile subsystem.*

(1) *φ_U is left σ_U-expansive if and only if for some $n \geq 1$, $\xi_{(U^{[n]})^*}$ is one-to-one ($(U^{[n]})^*$ is one-sided 1-1).*
(2) *φ_U is right σ_U-expansive if and only if for some $n \geq 1$, $\eta_{(U^{[n]})^*}$ is one-to-one.*
(3) [**21**] *φ_U is expansive if and only if for some $n \geq 1$, $(U^{[n]})^*$ is 1-1.*
(4) [**21**] *If both U and U^* are 1-1, then the homeomorphism $\chi_U : X_U \to X_{U^*}$ which maps $\xi_U((\alpha_{0,j})_{j\in\mathbb{Z}})$ to $\xi_{U^*}((\alpha_{i,0})_{i\in\mathbb{Z}})$ with $(\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U$ can be defined and $(X_U, \sigma_U, \varphi_U)$ is topologically conjugate to $(X_{U^*}, \varphi_{U^*}, \sigma_{U^*})$ through χ_U.*

Proof. Assume that there exists no $n \geq 1$ such that $\xi_{(U^{[n]})^*}$ is one-to-one. Then for all $n \geq 1$, there exist a pair of textiles $(\alpha_{i,j}^{(n)})_{i,j\in\mathbb{Z}}$ and $(\beta_{i,j}^{(n)})_{i,j\in\mathbb{Z}}$ in U such that $\alpha_{i,j}^{(n)} = \beta_{i,j}^{(n)}$

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for all $i \in \mathbb{Z}$ and all $-n \le j \le -1$ but $\alpha^{(n)}_{0,0} \neq \beta^{(n)}_{0,0}$. Hence a standard compactness argument shows that there exists a pair of textiles $(\alpha_{i,j})_{i,j\in\mathbb{Z}}$ and $(\beta_{i,j})_{i,j\in\mathbb{Z}}$ in U such that $\alpha_{i,j} = \beta_{i,j}$ for all $i \in \mathbb{Z}$ and $j \le -1$ but $\alpha_{0,0} \neq \beta_{0,0}$. From this, it follows that φ_U is not left σ_U-expansive.

Assume that $\xi_{(U^{[n]})^*}$ is one-to-one with $n \ge 1$. Let $(x_i)_{i\in\mathbb{Z}}$ and $(y_i)_{i\in\mathbb{Z}}$ be two φ_U-orbits with $\Delta((\tau^{-j}(x_i))_{i\in\mathbb{Z}}, (\tau^{-j}(y_i))_{i\in\mathbb{Z}}) < \delta$ for all $j \ge 0$, where $\delta > 0$ is such that for any $x, y \in X_U$ if $d_{X_U}(x, y) < \delta$ then $d_{Z_U}(\xi_U^{-1}(x), \xi_U^{-1}(y)) < 1$. Then there exist textiles $(\alpha_{i,j})_{i,j\in\mathbb{Z}}$ and $(\beta_{i,j})_{i,j\in\mathbb{Z}}$ in U such that $\xi_U((\alpha_{i,j})_{j\in\mathbb{Z}}) = x_i$ and $\xi_U((\beta_{i,j})_{j\in\mathbb{Z}}) = y_i$ and hence $\alpha_{i,j} = \beta_{i,j}$ for all $i \in \mathbb{Z}$ and $j \le 0$. Therefore, since $\xi_{(U^{[n]})^*}$ is one-to-one, $(\alpha_{i,j})_{i,j\in\mathbb{Z}} = (\beta_{i,j})_{i,j\in\mathbb{Z}}$, so that $(x_i)_{i\in\mathbb{Z}} = (y_i)_{i\in\mathbb{Z}}$. Thus φ_U is left σ_U-expansive.

We have proved (1). The proofs of (2) and (3) are similar to the above, and (4) is easily seen. □

COROLLARY 7.3. *Let φ be an endomorphism of a subshift (X, σ). Then φ is expansive if and only if φ is both left σ-expansive and right σ-expansive.*

A graph homomorphism $h : \Gamma \to G$ is said to be *weakly right resolving* if each $\alpha \in A_\Gamma$ is uniquely determined by $i_\Gamma(\alpha)$ and $h_A(\alpha)$. It is said to be *weakly left resolving* if each $\alpha \in A_\Gamma$ is uniquely determined by $t_\Gamma(\alpha)$ and $h_A(\alpha)$.

A textile system $T = (p, q : \Gamma \to G)$ is said to be *weakly p-L* if p is weakly left resolving. We similarly define *weakly p-R*, *weakly q-L* and *weakly q-R textile systems*. A textile system is said to be *weakly LR* if it is both weakly p-L and weakly q-R, and *weakly RL* if it is both weakly p-R and weakly q-L. Similarly, a both weakly p-L and weakly q-L (respectively, both weakly p-R and weakly q-R) textile system is said to be *weakly LL* (respectively, *weakly RR*).

Let φ be an onto endomorphism of a subshift (X, σ). We say that φ is *weakly p-L* (respectively, *weakly p-R*, *weakly q-L*, *weakly q-R*) if there exists a one-sided 1-1 textile subsystem U of a weakly p-L (respectively, weakly p-R, weakly q-L, weakly q-R) textile system such that $(X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi)$. We say that φ is *weakly one-sided resolving* if it is weakly p-L, weakly p-R, weakly q-L or weakly q-R. We say that φ is *weakly LR* (respectively, *weakly RL*, *weakly LL*, *weakly RR*) if there exists a one-sided 1-1 textile subsystem U of a weakly LR (respectively, weakly RL, weakly LL, weakly RR) textile system such that $(X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi)$.

From the proof of Remark 7.1, we can make the following remark.

Remark 7.4. For a subshift (X, σ), the identity map of X is a weakly LR, weakly RL, weakly LL and weakly RR automorphism of (X, σ), and σ is a weakly LR automorphism of (X, σ).

In the following propositions of this section, the second statements are the symmetric versions of the first and hence follow from these by symmetry.

PROPOSITION 7.5. *Let φ be an onto endomorphism of a subshift (X, σ).*

(1) *φ has memory (respectively, anticipation) 0 if and only if φ is weakly p-L (respectively, p-R).*

(2) [**21**] *If φ is one-to-one and has memory 0 and φ^{-1} has anticipation 0, then $\varphi^{[n]}$ is a weakly LR automorphism of $(X^{[n]}, \sigma^{[n]})$ for some $n \ge 1$.*

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Proof. (1) If φ is of $(0, l)$ type and has a local rule $f : L_{l+1}(X) \to L_1(X)$, then $\varphi = \varphi_{U_{f,0,l}}$, where $U_{f,0,l}$ is the textile subsystem of the textile system $T_{f,0,l}$ defined in the proof of Remark 7.1. Since $T_{f,0,l}$ is weakly p-L, φ is weakly p-L.

Assume that U is a one-sided 1-1 textile subsystem of a textile system $T = (p, q : \Gamma \to G)$ with p weakly left resolving. There exist $m, n \geq 1$ and a local rule $g : L_{m+n+1}(X_U) \to L_1(Z_U)$ such that $\xi_U^{-1}((a_j)_{\in\mathbb{Z}}) = (g(a_{j-m} \dots a_{j+n}))_{j\in\mathbb{Z}}$. Since ξ_U is a restriction of ϕ_p with p weakly left resolving, for any $\alpha_{-m} \dots \alpha_n \in L_{m+n+1}(Z_U)$, $\alpha_{-m} \dots \alpha_{-1}$ is uniquely determined by $p_A(\alpha_{-m}) \dots p_A(\alpha_{-1})$ and $t_\Gamma(\alpha_{-1}) = i_\Gamma(\alpha_0) = i_\Gamma(g(p_A(\alpha_{-m}) \dots p_A(\alpha_n)))$. Hence we can define a local rule $g' : L_{m+n+1}(X_U) \to L_1(Z_U)$ which maps $p_A(\alpha_{-m}) \dots p_A(\alpha_n)$ to α_{-m}. With g', ξ_U^{-1} is of $(0, m + n + 1)$ type. With the local rule $q_A g'$, φ_U is of $(0, m + n + 1)$ type.

(2) This is proved by [**21,** Proposition 6.4]. The following is a direct proof.

There exists $n \geq 1$ such that φ is of $(0, n)$ type and φ^{-1} is of $(n, 0)$ type. Let Z' be the set of all elements of the form $(a_j/b_j)_{j\in\mathbb{Z}}$, where $(a_j)_{j\in\mathbb{Z}} \in X$ and $\varphi((a_j)_{j\in\mathbb{Z}}) = (b_j)_{j\in\mathbb{Z}}$ with $a_j, b_j \in L_1(X)$. Then Z' is the space of a subshift. Let $T' = (p', q' : \Gamma' \to G')$ be the textile system defined as follows: $A_{G'} = L_{n+1}(X)$, $V_{G'} = L_n(X)$ and for each arc $a_1 \dots a_{n+1} \in A_{G'}$, $i_{G'}$ and $t_{G'}$ map it to $a_1 \dots a_n$ and $a_2 \dots a_{n+1}$, respectively; $A_{\Gamma'} = L_{n+1}(Z')$, $V_{\Gamma'} = L_n(Z')$, and for each arc $(a_1/b_1) \dots (a_{n+1}/b_{n+1}) \in A_{\Gamma'}$, $i_{\Gamma'}$, $t_{\Gamma'}$, p'_A and q'_A map it to $(a_1/b_1) \dots (a_n/b_n)$, $(a_2/b_2) \dots (a_{n+1}/b_{n+1})$, $a_1 \dots a_{n+1}$ and $b_1 \dots b_{n+1}$, respectively. Then it is easily checked that T' is 1-1 and weakly LR. Since $Z'^{[n+1]} \subset Z_{T'}$ and $\xi_{T'}(Z'^{[n+1]}) = X^{[n+1]} = \eta_{T'}(Z'^{[n+1]})$, we have a 1-1 textile subsystem U' of T' such that $Z_{U'} = Z'^{[n+1]}$ and hence $(X_{U'}, \sigma_{U'}, \varphi_{U'}) = (X^{[n+1]}, \sigma^{[n+1]}, \varphi^{[n+1]})$. □

PROPOSITION 7.6. *Let φ be an onto endomorphism of a subshift (X, σ).*

(1) *If φ is weakly p-L (respectively, p-R) , then $\varphi\sigma$ (respectively, $\varphi\sigma^{-1}$) is weakly p-L (respectively, p-R) and right (respectively, left) σ-expansive.*

(2) *If φ is weakly q-R (respectively, q-L), then $\varphi\sigma$ (respectively, $\varphi\sigma^{-1}$) is weakly q-R (respectively, q-L) and left (respectively, right) σ-expansive.*

(3) *If φ is weakly LR (respectively, RL), then $\varphi\sigma$ (respectively, $\varphi\sigma^{-1}$) is weakly LR (respectively, RL) and expansive.*

Proof. Assume that φ is weakly p-L. Then there exists a one-sided 1-1 textile subsystem U of a textile system $T = (p, q : \Gamma \to G)$ with p weakly left resolving such that $(X_U, \varphi_U, \sigma_U) = (X, \varphi, \sigma)$. Let $T' = (p', q' : \Gamma' \to G)$ be the textile system defined as follows: $\Gamma' = \Gamma^{[2]}$, and p' and q' are given by

$$p'_A(\alpha_1\alpha_2) = p_A(\alpha_1) \quad \text{and} \quad q'_A(\alpha_1\alpha_2) = q_A(\alpha_2), \quad \alpha_1\alpha_2 \in A_{\Gamma'} = L_2(X_\Gamma).$$

Since p is weakly left resolving, it easily follows that p' is weakly left resolving. Since $Z_U^{[2]} \subset Z_{T'}$, $\xi_{T'}(Z_U^{[2]}) = X_U = \eta_{T'}(Z_U^{[2]})$ and $\xi_{T'}|Z_U^{[2]}$ is one-to-one, we have a one-sided 1-1 textile subsystem U' of T' such that $Z_{U'} = Z_U^{[2]}$. Clearly $\varphi_{U'} = \varphi_U\sigma_U = \varphi\sigma$. Hence $\varphi\sigma$ is weakly p-L.

To see that $\varphi\sigma$ is right σ-expansive, it suffices to show that $\eta_{T'^*}$ is one-to-one, by Proposition 7.2. Let $T'^* = (p'^*, q'^* : \Gamma'^* \to G'^*)$. Any point in $X_{G'^*}$ is of the form

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$(\alpha_i)_{i\in\mathbb{Z}}$ with $\alpha_i \in A_\Gamma$ and $q_A(\alpha_{i-1})p_A(\alpha_i) \in L_2(X_G)$ for all $i \in \mathbb{Z}$. Since p is weakly left resolving, an arc $\alpha'_i \in A_\Gamma$ with $p_A(\alpha'_i) = q_A(\alpha_{i-1})$ and $t_\Gamma(\alpha'_i) = i_\Gamma(\alpha_i)$ is unique, if any, for all $i \in \mathbb{Z}$, and hence $(\alpha'_i\alpha_i)_{i\in\mathbb{Z}}$ is a unique point in $X_{\Gamma'^*}$ whose image under $\phi_{q'^*}$ is $(\alpha_i)_{i\in\mathbb{Z}}$, if such a point exists. Therefore $\eta_{T'^*}$ is one-to-one and (1) is proved.

The proofs of (2) and (3) are similar to the above. □

PROPOSITION 7.7. *Let φ be an onto endomorphism of a subshift (X, σ).*

(1) *If φ is weakly p-L (respectively, p-R) and right (respectively, left) σ-expansive, then for some $k \geq 1$, $\varphi^k\sigma^{-1}$ (respectively, $\varphi^k\sigma$) is essentially weakly p-L (respectively, p-R).*

(2) *If φ is weakly q-R (respectively, q-L) and left (respectively, right) σ-expansive, then for some $k \geq 1$, $\varphi^k\sigma^{-1}$ (respectively, $\varphi^k\sigma$) is essentially weakly q-R (respectively, q-L).*

(3) *If φ is weakly LR (respectively, RL) and expansive, then for some $k \geq 1$, $\varphi^k\sigma^{-1}$ (respectively, $\varphi^k\sigma$) is essentially weakly LR (respectively, RL).*

Proof. (1) Assume that φ is weakly p-L and right σ-expansive. Then there exists a one-sided 1-1 textile subsystem U of a weakly p-L textile system T such that $\eta_{(U^{[n]})^*}$ is one-to-one for some $n \geq 1$ (by Proposition 7.2) and $(X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi)$. Replacing $U^{[n]}$ and $T^{[n]}$ by U and T, we know that there exists a one-sided 1-1 textile subsystem U of a textile system $T = (p, q : \Gamma \to G)$ with p weakly left resolving such that η_{U^*} is one-to-one and $(X_U, \sigma_U, \varphi_U)$ is conjugate to (X, σ, φ) (because if a textile system is weakly p-L, then its higher block system of any order is also weakly p-L). Let $T^* = (p^*, q^* : \Gamma^* \to G^*)$. Then q^* is weakly right resolving and the one-to-one block map η_{U^*} is the restriction of ϕ_{q^*} on X_{U^*}. Therefore it follows that $\eta_{U^*}^{-1}$ is a block map of $(k-1, 0)$ type for some $k \geq 1$ (see the proof of Proposition 7.5). Hence for any $(\alpha_i)_{1\leq i\leq k} \in L_k(Z_{U^*})$, $(q_A^*(\alpha_i))_{1\leq i\leq k}$ uniquely determines α_k and hence $q_A(\alpha_k)$, so that we can define a mapping $F : L_k(X_{U^*}) \to L_1(X_U)$ by

$$F((q_A^*(\alpha_i))_{1\leq i\leq k}) = q_A(\alpha_k) \quad \text{with } (\alpha_i)_{1\leq i\leq k} \in L_k(Z_{U^*}).$$

Let $T' = (p', q' : \Gamma' \to G)$ be the textile system such that the arc-set $A_{\Gamma'}$ is equal to $L_k(Z_{U^*})$ and $i_{\Gamma'}$, $t_{\Gamma'}$, p' and q' are given as follows: for $\alpha' = (\alpha_i)_{1\leq i\leq k} \in A_{\Gamma'}$,

$$i_{\Gamma'}(\alpha') = (p_A^*(\alpha_i))_{1\leq i\leq k}, \quad t_{\Gamma'}(\alpha') = (q_A^*(\alpha_i))_{1\leq i\leq k},$$
$$p'_A(\alpha') = p_A(\alpha_1), \quad q'_A(\alpha') = F((p_A^*(\alpha_i))_{1\leq i\leq k}).$$

Since each $\alpha' \in A_{\Gamma'}$ is uniquely determined by $p'_A(\alpha')$ and $t_{\Gamma'}(\alpha')$, p' is weakly left resolving. Let $Z' = Z_{((U^*)^{[k]})^*}$. Then Z' is a subshift space with $Z' \subset Z_{T'}$, $\xi_{T'}(Z') = X_U = \eta_{T'}(Z')$ and $\xi_{T'}|Z'$ is one-to-one. Hence we have a one-sided 1-1 textile subsystem U' of T' such that $Z_{U'} = Z'$. Clearly $\varphi_{U'} = \varphi_U^k\sigma_U^{-1}$. Thus $\varphi_U^k\sigma_U^{-1}$ is weakly p-L, so that $\varphi^k\sigma^{-1}$ is essentially weakly p-L.

(2) The proof is given by straightforward modifications in the arguments above. Instead of F, we have for some $k \geq 1$, $F' : L_k(X_{U^*}) \to L_1(X_U)$ such that

$$F'((p_A^*(\alpha_i))_{1\leq i\leq k}) = p_A(\alpha_1) \quad \text{with } (\alpha_i)_{1\leq i\leq k} \in L_k(Z_{U^*}).$$

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We change p' and q' above to those defined by $p'_A(\alpha') = F'((q^*_A(\alpha_i))_{1\le i\le k})$ and $q'_A(\alpha') = q_A(\alpha_k)$ for $\alpha' = (\alpha_i)_{1\le i\le k}$.

(3) The proof is also given by straightforward modifications in the arguments above. We have both F and F' in the above for some $k \ge 1$. We change T' above to that defined by $T' = (p', q' : \Gamma' \to G^{[2]})$ such that $p'_A(\alpha') = p_A(\alpha_1)F'((q^*_A(\alpha_i))_{1\le i\le k})$ and $q'_A(\alpha') = F((p^*_A(\alpha_i))_{1\le i\le k})q_A(\alpha_k)$ for $\alpha' = (\alpha_i)_{1\le i\le k}$. Then $\varphi_{U'} = (\varphi^k_U \sigma_U^{-1})^{[2]}$. □

Here we define a *textile relation system* to be an ordered pair of graph homomorphisms $p : \Gamma \to G_1$ and $q : \Gamma \to G_2$. (NB The additional condition of the one–one correspondence between $\alpha \in A_\Gamma$ and the quadruple $(i_\Gamma(\alpha), t_\Gamma(\alpha), p_A(\alpha), q_A(\alpha))$ imposed on the original definition in [**21**] is removed for reasons similar to those stated for the definition of a textile system in §3.) A textile relation system $T = (p : \Gamma \to G_1, q : \Gamma \to G_2)$ defines the relation $\{\xi(\alpha)/\eta(\alpha) \mid \alpha \in X_\Gamma\}$, where $\xi = \phi_p$ and $\eta = \phi_q$. Let $X_{1,T} = \xi(X_\Gamma)$ and let $X_{2,T} = \eta(X_\Gamma)$. Let $\xi_T : X_\Gamma \to X_{1,T}$ and $\eta_T : X_\Gamma \to X_{2,T}$ be the onto maps naturally induced by ξ and η, respectively. If T is *one-sided 1-1*, i.e. ξ_T is one-to-one, then we define a factor map (i.e. an onto homomorphism) ϕ_T of $(X_{1,T}, \sigma_{1,T})$ onto $(X_{2.T}, \sigma_{2,T})$ by $\phi_T = \eta_T \xi_T^{-1}$. We say that T is *1-1* if both ξ_T and η_T are one-to-one, and T is *non-degenerate* if both ξ and η are onto.

Note that a textile system is considered to be a textile relation system. The definitions of *p-L*, *p-R*, *q-L*, *q-R*, *LR*, *RL*, *LL* and *RR textile relation systems* together with their '*weakly*' versions should be clear.

For two textile relation systems $T = (p : \Gamma \to G_1, q : \Gamma \to G_2)$ and $T' = (p' : \Gamma' \to G'_1, q' : \Gamma' \to G'_2)$, we define their *composition* $T \circ T'$ to be the textile relation system $(\bar p : \bar\Gamma \to G_1, \bar q : \bar\Gamma \to G'_2)$, where the arcs of $\bar\Gamma$ are the elements of the form α/α' with $\alpha \in A_\Gamma$, $\alpha' \in A_{\Gamma'}$ and $q_A(\alpha) = p'_A(\alpha')$, and for each $\bar\alpha = \alpha/\alpha' \in A_{\bar\Gamma}$, $i_{\bar\Gamma}(\bar\alpha) = i_\Gamma(\alpha)/i_{\Gamma'}(\alpha')$, $t_{\bar\Gamma}(\bar\alpha) = t_\Gamma(\alpha)/t_{\Gamma'}(\alpha')$, $\bar p_A(\bar\alpha) = p_A(\alpha)$ and $\bar q_A(\bar\alpha) = q'_A(\alpha')$. If T and T' are weakly p-L textile relation systems, then $T \circ T'$ is weakly p-L. In this statement, 'p-L' can be replaced by each of 'q-R', 'p-R', 'q-L', 'LR' and 'RL'.

For a textile relation system $T = (p : \Gamma \to G_1, q : \Gamma \to G_2)$, a *textile relation subsystem* of T is defined to be a subshift space $Z \subset X_\Gamma$ with $\xi_Z : Z \to X_{1,Z}$ and $\eta_Z : Z \to X_{2,Z}$ which are the restrictions of ξ_T and η_T, respectively, where $X_{1,Z} = \xi_T(Z)$ and $X_{2,Z} = \eta_T(Z)$. If Z is *one-sided 1-1*, i.e. ξ_Z is one-to-one, then we define a factor map $\phi_Z = \eta_Z \xi_Z^{-1}$ of $(X_{1,Z}, \sigma_{1,Z})$ onto $(X_{2,Z}, \sigma_{2,Z})$. If Z is *1-1*, i.e. both ξ_Z and η_Z are one-to-one, then ϕ_Z is a conjugacy. For any factor map ϕ between subshifts, there exists a one-sided 1-1 textile relation subsystem Z such that $\phi_Z = \phi$. (This is clear by the proof of Remark 7.1).

A factor map ϕ between subshifts is said to be *weakly p-L* if there exists a one-sided 1-1 textile relation subsystem Z of a weakly p-L textile relation system with $\phi_Z = \phi$. Similar notions are also given by replacing 'p-L' in this definition by each of 'q-R', 'p-R', 'q-L', 'LR', 'RL', 'LL' and 'RR'.

If Z is a textile relation subsystem of a textile relation system T and Z' is another of another T' such that $X_{2,Z} = X_{1,Z'}$, we naturally define their *composition* $Z \circ Z'$ which is a textile relation subsystem of $T \circ T'$. If Z and Z' are one-sided 1-1, then so is $Z \circ Z'$ and we have $\phi_{Z\circ Z'} = \phi_{Z'}\phi_Z$.

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Remark 7.8. If $\phi_1 : (X, \sigma) \to (X', \sigma')$ and $\phi_2 : (X', \sigma') \to (X'', \sigma'')$ are weakly p-L factor maps between subshifts, then $\phi_2\phi_1$ is weakly p-L. In this statement, 'p-L' can be replaced by each of 'q-R', 'p-R', 'q-L', 'LR' and 'RL'.

A textile subsystem U of a textile system $T = (p, q : \Gamma \to G)$ is considered as the textile relation subsystem Z_U of the textile relation system $T = (p : \Gamma \to G, q : \Gamma \to G)$. If Z is a one-sided 1-1 textile relation subsystem of a textile relation system $T = (p : \Gamma \to G, q : \Gamma \to G)$ with $X_{1,Z} = X_{2,Z}$, then there exists a unique textile subsystem U of the textile system $T = (p, q : \Gamma \to G)$ with $Z_U = Z$. For two one-sided 1-1 textile subsystems U and U' with $X_U = X_{U'}$, we define $U \circ U'$ to be the one-sided 1-1 textile subsystem $\bar{U}$ such that $Z_{\bar{U}} = Z_U \circ Z_{U'}$. Then we have $\varphi_{U \circ U'} = \varphi_{U'}\varphi_U$.

Remark 7.9. If φ_1 and φ_2 are weakly p-L (onto) endomorphisms of a subshift (X, σ), then $\varphi_1\varphi_2$ is a weakly p-L (onto) endomorphism of a subshift (X, σ). If φ is a weakly p-L endomorphism of a subshift (X, σ), then $((\varphi)^{(s)})^r$ is a weakly p-L endomorphism of a subshift $(X^{(s)}, \sigma^{(s)})$, and hence φ^r is an essentially weakly p-L endomorphism of (X, σ^s), for all $r, s \geq 1$. In these statements, 'p-L' can be replaced by each of 'q-R', 'p-R', 'q-L', 'LR' and 'RL'.

PROPOSITION 7.10. *A factor map $\phi : (X, \sigma) \to (X', \sigma')$ between subshifts is right closing if and only if $\phi\sigma^n$ is weakly LR for some (all sufficiently large) $n \geq 0$.*

Proof. Since a weakly q-R factor map between subshifts is right closing, the 'if' part is clear. The 'only if' part is proved by a slight modification in [**22,** the proof of Proposition 4.10] (and Remarks 7.4 and 7.8). (In [**22,** the proof of Proposition 4.10], apply the argument in [**20,** the proof of Proposition 6.30] to the 'λ-graph' $\mathcal{G} = (G, \lambda)$ with $\lambda = \Phi''$ instead of $\lambda = \Phi'$, where Φ'' is such that if $\Phi'(\alpha) = a_1 \dots a_{s-k}$ with $a_j \in L_1(Y)$ for $\alpha \in A_G$, then $\Phi''(\alpha) = a_1$.) □

PROPOSITION 7.11. *Let φ be an onto endomorphism of a subshift (X, σ).*

(1) *If φ is essentially weakly p-L (respectively, p-R) and right (respectively, left) σ-expansive, then there exists $m \geq 1$ such that φ^n is weakly p-L (respectively, p-R) for all $n \geq m$.*
(2) *If φ is essentially weakly q-R (respectively, q-L) and left (respectively, right) σ-expansive, then there exists $m \geq 1$ such that φ^n is weakly q-R (respectively, q-L) for all $n \geq m$.*
(3) *If φ is essentially weakly LR (respectively, RL) and expansive, then there exists $m \geq 1$ such that φ^n is weakly LR (respectively, RL) for all $n \geq m$.*

Proof. The proof is similar to [**20,** the proof of Proposition 8.8].

To prove (1), assume that φ is essentially weakly p-L and is right σ-expansive. There exist a subshift (X_1, σ_1) and a topological conjugacy $\theta : (X, \sigma) \to (X_1, \sigma_1)$ such that $\varphi_1 = \theta\varphi\theta^{-1}$ is a weakly p-L endomorphism of (X_1, σ_1). Since θ and θ^{-1} are right closing, it may be assumed by Proposition 7.10 that θ is weakly p-L and $\theta^{-1}\sigma_1^l$ is weakly p-L for some $l \geq 0$.

Since φ_1 is weakly p-L and right σ_1-expansive, it follows from Proposition 7.7 that there exists $k \geq 1$ such that $\varphi_1^k\sigma_1^{-1}$ is an essentially weakly p-L endomorphism of (X_1, σ_1).

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Hence there exist a subshift (X_2, σ_2) and a topological conjugacy $\theta_1 : (X_1, \sigma_1) \to (X_2, \sigma_2)$ such that $\varphi_2 = \theta_1 \varphi_1^k \sigma_1^{-1} \theta_1^{-1}$ is a weakly p-L endomorphism of (X_2, σ_2). For a reason similar to the above, we may assume that θ_1 is weakly p-L and $\theta_1^{-1}\sigma_2^{l_1}$ is weakly p-L for some $l_1 \geq 0$.

Let $\psi = \theta_1^{-1}\sigma_2^{l_1}\varphi_2^{l_1+1}\theta_1$. It then follows from Remark 7.8 that ψ is weakly p-L, because so are $\theta_1^{-1}\sigma_2^{l_1}$, φ_2 and θ_1. Since $\varphi_1^k = \theta_1^{-1}\sigma_2\varphi_2\theta_1$, we have $\varphi_1^{k(l_1+1)} = \sigma_1\psi$. Let $m = k(l_1+1)l$ and let i be any non-negative integer. Then, since $\varphi^{m+i} = \theta^{-1}\varphi_1^{k(l_1+1)l}\varphi_1^i\theta = \theta^{-1}\sigma_1^l\psi^l\varphi_1^i\theta$, it follows from Remark 7.8 that φ^{m+i} is weakly p-L, because so are $\theta^{-1}\sigma_1^l$, ψ, φ_1 and θ. Hence (1) is proved.

The proofs of (2) and (3) are similar to the above. □

Let φ be an endomorphism of a dynamical system (X, τ). Let P be a property of endomorphisms of dynamical systems. We say that φ is *directionally* P if there exist $r, s \geq 1$ such that the endomorphism φ^r of (X, τ^s) is P. We remark that φ is directionally essentially P if and only if φ is essentially directionally P. Hence the property of being directionally essentially P is an invariant of topological conjugacy between endomorphisms of dynamical systems.

PROPOSITION 7.12. *Let φ_1 and φ_2 be commuting onto endomorphisms of a subshift (X, σ).*

(1) *If φ_1 and φ_2 are directionally essentially weakly p-L (respectively, p-R) with φ_2 right (respectively, left) σ-expansive, then $\varphi_1\varphi_2$ is directionally essentially weakly p-L (respectively, p-R) and right (respectively, left) σ-expansive.*

(2) *If φ_1 and φ_2 are directionally essentially weakly q-R (respectively, q-L) with φ_2 left (respectively, right) σ-expansive, then $\varphi_1\varphi_2$ is directionally essentially weakly q-R (respectively, q-L) and left (respectively, right) σ-expansive.*

(3) *If φ_1 and φ_2 are directionally essentially weakly LR (respectively, RL) with φ_2 expansive, then $\varphi_1\varphi_2$ is directionally essentially weakly LR (respectively, RL) and expansive.*

Proof. (1) For $i = 1, 2$, there exist $r_i, s_i \geq 1$ such that $\varphi_i^{r_i}$ is an essentially weakly p-L endomorphism of (X, σ^{s_i}). Let $r = r_1 r_2 s_1 s_2$. Then using Remark 7.9 we see that φ_i^r is an essentially weakly p-L endomorphism of (X, σ^r) for $i = 1, 2$. Hence there exist a subshift (X', σ') and a topological conjugacy $\theta : (X, \sigma^r) \to (X', \sigma')$ such that $\varphi_1' = \theta\varphi_1^r\theta^{-1}$ is a weakly p-L endomorphism of (X', σ'). Let $\varphi_2' = \theta\varphi_2^r\theta^{-1}$. Then φ_2' is an essentially weakly p-L endomorphism of (X', σ') and is right σ'-expansive (by Proposition 6.1). Hence it follows from Proposition 7.7 that there exists $m \geq 1$ such that $(\varphi_2')^m\sigma'^{-1}$ is an essentially weakly p-L endomorphism of (X', σ'). Therefore it follows from Remark 7.9 and Proposition 7.6 that $((\varphi_2')^m\sigma'^{-1})^2\sigma' = (\varphi_2')^{2m}\sigma'^{-1}$ is a right σ'-expansive, essentially weakly p-L endomorphism of (X', σ'). By Proposition 7.11, there exists $n \geq 1$ such that $((\varphi_2')^{2m}\sigma'^{-1})^n$ is a weakly p-L endomorphism of (X', σ'). Set $(\varphi_2')^{2mn}\sigma'^{-n} = \psi$. Then since $\varphi_1'\varphi_2' = \varphi_2'\varphi_1'$, we have $(\varphi_1'\varphi_2')^{2mn} = (\varphi_1')^{2mn}\psi\sigma'^n$. Therefore, since φ_1' and ψ are weakly p-L, $(\varphi_1'\varphi_2')^{2mn}$ is a right σ'-expansive, weakly p-L endomorphism of (X', σ') (by Proposition 7.6 and Remark 7.9). Hence $\theta(\varphi_1\varphi_2)^{2mnr}\theta^{-1}$ is a right $\theta\sigma^r\theta^{-1}$-expansive,

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weakly p-L endomorphism of $(X', \theta\sigma^r\theta^{-1})$. Thus $(\varphi_1\varphi_2)^{2mnr}$ is a right σ^r-expansive, essentially weakly p-L endomorphism of (X, σ^r), so that (1) is proved.

The proofs of (2) and (3) are similar to the above. □

The following is merely the 'directionally' version of Proposition 7.7.

PROPOSITION 7.13. *Let φ be an onto endomorphism of a subshift (X, σ).*

(1) *If φ is directionally essentially weakly p-L (respectively, p-R) and right (respectively, left) σ-expansive, then for some $k \geq 1$, $\varphi^k\sigma^{-1}$ (respectively, $\varphi^k\sigma$) is directionally essentially weakly p-L (respectively, p-R).*

(2) *If φ is directionally essentially weakly q-R (respectively, q-L) and left (respectively, right) σ-expansive, then for some $k \geq 1$, $\varphi^k\sigma^{-1}$ (respectively, $\varphi^k\sigma$) is directionally essentially weakly q-R (respectively, q-L).*

(3) *If φ is directionally essentially weakly LR (respectively, RL) and expansive, then for some $k \geq 1$, $\varphi^k\sigma^{-1}$ (respectively, $\varphi^k\sigma$) is directionally essentially weakly LR (respectively, RL).*

Proof. (1) Using Remark 7.9 we see that there exists $r \geq 1$ such that φ^r is an essentially weakly p-L endomorphism of (X, σ^r). Since φ^r is right σ^r-expansive, it follows from Proposition 7.7 that there exists $k \geq 1$ such that $\varphi^{kr}\sigma^{-r}$ is an essentially weakly p-L endomorphism of (X, σ^r), so that $\varphi^k\sigma^{-1}$ is a directionally essentially weakly p-L endomorphism of (X, σ).

The proofs of (2) and (3) are similar to the above. □

8. *One-sided resolving districts*

Let X be a zero-dimensional compact metric space, let $S(X)$ denote the monoid of surjective continuous maps of X and let $H(X)$ denote the group of homeomorphisms in $S(X)$. Let τ be in $H(X)$ and expansive. Let J be a commutative submonoid of $S(X)$.

Let $PL_J(\tau)$ denote the set of all directionally essentially weakly p-L endomorphisms of (X, τ) in J. Replacing 'p-L' by 'q-R' 'p-R' and 'q-L' in this definition, we respectively define $QR_J(\tau)$, $PR_J(\tau)$ and $QL_J(\tau)$. Clearly $PR_J(\tau) = PL_J(\tau^{-1})$ and $QL_J(\tau) = QR_J(\tau^{-1})$. Let $PL_J^\circ(\tau)$, $QR_J^\circ(\tau)$, $PR_J^\circ(\tau)$ and $QL_J^\circ(\tau)$ be the set of all right τ-expansive elements of $PL_J(\tau)$, the set of all left τ-expansive elements of $QR_J(\tau)$, the set of all left τ-expansive elements of $PR_J(\tau)$ and the set of all right τ-expansive elements of $QL_J(\tau)$, respectively. If $\tau \in J$, then we can imagine by Remark 7.4 and Propositions 7.12 and 7.13 that $PL_J(\tau)$ is a kind of 'cone' with $\{\tau^n \mid n = 0, 1, \ldots\}$ as an 'inner half-line of points' and with the 'interior' $PL_J^\circ(\tau)$, and can do similarly for $QR_J(\tau)$.

Let $C_J(\tau)$ denote the set of all directionally essentially weakly LR endomorphisms of (X, τ). Let $C_J^\circ(\tau)$ denote the set of all expansive elements of $C_J(\tau)$. If $\tau \in J$, then by Remark 7.4 and Propositions 7.12 and 7.13 we can also imagine that $C_J(\tau)$ is a kind of 'cone' with $\{\tau^n \mid n = 0, 1, \ldots\}$ as an 'inner half-line of points' and with the 'interior' $C_J^\circ(\tau)$.

By Proposition 7.10 we know that if $\tau, \varphi \in J$, then $\varphi^k\tau^n \in C_J(\tau)$ for some $k \geq 1$ and $n \in \mathbb{Z}$ if and only if φ is a right-closing endomorphism of (X, τ). We also know that if φ is an onto endomorphism of (X, τ) with $\varphi \notin \{\tau^n \mid n \in \mathbb{Z}\}$ and $J = \{\varphi^k\tau^n \mid k \geq 0, n \in \mathbb{Z}\}$, then $C_J(\tau) = \{\tau^n \mid n = 0, 1, \ldots\}$ if and only if φ is not a right-closing endomorphism of (X, τ).

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Later we shall see (by Theorem 8.8(4)) that for τ with POTP, if $J \subset H(X)$ or τ is topologically transitive, then $C_J(\tau) = PL_J(\tau) \cap QR_J(\tau)$. We do not know whether this generally holds. The inclusion relation between $PL_J(\tau)$ and $QR_J(\tau)$ can vary (see Example 5.6).

PROPOSITION 8.1. *Let (X, σ) be a subshift and J a commutative submonoid of $S(X)$. Then $C_J^\circ(\sigma) = PL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$.*

Proof. It suffices to show that $C_J^\circ(\sigma) \supset PL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$. Let $\varphi \in PL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$. Using Remark 7.9, we see that there exist $r, s \geq 1$ such that φ^r is an essentially weakly p-L, essentially weakly q-R endomorphism of (X, σ^s). Hence there exist a weakly q-R, essentially weakly p-L endomorphism φ_0 of a subshift (X_0, σ_0) with $(X_0, \sigma_0, \varphi_0)$ conjugate to (X, σ^s, φ^r). Since φ is right σ-expansive, φ_0 is right σ_0-expansive (by Proposition 6.1). It therefore follows from Proposition 7.11 and Remark 7.9 that there exists $r' \geq 1$ such that $\varphi_0^{r'}$ is a weakly p-L, weakly q-R endomorphism of (X_0, σ_0). Replacing $\varphi_0^{r'}$ by φ_0 and rr' by r, we have $r, s \geq 1$ and a weakly p-L, weakly q-R endomorphism φ_0 of (X_0, σ_0) such that $(X_0, \sigma_0, \varphi_0)$ is conjugate to (X, σ^s, φ^r). (Hence the proposition is proved by Proposition 7.5 for the case that $J \subset H(X)$.)

There exist a one-sided 1-1 textile subsystem U of a weakly q-R textile system T and a one-sided 1-1 textile subsystem U' of a weakly p-L textile system T' such that $(X_U, \sigma_U, \varphi_U) = (X_{U'}, \sigma_{U'}, \varphi_{U'}) = (X_0, \sigma_0, \varphi_0)$. Since φ_0 is both right σ_0-expansive and left σ_0-expansive, it is expansive by Corollary 7.3. Hence, passing through higher block systems of a sufficiently large order of U and U', we may assume that both U^* and U'^* are 1-1 (by Proposition 7.2). Therefore, since U and U' are one-sided 1-1, we have the one-to-one correspondence of each point $y \in X_{U^*}$ to the φ_0-orbit $(\xi_U((\alpha_{i,j})_{j\in\mathbb{Z}}))_{i\in\mathbb{Z}}$ appearing in the textile $(\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U$ with $\xi_{U^*}((\alpha_{i,0})_{i\in\mathbb{Z}}) = y$, and the one-to-one correspondence of each point $y' \in X_{U'^*}$ to the φ_0-orbit $(\xi_{U'}((\alpha'_{i,j})_{j\in\mathbb{Z}}))_{i\in\mathbb{Z}}$ appearing in the textile $(\alpha'_{i,j})_{i,j\in\mathbb{Z}} \in U'$ with $\xi_{U'^*}((\alpha'_{i,0})_{i\in\mathbb{Z}}) = y'$. If we define $\theta : X_{U^*} \to X_{U'^*}$ so that y and $\theta(y)$ may correspond to the same φ_0-orbit, then θ is a conjugacy of $(X_{U^*}, \sigma_{U^*}, \varphi_{U^*})$ onto $(X_{U'^*}, \sigma_{U'^*}, \varphi_{U'^*})$. Therefore, since U^* is a textile subsystem of a weakly p-L textile system T^* and U'^* is a textile subsystem of a weakly q-R textile system T'^*, φ_{U^*} is a weakly p-L, essentially weakly q-R automorphism of (X_{U^*}, σ_{U^*}). Since φ_{U^*} is left σ_{U^*}-expansive (because U is one-sided 1-1), it follows from Remark 7.9 and Proposition 7.11 that there exists $m \geq 1$ such that $\varphi_{U^*}^m$ is weakly p-L and weakly q-R. By Proposition 7.5, there exists $n \geq 1$ such that $(\varphi_{U^*}^m)^{[n]} = \varphi_{(U^*)^{[n]}}^m$ is a weakly LR automorphism of $(X_{(U^*)^{[n]}}, \sigma_{(U^*)^{[n]}})$. Hence there exists a 1-1 textile subsystem U_1^* of a weakly LR textile system T_1^* such that $(X_{U_1^*}, \varphi_{U_1^*}, \sigma_{U_1^*}) = (X_{(U^*)^{[n]}}, \varphi_{(U^*)^{[n]}}^m, \sigma_{(U^*)^{[n]}})$. Let $U_0 = (((U^*)^{[n]})^m)^*$. Then U_0 is a one-sided 1-1 textile subsystem of the weakly q-R textile system $T_0 = (((T^*)^{[n]})^m)^*$ with U_0^* 1-1, $(X_{U_0}, \sigma_{U_0}, \varphi_{U_0})$ conjugate to $(X_U, \sigma_U^m, \varphi_U)$ (cf. [**20,** Lemma 2.8]) and $(X_{U_0^*}, \varphi_{U_0^*}, \sigma_{U_0^*}) = (X_{U_1^*}, \varphi_{U_1^*}, \sigma_{U_1^*})$. Since U_0 is one-sided 1-1, $\varphi_{U_1^*} = \varphi_{U_0^*}$ is left $\sigma_{U_1^*}$-expansive. Hence there exists $n' \geq 1$ such that $((U_1^*)^{[n']})^*$ is one-sided 1-1, by Proposition 7.2. Since $\varphi_{U_0^*} = \varphi_{U_1^*}$ is weakly p-L and hence has memory zero, we can define the induced endomorphism $\tilde{\varphi}_{U_0^*} = \tilde{\varphi}_{U_1^*}$ of the induced one-sided subshift $(\tilde{X}_{U_0^*}, \tilde{\sigma}_{U_0^*}) = (\tilde{X}_{U_1^*}, \tilde{\sigma}_{U_1^*})$. Each point of X_{U_0} corresponds in a one-to-one

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way to an orbit of $\tilde{\varphi}_{U_0^*} = \tilde{\varphi}_{U_1^*}$. This also corresponds in a one-to-one way to a point of $X_{((U_1^*)^{[n']})^*}$ and hence the composition of the correspondences give the conjugacy of $(X_{U_0}, \sigma_{U_0}, \varphi_{U_0})$ onto $(X_{((U_1^*)^{[n']})^*}, \sigma_{((U_1^*)^{[n']})^*}, \varphi_{((U_1^*)^{[n']})^*})$. Since $((U_1^*)^{[n']})^*$ is a one-sided 1-1 textile subsystem of the LR textile system $((T_1^*)^{[n']})^*$, φ_{U_0} is an essentially weakly LR endomorphism of (X_{U_0}, σ_{U_0}), so that φ_U is an essentially weakly LR endomorphism of (X_U, σ_U^m). Thus φ is a directionally essentially weakly LR endomorphism of (X, σ). □

PROPOSITION 8.2. *Let (X, σ) be a subshift with infinitely many points. Let J be a commutative submonoid of $S(X)$. Then:*

(1) $PL_J^\circ(\sigma) \cap PR_J(\sigma) = PL_J(\sigma) \cap PR_J^\circ(\sigma) = \emptyset$;
(2) $PL_J^\circ(\sigma) \cap QL_J(\sigma) = PL_J(\sigma) \cap QL_J^\circ(\sigma) = \emptyset$;
(3) $PR_J^\circ(\sigma) \cap QR_J(\sigma) = PR_J(\sigma) \cap QR_J^\circ(\sigma) = \emptyset$; *and*
(4) *if $J \subset H(X)$, then* $QL_J^\circ(\sigma) \cap QR_J(\sigma) = QL_J(\sigma) \cap QR_J^\circ(\sigma) = \emptyset$.

Proof. (1) It suffices to prove that $PL_J^\circ(\sigma) \cap PR_J(\sigma) = \emptyset$, by symmetry.

Assume that $\varphi \in PL_J^\circ(\sigma) \cap PR_J(\sigma)$. Then by the discussion of the first paragraph of the proof of Proposition 8.1 (replace 'q-R' by 'p-R'), we see that there exist $r, s \geq 1$ and a weakly p-L, weakly p-R endomorphism φ_0 of (X_0, σ_0) with $(X_0, \sigma_0, \varphi_0)$ conjugate to (X, σ^s, φ^r). By Proposition 7.5, φ_0 is of $(0, k)$ type and of $(l, 0)$ type for some $k, l \geq 0$. This implies that the endomorphism $\varphi_0^{[k+l]}$ of $(X_0^{[k+l]}, \sigma_0^{[k+l]})$ is of $(0, 0)$ type, which cannot be the case because $\varphi_0^{[k+l]}$ is right $\sigma_0^{[k+l]}$-expansive and $X_0^{[k+l]}$ has infinitely many points.

(2) Assume the contrary and let $\varphi \in PL_J^\circ(\sigma) \cap QL_J^\circ(\sigma)$, noting that $PL_J^\circ(\sigma) \cap QL_J(\sigma) = PL_J^\circ(\sigma) \cap QL_J^\circ(\sigma) = PL_J(\sigma) \cap QL_J^\circ(\sigma)$. Then by the discussion of the first paragraph of the proof of Proposition 8.1 (replace 'q-R' by 'q-L'), we see that there exist $r, s \geq 1$ and a weakly p-L, weakly q-L endomorphism φ_0 of (X_0, σ_0) with $(X_0, \sigma_0, \varphi_0)$ conjugate to (X, σ^s, φ^r). There exists a one-sided 1-1 textile subsystem U of a textile system $T = (p, q : \Gamma \to G)$ such that φ_U is weakly p-L, q is weakly left resolving and $(X_U, \sigma_U, \varphi_U) = (X_0, \sigma_0, \varphi_0)$.

Suppose ξ_U^{-1} is of (k, l) type with $k, l \geq 1$. Since φ_U is weakly p-L, φ_U is of $(0, l')$ type for some $l' \geq l$, by Proposition 7.5. For any $\alpha_1 \dots \alpha_{k+l'+1} \in L_{k+l'+1}(Z_U)$ with $\alpha_j \in A_\Gamma$, $p_A(\alpha_1) \dots p_A(\alpha_{k+l'+1})$ uniquely determines α_{k+1} because ξ_U^{-1} is of (k, l) type. It also uniquely determines $q_A(\alpha_1) \dots q_A(\alpha_k)$ because φ_U is of $(0, l')$ type, and α_{k+1} and $q_A(\alpha_1) \dots q_A(\alpha_k)$ determine $\alpha_1 \dots \alpha_k$ because q is weakly left resolving. Therefore ξ_U^{-1} is of $(0, k + l')$ type.

Since φ is right σ-expansive, φ_U is right σ_U-expansive. Hence there exists $m \geq 2$ such that $\eta_{(U^{[m]})^*}$ is one-to-one, by Proposition 7.2. Since T is weakly q-L, so is $(T^{[m]})^*$. Since $(U^{[m]})^*$ is a textile subsystem of $(T^{[m]})^*$, it follows that $\eta_{(U^{[m]})^*}^{-1}$ is $(0, n)$ type for some $n \geq 0$ by the proof of Proposition 7.5(1).

Let $N = m - 1 + (k + l')(n + 1)$. Let $(\alpha_{i,j})_{i,j \in \mathbb{Z}}$ be any textile in U with $\alpha_{i,j} \in A_\Gamma$. Since ξ_U^{-1} is of $(0, k + l')$ type, $p_A(\alpha_{0,1}) \dots p_A(\alpha_{0,N})$ uniquely determines $(\alpha_{i,j})_{0 \leq i \leq n, 1 \leq j \leq m-1}$, which uniquely determines $\alpha_{0,0}$ because $\eta_{(U^{[m]})^*}^{-1}$ is of $(0, n)$ type. Therefore, for any textile $(\alpha_{i,j})_{i,j \in \mathbb{Z}} \in U$, $p_A(\alpha_{0,1}) \dots p_A(\alpha_{0,N})$ uniquely determines $p_A(\alpha_{0,0})$. This means that for any $a_0 \dots a_N \in L_{N+1}(X_U)$ with $a_j \in A_G$, a_0

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is uniquely determined by $a_1 \dots a_N$, which cannot be the case because X_U has infinitely many points.

By symmetry (3) follows from (2), and (4) follows from (1). □

Let X be a zero-dimensional compact metric space and J a commutative submonoid of $S(X)$. For an expansive element τ of $H(X)$, define

$$D_J(\tau) = PL_J(\tau) \cup QR_J(\tau) \quad \text{and} \quad D_J^\circ(\tau) = PL_J^\circ(\tau) \cup QR_J^\circ(\tau).$$

Then

$$D_J(\tau) \cup D_J(\tau^{-1}) = PL_J(\tau) \cup QR_J(\tau) \cup PR_J(\tau) \cup QL_J(\tau)$$

is the set of all directionally essentially weakly one-sided resolving endomorphisms of (X, τ) in J. We call $D_J(\tau)$ the *district of* τ *in* J, whereas we call $C_J(\tau)$ the *cone of* τ *in* J. By Proposition 8.2, we know that if X has infinitely many points, then $D_J^\circ(\tau^{-1}) \cap D_J(\tau) = QL_J^\circ(\tau) \cap QR_J(\tau)$ and $D_J(\tau^{-1}) \cap D_J^\circ(\tau) = QL_J(\tau) \cap QR_J^\circ(\tau)$, and that if $J \subset H(X)$ in addition, then $D_J^\circ(\tau^{-1}) \cap D_J(\tau) = D_J(\tau^{-1}) \cap D_J^\circ(\tau) = \emptyset$.

PROPOSITION 8.3. *Let (X, σ) be a subshift, J a commutative submonoid of $S(X)$.*

(1) *If $\sigma \in J$ and $\varphi \in QL_J^\circ(\sigma) \cap QR_J(\sigma)$ (respectively, $\sigma^{-1} \in J$ and $\varphi \in QL_J(\sigma) \cap QR_J^\circ(\sigma)$), then there exists $k \geq 1$ such that $\varphi^l\sigma$ (respectively, $\varphi^l\sigma^{-1}$) is in $QL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$ for all $l \geq k$.*

(2) *$QL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$ is equal to the set of all positively expansive onto endomorphisms of (X, σ) in J.*

Proof. (1) Let $\varphi \in QL_J^\circ(\sigma) \cap QR_J(\sigma)$. Let $i \geq 0$. Since $\varphi \in QL_J^\circ(\sigma)$, it follows from Proposition 7.13(2) that there exists $k \geq 1$ such that $\varphi^k\sigma \in QL_J(\sigma)$. By Proposition 7.12(2), $\varphi^{k+i}\sigma \in QL_J^\circ(\sigma)$. Since $\varphi \in QR_J(\sigma)$, we have $\varphi^{k+i} \in QR_J(\sigma)$. Hence, by Proposition 7.12(2) we have $\varphi^{k+i}\sigma \in QR_J^\circ(\sigma)$, because $\sigma \in QR_J^\circ(\sigma)$. Thus we have $\varphi^{k+i}\sigma \in QL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$.

By symmetry, the second statement is proved.

(2) Let $\varphi \in QL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$. Then by the discussion of the first paragraph of the proof of Proposition 8.1 (replace 'p-L' by 'q-L'), we see that there exist $r, s \geq 1$ and a weakly q-L, weakly q-R endomorphism φ_0 of (X_0, σ_0) with $(X_0, \sigma_0, \varphi_0)$ conjugate to (X, σ^s, φ^r). There exist a one-sided 1-1 textile subsystem U of a weakly q-L textile system T and a one-sided 1-1 textile subsystem U' of a weakly q-R textile system T' such that $(X_U, \sigma_U, \varphi_U) = (X_{U'}, \sigma_{U'}, \varphi_{U'}) = (X_0, \sigma_0, \varphi_0)$. Since φ_U is right σ_U-expansive and $\varphi_{U'}$ is left $\sigma_{U'}$-expansive, there exists $n \geq 1$ such that $\eta_{(U^{[n]})^*}$ and $\xi_{(U'^{[n]})^*}$ are one-to-one (by Proposition 7.2). Since T is weakly q-L, $(T^{[n]})^*$ is weakly q-L, and since T' is weakly q-R, $(T'^{[n]})^*$ is weakly p-L. Therefore, for the same reason as in the proof of Proposition 8.2(2), $\eta^{-1}_{(U^{[n]})^*}$ and $\xi^{-1}_{(U'^{[n]})^*}$ have memory 0. Let $l \in \mathbb{Z}$ with $2l + 1 \geq n$. There exists $m \geq 1$ such that for any $x, y \in X_0$, if $d_{X_0}(x, y) < 1/(m+1)$, then $d_{Z_U}(\xi_U^{-1}(x), \xi_U^{-1}(y)) < 1/(l+1)$ and $d_{Z_{U'}}(\xi_{U'}^{-1}(x), \xi_{U'}^{-1}(y)) < 1/(l+1)$. Let $(a_{i,j})_{i,j\in\mathbb{Z}}$ be any orbit of φ_0, i.e. for all $i \in \mathbb{Z}$, $(a_{i,j})_{j\in\mathbb{Z}} \in X_0$ with $a_{i,j} \in L_1(X_0)$ and $\varphi_0((a_{i,j})_{j\in\mathbb{Z}}) = (a_{i+1,j})_{j\in\mathbb{Z}}$. To prove that φ_0 is positively expansive, it suffices to show that $(a_{i,j})_{i\geq 0, -m\leq j\leq m}$ uniquely determines $(a_{0,j})_{j\in\mathbb{Z}}$.

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There exists $(\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U$ with $\alpha_{i,j} \in L_1(Z_U)$ such that $\xi_U((\alpha_{i,j})_{j\in\mathbb{Z}}) = (a_{i,j})_{j\in\mathbb{Z}}$ for all $i \in \mathbb{Z}$. The configuration $(a_{i,j})_{i\geq 0,-m\leq j\leq m}$ uniquely determines $(\alpha_{i,j})_{i\geq 0,-l\leq j\leq l}$ with $2l+1 \geq n$. This determines $(\alpha_{i,j})_{i\geq 0,-\infty<j\leq l}$ since $\eta^{-1}_{(U^{[m]})^*}$ has memory 0. Hence $(a_{i,j})_{i\geq 0,-m\leq j\leq m}$ determines $(a_{i,j})_{i\geq 0,-\infty<j\leq l}$. A symmetric argument (using U' and $\xi^{-1}_{(U'^{[n]})^*}$) shows that $(a_{i,j})_{i\geq 0,-m\leq j\leq m}$ determines $(a_{i,j})_{i\geq 0,-l\leq j<\infty}$.

Thus φ_0 is positively expansive, so that φ is positively expansive.

Conversely, assume that $\varphi \in J$ is a positively expansive endomorphism of a subshift (X, σ). There exists a one-sided 1-1 textile subsystem U of a textile system T such that $(X_U, \sigma_U, \varphi_U) = (X, \sigma, \varphi)$. Since φ_U is positively expansive, an argument similar to that in the proof of Proposition 7.2 shows that there exists $m \geq 1$ such that $\tilde{\xi}_{(U^{[m]})^*}$ and $\tilde{\eta}_{(U^{[m]})^*}$ of $\tilde{Z}_{(U^{[m]})^*}$ onto $\tilde{X}_{(U^{[m]})^*}$ are one-to-one (see §2 for the notation). Therefore, there exists $n \geq 0$ such that both $\xi^{-1}_{(U^{[m]})^*}$ and $\eta^{-1}_{(U^{[m]})^*}$ are of $(0, n)$ type. Let $T_1^* = ((T^{[m]})^*)^{[n+1]} = (p_1^*, q_1^* : \Gamma_1^* \to G_1^*)$. Then $((U^{[m]})^*)^{[n+1]}$ is a textile subsystem of T_1^*. Let $T_2^* = (p_2^*, q_2^* : \Gamma_2^* \to G_2^*)$ be the textile system defined as follows: $G_2^* = G_1^*$; Γ_2^* is the subgraph of Γ_1^* with $A_{\Gamma_2^*} = L_{n+1}(Z_{(U^{[m]})^*})$ and $V_{\Gamma_2^*} = L_n(Z_{(U^{[m]})^*})$; p_2^* and q_2^* are the restrictions of p_1^* and q_1^* on Γ_2^*, respectively. Clearly $((U^{[m]})^*)^{[n+1]}$ is a textile subsystem of T_2^*. Since both $\xi^{-1}_{(U^{[m]})^*}$ and $\eta^{-1}_{(U^{[m]})^*}$ are of $(0, n)$ type, it follows that T_2^* is weakly LL. Therefore, if we let $\bar{U} = (((U^{[m]})^*)^{[n+1]})^*$, then $\bar{U}$ is a textile subsystem of the weakly q-biresolving textile system $T_2 = (T_2^*)^*$. Since U is one-sided 1-1, so also is $\bar{U}$ and $(X_{\bar{U}}, \sigma_{\bar{U}}, \varphi_{\bar{U}})$ is conjugate to $(X_U, \sigma_U, \varphi_U)$. Hence $\varphi = \varphi_U$ is an essentially weakly q-biresolving endomorphism of (X, σ), such that $\varphi \in QL_J^\circ(\sigma) \cap QR_J^\circ(\sigma)$. □

Let (X, σ) be a transitive SFT and J a commutative submonoid of $S(X)$. Then we shall see by Theorem 8.8(5) that $QL_J(\sigma) \cap QR_J(\sigma)$ is the set of all (onto) essentially q-biresolving endomorphisms of (X, σ) in J. Hence by Proposition 8.3 we see that if φ is a right (respectively, left) σ-expansive, essentially q-biresolving endomorphism of (X, σ), then $\varphi^l\sigma$ (respectively, $\varphi^l\sigma^{-1}$) is positively expansive for all sufficiently large $l \geq 1$ (cf. [**25,** Proposition 3.10]). With the same hypothesis, φ has POTP and if σ is topologically mixing, then so is φ (by Corollary 6.6). If σ is topologically mixing and φ is an expansive, essentially q-biresolving endomorphism of (X, σ), then (X, φ) is conjugate to a one-sided full shift (see [**23,** p. 4083]) and (X, σ) is shift equivalent to a full shift [**12**]. Arithmetic constraints are provided for expansive and left (or right) σ-expansive q-biresolving endomorphisms of mixing SFTs (X, σ) in [**12, 25**].

Example 8.4. Let (X, σ) be the full 2-shift over the alphabet $\{0, 1\}$. Let $n \geq 2$. Suppose $f : \{0, 1\}^n \to \{0, 1\}$ is a local rule which gives an automorphism of (X, σ) and is left non-redundant, i.e. there exists $v \in \{0, 1\}^{n-1}$ with $f(0v) \neq f(1v)$. Suppose $g : \{0, 1\}^{n+1} \to \{0, 1\}$ is bipermutive, i.e. $g(0w) \neq g(1w)$ and $g(w0) \neq g(w1)$ for all $w \in \{0, 1\}^n$. Let $T = (p, q : G_{\{0,1\}}^{[n+1]} \to G_{\{0,1\}})$ be the textile system such that $p_A(w0) = p_A(w1) = f(w)$ for all $w \in \{0, 1\}^n$ and $q_A = g$, where $G_{\{0,1\}}$ is the one-vertex graph with arc-set $\{0, 1\}$. Then, as is easily seen, T is one-sided 1-1, non-degenerate and q-biresolving with T^* one-sided 1-1. Hence φ_T is a left σ-expansive, q-biresolving endomorphism of (X, σ). Using the existence of $v \in \{0, 1\}^{n-1}$ with $f(0v) \neq f(1v)$, one can show that φ_T is not right σ-expansive. As is known [**14,** Theorem 6.9], f is not leftmost permutive, i.e. there

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exists $v' \in \{0, 1\}^{n-1}$ with $f(0v') = f(1v')$. Using this, one can show that φ_T is not an essentially p-L endomorphism of (X, σ). Thus one can see that φ_T is a non-expansive, left σ-expansive, essentially non-p-L, q-biresolving endomorphism of (X, σ).

PROPOSITION 8.5. *Let (X, σ) be an SFT. If φ is a directionally essentially p-L endomorphism of (X, σ), then φ is an essentially p-L endomorphism of (X, σ). In this statement, 'p-L' can be replaced by each of 'q-R', 'p-R' and 'q-L'.*

Proof. We may assume that (X, σ) is a topological Markov shift. It follows from [**20,** Theorems 7.22(1) and 7.23(1)] that if $((\varphi)^{(s)})^r$ is an essentially p-L endomorphism of $(X, \sigma^{(s)})$ with $r, s \geq 1$, then φ is an essentially p-L endomorphism of (X, σ).

Each of 'q-R', 'p-R' and 'q-L' versions is proved by each corresponding version of [**20,** Theorems 7.22(1) and 7.23(1)], which is proved by easy modifications in [**20,** the proofs of Theorems 7.22 and 7.23]. □

PROPOSITION 8.6. *Let φ be an onto endomorphism of an SFT (X, σ).*

(1) *If φ is essentially p-L and essentially p-R, then it is essentially a 1-block automorphism of (X, σ).*
(2) *If φ is essentially p-L and essentially q-R, then it is an essentially LR endomorphism of (X, σ).*
(3) *If φ is essentially q-L and essentially q-R, then it is an essentially q-biresolving endomorphism of (X, σ).*

Proof. We follow the notation and terminology of [**20**].

(1) There exists a one-sided 1-1, non-degenerate, p-L textile system T such that φ_T is an essentially p-R endomorphism of (X_T, σ_T) and $(X_T, \sigma_T, \varphi_T)$ is conjugate to (X, σ, φ). There exists a one-sided 1-1, non-degenerate, p-R textile system T_0 with $(X_{T_0}, \sigma_{T_0}, \varphi_{T_0})$ conjugate to $(X_T, \sigma_T, \varphi_T)$. Therefore, by [**20,** Theorem 7.19 with Lemma 7.8(i)], $T^{[n]}$ is 'resolvable' for some $n \geq 1$. Since $T^{[n]}$ is p-L, $(T^{[n]})^{(-1)}$ is '0 resolvable'. Hence, by [**20,** Theorem 7.20(3)], there exists a one-sided 1-1, non-degenerate textile system $T_0' = (p_0', q_0' : \Gamma_0' \to G_0')$ with p_0' biresolving and $(X_{T_0'}, \sigma_{T_0'}, \varphi_{T_0'})$ conjugate to $(X_T, \sigma_T, \varphi_T)$.

Since p_0' is biresolving and the 1-block map $\phi_{p_0'}$ is a conjugacy, it follows that p_0' is a graph-automorphism of G_0' and hence $q' p'^{-1}$ is a graph-endomorphism of G_0'. Therefore $\varphi_{T_0'}$ is an onto 1-block endomorphism and hence a 1-block automorphism of $(X_{T_0'}, \sigma_{T_0'})$. Thus φ is essentially a 1-block automorphism of (X, σ).

The proofs of (3) and (2) are similar to the first paragraph in the proof of (1) above, but use [**20,** Theorem 7.20(2)] instead of [**20,** Theorem 7.20(3)] for the proof of (2). □

The following lemma is well known.

LEMMA 8.7. *Let Γ and G be graphs with i_Γ, t_Γ, i_G and t_G onto. Let $h : \Gamma \to G$ be a weakly right (respectively, left) resolving graph-homomorphism.*

(1) *If ϕ_h is a conjugacy, then h is right (respectively, left) resolving.*
(2) *If G and Γ are irreducible and have the same spectral radius, then h is right (respectively, left) resolving.*

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THEOREM 8.8. *Let φ be an onto endomorphism of an SFT (X, σ).*

(1) *If φ is directionally essentially weakly p-L (respectively, p-R) , then φ is essentially p-L (respectively, p-R).*

(2) *Suppose φ is an automorphism or σ is topologically transitive. Then if φ is directionally essentially weakly q-L (respectively, q-R) , then φ is essentially q-L (respectively, q-R).*

(3) *If φ is directionally essentially weakly p-L and directionally essentially weakly p-R, then φ is essentially a 1-block automorphism.*

(4) *Suppose φ is an automorphism or σ is topologically transitive. Then if φ is directionally essentially weakly p-L and directionally essentially weakly q-R, then φ is essentially LR.*

(5) *Suppose σ is topologically transitive. Then if φ is directionally essentially weakly q-L and directionally essentially weakly q-R, then φ is essentially q-biresolving.*

Proof. (1) There exist a one-sided 1-1 textile subsystem U of a weakly p-L textile system T and integers r, $s \geq 1$ such that $(X_U, \sigma_U, \varphi_U)$ is conjugate to (X, σ^s, φ^r). Since (X, σ) is an SFT, so is (X_U, σ_U) and hence (Z_U, ς_U) is an SFT as U is one-sided 1-1. There exists $n \geq 1$ such that $(Z_U^{[n]}, \varsigma_U^{[n]})$ and $(X_U^{[n]}, \sigma_U^{[n]})$ are topological Markov shifts. There exists a one-sided 1-1, non-degenerate textile system $T' = (p', q' : \Gamma' \to G')$ with $i_{\Gamma'}$, $t_{\Gamma'}$, $i_{G'}$ and $t_{G'}$ onto such that $U_{T'} = U^{[n]}$. Since $T^{[n]}$ is weakly p-L, so is T'. Since $\phi_{p'}$ is a conjugacy, it follows from Lemma 8.7(1) that p' is left resolving. Since (X, σ^s, φ^r) is conjugate to $(X_{T'}, \sigma_{T'}, \varphi_{T'})$, φ is a directionally essentially p-L endomorphism of (X, σ). Hence the result follows from Proposition 8.5. (The second version is proved by symmetry.)

(2) If φ is an automorphism, then apply (1) to φ^{-1}. If σ is transitive, then the proof is similar to that of (1), but use Lemma 8.7(2) instead of Lemma 8.7(1).

(3) By (1) and Proposition 8.6(1).

(4) By (1) and (2), φ is essentially p-L and essentially q-R. Hence (4) is proved by Proposition 8.6(2).

(5) By (2) and Proposition 8.6(3). □

PROPOSITION 8.9. *Let φ be an onto endomorphism of a subshift (X, σ) with infinitely many points. Let $J = \{\varphi^k \sigma^n \mid k \geq 0, n \in \mathbb{Z}\}$. If $D_J(\sigma) \cap D_J(\sigma^{-1}) \neq \{i_X\}$, then the following hold.*

(1) *$J = D_J(\sigma) \cup D_J(\sigma^{-1})$.*

(2) *If (X, σ) is an SFT, then all elements of J are right or left closing.*

(3) *If (X, σ) is a topologically-transitive SFT, then all expansive elements of J have POTP.*

Proof. Since $D_J(\sigma) = PL_J(\sigma) \cup QR_J(\sigma)$ and $D_J(\sigma^{-1}) = PR_J(\sigma) \cup QL_J(\sigma)$, at least one of the following holds: (a) $PL_J(\sigma) \cap PR_J(\sigma) \neq \{i_X\}$; (b) $PL_J(\sigma) \cap QL_J(\sigma) \neq \{i_X\}$; (c) $PR_J(\sigma) \cap QR_J(\sigma) \neq \{i_X\}$; (d) $QL_J(\sigma) \cap QR_J(\sigma) \neq \{i_X\}$. We note that $\sigma, \sigma^{-1} \notin D_J(\sigma) \cap D_J(\sigma^{-1})$ by Proposition 8.2 (since $\sigma \in PL_J^\circ(\sigma)$ and $\sigma^{-1} \in PR_J^\circ(\sigma)$).

(1) Assume (a) and let $\psi \in PL_J(\sigma) \cap PR_J(\sigma) - \{i_X\}$. Then $\psi = \varphi^i \sigma^j$ for some $i \geq 1$, $j \in \mathbb{Z}$, because $\psi \neq \sigma^j$ for all $j \in \mathbb{Z}$. Using Remark 7.9 we see that there exist r, $s \geq 1$ such that ψ^r is an essentially weakly p-L and essentially weakly p-R endomorphism of

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(X, σ^s). Using Remark 7.9 and Proposition 7.6 we see that for all $i' \geq 0$ and $j' \in \mathbb{Z}$, $(\psi^{i'}\sigma^{j'})^{rs} = (\psi^r)^{si'}(\sigma^s)^{rj'}$ is an essentially weakly p-L or essentially weakly p-R endomorphism of (X, σ^s). Hence for all $i' \geq 0$ and $j' \in \mathbb{Z}$, $\psi^{i'}\sigma^{j'} \in PL_J(\sigma) \cup PR_J(\sigma)$. Therefore, for all $k \geq 0$ and $n \in \mathbb{Z}$, $(\varphi^k\sigma^n)^i = \psi^k\sigma^{ni-kj}$ is in $PL_J(\sigma) \cup PR_J(\sigma)$, so that $\varphi^k\sigma^n \in PL_J(\sigma) \cup PR_J(\sigma)$. Thus $J \subset PL_J(\sigma) \cup PR_J(\sigma)$.

Similarly we see the following: if (b) holds, then $J \subset PL_J(\sigma) \cup QL_J(\sigma)$; if (c) holds, then $J \subset PR_J(\sigma) \cup QR_J(\sigma)$; if (d) holds, then $J \subset QL_J(\sigma) \cup QR_J(\sigma)$.

(2) If (b), (c) or (d) holds, then there exists $\psi \in QL_J(\sigma) \cup QR_J(\sigma)$ with $\psi = \varphi^i\sigma^j$ for some $i \geq 1$, $j \in \mathbb{Z}$. Since ψ is left or right closing and $(\varphi^k\sigma^n)^i = \psi^k\sigma^{ni-kj}$ for all $k \geq 0$ and $n \in \mathbb{Z}$, all elements of J are left or right closing.

If (a) holds, then there exists $\psi \in PL_J(\sigma) \cap PR_J(\sigma)$ with $\psi = \varphi^i\sigma^j$ for some $i \geq 1$, $j \in \mathbb{Z}$. Since (X, σ) is an SFT, it follows from Theorem 8.8(3) that ψ is an automorphism of (X, σ), implying φ also is. Hence the conclusion of (2) also follows.

(3) The result follows from (1), (2) and Corollary 6.6. □

COROLLARY 8.10. *Let φ be an onto endomorphism of a topologically-transitive SFT (X, σ). If φ is essentially p-L and essentially p-R, essentially p-L and essentially q-L (or essentially p-R and essentially q-R), or essentially q-L and essentially q-R, then all expansive elements of $\{\varphi^k\sigma^n \mid k \geq 0, n \in \mathbb{Z}\}$ have POTP.*

This contains a generalization of Corollary 5.4. We remark that if φ is an essentially p-L and essentially p-R endomorphism of an SFT (X, σ), the conclusion of this corollary holds without the condition that (X, σ) is topologically transitive. This follows from Proposition 8.6(1) and the known result [**20,** Proposition 8.3] that the conclusion of the corollary holds for every automorphism φ of finite order of every SFT (X, σ).

We shall be successful in checking that a given endomorphism φ of an SFT (X, σ) is not an essentially one-sided resolving one, if we can find the 'boundary' $D_J(\sigma) - D_J^\circ(\sigma)$ of $D_J(\sigma)$ and that of $D_J(\sigma^{-1})$ for $J = \{\varphi^k\sigma^n \mid k \geq 0, n \in \mathbb{Z}\}$.

PROPOSITION 8.11. *Let φ be an endomorphism of a subshift (X, σ) with infinitely many points. Let $J = \{\varphi^k\sigma^n \mid k \geq 0, n \in \mathbb{Z}\}$. Suppose that*

$$\psi_+ = \varphi^i\sigma^j \in D_J(\sigma) - D_J^\circ(\sigma) - \{i_X\} \quad \textit{and}$$
$$\psi_- = \varphi^{i'}\sigma^{j'} \in D_J(\sigma^{-1}) - D_J^\circ(\sigma^{-1}) - \{i_X\}$$

with $i, i' \geq 1$ and $j, j' \in \mathbb{Z}$. Then

(1) *if $ij' - ji' \geq 0$, then $J = D_J(\sigma^{-1}) \cup D_J(\sigma)$, and*
(2) *if $ij' - ji' < 0$, then any element $\psi \in J$ such that $\psi^m = \psi_+^r\psi_-^s$ for some $m, r, s \geq 1$ is not in $D_J(\sigma^{-1}) \cup D_J(\sigma)$.*

Proof. We have $\psi_-^i = \psi_+^{i'}\sigma^{ij'-ji'}$.

(1) If $ij' - ji' = 0$, then $\psi_-^i = \psi_+^{i'}$. Hence $\psi_- \in D_J(\sigma) \cap D_J(\sigma^{-1}) - \{i_X\}$. Thus by Proposition 8.9, $J = D_J(\sigma^{-1}) \cup D_J(\sigma)$. Assume that $ij' - ji' > 0$. Then it follows from Remark 7.9 and Proposition 7.6 that if $\psi_+ \in PL_J(\sigma)$ then $\psi_- \in PL_J^\circ(\sigma)$, which is not the case by Proposition 8.2, and if $\psi_+ \in QR_J(\sigma)$ then $\psi_- \in QR_J^\circ(\sigma)$. Hence $\psi_- \in D_J(\sigma^{-1}) \cap D_J^\circ(\sigma)$, so that $J = D_J(\sigma^{-1}) \cup D_J(\sigma)$ by Proposition 8.9.

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(2) We have $j'/i' < j/i$. Let $i'' = ir + i's$ and let $j'' = jr + j's$. Then $\psi^m = \varphi^{i''}\sigma^{j''}$ and $j'/i' < j''/i'' < j/i$. We have $\psi_+^{i''} = \psi^{mi}\sigma^{ji''-ij''}$ and $\psi_-^{i''} = \psi^{mi'}\sigma^{i''j'-i'j''}$. Therefore, using Remark 7.9 and Proposition 7.6, we see the following: if $\psi \in PL_J(\sigma)$, then ψ_+ would be in $PL_J^\circ(\sigma)$; if $\psi \in QR_J(\sigma)$, then ψ_+ would be in $QR_J^\circ(\sigma)$; if $\psi \in PR_J(\sigma)$, then ψ_- would be in $PR_J^\circ(\sigma)$; if $\psi \in QL_J(\sigma)$, then ψ_- would be in $QL_J^\circ(\sigma)$. Since every conclusion contradicts our hypotheses, ψ is not in $D_J(\sigma) \cup D_J(\sigma^{-1})$. □

Example 8.12. (1) The reader is referred to Example 1 in [**20,** §10]. The 1-1, LR textile systems T and T' and the automorphism φ of the topological Markov shift (X_M, σ_M) found there are such that $(X_T, \sigma_T) = (X_{T'}, \sigma_{T'}) = (X_M, \sigma_M)$, $\varphi_T = \varphi$ and $\varphi_{T'} = \varphi^{-1}\sigma_M$. It is easy to check that neither ξ_{T^*} nor η_{T^*} is one-to-one and so is neither $\xi_{T'^*}$ nor $\eta_{T'^*}$. Hence φ_T is LR, not right σ_M-expansive and not left σ_M-expansive, and $\varphi_{T'}^{-1}$ is RL, not left σ_M-expansive and not right σ_M-expansive. Therefore, $\varphi \in D_J(\sigma_M) - D_J^\circ(\sigma_M) - \{i_{X_M}\}$ and $\varphi\sigma_M^{-1} \in D_J(\sigma_M^{-1}) - D_J^\circ(\sigma_M^{-1}) - \{i_{X_M}\}$ with $J = \{\varphi^k\sigma_M^n \mid k \geq 0, n \in \mathbb{Z}\}$. Thus by Proposition 8.11, $\varphi^2\sigma_M^{-1}$ is not an essentially one-sided resolving automorphism of (X_M, σ_M), i.e. every automorphism ψ of an SFT (X, σ) with (X, σ, ψ) conjugate to $(X_M, \sigma_M, \varphi^2\sigma_M^{-1})$ and its inverse ψ^{-1} have strictly positive memory and strictly positive anticipation.

(2) The reader is referred to Example 2 in [**20,** §10]. The automorphism φ of the full shift over the alphabet $\{0, 1\}$ considered there is of (1, 2) type, as is φ^{-1}. Hence $\varphi\sigma$ is p-L and $\varphi\sigma^{-1}$ is q-L (since $\varphi^{-1}\sigma$ is p-L). A textile system T_1 with $(X_{T_1}, \sigma_{T_1}, \varphi_{T_1}) = (X, \sigma, \varphi\sigma)$ and its dual T_1^* are represented by

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} \frac{0}{1} & \frac{0}{1} & & & \\ & & \frac{0}{1} & & \\ & & & \frac{0}{0} & \frac{0}{0} \\ & & \frac{1}{1} & & \\ \frac{1}{1} & \frac{1}{0} & & \frac{1}{1} & \frac{1}{0} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \frac{c}{d} + \frac{c}{e} & \frac{a}{a} + \frac{a}{b} + \frac{b}{c} \\ \frac{e}{b} + \frac{e}{e} & \frac{d}{c} + \frac{e}{a} + \frac{e}{d} \end{pmatrix} \end{array},$$

respectively. Observing the trace of the latter matrix, we see that the point $(c_j)_{j\in\mathbb{Z}} \in X_{T_1^*}$ with $c_j = c$ for all $j \in \mathbb{Z}$ appears on different (vertically) periodic textiles woven by T_1^*. Hence $\xi_{T_1^*}$ and $\eta_{T_1^*}$ are not one-to-one, so that $\varphi\sigma = \varphi_{T_1}$ is neither right σ-expansive nor left σ-expansive. By a similar method one can also see that $\varphi\sigma^{-1}$ is neither right σ-expansive nor left σ-expansive. Hence we have $\varphi\sigma \in D_J(\sigma) - D_J^\circ(\sigma) - \{i_X\}$ and $\varphi\sigma^{-1} \in D_J(\sigma^{-1}) - D_J^\circ(\sigma^{-1}) - \{i_X\}$ with $J = \{\varphi^k\sigma^n \mid k \geq 0, n \in \mathbb{Z}\}$. Thus by Proposition 8.11, we know that all ψ such that $\psi^m = (\varphi\sigma^{-1})^r(\varphi\sigma)^s$ for some $m, r, s \geq 1$, is not an essentially one-sided resolving automorphism of (X, σ).

Question 8.13. Let (X, σ) be an SFT with infinitely many points. Let φ be an automorphism of (X, σ) with $\varphi \neq i_X$. Let K be the subgroup of $H(X)$ generated by $\{\sigma, \varphi\}$. (1) Do there exist σ and φ such that $PL_K(\sigma) = PL_K^\circ(\sigma) \cup \{i_X\}$? (2) Is it true that for every element $\psi \in PL_K(\sigma)$ there exists $m \geq 1$ such that ψ^m is contained in the submonoid of K generated by $(PL_K(\sigma) - PL_K^\circ(\sigma)) \cup \{\sigma\}$? (3) Do there exist σ and φ such that $LR_K(\sigma) = LR_K^\circ(\sigma) \cup \{i_X\}$? (4) Is it true that for every element $\psi \in LR_K(\sigma)$

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there exists $m \geq 1$ such that ψ^m is contained in the submonoid of K generated by $(LR_K(\sigma) - LR_K^\circ(\sigma)) \cup \{\sigma\}$?

A question similar to (4) was asked in [**20,** Question 8.b]. A result of Boyle and Lind [**11,** Proposition 4.1] shows that there exists an automorphism φ of a subshift (X, σ) with infinitely many points such that $\varphi^i \sigma^j$ is expansive for all $i, j \in \mathbb{Z}$ with $(i, j) \neq (0, 0)$, and hence for automorphisms of general subshifts the answers to (1) and (3) are affirmative and hence the answers to (2) and (4) are negative.

9. *ELR cones*

Throughout this section, let X be a zero-dimensional compact metric space and $H(X)$ the group of homeomorphisms of X.

PROPOSITION 9.1. *Let (X, τ, φ) be a commuting system such that τ and φ are expansive elements of $H(X)$.*

(1) *φ is an essentially weakly p-L automorphism of (X, τ) if and only if τ is an essentially weakly q-R automorphism of (X, φ).*

(2) *φ is an essentially weakly LR automorphism of (X, τ) if and only if τ is an essentially weakly LR automorphism of (X, φ).*

Proof. Assume that φ is an essentially weakly p-L automorphism of (X, τ). Then there exists a 1-1 textile subsystem U of a weakly p-L textile system T such that $(U^{[n]})^*$ is 1-1 for some $n \geq 1$ and $(X_U, \sigma_U, \varphi_U)$ is conjugate to (X, τ, φ) (by Proposition 7.2). Let $U_1 = U^{[n]}$ and let $T_1 = T^{[n]}$. Then U_1 is a 1-1 textile subsystem of a weakly p-L textile system T_1 with U_1^* 1-1 and $(X_{U_1}, \sigma_{U_1}, \varphi_{U_1})$ is conjugate to (X, τ, φ). Hence U_1^* is a 1-1 textile subsystem of a weakly q-R textile system T_1^* and $(X_{U_1^*}, \sigma_{U_1^*}, \varphi_{U_1^*})$ is conjugate to $(X_{U_1}, \varphi_{U_1}, \sigma_{U_1})$, which is conjugate to (X, φ, τ). Hence τ is an essentially weakly q-R automorphism of (X, φ).

The proofs of the 'if part' and (2) are similar to the above. □

For a textile system $T = (p, q : \Gamma \to G)$, define $T^{-1} = (q, p : \Gamma \to G)$ and $T^{(-1)} = ((T^*)^{-1})^*$. For a textile subsystem U of T, U^{-1} and $U^{(-1)}$ denote the naturally defined textile subsystems of T^{-1} and $T^{(-1)}$, respectively.

PROPOSITION 9.2. *Let $\varphi : X \to X$ be an onto continuous map and let τ and τ' be expansive elements of $H(X)$ with every two of φ, τ and τ' commuting.*

(1) *If φ is a directionally essentially weakly p-L endomorphism of (X, τ) and τ is a directionally essentially weakly LR automorphism of (X, τ'), then φ is a directionally essentially weakly p-L endomorphism of (X, τ').*

(2) *If φ is a directionally essentially weakly q-R automorphism of (X, τ) and τ is a directionally essentially weakly LR automorphism of (X, τ'), then φ is a directionally essentially weakly q-R automorphism of (X, τ').*

(3) *If φ is a directionally essentially weakly LR automorphism of (X, τ) and τ is a directionally essentially weakly LR automorphism of (X, τ'), then φ is a directionally essentially weakly LR automorphism of (X, τ').*

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Proof. (1) It follows from Remark 7.9 that it suffices to show the conclusion under the stronger hypothesis that φ is an essentially weakly p-L endomorphism of (X, τ) and τ is an essentially weakly LR automorphism of (X, τ').

Since τ is an essentially weakly LR automorphism of (X, τ'), it follows from Proposition 9.1 that τ' is an essentially weakly LR automorphism of (X, τ). Since φ is an essentially weakly p-L endomorphism of (X, τ), there exist a one-sided 1-1 textile subsystem U of a weakly p-L textile system T and a conjugacy $\theta : (X, \tau, \varphi) \to (X_U, \sigma_U, \varphi_U)$. Since $\theta\tau'\theta^{-1}$ is an expansive, essentially weakly LR automorphism of (X_U, σ_U), it follows from Proposition 7.11 that for some $k \geq 1$, $(\theta\tau'\theta^{-1})^k$ is an expansive, weakly LR automorphism of (X_U, σ_U). Thus there exists a 1-1 textile subsystem U' of a weakly LR textile system T' such that $(X_{U'}, \sigma_{U'}, \varphi_{U'}) = (X_U, \sigma_U, \theta\tau'^k\theta^{-1})$ and $(U'^{[n]})^*$ is 1-1 for some $n \geq 1$ (by Proposition 7.2(3)). Let $\rho_n : X_U \to X_{U^{[n]}}$ be the homeomorphism which maps $(a_j)_{j\in\mathbb{Z}} \in X_U$ with $a_j \in L_1(X_U)$ to $(a_j \dots a_{j+n-1})_{j\in\mathbb{Z}}$.

Replacing $T^{[n]}$, $U^{[n]}$, $T'^{[n]}$ and $U'^{[n]}$ by T, U, T' and U', respectively, and replacing $\rho_n\theta$ by θ, we see that there exist a one-sided 1-1 textile subsystem U of a weakly p-L textile system T with $(X_U, \sigma_U, \varphi_U)$ conjugate to (X, τ, φ) through θ^{-1} and a 1-1 textile subsystem U' of a weakly LR textile system T' such that $(X_U, \sigma_U) = (X_{U'}, \sigma_{U'})$, $(X_{U'}, \sigma_{U'}, \varphi_{U'})$ is conjugate to (X, τ, τ'^k) through θ^{-1} and U'^* is 1-1. Since $\varphi\tau' = \tau'\varphi$, we have $\varphi_U\varphi_{U'} = \varphi_{U'}\varphi_U$. Let $\psi = \chi_{U'}\varphi_U\chi_{U'}^{-1}$ (see Proposition 7.2(4)). Since $\varphi_U\varphi_{U'} = \varphi_{U'}\varphi_U$ and $\chi_{U'}$ is a conjugacy of $(X_{U'}, \varphi_{U'})$ onto $(X_{U'^*}, \sigma_{U'^*})$, we see that ψ is an endomorphism of $(X_{U'^*}, \sigma_{U'^*})$.

Since U' is a 1-1 textile subsystem of a weakly p-L textile system T', $\xi_{U'}^{-1}$ has memory 0 (see the proof of Proposition 7.5). We can therefore define the conjugacy $(\xi_{U'}^{-1})^\sim : (\tilde{X}_{U'}, \tilde{\sigma}_{U'}) \to (\tilde{Z}_{U'}, \tilde{\zeta}_{U'})$ between the induced one-sided subshifts such that $s_{X_{U'}}\xi_{U'}^{-1} = (\xi_{U'}^{-1})^\sim s_{X_{U'}}$. Since U'^* is a 1-1 textile subsystem of a weakly p-L textile system T'^* (because T' is weakly q-R), we can define the conjugacy $(\xi_{U'^*}^{-1})^\sim : (\tilde{X}_{U'^*}, \tilde{\sigma}_{U'^*}) \to (\tilde{Z}_{U'^*}, \tilde{\zeta}_{U'^*})$ between the induced one-sided subshifts such that $s_{X_{U'^*}}\xi_{U'^*}^{-1} = (\xi_{U'^*}^{-1})^\sim s_{X_{U'^*}}$. The existence of $(\xi_{U'}^{-1})^\sim$ and $(\xi_{U'^*}^{-1})^\sim$ implies that for $(\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U'$, each of $\tilde{\xi}_{U'}((\alpha_{1,j})_{j\geq 1})$ and $\tilde{\xi}_{U'^*}((\alpha_{i,1})_{i\geq 1})$ determines $(\alpha_{i,j})_{i,j\geq 1}$. Hence we can define the homeomorphism $\hat{\chi}_{U'} : \tilde{X}_{U'} \to \tilde{X}_{U'^*}$ which maps $\tilde{\xi}_{U'}((\alpha_{1,j})_{j\geq 1})$ to $\tilde{\xi}_{U'^*}((\alpha_{i,1})_{i\geq 1})$ for $(\alpha_{i,j})_{i,j\in\mathbb{Z}} \in U'$. We have $\hat{\chi}_{U'}s_{X_{U'}} = s_{X_{U'^*}}\chi_{U'}$.

Since U is a one-sided 1-1 textile subsystem of a weakly p-L textile system, φ_U has memory zero and hence we can define the endomorphism $\tilde{\varphi}_U$ of the induced one-sided subshift $(\tilde{X}_U, \tilde{\sigma}_U)$ such that $s_{X_U}\varphi_U = \tilde{\varphi}_U s_{X_U}$.

Let $\tilde{\upsilon} = \hat{\chi}_{U'}\tilde{\varphi}_U\hat{\chi}_{U'}^{-1}$. Then we have

$$\begin{aligned} s_{X_{U'^*}}\psi &= s_{X_{U'^*}}\chi_{U'}\varphi_U\chi_{U'}^{-1} = \hat{\chi}_{U'}s_{X_{U'}}\varphi_U\chi_{U'}^{-1} = \hat{\chi}_{U'}s_{X_U}\varphi_U\chi_{U'}^{-1} \\ &= \hat{\chi}_{U'}\tilde{\varphi}_U s_{X_U}\chi_{U'}^{-1} = \hat{\chi}_{U'}\tilde{\varphi}_U s_{X_{U'}}\chi_{U'}^{-1} = \hat{\chi}_{U'}\tilde{\varphi}_U\hat{\chi}_{U'}^{-1}s_{X_{U'^*}} = \tilde{\upsilon}s_{X_{U'^*}}. \end{aligned}$$

This implies that ψ has memory zero. Hence by Proposition 7.5(1), ψ is a weakly p-L endomorphism of $(X_{U'^*}, \sigma_{U'^*})$.

Since $(X_U, \varphi_{U'}, \varphi_U)$ is conjugate to $(X_{U'^*}, \sigma_{U'^*}, \psi)$ (through $\chi_{U'}$) and since (X, τ'^k, φ) is conjugate to $(X_U, \varphi_{U'}, \varphi_U)$ (through θ), φ is a directionally essentially weakly p-L endomorphism of (X, τ').

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(2) It follows from the hypothesis that φ^{-1} is a directionally essentially weakly p-L automorphism of (X, τ^{-1}) and τ^{-1} is a directionally essentially weakly LR automorphism of (X, τ'^{-1}). Hence by (1), φ^{-1} is a directionally essentially weakly p-L automorphism of (X, τ'^{-1}). From this the conclusion follows.

(3) It follows from Remark 7.9 that it suffices to show the conclusion under the stronger hypothesis that φ is an essentially weakly LR endomorphism of (X, τ) and τ is an essentially weakly LR automorphism of (X, τ').

We follow the proof of (1). Since φ is an essentially weakly LR automorphism in this case, U and T in this case can be such that U is a 1-1 textile subsystem of a weakly LR textile system T. As we saw, $\psi = \chi_{U'}\varphi_U\chi_{U'}^{-1}$ has memory zero. Since T is weakly q-R, the textile system $(T^{(-1)})^{-1}$ is weakly p-L, and since T' is weakly LR, $(T'^{(-1)})^{-1}$ is weakly LR. Hence if we apply the same argument as in the proof of (1) to the textile subsystem $(U^{(-1)})^{-1}$ of $(T^{(-1)})^{-1}$ and the textile subsystem $(U'^{(-1)})^{-1}$ of $(T'^{(-1)})^{-1}$, we see that $\chi_{(U'^{(-1)})^{-1}}\varphi_{(U^{(-1)})^{-1}}\chi^{-1}_{(U'^{(-1)})^{-1}}$ has memory zero, so that ψ^{-1} has anticipation zero. Hence the conclusion is shown by using Proposition 7.5(2). □

PROPOSITION 9.3. *Let τ be an expansive element of $H(X)$. If φ_1 and φ_2 are commuting, directionally essentially weakly p-L (respectively, q-R, LR) automorphisms of (X, τ) with $\varphi_1^m\varphi_2^n$ expansive for some $m, n \geq 0$, then $\varphi_1\varphi_2$ is a right τ-expansive (respectively, left τ-expansive, expansive), directionally essentially weakly p-L (respectively, q-R, LR) automorphism of (X, τ).*

Proof. First we prove the following.

$(*)$ If φ_1 and φ_2 are commuting, directionally essentially weakly p-L automorphisms of (X, τ) with $\varphi_1\varphi_2$ expansive, then $\varphi_1\varphi_2$ is a directionally essentially weakly p-L automorphism of (X, τ).

Since τ is an expansive automorphism of $(X, \varphi_1\varphi_2)$ with $\varphi_1\varphi_2$ expansive, by Proposition 7.10, there exists $k \geq 1$ such that $(\varphi_1\varphi_2)^{k-1}\tau$ is an essentially weakly LR automorphism of $(X, \varphi_1\varphi_2)$. Hence, by Proposition 7.6, $(\varphi_1\varphi_2)^k\tau$ is an expansive, essentially weakly LR automorphism of $(X, \varphi_1\varphi_2)$. Therefore, by Proposition 9.1, $\varphi_1\varphi_2$ is an essentially weakly LR automorphism of $(X, \varphi_1^k\varphi_2^k\tau)$.

Using Remark 7.9, we see that for some $r \geq 1$, $(\varphi_1^k)^r = (\varphi_1^r)^k$ and $(\varphi_2^r)^k$ are essentially weakly p-L automorphisms of (X, τ^r). By Proposition 7.6, $(\varphi_2^k\tau)^r = (\varphi_2^r)^k\tau^r$ is a right τ^r-expansive, essentially weakly p-L automorphism of (X, τ^r). Hence by Proposition 7.12, $(\varphi_1^k)^r(\varphi_2^k\tau)^r$ is a directionally essentially weakly p-L automorphism of (X, τ^r), so that $\varphi_1^k\varphi_2^k\tau$ is a directionally essentially weakly p-L automorphism of (X, τ). Applying Proposition 9.2 to this and the conclusion of the preceding paragraph we see that $\varphi_1\varphi_2$ is a directionally essentially weakly p-L automorphism of (X, τ). Thus $(*)$ is proved.

Since φ_1^m and φ_2^n are directionally essentially weakly p-L automorphisms of (X, τ) with $\varphi_1^m\varphi_2^n$ expansive, it follows from $(*)$ that $\varphi_1^m\varphi_2^n$ is a directionally essentially weakly p-L automorphism of (X, τ). We may assume that $m > n$. Since φ_2^{m-n} is a directionally essentially weakly p-L automorphism of (X, τ), it follows from Proposition 7.12 that $(\varphi_1\varphi_2)^m = (\varphi_1^m\varphi_2^n)\varphi_2^{m-n}$ is a right τ-expansive, directionally essentially weakly p-L automorphism of (X, τ), and hence so is $\varphi_1\varphi_2$.

The proofs of the 'q-R' and 'LR' versions are similar to the above. □

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We do not know whether the statement (∗) in this proof is valid if 'with $\varphi_1\varphi_2$ expansive' is replaced by 'with $\varphi_1\varphi_2$ right τ-expansive' in it.

For $\varphi, \tau \in H(X)$, let us write $\varphi \rightharpoonup \tau$ (respectively, $\varphi \rightharpoondown \tau$) to mean that τ is expansive and φ is a directionally essentially weakly p-L (respectively, q-R) automorphism of (X, τ), and $\varphi \to \tau$ to mean that τ is expansive and φ is a directionally essentially weakly LR automorphism of (X, τ).

The following two propositions are summaries of the preceding results. The second one without (5) was stated without proof in [**21**].

PROPOSITION 9.4. *Let $\tau, \tau_1, \tau_2, \varphi, \varphi_1, \varphi_2 \in H(X)$.*

(1) *If τ_1 and τ_2 are expansive, then $\tau_1 \rightharpoonup \tau_2$ if and only if $\tau_2 \rightharpoondown \tau_1$.*
(2) *If $\tau\tau_2 = \tau_2\tau$, then $\tau \rightharpoonup \tau_1$ and $\tau_1 \to \tau_2$ imply $\tau \rightharpoonup \tau_2$. In this statement, '$\rightharpoonup$' can be replaced by '$\rightharpoondown$'.*
(3) *If $\varphi \rightharpoonup \tau$ (respectively, $\varphi \rightharpoondown \tau$) and φ is right (respectively, left) τ-expansive, then $\varphi^n\tau^{-1} \rightharpoonup \tau$ (respectively, $\varphi^n\tau^{-1} \rightharpoondown \tau$) for some $n \geq 1$.*
(4) *If $\varphi_1, \varphi_2 \rightharpoonup \tau$ (respectively, $\varphi_1, \varphi_2 \rightharpoondown \tau$) with $\varphi_1\varphi_2 = \varphi_2\varphi_1$ and if φ_1 is right (respectively, left) τ-expansive or $\varphi_1^m\varphi_2^n$ is expansive for some $m, n \geq 1$, then $\varphi_1\varphi_2 \rightharpoonup \tau$ (respectively, $\varphi_1\varphi_2 \rightharpoondown \tau$) with $\varphi_1\varphi_2$ right (respectively, left) τ-expansive.*

Proof. (1) holds by Proposition 9.1; (2) holds by Proposition 9.2; (3) holds by Proposition 7.13; (4) holds by Propositions 7.12 and 9.3. □

PROPOSITION 9.5. *Let $\tau, \tau_1, \tau_2, \varphi, \varphi_1, \varphi_2 \in H(X)$.*

(1) *If τ is expansive, then $\tau \to \tau$.*
(2) *If τ_1 is expansive and $\tau_1 \to \tau_2$, then $\tau_2 \to \tau_1$.*
(3) *If $\tau\tau_2 = \tau_2\tau$, then $\tau \to \tau_1$ and $\tau_1 \to \tau_2$ imply $\tau \to \tau_2$.*
(4) *If τ is expansive with $\varphi\tau = \tau\varphi$, then $\varphi\tau^n \to \tau$ for all sufficiently large n.*
(5) *If $\varphi \to \tau$ and φ is expansive, then $\varphi^n\tau^{-1} \to \tau$ for some $n \geq 1$.*
(6) *If $\varphi_1, \varphi_2 \to \tau$ with $\varphi_1\varphi_2 = \varphi_2\varphi_1$ and if φ_1 is expansive or $\varphi_1^m\varphi_2^n$ is expansive for some $m, n \geq 1$, then $\varphi_1\varphi_2 \to \tau$ with $\varphi_1\varphi_2$ expansive.*

Proof. (1) holds by Remark 7.4; (2) holds by Proposition 9.1; (3) holds by Proposition 9.2; (4) holds by Proposition 7.10; (5) holds by Proposition 7.13; (6) holds by Proposition 9.3. □

Throughout the remainder of this section let K be a commutative subgroup of $H(X)$. For an expansive element $\tau \in K$, $C_K(\tau) = \{\varphi \in K \mid \varphi \to \tau\}$ is the cone of τ in K, by definition. (This notion appears in [**21,** §6] with a different term.) We can call this *a cone in K*, since $C_K(\tau) = C_K(\tau')$ for every $\tau' \in C_K^\circ(\tau)$. Indeed, by Proposition 9.5(1)(2)(3), the relation $\to$ restricted on the set of all expansive elements of K, say $E(K)$, is an equivalence relation on $E(K)$ and the *interior* $C_K^\circ(\tau)$ of the cone $C_K(\tau)$ is the equivalence class containing τ.

The interior of a cone is closely related to the zero-dimensional case of an 'expansive component of 1-frames' for a $\mathbb{Z}^d$-action in the theory 'expansive subdynamics' developed by Boyle and Lind [**11**], which studies commuting homeomorphisms of (not necessarily zero-dimensional) compact metric spaces in a framework treating not only 'rational

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directions' but also 'irrational directions'. In fact, one can prove the following using [**11,** Lemma 6.28] and Proposition 7.5(2).

Remark 9.6. Let φ be an automorphism of a subshift (X, σ) with infinitely many points. Let K be the subgroup of $H(X)$ generated by $\{\sigma, \varphi\}$. Let $\alpha : \mathbb{Z}^2 \to K$ be the $\mathbb{Z}^2$-action with $\alpha^{(i,j)} = \varphi^i \sigma^j$. Then for $i, j, i', j' \in \mathbb{Z}$, $\varphi^i \sigma^j$ and $\varphi^{i'} \sigma^{j'}$ belong to the interior of the same cone in K if and only if the vectors (i, j) and (i', j') belong to the same expansive component of 1-frames for α in the sense of the definition of Boyle and Lind [**11**].

PROPOSITION 9.7. *Let φ and τ be expansive elements of K. If $\varphi \rightharpoonup \tau$, $\varphi' \in C_K^\circ(\varphi)$ and $\tau' \in C_K^\circ(\tau)$, then $\varphi' \rightharpoonup \tau'$. In this statement, '$\rightharpoonup$' can be replaced by '$\rightharpoondown$'.*

Proof. Since $\varphi \rightharpoonup \tau$ and $\tau \to \tau'$, we have $\varphi \rightharpoonup \tau'$ by Proposition 9.4(2). Hence by Proposition 9.4(1) we have $\tau' \rightharpoondown \varphi$. From this and $\varphi \to \varphi'$, it follows that $\tau' \rightharpoondown \varphi'$. Thus we have $\varphi' \rightharpoonup \tau'$.

The '$\rightharpoondown$' version is similarly proved. □

Therefore, the relations '$\rightharpoonup$' and '$\rightharpoondown$' can be defined on the set of all interiors C°'s of all cones C's in K as their quotient relations with respect to the equivalence relation $\to$ on $E(K)$, where C° denotes the interior of C. Let C and C' be cones in K. We write $C^\circ \rightharpoonup C'^\circ$ (respectively $C^\circ \rightharpoondown C'^\circ$) if there exist $\varphi \in C^\circ$ and $\tau \in C'^\circ$ with $\varphi \rightharpoonup \tau$ (respectively, $\varphi \rightharpoondown \tau$). We write $C \rightharpoonup C'$ (respectively, $C \rightharpoondown C'$) if $\varphi \rightharpoonup \tau$ (respectively, $\varphi \rightharpoondown \tau$) for all (not necessarily expansive) elements $\varphi \in C$ and some element $\tau \in C'^\circ$.

We do not know whether $C^\circ \rightharpoonup C'^\circ$ implies $C \rightharpoonup C'$ (or whether $\varphi \to \tau \rightharpoonup \tau'$ implies $\varphi \rightharpoonup \tau'$).

Example 9.8. Let (X_B, σ_B) and ψ be the same as in (3) of Example 5.6. Let $X = X_B$ and let $K = \{\psi^k \sigma_B^n \mid k, n \in \mathbb{Z}\}$. $\psi^k \sigma_B^n$ is non-expansive only on the three 'lines' $\{\psi^k \sigma_B^n \mid n = 0\}$, $\{\psi^k \sigma_B^n \mid k = rn\}$ and $\{\psi^k \sigma_B^n \mid k = -sn\}$. There are six cones in K which, in fact, are ELR cones (see below), given by

$$\Delta_1 = \{\psi^k \sigma_B^n \mid n \le 0,\ k \ge -sn\}, \qquad \Delta_2 = \{\psi^k \sigma_B^n \mid n \le 0,\ rn \le k \le -sn\},$$
$$\Delta_3 = \{\psi^k \sigma_B^n \mid n \le 0,\ k \le rn\}, \qquad \bar{\Delta}_i = \{\psi^k \sigma_B^n \mid \psi^{-k} \sigma_B^{-n} \in \Delta_i\} \quad \text{for } i = 1, 2, 3.$$

One can see that all the arrow relations between the different cones are given by

$$\bar{\Delta}_1 \rightharpoonup \bar{\Delta}_2, \quad \bar{\Delta}_3 \rightharpoonup \bar{\Delta}_2, \quad \Delta_2 \rightharpoonup \Delta_1, \quad \bar{\Delta}_3 \rightharpoonup \Delta_1, \quad \Delta_2 \rightharpoonup \Delta_3, \quad \bar{\Delta}_1 \rightharpoonup \Delta_3,$$
$$\bar{\Delta}_2 \rightharpoondown \bar{\Delta}_1, \quad \bar{\Delta}_2 \rightharpoondown \bar{\Delta}_3, \quad \Delta_1 \rightharpoondown \Delta_2, \quad \Delta_1 \rightharpoondown \bar{\Delta}_3, \quad \Delta_3 \rightharpoondown \Delta_2, \quad \Delta_3 \rightharpoondown \bar{\Delta}_1.$$

The following includes an extension of a zero-dimensional version of [**22,** Theorem 4.3].

PROPOSITION 9.9. (1) *An essentially LR endomorphism of an SFT has POTP.*
(2) *If an expansive element $\tau \in K$ has POTP, then so does every element of $C_K(\tau)$.*

Proof. (1) This follows from Proposition 3.4, because an LR textile system is finitely saturated.

(2) By Theorem 8.8(4), $C_K(\tau)$ is the set of all essentially LR automorphisms of (X, τ). Hence (2) is proved by (1). □

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For an expansive element $\tau \in K$ with POTP, $C_K(\tau)$ is called the *ELR cone of* τ *in* K or an *ELR cone in* K. (This notion appears in [**21,** §6]. A different definition corresponding to it with the same term was introduced in [**20,** §8] for a typical class of K.) Every element of an ELR cone has POTP, and every element of its interior defines a dynamical system which is conjugate to an SFT. Therefore, the interior of an ELR cone is closely related to the zero-dimensional case of a 'Markov component of 1-frames' for a $\mathbb{Z}^d$-action introduced by Boyle and Lind [**11**], in the same sense as stated in Remark 9.6.

Remark 9.10. Let φ be an automorphism of a subshift (X, σ) with infinitely many points. Let K be the subgroup of $H(X)$ generated by $\{\sigma, \varphi\}$. Let $\alpha : \mathbb{Z}^2 \to K$ be the $\mathbb{Z}^2$-action with $\alpha^{(i,j)} = \varphi^i \sigma^j$. Then for $i, j, i', j' \in \mathbb{Z}$, $\varphi^i \sigma^j$ and $\varphi^{i'} \sigma^{j'}$ belong to the interior of the same ELR cone in K if and only if the vectors (i, j) and (i', j') belong to the same Markov component of 1-frames for α in the sense of the definition of Boyle and Lind [**11**].

PROPOSITION 9.11. *Suppose that* τ *is an expansive, topologically-mixing element of* K *and that some expansive element of* $D_K(\tau) = PL_K(\tau) \cup QR_K(\tau)$ *has POTP. Then all left* τ*-expansive elements of* $PL_K(\tau)$ *and all right* τ*-expansive elements of* $QR_K(\tau)$ *have POTP and are topologically mixing, and in particular all expansive elements of* $D_K(\tau)$ *have POTP and are topologically mixing.*

Proof. Let τ' be an expansive element of $D_K(\tau)$ which has POTP. Then by Proposition 9.1(1), $\tau \in D_K(\tau')$. Therefore, since (X, τ') is conjugate to an SFT, it follows from Theorem 8.8 that τ is an essentially p-L or essentially q-R automorphism of (X, τ'). Since τ' is expansive, has POTP and commutes with a topologically-mixing element τ, τ' is topologically mixing by [**22,** Theorem 1.6]. Therefore by Theorem 5.2 (with Proposition 5.1), (X, τ) is conjugate to an SFT. Hence (X, τ) is conjugate to a mixing SFT. Thus using Theorems 8.8 and 6.3 we see that all left τ-expansive elements of $PL_K(\tau)$ and all right τ-expansive elements of $QR_K(\tau)$ have POTP and are topologically mixing. □

COROLLARY 9.12. *Let* C *and* C' *be any cones in* K *such that* $C^\circ \rightharpoonup C'^\circ$ *or* $C^\circ \rightharpoondown C'^\circ$. *Then if either one of* C *and* C' *is an ELR cone containing an expansive, topologically-mixing element, then so is the other.*

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