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On transversality

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The simplest definition of transversality in the PL or Top categories is the purely local one: Manifolds M and N are transverse in Q if, for each point x of their intersection, there is a (closed) neighbourhood U of x in Q and a (PL) homeomorphism

$$h\colon U, U \cap M, U \cap N \to I^p \times I^q \times I^r, \quad I^p \times 0 \times I^r, \quad 0 \times I^q \times I^r$$

for suitable p, q, and r

In (1), Armstrong and Zeeman prove that, in the PL case, one manifold may be shifted by an arbitrarily small PL ambient isotopy so that it is transverse with respect to another. However, if two manifolds are already transverse on some subset, we would like to keep them fixed there. Armstrong and Zeeman make the following conjecture.

Relative transversality conjecture: If M, N are PL manifolds in a PL manifold Q with $M \cap \partial Q = \partial M$, $N \cap \partial Q = \partial N$ and ∂M, ∂N transverse in ∂Q, then there is an ambient isotopy, fixed on the boundary, throwing M into a position transverse with respect to N.

THEOREM 1. *The relative transversality conjecture is false.*

In fact we have

THEOREM 2. *For all m, n, q with $m+n-q = 4k$, $m, n \geqslant 8k+2$, there are PL transverse PL spheres S^m, S^n in S^q which do not bound topological transverse balls B^{m+1}, B^{n+1} in B^{q+1}.*

Notation. $T_{q,m,n}$ will denote the 'standard triple'

$$(I^{m+n-q} \times I^{q-n} \times I^{q-m}, I^{m+n-q} \times I^{q-n} \times 0, I^{m+n-q} \times 0 \times I^{q-m}).$$

$\partial T_{q,m,n}$ will denote the triple of boundary spheres.

LEMMA 1. *Let B^m, $B^n \subset B^q$ be properly embedded PL balls [i.e. $B^m \cap \partial B^q = \partial B^n$, $B^n \cap \partial B^q = \partial B^n$]. Suppose that $B^m \cap B^n$ is a PL ball B^{m+n-q} in B^q. If $m, n \leqslant q-3$, then*

$$(B^q, B^m, B^n) \cong T_{q,m,n}$$

if and only if the inclusion

$$B^m - B^{m+n-q} \subset B^q - B^n$$

is a homotopy equivalence. ($\cong$ denotes PL homeomorphism.)

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Proof. The 'only if' is immediate. The 'if' part is proved by induction on dimension, making strong use of Lickorish's theorem on unknotting cones (2).

First, suppose S^p, S^q are disjoint PL embedded spheres in S^{p+q+1} with $p, q \geqslant 2$. Then Zeeman has shown (6) that the isotopy class of the embedding is determined by the linking number. So

$$S^{p+q+1}, S^p, S^q \cong \partial T_{p+q+2,\, p+1,\, q+1}$$

if and only if the inclusion $S^p \subset S^{p+q+1} - S^q$ is a homotopy equivalence. Now suppose $B^m, B^n \subset B^q$ are as in the conditions of this lemma. By Zeeman's theorem on unknotting balls (B^m, B^{m+n-q}) and (B^n, B^{m+n-q}) are unknotted ball pairs.

$$B^m \cup B^n \cong I^{m+n-q} \times [(I^{q-n} \times 0) \cup (0 \times I^{q-m})]$$

$$\cong \text{the cone on } (\partial B^m \cup \partial B^n).$$

So by Lickorish's theorem on unknotting cones

$$B^q, B^m, B^n \cong v\,.\,\partial B^q,\ v\,.\,\partial B^m,\ v\,.\,\partial B^n.$$

So a PL homeomorphism

$$\partial B^q, \partial B^m, \partial N^n \to \partial T_{q,m,n}$$

always extends to a PL homeomorphism

$$B^q, B^m, B^n \to T_{q,m,n} \quad (\text{provided } m, n \leqslant q-3).$$

Combining this with the results above proves Lemma 1 for the case $m+n=q$.

The inductive step. If $B^m, B^n \subset B^q$ satisfy the conditions of this lemma and $m+n-q \geqslant 1$, let x be a point of $\partial B^m \cap \partial B^n$. Assume B^q is triangulated with B^m, B^n as subcomplexes and x as a vertex. Then $\overline{\text{star}}\,(x, \partial B^m)$, $\overline{\text{star}}\,(x, \partial B^n)$ are properly embedded balls in $\overline{\text{star}}\,(x, \partial B^q)$. Moreover, in the commutative diagram of inclusions

$$\begin{array}{ccc}
\overline{\text{star}}(x,\partial B^m) - \overline{\text{star}}(x,\partial B^m \cap \partial B^n) & \xrightarrow{i_1} & B^m - B^m \cap B^n \\
\downarrow j_1 & & \downarrow j_2 \\
\overline{\text{star}}(x,\partial B^q) - \overline{\text{star}}(x,\partial B^n) & \xrightarrow{i_2} & B^q - B^n
\end{array},$$

both i_1 and i_2 are homotopy equivalences. So j_1 is a homotopy equivalence if j_2 is. Let $B_1^{m-1} = \overline{\text{star}}\,(x, \partial B^m)$, $B_2^{m-1} = \text{cl}\,[\partial B^m - B_1^{m-1}]$. Similarly, define $B_1^{n-1}, B_2^{n-1}, B_1^{q-1}, B_2^{q-1}$. Now, assuming the lemma to hold true in lower dimensions, there is a PL homeomorphism

$$h_1 \colon B_1^{q-1}, B_2^{m-1}, B_1^{n-1} \to T_{q-1,\, m-1,\, n-1}.$$

Now apply Lickorish's theorem on unknotting cones, just as before, to B_2^{m-1} and B_2^{n-1} in B_2^{q-1}. We get a PL homeomorphism

$$h_2 \colon B_2^{q-1}, B_2^{m-1}, B_2^{n-1} \to T_{q-1,\, m-1,\, n-1}$$

with $h_2 | \partial B_2^{q-1} = h_1 | \partial B_1^{q-1}$. Now $2T_{q-1,\, m-1,\, n-1} \cong \partial T_{q,m,n}$ where the left-hand side means taking two copies and identifying the boundary. So $\partial B^q, \partial B^m, \partial B^n \cong \partial T_{q,m,n}$. Applying Lickorish's theorem again shows that $B^q, B^m, B^n \cong T_{q,m,n}$.

Now suppose that W^{m+N}, Q^q are PL manifolds, $i \colon W \to Q \times R^N$ a PL embedding, and

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suppose that $i^{-1}(Q \times 0) = M^m$, a properly embedded m-manifold in M having a normal PL R^N-bundle γ in W.

LEMMA 2. *If $q-m \geqslant 3$, $N \geqslant 3$, then W is transverse to $Q \times 0$ in $Q \times R^N$ if and only if the map $p_2 i\colon E(\gamma) \to R^N$ is a fibre homotopy trivialization.*

$p_2\colon Q \times R^N \to R^N$ is the natural projection. $E(\gamma) \subset W$ is the total space of γ. A map $E(\gamma) \to R^N$ is said to be a fibre homotopy trivialization if for each fibre F of γ, $F - F \cap M$ is mapped into $R^N - 0$ by a map of degree ± 1.

Proof of Lemma 2. This is now a fairly direct application of Lemma 1. For let $x \in M$. Let F be the fibre of γ lying over x and let D be a closed PL N-ball in F containing the origin in its interior.

Triangulate $W, Q \times R^N, R^N$ so that i, p_2 are simplicial and $D, Q \times 0$ are subcomplexes. Let

$$B^{q+N} = \overline{\text{star}}\,(ix, Q \times R^N),$$

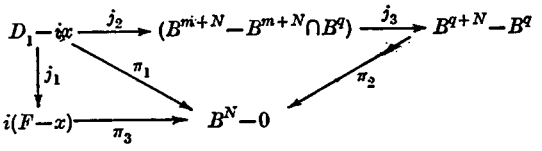
$$B^{m+N} = \overline{\text{star}}\,(ix, iW),$$

$$B^q = \overline{\text{star}}\,(ix, B^q \times 0),$$

$$D_1 = \overline{\text{star}}\,(ix, iD),$$

$$B^N = \overline{\text{star}}\,(0, R^N).$$

Then we have a commutative diagram of maps



π, π_2, π_3 being the appropriate restrictions of p_2. π_2, j_1, j_2 are homotopy equivalences. So π_3 is a homotopy equivalence if and only if j_3 is.

By Lemma 1, j_3 is a homotopy equivalence if and only if

$$(B^{q+N}, B^{m+N}, B^q) \cong T_{q+N,\, m+N,\, q},$$

which is true if and only if $W, Q \times 0$ are transverse at x. This holds for all points x in M and thus the lemma is proved.

Proof of Theorem 2. Let $m = 4k$ and let M be one of the Milnor–Kervaire manifolds with

(1) $\partial M \cong S^{m-1}$,

(2) M has a trivial stable normal bundle,

(3) index $(M) \neq 0$.

Let $M_1 = M \cup B^m$, the boundaries being identified. By (3) and (5), for $N \geqslant m+2$ there is an embedding of M_1 in S^{m+N} with a normal closed disc bundle, γ say, with $\gamma|M$ trivial, and so there is a PL map $\alpha\colon E(\gamma|M) \to R^N$ with $R^{-1}(0) = M$, and each fibre embedded. If $D =$ closure $[S^{m+N} - E(\gamma|B^m)]$, we can extend α to a PL map $\alpha'\colon D \to R^N$ with $(\alpha')^{-1}\,0 = M$, for the only obstructions would lie in

$$H^i[D - M, E(\gamma|M) - M;\, \pi_{i-1}(S^{N-1})]$$

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which will always vanish for $N > m$. Now $(E(\gamma|B^m), B^m)$ is an unknotted ball pair. So we can extend α' conically to get a map $\alpha'': S^{m+N} \to R^N$ with

$$(\alpha'')^{-1}\,0 = M \cup B^m = M_1$$

$\alpha''|E(\gamma)$: $E(\gamma) \to R^N$ is a fibre homotopy trivialization.

If $j: S^{m+N} \to S^q$ is a PL embedding with $q - m \geqslant 3$ define $i: S^{m+N} \to S^q \times E^N$ by $i(x) = jx \times \alpha'' x$. By Lemma 2, iS^{m+N} is transverse to $S^q \times 0$ in $S^q \times E^N$ and they intersect in M_1. Taking the natural PL homeomorphism

$$S^q \times E^N, S^q \times 0 \to S^{q+N} - S^{N-1}, S^q$$

we get S^{m+N} and S^q in S^{q+N} meeting transversely in M_1. Now S^{m+N}, S^q cannot bound transverse balls in B^{q+N+1}, since M_1 would then bound the intersection of these two balls, which would be a manifold. But index $(M_1) \neq 0$, and so M_1 is not a boundary.

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