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On the existence of ergodic continuous invariant measures for folding transformations

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Abstract. We present a sufficient condition for the existence of ergodic continuous invariant measures (ECIMs) for continuous folding transformations. We apply this result to prove the existence of ECIMs for general set expanding Markov transformations and n-dimensional tent maps.

1. *Introduction*

Let X be a Hausdorff topological space. By a measure we mean any regular probabilistic measure defined on the Borel σ-algebra of X.

Definition 1. We say that a transformation $T : X \to X$ is *folding* if there exists a collection of non-empty compact sets $A_1, \ldots, A_N \subset X$ such that

$$\bigcup_{i=1}^{N} A_i \subset \bigcap_{i=1}^{N} T A_i. \tag{1}$$

If, also, $A_1 \cap \cdots \cap A_N = \emptyset$, then T is called *N-adic*.

In 1979, Lasota and Pianigiani [**5**], using Avez's [**2**] method, proved that a continuous 2-adic transformation admits an ergodic continuous invariant measure (ECIM). Then, in 1981, Rudnicki [**7**], with the help of the technique developed by Lasota and Yorke [**6**], generalized this result to continuous N-adic transformations. But the condition of N-adicity for a given transformation (or one of its iterates) is usually more difficult to check than the folding condition which does not require the compact sets $A_1, \ldots, A_N$ to be disjoint.

In this paper, we present a further generalization of that result: we prove that under a mild condition, a folding transformation admits an ECIM. The idea of the proof is to construct a new collection of compact subsets $A_i' \subset A_i$ $(i = 1, \ldots, N)$ with respect to which some iterate T^p $(p \geq 1)$ is N-adic.

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In §§3 and 4, we apply our results to prove, respectively, the existence of ECIMs for two general classes of folding transformations: set expanding Markov transformations (satisfying certain mild conditions) on Hausdorff spaces and generalized n-dimensional tent maps. The set expanding condition is applied to sets rather than points and, as such, is much milder than the pointwise expansion condition, which ensures the existence of an absolutely continuous invariant measure [**1**].

2. *Main results*

First, we present in the following theorem a special case of our sufficient condition for the existence of an ECIM.

THEOREM 1. *Let $T : X \to X$ be a continuous folding transformation with respect to sets $A_1, \ldots, A_N$, and let $B = A_1 \cap \cdots \cap A_N$, $B^{\cap} = \bigcap_{k=0}^{\infty} T^{-k}B$. Suppose that $B^{\cap} = \emptyset$. Then T admits an ECIM.*

To prove this theorem, we need the following two lemmas.

LEMMA 1. *A Borel-measurable transformation $T : X \to X$ admits an ECIM if some iterate T^p $(p \geq 1)$ admits an ECIM.*

LEMMA 2. *Given any mapping f and any sets U, V:*

(a) $f(U \cap f^{-1}(V)) = f(U) \cap V$;

(b) $f(U \setminus V) \supset f(U) \setminus f(V)$.

The proof of Lemma 1 is easy; see, for instance, [**5**]. The proof of Lemma 2 is straightforward.

Proof of Theorem 1. Step 1. Consider the nested sequence of sets,

$$B \supset (B \cap T^{-1}B) \supset (B \cap T^{-1}B \cap T^{-2}B) \supset \cdots,$$

with empty intersection $B^{\cap}$. Since B is compact and T is continuous, all of these sets are compact. Therefore, starting with one of them, all the subsequent sets are empty, i.e.

$$\exists m \geq 0 : \bigcap_{k=0}^{m} T^{-k}B = \emptyset. \tag{2}$$

We shall show that the iterate T^{m+1} is N-adic with respect to the compact sets $D_1, \ldots, D_N$ defined by

$$D_i = \bigcap_{k=0}^{m} T^{-k}A_i, \quad \forall i = 1, \ldots, N.$$

Step 2. We prove that each $D_i \neq \emptyset$. Applying Lemma 2(a) m times and using the fact that $TA_i \supset A_i$, we obtain,

$$T^m D_i = T^m(A_i \cap T^{-1}A_i \cap \cdots \cap T^{-m}A_i) = A_i \neq \emptyset, \tag{3}$$

which implies that $D_i \neq \emptyset$.

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Step 3. We prove that $D_1 \cap \cdots \cap D_N = \emptyset$. Using (2), we get

$$\bigcap_{i=1}^{N} D_i = \bigcap_{i=1}^{N} \bigcap_{k=0}^{m} T^{-k} A_i = \bigcap_{k=0}^{m} \bigcap_{i=1}^{N} T^{-k} A_i = \bigcap_{k=0}^{m} T^{-k} \bigcap_{i=1}^{N} A_i = \bigcap_{k=0}^{m} T^{-k} B = \emptyset.$$

Step 4. We prove that condition (1) holds for T^{m+1} and the sets $D_1, \ldots, D_N$, i.e. that

$$\bigcup_{i=1}^{N} D_i \subset \bigcap_{i=1}^{N} T^{m+1} D_i.$$

Indeed, from (3) and (1), we obtain for any $i, j = 1, \ldots, N$,

$$T^{m+1} D_i = T(T^m D_i) = T A_i \supset A_j \supset D_j.$$

Thus, T^{m+1} is an N-adic continuous transformation and, as such, admits an ECIM. Hence, by Lemma 1, T also admits an ECIM. □

Definition 2. Let $T : X \to X$ be a transformation and $A_1, \ldots, A_N$ be a collection of non-empty compact sets in X. Denote $B = A_1 \cap \cdots \cap A_N$ and $B^{\cap} = \bigcap_{k=0}^{\infty} T^{-k} B$. We say that a transformation T is *strict* with respect to $A_1, \ldots, A_N$ iff there exists a neighborhood $\mathcal{O}$ (possibly, empty set) of the set $B^{\cap}$ such that

$$\bigcup_{i=1}^{N} A_i \not\subset \bigcup_{k=0}^{\infty} T^k \mathcal{O}. \tag{4}$$

The following theorem is more general than Theorem 1, covering cases where $B^{\cap} \neq \emptyset$.

THEOREM 2. *Let $T : X \to X$ be a continuous transformation which is folding and strict with respect to sets $A_1, \ldots, A_N \subset X$. Then T admits an ECIM.*

Proof. Step 1. Let the sets B, $B^{\cap}$, $\mathcal{O}$ be as defined in Definition 2. Without loss of generality, assume that $\mathcal{O}$ is open. Consider the nested sequence of sets:

$$B \setminus \mathcal{O} \supset (B \cap T^{-1} B) \setminus \mathcal{O} \supset (B \cap T^{-1} B \cap T^{-2} B) \setminus \mathcal{O} \supset \cdots$$

with empty intersection $B^{\cap} \setminus \mathcal{O}$. Since B is compact, $\mathcal{O}$ is open, and T is continuous, it follows that all the sets in the sequence are compact. Therefore, starting with one of them, all the subsequent sets are empty, i.e.

$$\exists m \geq 0 : \bigcap_{k=0}^{m} T^{-k} B \subset \mathcal{O}. \tag{5}$$

We shall show that the iterate T^{m+1} is N-adic with respect to the compact sets $\bar{D}_1, \ldots, \bar{D}_N$ (where $\bar{D}_i$ denotes the closure of D_i) defined by

$$D_i = \left(\bigcap_{k=0}^{m} T^{-k} A_i \right) \setminus \mathcal{O}^{\cup}, \quad \forall i = 1, \ldots, N,$$

where $\mathcal{O}^{\cup} = \bigcup_{k=0}^{\infty} T^k \mathcal{O}$. Note that the set $\mathcal{O}^{\cup}$ is not necessarily open and, for any $p \geq 0$,

$$T^p \mathcal{O}^{\cup} = \bigcup_{k=p}^{\infty} T^k \mathcal{O} \subset \mathcal{O}^{\cup}.$$

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Step 2. We prove that each $D_i \neq \emptyset$. Applying Lemma 2(b), then Lemma 2(a) m times, and finally, the inclusions $TA_i \supset A_i$ and $T^{m+1}\mathcal{O}^{\cup} \subset \mathcal{O}^{\cup}$, we have

$$T^{m+1}D_i \supset T^{m+1}\left(\bigcap_{k=0}^{m} T^{-k}A_i\right) \setminus T^{m+1}\mathcal{O}^{\cup}$$

$$= TA_i \setminus T^{m+1}\mathcal{O}^{\cup} \supset TA_i \setminus \mathcal{O}^{\cup} \supset \left(\bigcup_{i=1}^{N} A_i\right) \setminus \mathcal{O}^{\cup}. \tag{6}$$

Together with assumption (4) of the theorem, it implies that $T^{m+1}D_i \neq \emptyset$, which in turn implies $D_i \neq \emptyset$.

Step 3. We prove that $\bar{D}_1 \cap \cdots \cap \bar{D}_N = \emptyset$. Using the obvious inclusion $\bar{D}_i \subset \left(\bigcap_{k=0}^{m} T^{-k}A_i\right) \setminus \mathcal{O}$ together with (5), we have

$$\bigcap_{i=1}^{N} \bar{D}_i \subset \bigcap_{i=1}^{N} \left(\left(\bigcap_{k=0}^{m} T^{-k}A_i\right) \setminus \mathcal{O}\right) = \left(\bigcap_{k=0}^{m}\bigcap_{i=1}^{N} T^{-k}A_i\right) \setminus \mathcal{O} = \left(\bigcap_{k=0}^{m} T^{-k}\bigcap_{i=1}^{N} A_i\right) \setminus \mathcal{O}$$

$$= \left(\bigcap_{k=0}^{m} T^{-k}B\right) \setminus \mathcal{O} = \emptyset.$$

Step 4. We show that condition (1) holds for T^{m+1} and the sets $\bar{D}_1, \ldots, \bar{D}_N$, i.e. that

$$\bigcup_{i=1}^{N} \bar{D}_i \subset \bigcap_{i=1}^{N} T^{m+1}\bar{D}_i. \tag{7}$$

Indeed, from (6) we obtain for any $p, q = 1, \ldots, N$:

$$T^{m+1}D_p \supset \left(\bigcup_{i=1}^{N} A_i\right) \setminus \mathcal{O}^{\cup} \supset A_q \setminus \mathcal{O}^{\cup} \supset D_q.$$

Next, since the set $T^{m+1}\bar{D}_p$ is the image of a compact set under a continuous transformation, it is closed. Therefore, since $T^{m+1}\bar{D}_p$ contains D_q, it also contains $\bar{D}_q$, which implies (7).

Thus, T^{m+1} is an N-adic continuous transformation and, as such, admits an ECIM. Hence, by Lemma 1, T also admits an ECIM. □

3. *Applications to Markov transformations*

Let $\mathcal{P} = \{A_1, A_2, \ldots, A_N\}$ be a collection of non-empty compact sets in X. Given a transformation $T : X \to X$, define the adjacency matrix $\mathbb{M}_T = (m_{ij})_{1 \leq i,j \leq n}$ in which

$$m_{ij} = \begin{cases} 1, & \text{if } T(A_i) \supset A_j \\ 0, & \text{otherwise.} \end{cases}$$

Possibly renumbering the sets A_i, we can assume that the adjacency matrix is represented in a normal block form,

$$\mathbb{M}_T = \begin{pmatrix} M_{11} & 0 & \ldots & 0 \\ M_{21} & M_{22} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ M_{s1} & M_{s2} & \ldots & M_{ss} \end{pmatrix},$$

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where each block M_{ii} $(i = 1, \ldots, s)$, is either irreducible, or a 1×1 null matrix.

Definition 3. A real matrix M is called *primitive (or ℓ-primitive)*, if there exists an integer $\ell \geq 1$ such that all entries of M^ℓ are strictly positive, i.e. $M^\ell > 0$.

PROPOSITION 1. *Let $T : X \to X$ be a continuous transformation. Suppose that the adjacency matrix $\mathbb{M}_T$ has a primitive block M_{ii} $(i = 1, \ldots, s)$ of size at least 2×2, such that T is strict with respect to the sets in $\mathcal{P}$ which form M_{ii}. Then T admits an ECIM.*

Proof. This follows easily from Theorem 2 and the obvious observation that an iterate T^ℓ is folding with respect to sets in $\mathcal{P}$ which form the ℓ-primitive block M_{ii}. □

Definition 4. A continuous transformation $T : X \to X$ is called *Markov* (with respect to the collection $\mathcal{P}$) if, for any $i = 1, \ldots, N$,

$$T(A_i) = \bigcup_{j \in J_i} A_j,$$

where J_i is a subset of $\{1, 2, \ldots, N\}$. We say that a Markov transformation T is *set expanding*, if each set J_i consists of more than one element.

The following theorem is a consequence of Proposition 1.

THEOREM 3. *Let $T : X \to X$ be a set expanding Markov transformation with respect to $\mathcal{P}$. Suppose that T has a periodic point and is strict with respect to each subfamily of $\mathcal{P}$ which forms a block M_{ii} $(i = 1, \ldots, s)$ of $\mathbb{M}_T$. Then T admits an ECIM.*

4. *Applications to n-dimensional tent maps*

An n-dimensional analogue $\hat{\psi} : [0, 1]^n \to [0, 1]^n$ of the one-dimensional tent map $\hat{\psi}_b : [0, 1] \to [0, 1]$ defined by

$$\hat{\psi}_b(x) = \min\left\{\frac{x}{b}, \frac{1 - x}{1 - b}\right\}, \tag{8}$$

where $b \in (0, 1)$ denotes the turning point, was studied in [**3**]. It was proved that under certain assumptions, the 2-dimensional map $\hat{\psi}$ preserves Lebesgue measure. We define the map $\hat{\psi}$ in a more general setting and show that it satisfies the conditions of Theorem 1 and, therefore, admits ECIMs.

Definition 5. Let $\psi : [0, 1]^n \to (0, 1)^n$ be a continuous map, which we shall refer to as a *spine map*. Define a *spine tent map* $\hat{\psi} : [0, 1]^n \to [0, 1]^n$ by

$$\hat{\psi}_i(x) = \min\left\{\frac{x_i}{\psi_i(x)}, \frac{1 - x_i}{1 - \psi_i(x)}\right\}, \quad \forall x \in [0, 1]^n \ \forall i = 1, \ldots, n,$$

where $\hat{\psi} = (\hat{\psi}_1, \ldots, \hat{\psi}_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$.

Note that this definition coincides with (8), when $n = 1$ and $\psi = \psi_b \equiv b$. Obviously, a spine tent map $\hat{\psi}$ is continuous because each of its components $\hat{\psi}_i$ is continuous, being the minimum of two continuous functions. It is also easy to see that any spine map ψ yields

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a covering of the n-dimensional cube $[0,1]^n$ by 2^n sets R_v defined for each of its vertices $v \in \{0,1\}^n$ as follows:

$$R_v = \{x \in [0,1]^n \mid \forall i = 1,\ldots,n;\ x_i \le \psi_i(x) \text{ if } v_i = 0 \text{ and } x_i \ge \psi_i(x) \text{ if } v_i = 1\}.$$

Moreover, each set R_v is non-empty and closed, since $v \in R_v$ and R_v is the preimage of the closed subset $[-v_1, 1-v_1] \times \cdots \times [-v_n, 1-v_n]$ under the continuous map $x \mapsto \psi(x) - x$ acting from $[0,1]^n$ to $\mathbb{R}^n$. Note that the intersection of all sets R_v is exactly the set of fixed points of the spine map ψ, i.e.

$$\bigcap_{v \in \{0,1\}^n} R_v = \{x \in [0,1]^n \mid x = \psi(x)\}. \tag{9}$$

It is easy to check that the restriction of $\hat{\psi}$ to a set R_v can be defined as follows:

$$\hat{\psi}_i(x) = \frac{v_i - x_i}{v_i - \psi_i(x)}, \quad \forall x \in R_v\ \forall i = 1,\ldots,n. \tag{10}$$

PROPOSITION 2. *A spine tent map $\hat{\psi}$ is folding with respect to the sets R_v, $v \in \{0,1\}^n$.*

Proof. We need to prove that each set R_v is mapped onto the entire cube $[0,1]^n$. To this end, let us fix any vertex $v \in \{0,1\}^n$, any point $a \in [0,1]^n$, and show that there exists $x \in R_v$ such that $\hat{\psi}(x) = a$. By (10), this is equivalent to the existence of a solution $x \in [0,1]^n$ of the system of equations:

$$\frac{v_i - x_i}{v_i - \psi_i(x)} = a_i, \quad \forall i = 1,\ldots,n. \tag{11}$$

Indeed, if x is a solution of the system, it must satisfy the inequalities:

$$\frac{v_i - x_i}{v_i - \psi_i(x)} \le 1, \quad \forall i = 1,\ldots,n,$$

which is equivalent to $x \in R_v$. Now rewrite the system (11) in the form:

$$x_i = v_i - a_i(v_i - \psi_i(x)), \quad \forall i = 1,\ldots,n.$$

The set of solutions of this system clearly coincides with the set of fixed points of the continuous transformation $x \mapsto v - a \cdot (v - \psi(x))$ acting from the cube $[0,1]^n$ to itself. But the existence of a fixed point of this transformation is guaranteed by Brouwer's theorem [**4**, p. 468], which states (in the extended form) that any continuous map from a compact convex subset of $\mathbb{R}^n$ to itself has a fixed point. This completes the proof. □

Thus, by Proposition 2, a spine tent map $\hat{\psi}$ belongs to the larger class of spine folding maps associated with ψ in the following sense.

Definition 6. We say that a continuous map $\tau : [0,1]^n \to [0,1]^n$ is a *spine folding map* associated with a spine map ψ if

(i) $\forall i = 1,\ldots,n;\ \tau_i(x) = \begin{cases} 0 & \text{if } x_i = 0, 1 \\ 1 & \text{if } x_i = \psi_i(x), \end{cases}$ where $\tau = (\tau_1, \ldots, \tau_n)$;

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(ii) $\forall v \in \{0,1\}^n;\ \tau R_v = [0,1]^n$.

PROPOSITION 3. *A spine folding map (and, in particular, a spine tent map) satisfies the conditions of Theorem 1 and, therefore, admits an ECIM.*

Proof. Since a spine folding map τ associated with ψ is folding with respect to the sets R_v, it suffices to show that $B \cap \tau^{-1} B = \emptyset$ or, equivalently, $B \cap \tau B = \emptyset$, where $B = \cap_{v \in \{0,1\}^n} R_v$. Let $x \in B$. By (9), we have $x = \psi(x)$, and therefore, by property (i), $\tau(x) = (1, \ldots, 1)$. But B is the set of fixed points of ψ and, as such, is entirely in $(0,1)^n$. Hence, $\tau(x) \notin B$. Thus, $B \cap \tau B = \emptyset$, which is what we needed to show. □

REFERENCES

[1] R. Adler and L. Flatto. Geodesic flows, interval maps, and symbolic dynamics. *Bull. Amer. Math. Soc.* **25** (1991), 229–334.

[2] A. Avez. Propriétés ergodiques des endomorphismes dilatants des variétés compactes. *C. R. Acad. Sci., Paris Sér. A* **266** (1968), 610–612.

[3] A. Boyarsky, W. Byers and P. Gauthier. Higher dimensional analogues of the tent maps. *Nonlinear Analysis* **11** (1987), 1317–1324.

[4] N. Dunford and J. T. Schwartz. *Linear Operators, Part I.* Interscience, New York, 1967.

[5] A. Lasota and G. Pianigiani. Invariant measures on topological spaces. *Bolletino, U.M.I.* (5) **14-B** (1977), 592–603.

[6] A. Lasota and J. Yorke. On the existence of invariant measures for transformations with strictly turbulent trajectories. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **25** (1977), 233–238.

[7] R. Rudnicki. Ergodic measures on topological spaces. *Univ. Iag. Acta Math.* **26** (1987), 231–237.

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