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Positive processes

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Abstract. In the present paper we introduce positive flows and processes, which generalize ordinary dynamical systems and stochastic processes. The theory of positive flows and processes is based on the methods of positive operator theory and operator algebras theory. Its basic concepts are the phase and positive algebras, the spectral potential, dual entropy, equilibrium measures, action functional, sensitive states, and empirical measures. Within the frames of the theory we prove the law of large numbers with respect to the sensitive states and calculate asymptotics for the probabilities of large deviations in terms of the action functional.

0. *Declaration*

In this paper we introduce the notions of a positive flow and a positive process. They generalize the ordinary dynamical systems and stochastic processes. Their main distinctive feature is the presence of not a unique probability measure on the phase space, but a family of measures which are defined as follows. For each positive flow or process with phase space X, we fix a certain convex functional on the algebra $C(X)$. This functional is called the spectral potential and is nothing else but the logarithm of the spectral radius of the corresponding weighted shift operator. It is analogous to the free energy (the Helmholtz potential) in thermodynamics. Any subgradient of the spectral potential automatically turns out to be an invariant probability measure on X. As in thermodynamics, we call these measures equilibrium measures. All of them have equal rights because there is no reason to prefer one of them rather than another.

Another peculiarity of the positive flows and processes is the fact that we treat the concept of motion mostly from the algebraic point of view. We declare an abstract semigroup of positive operators (for instance, the family of weighted shift operators) as the source of motion; first we investigate the evolution of operators instead of individual trajectories in the phase space. This approach is borrowed from quantum mechanics. Within it, the very concept of a phase space becomes secondary. This can be seen, in particular, from fact that for one and the same law of motion (evolution) one can consider

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different phase spaces. They are determined by the set of those functions whose values an observer wishes to measure. In quantum mechanics similar sets of functions are called Abelian algebras of observables. However, for a number of reasons we call them the phase algebras in the present paper.

The concepts of a positive flow and a positive process are equivalent in the sense that they are particular cases of each other. To define them one should specify some semiordered operator algebra with one distinguished element (evolution operator) and fix in this algebra some Abelian subalgebra isomorphic to the algebra of continuous functions on a Hausdorff compact set. The latter set is called the phase space of the positive flow (process).

Despite the great generality of the described approach, we succeed to prove within it some quite concrete probabilistic results. For instance, if the spectral potential is differentiable (in the sense of Gâteaux) at some point, then the large numbers law is valid for the corresponding equilibrium measure. Moreover, we obtain exponential asymptotics for the probabilities of observation of various empirical measures.

Every discrete time dynamical system can be represented as a positive flow with a fixed equilibrium measure. On the other hand, if we consider a dynamical system without a distinguished measure on the phase space, then we can assign to it many different positive flows with different families of equilibrium measures. In the same way positive processes correlate with stochastic processes.

Finally, we note that the positive flows and processes allow us to introduce in a natural way the ideology and categories of the classical equilibrium thermodynamics into the theory of dynamical systems and stochastic processes (although this is not discussed in the text).

The paper is organized as follows. In §1 we introduce the notion of a dynamical potential. We prove that both the topological pressure, which is well known in the theory of dynamical systems, and the logarithm of the spectral radius of a weighted shift operator are, in fact, dynamical potentials. For any dynamical potential by means of the Fenchel–Legendre transform we define the corresponding dual entropy. In §2 we introduce the phase algebras, which are isomorphic to the algebras of continuous functions on the phase space, and formulate a criterion for an algebra to be phase (a certain analogue to the Gelfand–Naimark theorem). Section 3 has a rather technical character and is entirely devoted to the proof of the criterion. In §4 we define the basic element of all of the subsequent constructions—the positive algebras. They inherit the most important features of operator algebras on the real spaces $C(X)$ and $L^p(X, \mu)$. It is proved that for any positive element of a positive algebra there exists a linear eigenfunctional on the algebra with eigenvalue equal to the spectral radius of the element. In §5 the definitions of a positive flow and a spectral potential are presented. It is proved that every spectral potential is a dynamical potential. From this fact it follows that the spectral potential is convex, and that its subgradient is an invariant probability measure on the phase space. In §6 we define the eigen-states of a positive flow and prove the corresponding large numbers law. In §7 we calculate asymptotics for the probabilities of large deviations by means of an action functional. Section 8 deals with positive processes. It contains a construction that enables us to identify naturally each positive process with a certain positive flow called

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the suspension over the process. On the other hand, by definition each positive flow is a positive process. So it possesses the corresponding suspension as well. In §9 we prove that in essence a positive flow and the suspension over it coincide. The concluding §10, contains several examples illustrating the fact that all of the dynamical systems and positive processes are special cases of positive flows and positive processes.

1. *Dynamical potentials and equilibrium measures*

Let X be a Hausdorff compact set. Denote by $C(X)$ the space of all real-valued continuous functions on X supplied with the supremum norm. By the Riesz theorem [**7**, Theorem IV.6.3] each linear continuous functional $\mu : C(X) \to \mathbb{R}$ is identified with a real-valued Borel measure on X, which we will denote by the same letter μ, so that $\mu(f) = \int_X f\, d\mu$ for all $f \in C(X)$. In particular, if the functional μ is positive (i.e. non-negative over non-negative functions), then the corresponding measure μ is positive as well. In addition if the functional μ is not only positive but also normalized (takes the unit value on the unit function), then the corresponding measure is a probability one.

Let $\alpha : X \to X$ be an arbitrary continuous mapping. By (X, α) we denote the discrete time dynamical system generated by α. For each $\varphi \in C(X)$ we define the weighted shift operator A_φ in $C(X)$ by the rule $A_\varphi f = e^\varphi \cdot f \circ \alpha$. Let $\lambda(\varphi)$ be the logarithm of the spectral radius of A_φ. It can be easily shown (and is shown below) that the functional $\lambda(\varphi)$ possesses a number of remarkable properties. Firstly, it is *monotone:* if $\varphi \le \psi$, then $\lambda(\varphi) \le \lambda(\psi)$. Secondly, it is *additive homogeneous:* $\lambda(\varphi + t) = \lambda(\varphi) + t$ for any $t \in \mathbb{R}$. Thirdly, $\lambda(\varphi + \psi) = \lambda(\varphi + \psi \circ \alpha)$ for all $\varphi, \psi \in C(X)$. We call this property the *strong α-invariance* of $\lambda(\varphi)$. Finally, $\lambda(\varphi)$ is *convex* with respect to φ.

If we fix a positive Borel measure μ on X, then we may consider the action of the weighted shift operator A_φ in the space $L^p(X, \mu)$, where $p \ge 1$. In this case the four properties of $\lambda(\varphi)$ mentioned above will be true as well. It is easy to check that one more known invariant of the dynamical system (X, α), the so-called topological pressure, also possesses the same properties. So we are motivated to make the following definition.

We say that a functional $\lambda(\varphi)$ on the $C(X)$ is a *dynamical potential* of the dynamical system (X, α) if $\lambda(\varphi)$ is monotone, additive homogeneous, strongly α-invariant, and convex with respect to φ. It follows from the definition that any dynamical potential satisfies the Lipschitz condition with the unit Lipschitz constant. Indeed, the monotonicity and the additive homogeneity imply that $\lambda(\varphi) - \lambda(\psi) \le \lambda(\psi + \max|\varphi - \psi|) - \lambda(\psi) = \max|\varphi - \psi|$.

Let us define the *dual entropy* of a dynamical potential $\lambda(\varphi)$ as the function of measure

$$S(\mu) = \inf_{\varphi \in C(X)} (\lambda(\varphi) - \mu(\varphi)), \quad \mu \in C^*(X). \tag{1.1}$$

The thermodynamical sense of this notion is discussed in [**4**]. Obviously, $S(\mu)$ is bounded above (because $S(\mu) \le \lambda(0)$) and differs from the Fenchel–Legendre transform of the function $\lambda(\varphi)$ only in sign. Hence, the dual entropy is always concave and upper semicontinuous (relative to the *-weak topology). Since the Legendre transform is involutive, we have

$$\lambda(\varphi) = \sup_{\mu \in C^*(X)} (\mu(\varphi) + S(\mu)). \tag{1.2}$$

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We call any subgradient of the dynamical potential $\lambda(\varphi)$ at a point $\varphi \in C(X)$ an *equilibrium measure* corresponding to φ (in other words, a linear functional $\mu : C(X) \to \mathbb{R}$ is an equilibrium measure if and only if $\lambda(\varphi + \psi) - \lambda(\varphi) \geq \mu(\psi)$ for all $\psi \in C(X)$). Evidently, the set of all equilibrium measures corresponding to any fixed φ is convex and closed (in the *-weak topology). It follows from the supporting hyperplane theorem that this set is non-empty. It consists of a unique measure μ if and only if there exists the Gâteaux derivative $\lambda'(\varphi)$. In this case $\mu = \lambda'(\varphi)$.

The definitions of the dual entropy and an equilibrium measure imply the following.

PROPOSITION 1.1. *For any $\varphi \in C(X)$ and $\mu \in C^*(X)$ the Young inequality $S(\mu) \leq \lambda(\varphi) - \mu(\varphi)$ holds. It turns into equality if and only if μ is an equilibrium measure corresponding to φ.*

PROPOSITION 1.2. *If $S(\mu) > -\infty$, then the measure μ is probability and α-invariant. In particular, this is true for all equilibrium measures.*

Proof. Let us prove it by contradiction. Firstly, in view of the additive homogeneity of the dynamical potential, $S(\mu) \leq \lambda(t) - \mu(t) = \lambda(0) + t(1 - \mu(1))$ for all $t \in \mathbb{R}$. Hence, if $\mu(1) \neq 1$, then $S(\mu) = -\infty$. Secondly, suppose that $\mu(\varphi) < 0$ for a certain non-negative function $\varphi \in C(X)$. Then, since the dynamical potential is monotone, we see that $S(\mu) \leq \lambda(-t\varphi) - \mu(-t\varphi) \leq \lambda(0) + t\mu(\varphi)$ for all $t > 0$ and so $S(\mu) = -\infty$. Thirdly, let us prove that μ is α-invariant. This is equivalent to the identity $\mu(\varphi) = \mu(\varphi \circ \alpha)$, $\varphi \in C(X)$. Suppose that $\mu(\varphi) \neq \mu(\varphi \circ \alpha)$ for a certain $\varphi \in C(X)$. The strong α-invariance of the dynamical potential implies $\lambda(t\varphi - t\varphi \circ \alpha) \equiv \lambda(0)$. Therefore, $S(\mu) \leq \lambda(t\varphi - t\varphi \circ \alpha) - \mu(t\varphi - t\varphi \circ \alpha) = \lambda(0) - t(\mu(\varphi) - \mu(\varphi \circ \alpha))$ and, hence, $S(\mu) = -\infty$. Finally, if μ is an equilibrium measure corresponding to $\varphi \in C(X)$, then by Proposition 1.1 we have $S(\mu) = \lambda(\varphi) - \mu(\varphi) > -\infty$. □

PROPOSITION 1.3. *If $S(\mu) > -\infty$, then μ belongs to the closure of all equilibrium measures (in the norm of $C^*(X)$).*

Proof. A variant of Bishop–Phelps' theorem [**6**] states that if $\lambda(\varphi)$ is a continuous convex functional on a Banach space B and $\mu \in B^*$ and the difference $\lambda - \mu$ is bounded from below, then μ lies at zero distance from the set of subgradients of λ. □

Let us define on $C^*(X)$ the *action functional* $\tau_\varphi(\mu) = \mu(\varphi) + S(\mu) - \lambda(\varphi)$. By Proposition 1.1 it is always non-positive and vanishes only on the equilibrium measures. In view of Proposition 1.3, it may be finite (not equal to $-\infty$) only on the closure of all equilibrium measures.

Denote by $M_\alpha(X)$ the set of all α-invariant Borel probability measures on X. From Proposition 1.2 it follows that the supremum in (1.2) is attained on $M_\alpha(X)$. Hence, any dynamical potential has the form

$$\lambda(\varphi) = \sup_{\mu \in M_\alpha(X)} (\mu(\varphi) + S(\mu)), \tag{1.3}$$

where $S(\mu)$ is the corresponding dual entropy. By Proposition 1.1, the supremum in (1.3) is attained exactly on the equilibrium measures corresponding to φ. Therefore, the

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uniqueness of the extremal measure in (1.3) is equivalent to the existence of the Gâteaux derivative $\lambda'(\varphi)$.

Now we present one more statement explaining the structure of dynamical potentials.

PROPOSITION 1.4. *If a function $S(\mu)$ is bounded from above on $M_\alpha(X)$, then (1.3) determines a dynamical potential.*

Proof of this fact reduces to the trivial verification of all conditions defining a dynamical potential and so we leave it to the reader.

Since the dual entropy satisfies the inequality $S(\mu) \leq \lambda(0)$, we see from (1.3) that $\lambda(\varphi) \leq \sup_{M_\alpha(X)} \mu(\varphi) + \lambda(0)$. On the other hand, if $S(\mu) \equiv \lambda(0)$, then by means of (1.3) we obtain the dynamical potential of the form $\lambda(\varphi) = \sup_{M_\alpha(X)} \mu(\varphi) + \lambda(0)$. The previous estimate shows that this $\lambda(\varphi)$ is maximal among all of the dynamical potentials with the fixed value of $\lambda(0)$.

2. *Phase algebras*

Let L be a vector space over $\mathbb{R}$. Recall that a *cone* in L is a subset $K \subset L$ that is convex, homogeneous (if $v \in K$ and $t \geq 0$, then $tv \in K$), and antisymmetric (if $v \in K$ and $-v \in K$, then $v = 0$). Every cone K determines a partial order on L: we say that $u \leq v$ if and only if $v - u \in K$. All of the vectors from the cone are called *positive.* A real vector space with fixed cone is said to be *semiordered.* A linear functional on the semiordered space is called positive if it is non-negative on the cone. A linear operator on the semiordered space is called positive if it maps the cone into itself.

A *semiordered algebra* is an algebra $\mathcal{A}$ with fixed cone $\mathcal{A}_+ \subset \mathcal{A}$ such that the product of any two positive elements is also positive. For example, if $\mathcal{A}$ is an arbitrary algebra of functions on a set X, then it is naturally semiordered by the cone of non-negative functions. In particular, for any compact set X the corresponding algebra $C(X)$ is semiordered. Evidently, each subalgebra $\mathcal{B}$ of a semiordered algebra $\mathcal{A}$ is semiordered as well (by the cone $\mathcal{B} \cap \mathcal{A}_+$).

A semiordered algebra $\mathcal{C}$ with a unit will be called a *phase algebra* if there exists a Hausdorff compact set X such that $\mathcal{C}$ is isomorphic (including the order) to some dense (relative to the supremum norm) subalgebra in $C(X)$. We call this X the *support* or the *phase space* of the phase algebra $\mathcal{C}$.

PROPOSITION 2.1. *The support of a phase algebra is defined uniquely up to a homeomorphism.*

Proof. Assume that a phase algebra $\mathcal{C}$ is isomorphic (including the order) to some dense subalgebras $\mathcal{C}_X \subset C(X)$ and $\mathcal{C}_Y \subset C(Y)$, where X and Y are compact. Then $\mathcal{C}_X$ is isomorphic to $\mathcal{C}_Y$ including the order. Under this isomorphism the unit function has to go to the unit function. Hence, every constant function goes to the same constant function. Since the isomorphism between $\mathcal{C}_X$ and $\mathcal{C}_Y$ preserves the order, it follows that it preserves the supremum norm. By continuity it extends up to the metric isomorphism between $C(X)$ and $C(Y)$. By the Gelfand–Naimark theorem, each compact set X is homeomorphic to the space of maximal ideals in $C(X)$. Therefore, the isomorphism between $C(X)$ and $C(Y)$ generates a homeomorphism between X and Y. □

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PROPOSITION 2.2. *Every positive linear functional on a phase algebra is generated by a certain Borel measure on the support of the algebra.*

Proof. Without loss of generality it is sufficient to consider a positive linear functional μ on a phase algebra $\mathcal{C}$ dense in $C(X)$. In this case for any $f \in \mathcal{C}$ we have $\mu(f) \le \mu(\max f) = \mu(1)\max f$ and, on the other hand, $\mu(f) \ge \mu(\min f) = \mu(1)\min f$. So μ is continuous relative to the supremum norm on $\mathcal{C}$. By the continuity it extends to $C(X)$ preserving the positiveness. Finally, by the Riesz theorem each positive linear functional μ on $C(X)$ is identified with a certain Borel measure μ on X so that $\mu(f) \equiv \int_X f\,d\mu$. □

PROPOSITION 2.3. *Suppose that μ is a positive linear functional on a phase algebra $\mathcal{C}$. Then the functional $\lambda(\varphi) = \ln \mu(e^{\varphi})$ is convex with respect to $\varphi \in \mathcal{C}$.*

Proof. Without loss of generality it can be assumed that $\mathcal{C}$ is a dense subalgebra in $C(X)$ and μ is a Borel measure on X. So the expression $\mu(e^{\varphi})$ is well defined even if $e^{\varphi} \notin \mathcal{C}$. It is easy to see that the function $\mu(fg)$ of the pair $(f, g) \in \mathcal{C}^2$ is bilinear, symmetric, and positive definite. Therefore, it satisfies the Cauchy–Bunyakovskii inequality $\mu(fg)^2 \le \mu(f^2)\mu(g^2)$. To prove the convexity of $\lambda(\varphi)$ it is enough to check that the restriction of $\lambda(\varphi)$ to any straight line $\{\varphi_t = \varphi_0 + t\varphi\}$ is convex with respect to $t \in \mathbb{R}$. Let us differentiate this restriction twice in t:

$$\frac{d\lambda(\varphi_t)}{dt} = \frac{\mu(e^{\varphi_t}\varphi)}{\mu(e^{\varphi_t})}, \qquad \frac{d^2\lambda(\varphi_t)}{dt^2} = \frac{\mu(e^{\varphi_t}\varphi^2)\mu(e^{\varphi_t}) - \mu(e^{\varphi_t}\varphi)^2}{\mu(e^{\varphi_t})^2}.$$

By the Cauchy–Bunyakovskii inequality, $\mu(e^{\varphi_t/2}\cdot e^{\varphi_t/2}\varphi)^2 \le \mu(e^{\varphi_t})\mu(e^{\varphi_t}\varphi^2)$. It is seen from here that $d^2\lambda(\varphi_t)/dt^2 \ge 0$. Hence, $\lambda(\varphi)$ is convex. □

Let us consider now the question: what algebras are the phase ones? One important class of phase algebras is provided by the following proposition.

PROPOSITION 2.4. *Suppose that Y is a Hausdorff compact set and $\mathcal{I}$ is a closed ideal in $C(Y)$. Then the quotient algebra $C(Y)/\mathcal{I}$ is phase.*

Proof. Denote by X the annihilator of $\mathcal{I}$. In other words, $X = \bigcap_{f\in\mathcal{I}} f^{-1}(0)$. This X is closed and so compact. The restriction of functions from Y to X generates the algebraic homomorphism $\pi : C(Y) \to C(X)$. By the Brauer–Urysohn theorem, any function $f \in C(X)$ can be extended up to a continuous function on Y. Therefore, π is a surjection. On the other hand, any closed ideal $\mathcal{I} \subset C(Y)$ coincides with the set of functions vanishing on its annihilator X. Hence, $\mathcal{I}$ is the kernel of π. This implies $C(Y)/\mathcal{I} \cong C(X)$. □

In fact we are able to formulate a necessary and sufficient condition for a semiordered algebra to be phase. To this end consider a semiordered algebra $\mathcal{C}$ with cone $\mathcal{C}_+$ and positive unit $I \in \mathcal{C}_+$. We call an element $a \in \mathcal{C}$ *bounded* if there exists a number $t > 0$ such that $-tI \le a \le tI$. From the definition of the phase algebra we see that it consists of bounded elements and is commutative and the intersection of the positive cone with any affine straight line is closed. It turns out that the inverse is valid as well.

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THEOREM 2.5. *If a semiordered Abelian algebra $\mathcal{C}$ has a positive unit $I \in \mathcal{C}_+$ and consists of bounded elements and the intersection of $\mathcal{C}_+$ with any affine straight line is closed, then the algebra $\mathcal{C}$ is phase.*

This theorem can be treated as a real analog of Gelfand–Naimark's theorem about the structure of an Abelian C^*-algebra with unit [**8**]. It is proved in the next section. Theorem 2.5 provides at least two important examples of phase algebras. The first of them is any algebra $\mathcal{C}$ consisting of bounded real-valued functions on an arbitrary set X and containing the identical unit. The second is the set of essentially bounded functions $L^\infty(X, \mu)$ on any measure space (X, μ).

3. *Proof of the real analog of Gelfand–Naimark's theorem*

Consider the algebra $\mathcal{C}$ from Theorem 2.5. Set

$$\|a\| = \inf\{t > 0 \mid -tI \le a \le tI\}, \quad a \in \mathcal{C}. \tag{3.1}$$

LEMMA 3.1. *Under the conditions of Theorem 2.5, the function (3.1) is a norm on $\mathcal{C}$. The completion of the cone $\mathcal{C}_+$ in this norm is a cone (in the completion of $\mathcal{C}$) as well.*

Proof. If we regard $\mathcal{C}$ as a vector space then, evidently, (3.1) is a seminorm on it. Suppose that $\|a\| = 0$. In this case it follows from (3.1) that $tI + a \in \mathcal{C}_+$ and $tI - a \in \mathcal{C}_+$ for all $t > 0$. Then, by the condition of Theorem 2.5, we have $a \in \mathcal{C}_+$ and $-a \in \mathcal{C}_+$. Therefore, $a = 0$. This proves that $\|a\|$ is a norm on the vector space $\mathcal{C}$.

Further, assume that $\|a\| < t$ and $\|b\| < s$. Then $tI \pm a \in \mathcal{C}_+$ and $sI \pm b \in \mathcal{C}_+$. Therefore,

$$2tsI - 2ab = (tI - a)(sI + b) + (tI + a)(sI - b) \in \mathcal{C}_+,$$
$$2tsI + 2ab = (tI + a)(sI + b) + (tI - a)(sI - b) \in \mathcal{C}_+.$$

So we see that $-tsI \le ab \le tsI$. Hence, $\|ab\| \le \|a\|\|b\|$ for all $a, b \in \mathcal{C}$. This proves that $\|\cdot\|$ is a norm on the algebra $\mathcal{C}$.

Let $[\mathcal{C}_+]$ be the completion of $\mathcal{C}_+$ in the norm (3.1). Obviously, $[\mathcal{C}_+]$ is convex and homogeneous. By (3.1), we have $\|a\| \le \|a+b\|$ for all $a, b \in \mathcal{C}_+$. This inequality remains true for $a, b \in [\mathcal{C}_+]$ as well. Therefore, if $a \in [\mathcal{C}_+]$ and also $-a \in [\mathcal{C}_+]$, then we have $\|a\| \le \|a - a\| = 0$. Hence, $[\mathcal{C}_+]$ is a cone. □

LEMMA 3.2. *For each natural n and any $x \in \mathbb{R}$,*

$$x^2 = 4^{-n} \sum_{i=-n}^{n} \frac{i^2}{n^2} C_{2n}^{n+i} (1+x)^{n+i} (1-x)^{n-i} - \frac{1-x^2}{2n}.$$

Proof. The proof is by direct calculation. Note that

$$4^n = ((1+x) + (1-x))^{2n} = \sum_{i=-n}^{n} C_{2n}^{n+i} (1+x)^{n+i} (1-x)^{n-i}.$$

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Therefore,

$$\begin{aligned}
&4^n - \sum_{i=-n}^{n} \frac{i^2}{n^2} C_{2n}^{n+i} (1+x)^{n+i} (1-x)^{n-i} \\
&= \sum_{i=-n}^{n} \left(1 - \frac{i^2}{n^2}\right) C_{2n}^{n+i} (1+x)^{n+i} (1-x)^{n-i} \\
&= \sum_{i=-n+1}^{n-1} \frac{(n+i)(n-i)}{n^2} \frac{(2n)!}{(n+i)!(n-i)!} (1+x)^{n+i} (1-x)^{n-i} \\
&= \frac{2n(2n-1)}{n^2} (1-x^2) \sum_{i=-n+1}^{n-1} C_{2n-2}^{n+i-1} (1+x)^{n+i-1} (1-x)^{n-i-1} \\
&= \left(4 - \frac{2}{n}\right)(1-x^2) 4^{n-1} = 4^n \left(1 - x^2 - \frac{1-x^2}{2n}\right). \qquad \square
\end{aligned}$$

Denote by $[\mathcal{C}]$ the completion of $\mathcal{C}$ in the norm (3.1) and denote by $[\mathcal{C}_+]$ the closure of $\mathcal{C}_+$ in the same norm. For each $a \in \mathcal{C}$ define two numbers: $\sup a = \min\{t \in \mathbb{R} \mid a \le tI\}$, and $\inf a = \max\{t \in \mathbb{R} \mid a \ge tI\}$, and let $I_a = [\inf a, \sup a]$.

LEMMA 3.3. *Under the conditions of Theorem 2.5 for each $a \in \mathcal{C}$ there exists a unique algebraic homomorphism $\pi_a : C(I_a) \to [\mathcal{C}]$ that maps non-negative functions to $[\mathcal{C}_+]$ and takes any polynomial $P(x) = c_0 + c_1 x + \cdots + c_n x^n$ to $P(a) = c_0 I + c_1 a + \cdots + c_n a^n$.*

Proof. First let us check that the square of any element $b \in \mathcal{C}$ is positive. Without loss of generality, it can be assumed that $-I \le b \le I$. Then by Lemma 3.2,

$$b^2 + \frac{I - b^2}{2n} = 4^{-n} \sum_{i=-n}^{n} \frac{i^2}{n^2} C_{2n}^{n+i} (I+b)^{n+i} (I-b)^{n-i} \in \mathcal{C}_+.$$

By condition of Theorem 2.5, the intersection of $\mathcal{C}_+$ with any affine straight line is closed. Therefore, we get $b^2 \in \mathcal{C}_+$ as $n \to \infty$.

Thereupon let us prove that if a polynomial $P(x)$ is non-negative on the segment $I_a = [\inf a, \sup a]$, then $P(a) \in \mathcal{C}_+$. Indeed, in this case there exists a factorization $P(x) = c \prod_i (x - \alpha_i) \prod_j (\beta_j - x) \prod_k ((x - \gamma_k)^2 + \delta_k^2)$ in which $c \ge 0$, $\alpha_i \le \inf a$, and $\beta_j \ge \sup a$. Hence, $P(a) = c \prod_i (a - \alpha_i I) \prod_j (\beta_j I - a) \prod_k ((a - \gamma_k I)^2 + \delta_k^2 I)$ is a product of factors lying in $\mathcal{C}_+$ and so it also belongs to $\mathcal{C}_+$.

From here it follows that if $-t \le P(x) \le t$ on the segment I_a for some $t > 0$, then $-tI \le P(a) \le tI$. Thus, $\|P(a)\| \le \sup_{I_a} |P(x)|$. This proves the continuity of the mapping $P(x) \mapsto P(a)$. By the continuity this mapping extends up to a positive algebraic homomorphism $\pi_a : C(I_a) \to [\mathcal{C}]$. $\square$

Proof of Theorem 2.5. For each $a \in \mathcal{C}$ define the segment $I_a = [\inf a, \sup a]$ and the corresponding homomorphism $\pi_a : C(I_a) \to [\mathcal{C}]$ from Lemma 3.3. Consider the Cartesian product $Y = \prod_{a \in \mathcal{C}} I_a$. By Tychonoff's theorem it is compact in the cylinder topology. Let $\mathcal{A}$ be the subalgebra in $C(Y)$ consisting of various finite sums of products $\sum_{i=1}^{m} f_1^i(x_1) \cdots f_n^i(x_n)$, where $x_j \in I_{a_j}$ and $f_j^i \in C(I_{a_j})$. This subalgebra contains the identical unit and separates points of Y. By Stone's theorem $\mathcal{A}$ is dense in $C(Y)$.

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Define an algebraic homomorphism $\pi : \mathcal{A} \to [\mathcal{C}]$ by means of the formula $\pi\left(\sum_{i=1}^m f_1^i(x_1)\cdots f_n^i(x_n)\right) = \sum_{i=1}^m \pi_{a_1}(f_1^i)\cdots\pi_{a_n}(f_n^i)$, where π_{a_j} are the homomorphisms from Lemma 3.3.

Let $F \in \mathcal{A}$ be a non-negative function. Then for any $\varepsilon > 0$ the sum $F + \varepsilon$ can be represented as $F + \varepsilon = \sum_{i=1}^k g_1^i \cdots g_n^i$, where all $g_j^i \in C(I_{a_j})$ are non-negative. This will be proved below in Theorem 3.4. Combining this representation with Lemma 3.3, we obtain $\pi(F+\varepsilon) \in [\mathcal{C}_+]$, which implies $\pi(F) \in [\mathcal{C}_+]$ as $\varepsilon \to 0$. Hence, the homomorphism π is positive. If $F \in \mathcal{A}$ satisfies $-t \le F \le t$, $t > 0$, then $-tI \le \pi(F) \le tI$. Therefore, $\|\pi(F)\|$ does not exceed the supremum norm of the F. By continuity, π uniquely extends from $\mathcal{A}$ to $C(Y)$ preserving positiveness.

Denote by $\mathcal{I}$ the kernel of the homomorphism $\pi : C(Y) \to [\mathcal{C}]$. Obviously, $\mathcal{I}$ is a closed ideal in $C(Y)$. Let X be its annihilator: $X = \bigcap_{f\in\mathcal{I}} f^{-1}(0)$. The set X is closed and so compact. Let us define the homomorphism $\rho : C(X) \to [\mathcal{C}]$ in the following way. By the Brauer–Urysohn theorem any $f \in C(X)$ can be extended up to a continuous function F on Y. Set $\rho(f) = \pi(F)$. It is well known that every closed ideal $\mathcal{I} \subset C(Y)$ coincides with the set of functions vanishing on its annihilator X. Therefore, the homomorphism ρ is well defined and injective.

For any $b \in \mathcal{C}$ define the function $F_b \in C(Y)$ by the rule $F_b(\{x_a \in I_a\}_{a\in\mathcal{C}}) = x_b$. Let f_b be the restriction of F_b to X. Then $\rho(f_b) = \pi(F_b) = b$ and, hence, $\mathcal{C} \subset \rho(C(X))$. On the other hand, the functions f_b, $b \in \mathcal{C}$, separate points of X. So by Stone's theorem the algebra $\rho^{-1}(\mathcal{C})$ is dense in $C(X)$. Since the homomorphism ρ is injective, its inverse $\rho^{-1} : b \mapsto f_b$ establishes an isomorphism between $\mathcal{C}$ and $\rho^{-1}(\mathcal{C})$. It is easy to see that this isomorphism is positive. Indeed, by construction the functions F_b and f_b take values in the segment $I_b = [\inf b, \sup b]$. In addition, if $b \in \mathcal{C}_+$, then $\inf b \ge 0$. This completes the proof of Theorem 2.5. □

THEOREM 3.4. *Suppose that on the Cartesian product $X_1 \times \cdots \times X_n$ of Hausdorff compact sets X_i there is a non-negative function F of the form $F = \sum_{i=1}^m f_1^i \cdots f_n^i$, where $f_j^i \in C(X_j)$. Then for any $\varepsilon > 0$ there exist non-negative functions $g_j^i \in C(X_j)$ such that $F + \varepsilon = \sum_{i=1}^k g_1^i \cdots g_n^i$.*

Let us denote by $\mathcal{A}_\#$ the set of all finite sums of the form $\sum_{i=1}^k g_1^i \cdots g_n^i$, where all $g_j^i \in C(X_j)$ are non-negative. We start the proof of Theorem 3.4 with the following.

LEMMA 3.5. *If under the conditions of Theorem 3.4 the function F has the form $F = f_1 \cdots f_n$, where $f_j \in C(X_j)$, and $\varepsilon \ge \|f_1\| \times \cdots \times \|f_n\|$, then $F + \varepsilon \in \mathcal{A}_\#$.*

Proof. Without loss of generality it can be assumed that $\varepsilon = \|f_1\| \times \cdots \times \|f_n\|$. Set $f_j^+ = \max\{0, f_j\}$ and $f_j^- = \max\{0, -f_j\}$. Then $f_j = f_j^+ - f_j^-$. Accordingly, F can be presented as a sum of 2^n terms of the form $\pm f_1^\pm \cdots f_n^\pm$. Obviously, the functions f_j^+, f_j^- and $g_j = \|f_j\| - f_j^+ - f_j^-$ are non-negative and continuous. Hence, by distributivity $\varepsilon = (f_1^+ + f_1^- + g_1) \times \cdots \times (f_n^+ + f_n^- + g_n)$ is a sum of 3^n terms, each of them being the product of n non-negative functions. In addition, all of the products $f_1^\pm \cdots f_n^\pm$ are contained among these 3^n terms. Therefore, $F + \varepsilon \in \mathcal{A}_\#$. □

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Proof of Theorem 3.4. Take any $\varepsilon > 0$. For each $j = 1, \dots, n$ choose a continuous partition of unity $\varphi_j^1, \dots, \varphi_j^{N_j}$ on X_j such that the oscillations of the functions f_j^i on the supports of φ_j^k do not exceed ε. Select points $x_j^k \in X_j$ belonging to the supports of the corresponding φ_j^k. Put

$$\eta_\varepsilon(x) = \begin{cases} -\varepsilon, & x < -\varepsilon, \\ x, & -\varepsilon \le x \le \varepsilon, \\ \varepsilon, & x > \varepsilon. \end{cases}$$

Let us consider the functions $g_j^{ik}(x) = \eta_\varepsilon(f_j^i(x) - f_j^i(x_j^k))$ on X_j. By construction, $\|g_j^{ik}\| \le \varepsilon$ and $f_j^i(x) = g_j^{ik}(x) + f_j^i(x_j^k)$ on the $\operatorname{supp} \varphi_j^k$. Therefore, $f_j^i \equiv \sum_{k=1}^{N_j} (g_j^{ik} + f_j^i(x_j^k))\varphi_j^k$.

Suppose $t = \max_{i,j} \|f_j^i\|$. Let us fix indices $k_1, \dots, k_n$ and consider the product $(g_1^{ik_1} + f_j^i(x_1^{k_1})) \times \cdots \times (g_n^{ik_n} + f_n^i(x_n^{k_n}))$. If we decompose it by distributivity into the sum of 2^n terms and apply Lemma 3.5 to each of them (except the final one), then we obtain

$$\prod_{j=1}^n (g_j^{ik_j} + f_j^i(x_j^{k_j})) - \prod_{j=1}^n f_j^i(x_j^{k_j}) + (2^n - 1)\varepsilon(t + \varepsilon)^{n-1} \in \mathcal{A}_\#.$$

Multiply the latter expression by $\varphi_1^{k_1} \cdots \varphi_n^{k_n}$ and sum up over all indices i, $k_1, \dots, k_n$. By construction the sum will lie in $\mathcal{A}_\#$:

$$\begin{aligned} &\sum_{i=1}^m \prod_{j=1}^n \sum_{k_j=1}^{N_j} (g_j^{ik_j} + f_j^i(x_j^{k_j}))\varphi_j^{k_j} - \sum_{i=1}^m \sum_{k_1 \dots k_n} \prod_{j=1}^n f_j^i(x_j^{k_j})\varphi_j^{k_j} \\ &+ (2^n - 1)\varepsilon(t + \varepsilon)^{n-1} \sum_{i=1}^m \prod_{j=1}^n \sum_{k_j=1}^{N_j} \varphi_j^{k_j} \\ &\quad = \sum_{i=1}^m f_1^i \cdots f_n^i - \sum_{k_1 \dots k_n} \sum_{i=1}^m f_1^i(x_1^{k_1}) \cdots f_n^i(x_n^{k_n})\varphi_1^{k_1} \cdots \varphi_n^{k_n} + (2^n - 1)\varepsilon(t + \varepsilon)^{n-1} m \\ &\quad = F - \sum_{k_1 \dots k_n} F(x_1^{k_1}, \dots, x_n^{k_n})\varphi_1^{k_1} \cdots \varphi_n^{k_n} + (2^n - 1)\varepsilon(t + \varepsilon)^{n-1} m \in \mathcal{A}_\#. \end{aligned}$$

Since F and φ_j^k are non-negative and $\varepsilon > 0$ is arbitrary this implies Theorem 3.4. □

4. *Positive algebras*

Assume that a semiordered vector space L with a cone K is supplied with a norm. We call this norm *monotone* if $\|u\| \le \|u + v\|$ for all $u, v \in K$. The semiordered vector space with monotone norm will be called *positive*.

Note at once that the definition of the positive space L remains valid if we replace the positive cone K by its closure $[K]$. In this case the closure of K will stay antisymmetric: if $v \in [K]$ and $-v \in [K]$, then $\|v\| \le \|v - v\| = 0$ (by monotonicity of the norm).

Examples of positive spaces are the spaces of integrable functions $L^p(X, \mu)$, where $1 \le p \le \infty$. On the other hand, the space of differentiable functions with the cone of non-negative functions is not positive.

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PROPOSITION 4.1. *If v is a positive vector in a positive space L, then there exists a positive linear functional $\nu : L \to \mathbb{R}$ such that $\|\nu\| = 1$ and $\nu(v) = \|v\|$.*

Proof. Without loss of generality it can be assumed that $\|v\| = 1$. Denote by K the positive cone in L. Consider two convex sets: the open unit ball $B = \{u \in L : \|u\| < 1\}$ and the shifted cone $K_v = v + K$. From the definition of the positive space it follows that $B \cap K_v = \emptyset$. By the separation theorem there exists a linear functional ν on L such that $\nu(u) \leq 1$ for all $u \in B$ and $\nu(u) \geq 1$ for all $u \in K_v$. Obviously, $\|\nu\| = \nu(v) = 1$. In addition, for any $u \in K$ we have $\nu(u) = \nu(v + u) - \nu(v) \geq 0$. □

Recall that an algebra $\mathcal{A}$ is said to be normed if $\mathcal{A}$ is a normed linear space and $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. If $\mathcal{A}$ contains a unit I, then in addition $\|I\| = 1$ is required (which always can be provided by mere replacement of the norm by an equivalent one).

Definition. We call a real algebra $\mathcal{A}$ *positive* if it is normed, semiordered, has a positive unit, and the norm on $\mathcal{A}$ is monotone.

Example 1. Any algebra of functions on an arbitrary domain that contains the identical unit and is supplied with the sup norm is positive.

Example 2. Let X be a topological space. Then the algebra of bounded linear operators on $C(X)$ is positive. Indeed, if two linear operators A, B on $C(X)$ are positive (take non-negative functions to non-negative ones), then $\|A\| = \sup A(1) \leq \sup(A(1) + B(1)) = \|A + B\|$.

Example 3. Let (X, μ) be a measure space. Then the algebra of bounded linear operators on the space $L^p(X, \mu)$, where $1 \leq p \leq \infty$, is positive.

Proof. For any $f \in L^p(X, \mu)$ define the functions $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Suppose that A is a positive linear operator on $L^p(X, \mu)$. Obviously, $Af^+ \geq 0$ and $Af^- \geq 0$, and hence $|Af| = |Af^+ - Af^-| \leq Af^+ + Af^- = A|f|$. On the other hand, $\|f\| = \||f|\|$. Therefore,

$$\|A\| = \sup_{f \in L^p(X,\mu)} \frac{\|Af\|}{\|f\|} = \sup_{f \geq 0} \frac{\|Af\|}{\|f\|}.$$

From the latter equality we see that if operators A and B are positive, then $\|A\| \leq \|A + B\|$. □

Example 4. Suppose that L is a semiordered and normed vector space and $\mathcal{A}_+$ is the set of all positive bounded linear operators on L. Denote by $\mathcal{A}$ the linear hull of $\mathcal{A}_+$. In other words, $\mathcal{A} = \{A - B \mid A, B \in \mathcal{A}_+\}$. For each $A \in \mathcal{A}$ define the seminorm

$$|A| = \inf\{\|A'\| \mid A' \pm A \in \mathcal{A}_+\} = \inf\{\|A'\| \mid -A' \leq A \leq A'\}.$$

Let $\mathcal{I} = \{A \in \mathcal{A} \mid |A| = 0\}$. Then the quotient algebra $\mathcal{A}/\mathcal{I}$ with the norm $|\cdot|$ is positive.

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Proof. If $A, B \in \mathcal{A}_+$, then $|A| \le |A + B|$ by the definition of the seminorm. Suppose that for some $A, B, A', B' \in \mathcal{A}$ we have $A' \pm A \in \mathcal{A}_+$ and $B' \pm B \in \mathcal{A}_+$. Then

$$2A'B' - 2AB = (A' - A)(B' + B) + (A' + A)(B' - B) \in \mathcal{A}_+,$$
$$2A'B' + 2AB = (A' + A)(B' + B) + (A' - A)(B' - B) \in \mathcal{A}_+.$$

Hence, $A'B' \pm AB \in \mathcal{A}_+$ and it follows that $|AB| \le |A||B|$ for all $A, B \in \mathcal{A}$. Therefore, $|\cdot|$ is a seminorm on the algebra $\mathcal{A}$ and is a norm on the algebra $\mathcal{A}/\mathcal{I}$. □

Example 5. We say that a norm on a semiordered vector space L is pseudomonotone if there exists a positive constant C such that $\|v\| \le C\|u\|$, provided that $u \ge v \ge 0$. It is known that for any pseudomonotone norm there exists an equivalent monotone norm. It can be given by $\|u\| = \sup_{\nu \in V} |\nu(u)|$, where V is the set of positive linear functionals on L with unit norm (see [**14**]). Unfortunately, in the case of semiordered algebras the estimate $\|uv\| \le \|u\|\|v\|$ may fail for this norm. So it would be interesting to know whether there is a monotone algebraic norm equivalent to the pseudomonotone norm on a semiordered algebra.

Suppose that $\mathcal{A}$ is a positive algebra with closed cone $\mathcal{A}_+$. Obviously, every subalgebra $\mathcal{C} \subset \mathcal{A}$ containing the unit is also positive (with the cone $\mathcal{C}_+ = \mathcal{C} \cap \mathcal{A}_+$). A subalgebra $\mathcal{C} \subset \mathcal{A}$ will be called *phase* if it is commutative, contains the unit of $\mathcal{A}$, and consists of bounded elements (in the sense of the definition given in §2). From Theorem 2.5 it follows that any phase subalgebra of a positive algebra $\mathcal{A}$ with closed cone $\mathcal{A}_+$ is phase in the sense of definition from §2 as well. In other words, it is isomorphic to a dense subalgebra in $C(X)$, where X is a Hausdorff compact set, which is uniquely defined by the phase algebra and named its support.

Example 6. Let $\mathcal{A}$ be the set of bounded linear operators on $C(Y)$, where $C(Y)$ is the algebra of bounded continuous functions on a certain topological space Y. Then $C(Y)$ can be considered as a phase subalgebra of $\mathcal{A}$ (under the condition that each $f \in C(Y)$ is identified with the operator of multiplication by f).

Example 7. Suppose that $\mathcal{A}$ is the set of bounded linear operators on $L^p(Y, \mu)$, where (Y, μ) is a measure space. Then $L^\infty(Y, \mu)$ is a phase subalgebra of $\mathcal{A}$.

PROPOSITION 4.2. *If $\mathcal{A}$ is a complete in norm positive algebra with closed cone $\mathcal{A}_+$ and $\mathcal{C}$ is a phase subalgebra in it, then for any $a \in \mathcal{A}_+$ the functional $\lambda(\varphi) = \ln \|e^\varphi a\|$ is convex with respect to $\varphi \in \mathcal{C}$.*

Proof. The phase subalgebra $\mathcal{C}$ is isomorphic to a dense subalgebra in $C(X)$, where X is the support of $\mathcal{C}$. This isomorphism extends up to a positive continuous embedding of $C(X)$ into $\mathcal{A}$. So it follows that for any $\varphi \in \mathcal{C}$ the element $e^\varphi \in \mathcal{A}$ is positive. Let V be the set of all positive linear functionals on $\mathcal{A}$ with unit norm. From Proposition 4.1 it follows that $\|e^\varphi a\| = \sup_{\nu \in V} \nu(e^\varphi a)$. For each $\nu \in V$ the functional $\nu(\cdot\, a)$ is positive. Also, by Proposition 2.3, the functional $\lambda_\nu(\varphi) = \ln \nu(e^\varphi a)$ is convex with respect to $\varphi \in \mathcal{C}$. Therefore, $\lambda(\varphi) = \sup_{\nu \in V} \lambda_\nu(\varphi)$ is convex with respect to φ too. □

Let ν be a linear functional on a positive algebra $\mathcal{A}$. We can multiply it by elements $a \in \mathcal{A}$ from the left and from the right. Namely, set $\nu a(\cdot) = \nu(a\,\cdot)$ and $a\nu(\cdot) = \nu(\cdot\, a)$.

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If ν and a are positive here, then the functionals νa and $a\nu$ are also positive. For any $a \in \mathcal{A}$ define its spectral radius $r(a) = \lim_{n\to\infty} \sqrt[n]{\|a^n\|}$.

THEOREM 4.3. *If a is a positive element of a positive algebra $\mathcal{A}$, then there exists a non-zero positive linear functional $\mu : \mathcal{A} \to \mathbb{R}$ such that $a\mu = r(a)\mu$ and $\|\mu\| = \mu(I)$. Similarly, there exists a positive linear functional μ' such that $\mu' a = r(a)\mu'$ and $\|\mu'\| = \mu'(I)$.*

Proof. Without loss of generality we can assume that $\mathcal{A}$ is complete in norm and that the cone $\mathcal{A}_+$ is closed. Consider $b = r(a)^{-1}a$. The spectral radius of b is equal to one. Hence, for any $\theta \in (0, 1)$ the series $R_\theta = \sum_{i=0}^{\infty} \theta^i b^i$ converges in norm and R_θ is a positive element of $\mathcal{A}$. By the positiveness of all of the summands in R_θ, the norm of R_θ does not decrease as θ grows. Let $N = \lim_{\theta\nearrow 1} \|R_\theta\|$. Suppose first that $N = \infty$. By Proposition 4.1 for any $\theta \in (0, 1)$ there exists a positive linear functional ν on $\mathcal{A}$ such that $\|\nu\| = 1$ and $\nu(R_\theta) = \|R_\theta\|$. Set $\nu_\theta = \|R_\theta\|^{-1} R_\theta \nu$. Then $\|\nu_\theta\| \le 1$ and at the same time $\nu_\theta(I) = \|R_\theta\|^{-1}\nu(R_\theta) = 1$. Hence, $\|\nu_\theta\| = 1$. Further, $R_\theta - \theta b R_\theta = I$. Therefore, the norm $\|\nu_\theta - \theta b \nu_\theta\| = \|R_\theta\|^{-1}\|\nu\|$ tends to zero as $\theta \nearrow 1$. Since the unit ball in the dual space to $\mathcal{A}$ is compact in the *-weak topology, the family ν_θ has a limit point as $\theta \nearrow 1$. This limit point is the positive functional μ we are looking for.

Suppose now that $N < \infty$. Consider the sequence $P_n = \sum_{i=0}^{2n} i(2n - i)b^i$. Evidently,

$$\begin{aligned} P_n - bP_n &= \sum_{i=1}^{2n} [i(2n - i) - (i - 1)(2n - i + 1)]b^i \\ &= \sum_{i=1}^{n} (2n - 2i + 1)b^i - \sum_{i=n+1}^{2n} (2i - 2n - 1)b^i. \end{aligned} \tag{4.1}$$

From the positiveness of $\mathcal{A}$ it follows that for any natural n and $m > n$ the inequality $\|\sum_{i=n}^{m} b^i\| \le N$ holds. So the norms of both sums in the right-hand side of (4.1) do not exceed $2nN$. Thus, $\|P_n - bP_n\| \le 4nN$. On the other hand, $\|b^n\| \ge 1$ by the definition of the spectral radius. Therefore, $\|P_n\| \ge \|n^2 b^n\| \ge n^2$. The rest of the proof can be carried out in a way similar to the case $N = \infty$, one should simply replace the family R_θ by P_n and the parameter $\theta \nearrow 1$ by $n \to \infty$.

In fact, once the eigenfunctional μ is found it becomes clear that the case $N < \infty$ is impossible. Indeed, by construction $\mu(b^i) = (b^i\mu)(I) = 1$ and hence the norm $\|R_\theta\| \ge \mu(R_\theta) = \sum_{i=0}^{\infty} \theta^i$ goes to infinity as $\theta \nearrow 1$.

The existence of the functional μ' can be proved in the same way. □

5. *Positive flows*

Suppose that $\alpha : X \to X$ is a continuous mapping of a Hausdorff compact set. It generates the shift $\alpha^*(f) = f \circ \alpha$ on $C(X)$. Evidently, this shift is positive and maps the unit function into itself. Now we show that the inverse statement is true as well.

PROPOSITION 5.1. (Lô [**17**]) *If a homomorphism $\delta : C(X) \to C(X)$, where X is a Hausdorff compact set, is positive and maps the unit function into itself, then there exists a continuous mapping $\alpha : X \to X$ such that $\delta(f) \equiv f \circ \alpha$.*

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This statement is not new, but for the sake of completeness of the presentation we still give the proof. To each point $x \in X$ assign the set $\mathcal{I}_x = \{f \in C(X) \mid [\delta(f)](x) = 0\}$. Obviously, $\mathcal{I}_x$ is a maximal ideal in $C(X)$. Hence, there exists $y \in X$ such that $\mathcal{I}_x = \{f \in C(X) \mid f(y) = 0\}$. Set $\alpha(x) = y$. If $f(y) = c$, then $f - c \in \mathcal{I}_x$ and so $[\delta(f)](x) - c = [\delta(f - c)](x) = 0$. Therefore, $\delta(f) \equiv f \circ \alpha$. This identity automatically yields the continuity of α. □

Let $\mathcal{C}$ be a phase algebra. We call any positive homomorphism $\delta : \mathcal{C} \to \mathcal{C}$ that maps the unit function into itself the *shift*. Without loss of generality we can assume that $\mathcal{C}$ is a dense subalgebra in $C(X)$, where X is the support of $\mathcal{C}$. By continuity the shift δ extends uniquely from $\mathcal{C}$ to $C(X)$ and by Proposition 5.1 it has the form $\delta(f) = f \circ \alpha$, where α is a certain continuous mapping of X to itself. Thus, every shift $\delta : \mathcal{C} \to \mathcal{C}$ generates automatically a dynamical system (X, α).

Definition. A (discrete time) *positive flow* is a quadruple $(\mathcal{A}, \mathcal{C}, E, \delta)$ consisting of a positive Banach (i.e. complete in norm) algebra $\mathcal{A}$ with closed cone $\mathcal{A}_+$, a phase subalgebra $\mathcal{C} \subset \mathcal{A}$, a fixed element $E \in \mathcal{A}_+$, and a shift $\delta : \mathcal{C} \to \mathcal{C}$ satisfying one of the following two *homological identities*: $Ef = \delta(f)E$ (the covariant form) or $fE = E\delta(f)$ (the contravariant form) for all $f \in \mathcal{C}$.

There is not much difference between the covariant and contravariant forms of the homological identity. Either of them one can convert into the other by redefining the multiplication in $\mathcal{A}$ by the rule $a * b = ba$. So in the sequel we use mostly the covariant form.

If a positive flow $(\mathcal{A}, \mathcal{C}, E, \delta)$ is fixed, then we call the elements of $\mathcal{A}$ *operators* and we call the distinguished positive element E the *evolution operator*. We call the support of the phase algebra $\mathcal{C}$ the *phase space* of the positive flow and denote it by X. Once and for all let us assume that the algebra $\mathcal{C}$ is naturally embedded in $C(X)$. The shift δ automatically generates a continuous mapping $\alpha : X \to X$ so that $\delta(f) = f \circ \alpha$ for all $f \in \mathcal{C} \subset C(X)$. Therefore, the (covariant) homological identity takes the form $Ef = (f \circ \alpha)E$. Below, if we wish to emphasize that the shift $\delta : \mathcal{C} \to \mathcal{C}$ is generated by a continuous mapping $\alpha : X \to X$, we use the notation α^* instead of δ.

Let us give several examples of positive flows.

Example 1. Suppose that $\alpha : X \to X$ is a continuous mapping of a topological space X to itself. Then one can regard the algebra of bounded linear operators on $C(X)$ as $\mathcal{A}$, the $C(X)$ as $\mathcal{C}$, the weighted shift operator $Ef = e^{\varphi} f \circ \alpha$, where $\varphi \in C(X)$ is fixed, as E, and the shift homomorphism $\alpha^*(f) = f \circ \alpha$ as δ.

Example 2. Suppose that $\alpha : X \to X$ is a measurable mapping of a measurable space supplied with an α-invariant measure μ. Then one can regard the algebra of bounded linear operators on $L^p(X, \mu)$, where $1 \leq p \leq \infty$, as $\mathcal{A}$, the algebra $L^\infty(X, \mu)$ as $\mathcal{C}$, the weighted shift operator $Ef = e^{\varphi} f \circ \alpha$, provided that $\varphi \in L^\infty(X, \mu)$ is fixed, as E, and the shift homomorphism $\alpha^*(f) = f \circ \alpha$ as δ.

Example 3. One may think of any stochastic process with discrete time and phase space X as the trajectory space $\bar{X} = \{(x_1, x_2, x_3, \dots) \mid x_i \in X\}$ supplied with the cylinder σ-algebra and the probability distribution $\mathbf{P}$. Then one can regard the algebra of continuous

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linear operators on $L^1(\bar{X}, \mathbf{P})$ as $\mathcal{A}$, $L^\infty(\bar{X}, \mathbf{P})$ as $\mathcal{C}$, and the shift $Ef(x_1, x_2, x_3, \dots) = \delta(f)(x_1, x_2, x_3, \dots) = f(x_2, x_3, \dots)$ as E and δ. In essence, this example is a special case of the previous one.

Example 4. Often more profound results come out if one considers a stochastic process with a stationary (invariant under shifts of the time axis) probability distribution $\mathbf{P}$ on the trajectory space $\bar{X} = X^{\mathbb{N}}$. Then one can define E as the conditional mathematical expectation $Ef(x_1, x_2, \dots) = \mathbf{E}(f(x_0, x_1, x_2, \dots) \mid x_1, x_2, \dots)$ on the space $L^1(\bar{X}, \mathbf{P})$ and take the shift δ on $\mathcal{C} = L^\infty(\bar{X}, \mathbf{P})$ from the previous example. In this case the homological identity takes the contravariant form $E(\delta(f)g) = fEg$.

PROPOSITION 5.2. *Suppose that $(\mathcal{A}, \mathcal{C}, E, \delta)$ is a positive flow with phase space X. Then there exists a unique algebraic homomorphism $i_{\mathcal{C}} : C(X) \to \mathcal{A}$ that is continuous, injective, and coincides with the identical embedding on $\mathcal{C} \subset C(X)$.*

Proof. Consider an arbitrary function $f \in \mathcal{C} \subset C(X)$. Put $t = \max |f|$. Then $0 \le t \pm f \le 2t$, and by monotonicity of the norm on $\mathcal{A}$ we have

$$2\|f\| = \|(t + f) - (t - f)\| \le \|t + f\| + \|t - f\| \le \|2t\| + \|2t\| = 4 \max |f|.$$

Therefore, the identical embedding $i_{\mathcal{C}} : \mathcal{C} \to \mathcal{A}$ is continuous with respect to the sup norm on $\mathcal{C}$. By the continuity it extends uniquely onto $C(X)$. Obviously, the extended homomorphism is positive. Suppose that $f \in C(X)$ and $i_{\mathcal{C}}(f) = 0$. Since $\mathcal{C}$ is dense in $C(X)$, for any $\varepsilon > 0$ there is a function $g \in \mathcal{C}$ such that $-\varepsilon \le g - f \le \varepsilon$. Applying $i_{\mathcal{C}}$ to these inequalities we obtain $-\varepsilon \le g \le \varepsilon$. Hence, $-2\varepsilon \le f \le 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, f is zero and so $i_{\mathcal{C}}$ is injective. □

Now consider a positive flow $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ with phase space X. Let $i_{\mathcal{C}} : C(X) \to \mathcal{A}$ be the homomorphism from Proposition 5.2. In the sequel we identify functions $f \in C(X)$ with their images $i_{\mathcal{C}}(f) \in \mathcal{A}$ and presuppose that $C(X)$ is naturally embedded into $\mathcal{A}$. For each $\varphi \in C(X)$ let us define the operator $E_\varphi = e^\varphi E \in \mathcal{A}$. It is easy to see that for all $f \in C(X)$ we have the homological identity

$$E_\varphi f = e^\varphi Ef = (f \circ \alpha)e^\varphi E = (f \circ \alpha)E_\varphi. \tag{5.1}$$

Introduce the notation $S_n\varphi = \varphi + \varphi \circ \alpha + \cdots + \varphi \circ \alpha^{n-1}$. Then (5.1) yields $E_\varphi^n = e^\varphi E e^\varphi E \cdots e^\varphi E = e^{S_n\varphi} E^n$. For any function $\varphi \in C(X)$ define the *spectral potential*

$$\lambda(\varphi) = \lim_{n\to\infty} \frac{1}{n} \ln \|E_\varphi^n\| = \lim_{n\to\infty} \frac{1}{n} \ln \|e^{S_n\varphi} E^n\|. \tag{5.2}$$

In other words, $\lambda(\varphi)$ is the logarithm of the spectral radius of E_φ.

THEOREM 5.3. *The spectral potential of any positive flow is monotone (if $\varphi \le \psi$, then $\lambda(\varphi) \le \lambda(\psi)$), additive homogeneous (if $t \in \mathbb{R}$, then $\lambda(\varphi + t) = \lambda(\varphi) + t$), continuous (satisfies the Lipschitz condition $|\lambda(\varphi) - \lambda(\psi)| \le \max |\varphi - \psi|$), strongly α-invariant (for all $\varphi, \psi \in C(X)$ satisfies $\lambda(\varphi + \psi) = \lambda(\varphi + \psi \circ \alpha)$), and convex with respect to $\varphi \in C(X)$.*

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Proof. The monotonicity and additive homogeneity follow from (5.2) and from the positiveness of $\mathcal{A}$. They imply the Lipschitz condition because $\lambda(\varphi) - \lambda(\psi) \leq \lambda(\psi + \max|\varphi - \psi|) - \lambda(\psi) = \max|\varphi - \psi|$. Note that always $|S_n(\varphi + \psi) - S_n(\varphi + \psi \circ \alpha)| = |\psi - \psi \circ \alpha^n| \leq 2\sup|\psi|$. Therefore, (5.2) yields the strong invariance of the spectral potential. Further, $S_n\varphi$ depends linearly on $\varphi \in C(X)$. By Proposition 4.2 the functional $\ln \|e^{S_n\varphi} E^n\|$ is convex with respect to $S_n\varphi$ and, hence, with respect to φ as well. Passing to the limit in (5.2) we see that $\lambda(\varphi)$ is also convex. □

It follows from the Lipschitz condition for the spectral potential that it is either finite on all of $C(X)$ or $\lambda(\varphi) \equiv -\infty$. In the sequel we always assume that the first of the two mentioned possibilities takes place.

In essence Theorem 5.3 states that the spectral potential is a particular case of the dynamical potential of a dynamical system (X, α). So the concepts of dual entropy, equilibrium measures, and the action functional are already defined for it (see §1).

6. *The law of large numbers*

We study the positive flow $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ and the corresponding dynamical system (X, α). We call any positive continuous linear functional on $\mathcal{A}$ a *state* of the flow. The restriction of a state to the subalgebra $C(X) \subset \mathcal{A}$ is also a positive linear functional. By the Riesz theorem, the latter is identified with a Borel measure on X. So every state induces a measure on the phase space X. A state ν is called normalized if $\nu(1) = 1$. In this case the corresponding measure on X is probability.

Let $\mathbf{P}$ be a probability measure on X. For any $f \in C(X)$ the sequence $\{f \circ \alpha^n\}_{n=0}^{\infty}$ may be considered as a sequence of random variables with respect to $\mathbf{P}$. This sequence is said to satisfy the law of large numbers if there exists a constant a such that

$$\lim_{n\to\infty} \mathbf{P}\left\{\left|\frac{f + f\circ\alpha + \cdots + f\circ\alpha^{n-1}}{n} - a\right| > \varepsilon\right\} = 0 \quad \text{for all } \varepsilon > 0.$$

This is equivalent to the convergence of $(1/n)S_n f$ to a with respect to the measure $\mathbf{P}$. In addition, if $(1/n)S_n f$ converges to a almost everywhere, it is said to satisfy the strong law of large numbers. Below, we consider one more modification of the law of large numbers. Let $\{\mathbf{P}_n\}$ be a sequence of probability distributions on X. We say that the sequence $\{f \circ \alpha^n\}$ satisfies the law of large numbers with respect to the distributions $\mathbf{P}_n$ if there exists a constant a such that

$$\lim_{n\to\infty} \mathbf{P}_n\left\{\left|\frac{f + f\circ\alpha + \cdots + f\circ\alpha^{n-1}}{n} - a\right| > \varepsilon\right\} = 0 \quad \text{for all } \varepsilon > 0.$$

Our next purpose is to prove the (strong) law of large numbers with respect to certain states of a positive flow. To be definite we consider below only covariant positive flows (although, of course, each of the statements proved below has a contravariant analog).

Recall that we can multiply any state ν of the flow $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ by operators $A \in \mathcal{A}$ from the right and from the left: by definition, $\nu A(\,\cdot\,) = \nu(A\,\cdot\,)$ and $A\nu(\,\cdot\,) = \nu(\,\cdot\, A)$. If A is positive, then the products νA and $A\nu$ are states too. A *right eigen-state* of the positive flow corresponding to a function $\varphi \in C(X)$ is a state ν satisfying $\nu(\,\cdot\, E_\varphi) = e^{\lambda(\varphi)}\nu(\,\cdot\,)$. Similarly, a *left* eigen-state of the flow is defined by the condition $\nu(E_\varphi\,\cdot\,) = e^{\lambda(\varphi)}\nu(\,\cdot\,)$.

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If both of these conditions take place simultaneously, then the state ν will be referred to as a *two-sided* eigen-state. In Theorem 4.3 it was proved that for any $\varphi \in C(X)$ there exist left and right eigen-states satisfying $\|\nu\| = \nu(1)$. However, the existence of two-sided eigen-states is still an open problem. Unfortunately, the author does not know of any conditions that guarantee the existence of two-sided eigen-states for an arbitrary positive flow.

A state ν will be called *sensitive* with respect to $\varphi \in C(X)$ (or φ-*sensitive*), if

$$\lim_{n\to\infty} \frac{1}{n} \ln \nu(E_\varphi^n) = \lambda(\varphi). \tag{6.1}$$

For example, the left and right eigen-states corresponding to φ are φ-sensitive. If a set $V \subset C(X)$ is separable, then it is easy to produce a state that is sensitive with respect to all $\varphi \in V$ (it has the form of a series of one-sided eigen-states corresponding to the elements of a countable dense subset in V).

PROPOSITION 6.1. *Suppose that a state ν of a positive flow is φ-sensitive. Then for any $\psi \in C(X)$ and $\varepsilon > 0$ there exists N so large that for all $n > N$,*

$$\frac{\nu(e^{S_n\psi} E_\varphi^n)}{\nu(E_\varphi^n)} \le e^{n(\lambda(\varphi+\psi)-\lambda(\varphi)+\varepsilon)}. \tag{6.2}$$

Proof. By the homological identity, $e^{S_n\psi} E_\varphi^n = e^{S_n(\varphi+\psi)} E^n = (E_{\varphi+\psi})^n$. Therefore, $\nu(e^{S_n\psi} E_\varphi^n) \le \|\nu\| \|(E_{\varphi+\psi})^n\|$. Note that the spectral radius of $E_{\varphi+\psi}$ is equal to $e^{\lambda(\varphi+\psi)}$. Therefore, combining the latter estimate with (6.1) we obtain (6.2). □

Since the spectral potential $\lambda(\varphi)$ is a particular case of dynamical potential, we can define on $C^*(X)$ the corresponding dual entropy $S(\mu) = \inf_{\varphi\in C(X)}(\lambda(\varphi) - \mu(\varphi))$ and the action functional $\tau_\varphi(\mu) = \mu(\varphi) + S(\mu) - \lambda(\varphi)$.

Denote by $M(X)$ the set of all probability distributions on X. For each point $x \in X$ define the *empirical measures* $\delta_{x,n} \in M(X)$:

$$\delta_{x,n}(f) = \frac{f(x) + f(\alpha(x)) + \cdots + f(\alpha^{n-1}(x))}{n}, \quad f \in C(X).$$

PROPOSITION 6.2. *Suppose that a state ν of a positive flow is φ-sensitive. Then for any measure $\mu \in C^*(X)$ and any number $t > \tau_\varphi(\mu)$ there exists a small neighborhood $O(\mu)$ in the *-weak topology such that for all n large enough*

$$\mathbf{P}_n\{x \in X \mid \delta_{x,n} \in O(\mu)\} < e^{nt}, \quad \text{where } \mathbf{P}_n = \frac{\nu(\cdot\, E_\varphi^n)}{\nu(E_\varphi^n)}. \tag{6.3}$$

Proof. Choose $\varepsilon > 0$ so small that $\tau_\varphi(\mu) < t - \varepsilon$. Then $S(\mu) = \lambda(\varphi) - \mu(\varphi) + \tau_\varphi(\mu) < \lambda(\varphi) - \mu(\varphi) + t - \varepsilon$. By the definition of $S(\mu)$ there exists a function $\psi \in C(X)$ such that $\lambda(\psi) - \mu(\psi) < \lambda(\varphi) - \mu(\varphi) + t - \varepsilon$ or, equivalently, $\mu(\psi - \varphi) < \lambda(\psi) - \lambda(\varphi) - t + \varepsilon$. Set $O(\mu) = \{\nu \in C^*(X) \mid \nu(\psi-\varphi) > \lambda(\psi)-\lambda(\varphi)-t+\varepsilon\}$. Obviously, $[S_n(\psi-\varphi)](x) = n\delta_{x,n}(\psi - \varphi)$. So if $\delta_{x,n} \in O(\mu)$, then $[S_n(\psi - \varphi)](x) - n(\lambda(\psi) - \lambda(\varphi) - t + \varepsilon) > 0$. Now by Proposition 6.1 for all sufficiently large n we have

$$\mathbf{P}_n\{x \in X \mid \delta_{x,n} \in O(\mu)\} \le \mathbf{P}_n(e^{S_n(\psi-\varphi)-n(\lambda(\psi)-\lambda(\varphi)-t+\varepsilon)}) < e^{nt}. \qquad \square$$

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Recall that any equilibrium measure corresponding to $\varphi \in C(X)$ is a subgradient of the functional $\lambda(\varphi)$ at the point φ. In Proposition 1.2 it was proved that each equilibrium measure is probability and α-invariant.

PROPOSITION 6.3. *Suppose that an open subset $U \subset M(X)$ contains all equilibrium measures corresponding to $\varphi \in C(X)$ and that states $\mathbf{P}_n$ have the form $\mathbf{P}_n = \nu(\cdot\, E_\varphi^n)/\nu(E_\varphi^n)$, where ν is some φ-sensitive state. Then there exists $\varkappa$ so small that for all n large enough the inequality $\mathbf{P}_n\{x \in X \mid \delta_{x,n} \notin U\} < e^{-n\varkappa}$ holds.*

Proof. Consider any measure $\mu \in M(X) \setminus U$. By Proposition 1.1 it satisfies the inequality $\tau_\varphi(\mu) < 0$. By Proposition 6.2 there exists a small neighborhood $O(\mu)$ in $M(X)$ such that $\mathbf{P}_n\{x \in X \mid \delta_{x,n} \in O(\mu)\} < e^{n\tau_\varphi(\mu)/2}$ as $n \to \infty$. The set $M(X) \setminus U$ is compact in the *-weak topology. Hence, it can be covered by finitely many neighborhoods $O(\mu_1), \ldots, O(\mu_k)$ of the form indicated above. Thus, $\mathbf{P}_n\{x \in X \mid \delta_{x,n} \notin U\} < \sum_{i=1}^k e^{n\tau_\varphi(\mu_i)/2}$, provided that n is large. □

THEOREM 6.4. *Suppose that $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ is a positive flow with phase space X, and for certain functions $\varphi, f \in C(X)$ the derivative*

$$\left.\frac{d\lambda(\varphi + tf)}{dt}\right|_{t=0} = a \tag{6.4}$$

does exist, and a state ν is φ-sensitive. Then the sequence $\{f \circ \alpha^n\}$ satisfies the law of large numbers with respect to the measures $\mathbf{P}_n = \nu(\cdot\, E_\varphi^n)/\nu(E_\varphi^n)$. Namely,

$$\lim_{n\to\infty} \mathbf{P}_n\{x \in X \mid |\delta_{x,n}(f) - a| > \varepsilon\} = 0 \quad \textit{for all } \varepsilon > 0. \tag{6.5}$$

If, in addition, ν is a right eigen-state for E_φ then the following strong law of large numbers is valid:

$$\lim_{n\to\infty} \delta_{x,n}(f) = a \quad \textit{for } \nu\textit{-almost all } x. \tag{6.6}$$

Proof. It follows from (6.4) that any equilibrium measure μ corresponding to φ satisfies the equality $\mu(f) = a$. Let us fix $\varepsilon > 0$ and denote by U_ε the set of all measures $m \in M(X)$ such that $|m(f) - a| < \varepsilon$. Thereupon define the sets $X_{\varepsilon,n} = \{x \in X \mid \delta_{x,n} \notin U_\varepsilon\}$. By Proposition 6.3 there exists $\varkappa > 0$ such that $\mathbf{P}_n(X_{\varepsilon,n}) < e^{-n\varkappa}$ for all n large enough. This estimate implies the first part of Theorem 6.4.

Now suppose that ν is a right eigen-state. Without loss of generality we can assume that it is normalized. Then all $\mathbf{P}_n$ coincide with ν. Hence, $\nu\big(\bigcup_{n>N} X_{\varepsilon,n}\big) < \sum_{n>N} e^{-n\varkappa} = e^{-N\varkappa}/(e^\varkappa - 1)$. Consider the set $X_\varepsilon = \bigcap_{N>0} \bigcup_{n>N} X_{\varepsilon,n}$. Evidently, $\nu(X_\varepsilon) = 0$. Therefore, $\nu\big(\bigcup_{\varepsilon>0} X_\varepsilon\big) = 0$. So, if $x \notin \bigcup_{\varepsilon>0} X_\varepsilon$, then $\lim_{n\to\infty} \delta_{x,n}(f) = a$. □

COROLLARY 6.5. *If the spectral potential $\lambda(\varphi)$ is Gâteaux differentiable at a point $\varphi \in C(X)$, then the laws of large numbers (6.5) and (6.6) are valid for all functions $f \in C(X)$.*

In connection with Theorem 6.4 the question of existence of the derivative (6.4) arises. In convex analysis they give the following answer to it. Let $\lambda(\varphi)$ be an arbitrary convex functional on $C(X)$. Then its restriction to any affine straight line in $C(X)$ can be non-differentiable on at most a countable set. If L is a finite-dimensional subspace in $C(X)$,

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then the restriction of $\lambda(\varphi)$ to L will be differentiable in the Gâteaux sense almost everywhere (with respect to Lebesgue's measure). Finally, if L is a separable vector subspace in $C(X)$, then the restriction of $\lambda(\varphi)$ to L will be Gâteaux differentiable on a residual set (a residual set is a countable intersection of some open dense subsets of L).

Another important question is how the probability distributions $\mathbf{P}_n$ in (6.5) might look. We present only one rather abstract example. Suppose that L is a separable vector subspace in $C(X)$ and $\{\varphi_1, \varphi_2, \dots\}$ is a countable dense subset in L. By Theorem 4.3 for each φ_i there exists a right eigen-state ν_i such that $\|\nu_i\| = \nu_i(I) = 2^{-i}$. Define the new state $\nu = \sum_{i=1}^{\infty} \nu_i$. It is sensitive with respect to all $\varphi \in L$. It is easy to see that in this case

$$\nu_i(fE_\varphi^n) = \nu_i(fe^{S_n(\varphi-\varphi_i)}E_{\varphi_i}^n) = \nu_i(fe^{S_n(\varphi-\varphi_i)})e^{n\lambda(\varphi_i)}$$

and, respectively,

$$\mathbf{P}_n(f) = \frac{\nu(fE_\varphi^n)}{\nu(E_\varphi^n)} = \frac{\sum_{i=1}^{\infty} \nu_i(fe^{S_n(\varphi-\varphi_i)})e^{n\lambda(\varphi_i)}}{\sum_{i=1}^{\infty} \nu_i(e^{S_n(\varphi-\varphi_i)})e^{n\lambda(\varphi_i)}}. \tag{6.7}$$

Now let us consider the case of E_φ having a two-sided eigen-state.

PROPOSITION 6.6. *If ν is a two-sided eigen-state corresponding to $\varphi \in C(X)$, then it induces an α-invariant measure on X.*

Proof. This follows from the fact that for any $f \in C(X)$ by the homological identity we have $\nu(f \circ \alpha) = e^{-\lambda(\varphi)}\nu((f \circ \alpha)E_\varphi) = e^{-\lambda(\varphi)}\nu(E_\varphi f) = \nu(f)$. □

PROPOSITION 6.7. *Suppose that ν is a normalized two-sided eigen-state corresponding to $\varphi \in C(X)$. Then ν induces an equilibrium measure on X. If, in addition, this measure is a unique equilibrium measure corresponding to φ, then it is ergodic.*

Proof. Consider the sequence of functionals $\lambda_n(\psi) = n^{-1} \ln \nu(E_\psi^n)$. It follows from the definition of the normalized eigen-state that $\lambda_n(\varphi) = \lambda(\varphi)$. Recall that by the homological identity $E_\psi^n = e^{S_n\psi}E^n$. Therefore, Proposition 2.3 yields the convexity of $\lambda_n(\psi)$. The following calculation shows that the restriction of ν to $C(X)$ coincides with the derivative $\lambda_n'(\varphi)$:

$$\left.\frac{d\lambda_n(\varphi + t\psi)}{dt}\right|_{t=0} = \frac{1}{n}e^{-n\lambda(\varphi)} \sum_{i=1}^{n} \nu(E_\varphi^{n-i}\psi E_\varphi^i) = \nu(\psi).$$

Hence, $\lambda_n(\varphi + \psi) - \lambda(\varphi) \geq \nu(\psi)$ for all $\psi \in C(X)$. From the latter inequality and the definition of $\lambda_n(\psi)$ it follows that $\lambda(\varphi + \psi) - \lambda(\varphi) \geq \nu(\psi)$. This exactly means that ν determines an equilibrium measure. If, in addition, this equilibrium measure is unique for φ, then its ergodicity follows from Corollary 6.5. □

Concluding the section we formulate a problem that seems to be of great significance. It is well known that in classical probability theory two theorems have the supreme importance, namely, the law of large numbers and the central limit theorem. As we have already seen, the restriction of the spectral potential to any finite-dimensional subspace $L \subset C(X)$ is differentiable almost everywhere. This differentiability also implies the law of large numbers by means of Theorem 6.4. However, furthermore, there is Alexandroff's

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theorem [**10**] asserting that any convex functional on a finite-dimensional vector space is *twice* differentiable almost everywhere. In this connection it would be desirable to know whether the central limit theorem is valid there where the spectral potential is twice differentiable.

7. *Probabilities of large deviations*

We continue to study the stochastic properties of the positive flow with phase space X. Suppose that L is a vector subspace in $C(X)$. For any $\varphi \in L$, let us define on L^* the *conditional action functional*

$$\tau_\varphi^L(u) = \inf_{\psi \in L} \{\lambda(\psi) - \lambda(\varphi) - u(\psi - \varphi)\}, \quad u \in L^*. \tag{7.1}$$

Obviously, in the case $L = C(X)$ it coincides with the ordinary action functional $\tau_\varphi(\mu)$. From the definition it follows that the conditional action functional is upper semicontinuous over L^* (relative to the *-weak topology).

PROPOSITION 7.1. *The conditional action functional satisfies the equality*

$$\tau_\varphi^L(u) = \sup\{\tau_\varphi(\mu) \mid \mu \in C^*(X), \mu|_L = u\}, \quad u \in L^*. \tag{7.2}$$

Proof. We have the Young inequality $S(\mu) \le \lambda(\psi) - \mu(\psi)$. It implies that if $\varphi, \psi \in L$ and $\mu|_L = u$, then $\tau_\varphi(\mu) = S(\mu) + \mu(\varphi) - \lambda(\varphi) \le \lambda(\psi) - \lambda(\varphi) - u(\psi - \varphi)$. Thus, the right-hand side of (7.2) does not exceed $\tau_\varphi^L(u)$.

Conversely, (7.1) implies $\tau_\varphi^L(u) + u(\psi) \le \lambda(\psi) - \lambda(\varphi) + u(\varphi)$ for all $\psi \in L$. By the Hahn–Banach theorem the linear functional u extends to the whole of $C(X)$ in such a way that $\tau_\varphi^L(u) + u(\psi) \le \lambda(\psi) - \lambda(\varphi) + u(\varphi)$ for all $\psi \in C(X)$. Therefore,

$$\tau_\varphi^L(u) \le \inf_{\psi \in C(X)} \{\lambda(\psi) - u(\psi) + u(\varphi) - \lambda(\varphi)\} = S(u) + u(\varphi) - \lambda(\varphi) = \tau_\varphi(u).$$

Hence, $\tau_\varphi^L(u)$ does not exceed the right-hand side of (7.2). □

PROPOSITION 7.2. *If $\tau_\varphi^L(u) > -\infty$, then u belongs to the closure (in the norm of L^*) of all subgradients of the restriction $\lambda(\cdot)|_L$.*

Proof. The proof is just the same as that of Proposition 1.3. □

Denote by D_ψ^L the set of all subgradients of the restriction $\lambda(\cdot)|_L$ at a point $\psi \in L$. Obviously, D_ψ^L is non-empty, convex, and closed in the *-weak topology. From (7.1) it follows that

$$\tau_\varphi^L(u) \le \lambda(\psi) - \lambda(\varphi) - u(\psi - \varphi), \quad u \in L^*. \tag{7.3}$$

Like the Young inequality, (7.3) turns into equality if and only if $u \in D_\psi^L$.

THEOREM 7.3. *Suppose that we are given a vector subspace $L \subset C(X)$, a function $\varphi \in L$, and a certain φ-sensitive state ν. Then for any $u \in L^*$ and any number $t_1 > \tau_\varphi^L(u)$ there exists a small neighborhood $O(u) \subset L^*$ (in the *-weak topology) such that for all sufficiently large n we have*

$$\mathbf{P}_n\{x \in X \mid \delta_{x,n} \in O(u)\} < e^{nt_1}, \quad \text{where } \mathbf{P}_n = \frac{\nu(\cdot\, E_\varphi^n)}{\nu(E_\varphi^n)}. \tag{7.4}$$

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In addition, if the state ν is simultaneously sensitive with respect to two functions $\varphi, \psi \in L$, then for any open subset $U \subset L^*$ containing D_ψ^L and any number $t_2 < \inf\{\tau_\varphi^L(u) \mid u \in D_\psi^L\}$ we have

$$\mathbf{P}_n\{x \in X \mid \delta_{x,n} \in U\} > e^{nt_2} \quad \text{where } \mathbf{P}_n = \frac{\nu(\cdot\, E_\varphi^n)}{\nu(E_\varphi^n)}, \tag{7.5}$$

provided that n is large enough.

The proof of the theorem is given below.

Remark. By Proposition 7.2, if a functional $u \in L^*$ lies at a positive distance from the set of all subgradients of the restriction $\lambda(\cdot)|_L$, then $\tau_\varphi^L(u) = -\infty$. In this case (7.4) is valid for any $t_1 < 0$.

COROLLARY 7.4. *If under the conditions of Theorem 7.3 the set D_ψ^L consists of only one element u, then for any $\varepsilon > 0$ there exists a small neighborhood $O(u) \subset L^*$ such that*

$$e^{(\tau-\varepsilon)} < \mathbf{P}_n\{x \in X \mid \delta_{x,n} \in O(u)\} < e^{(\tau+\varepsilon)} \quad \text{as } n \to \infty, \tag{7.6}$$

where $\tau = \tau_\varphi^L(u) = \lambda(\psi) - \lambda(\varphi) - u(\psi - \varphi)$.

The inequalities (7.6) represent the main result of the section. In essence, they mean that the number $\tau_\varphi^L(u)$ determines a rough exponential asymptotics for the probability of observing the empirical measures whose restriction to L is close to u.

Let us illustrate Corollary 7.4 in the case of finite-dimensional L. Suppose that functions $f_1, \dots, f_k \in C(X)$ are linearly independent and $L = \{t^1 f_1 + \cdots + t^k f_k \mid t^i \in \mathbb{R}\}$. Denote by $u = (u_1, \dots, u_k)$ the coordinates in L^* dual to $t = (t^1, \dots, t^k)$. Then formula (7.1) takes the form

$$\tau_{t_0}^L(u) = \inf_{t \in \mathbb{R}^k} \{\Lambda(t) - \Lambda(t_0) - \langle u, t - t_0\rangle\},$$

where $\Lambda(t) = \lambda(t^1 f_1 + \cdots + t^k f_k)$ and $t_0^1 f_1 + \cdots + t_0^k f_k = \varphi$.

The functional $\Lambda(t)$ is convex and so is Gâteaux differentiable almost everywhere. At the points of differentiability Corollary 7.4 looks as follows: for any $\varepsilon > 0$ there exists a small $\delta > 0$ such that for all sufficiently large n we have

$$e^{n(\tau-\varepsilon)} < \mathbf{P}_n\{x \in X \mid |\delta_{x,n}(f_i) - u_i| < \delta,\ i = 1, \dots, k\} < e^{n(\tau+\varepsilon)},$$

where $u = \Lambda'(t)$ and $\tau = \tau_{t_0}^L(u) = \Lambda(t) - \Lambda(t_0) - \langle \Lambda'(t), t - t_0\rangle$.

The finite-dimensional case has the additional advantage that the state ν, which generates the probability distributions $\mathbf{P}_n$, can be chosen the same for all t and t_0. The only condition for this is the sensitivity of ν with respect to all $\varphi \in L$.

Proof of Theorem 7.3. We prove the estimate (7.4) in just the same way as (6.3). Choose a small positive ε such that $\tau_\varphi^L(u) < t_1 - \varepsilon$. By (7.1) there exists $\psi \in L$ satisfying $u(\psi - \varphi) > \lambda(\psi) - \lambda(\varphi) - t_1 + \varepsilon$. Set $O(u) = \{v \in L^* \mid v(\psi - \varphi) > \lambda(\psi) - \lambda(\varphi) - t_1 + \varepsilon\}$.

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Recall that $[S_n(\psi-\varphi)](x) = n\delta_{x,n}(\psi-\varphi)$. Hence, the condition $\delta_{x,n} \in O(u)$ yields the inequality $[S_n(\psi-\varphi)](x) - n(\lambda(\psi)-\lambda(\varphi)-t_1+\varepsilon) > 0$. Then, in view of Proposition 6.1,

$$\mathbf{P}_n\{x \in X \mid \delta_{x,n} \in O(u)\} \le \mathbf{P}_n(e^{S_n(\psi-\varphi)-n(\lambda(\psi)-\lambda(\varphi)-t_1+\varepsilon)}) < e^{nt_1},$$

provided that n is large. Thus, the first part of Theorem 7.3 is proved.

Now we proceed to the second part. Choose $\varepsilon > 0$ such that $t_2 + \varepsilon < \inf\{\tau_\varphi^L(u) \mid u \in D_\psi^L\}$, and define the set

$$U' = \{u \in U \mid u(\psi-\varphi) < \lambda(\psi) - \lambda(\varphi) - t_2 - \varepsilon\}. \tag{7.7}$$

Obviously, it is open. In addition, in view of (7.3) and the choice of ε, it contains D_ψ^L.

Let B be the unit ball in L^*. By Alaoglu's theorem B is compact in the *-weak topology. Hence the set $B \setminus U'$ is compact too. By construction, it has no common points with D_ψ^L and so the inequality (7.3) is strict on $B \setminus U'$. In addition, the functional $\tau_\varphi^L(u)$ is upper semicontinuous. Therefore, there exists $\varepsilon > 0$ so small that

$$\tau_\varphi^L(u) < \lambda(\psi) - \lambda(\varphi) - u(\psi-\varphi) - 3\varepsilon \quad \text{for all } u \in B \setminus U'. \tag{7.8}$$

Using the already proved first part of Theorem 7.3, choose for each $u \in B \setminus U'$ a neighborhood $O(u) \subset L^*$ so small that

$$\mathbf{P}_n\{x \in X \mid \delta_{x,n} \in O(u)\} < e^{n(\tau_\varphi^L(u)+\varepsilon)} \quad \text{as } n \to \infty \tag{7.9}$$

and at the same time

$$\sup\{v(\psi-\varphi) \mid v \in O(u)\} < u(\psi-\varphi) + \varepsilon. \tag{7.10}$$

Choose from these neighborhoods a finite covering $O(u_1), \dots, O(u_k)$ of the set $B \setminus U'$. Then

$$\begin{aligned}\mathbf{P}_n(e^{S_n(\psi-\varphi)}) &\le \mathbf{P}_n\{x \mid \delta_{x,n} \in U'\} \sup\{e^{[S_n(\psi-\varphi)](x)} \mid \delta_{x,n} \in U'\} \\ &\quad + \sum_{i=1}^k \mathbf{P}_n\{x \mid \delta_{x,n} \in O(u_i)\} \sup\{e^{[S_n(\psi-\varphi)](x)} \mid \delta_{x,n} \in O(u_i)\}.\end{aligned}$$

Substituting here the estimates (7.7)–(7.10) and using the equality $[S_n(\psi-\varphi)](x) = n\delta_{x,n}(\psi-\varphi)$, we obtain

$$\begin{aligned}\mathbf{P}_n(e^{S_n(\psi-\varphi)}) &\le \mathbf{P}_n\{x \mid \delta_{x,n} \in U'\} e^{n(\lambda(\psi)-\lambda(\varphi)-t_2-\varepsilon)} + \sum_{i=1}^k e^{n(\tau_\varphi^L(u_i)+\varepsilon)} e^{n(u_i(\psi-\varphi)+\varepsilon)} \\ &\le e^{n(\lambda(\psi)-\lambda(\varphi))}(\mathbf{P}_n\{x \mid \delta_{x,n} \in U'\} e^{-n(t_2+\varepsilon)} + e^{-n\varepsilon}).\end{aligned} \tag{7.11}$$

On the other hand,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}_n(e^{S_n(\psi-\varphi)}) &= \lim_{n\to\infty} \frac{1}{n} \ln \frac{\nu(e^{S_n(\psi-\varphi)} E_\varphi^n)}{\nu(E_\varphi^n)} \\ &= \lim_{n\to\infty} \frac{1}{n} \ln \frac{\nu(E_\psi^n)}{\nu(E_\varphi^n)} = \lambda(\psi) - \lambda(\varphi).\end{aligned} \tag{7.12}$$

Comparison of (7.11) and (7.12) gives (7.5). □

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8. *Positive processes*

A (discrete time) *positive process* is a triple $(\mathcal{A}, \mathcal{C}, E)$ consisting of a positive Banach algebra $\mathcal{A}$ with closed cone $\mathcal{A}_+$, a phase subalgebra $\mathcal{C} \subset \mathcal{A}$, and an element $E \in \mathcal{A}_+$ called the evolution operator. For instance, one can take the algebra of bounded linear operators on the space $L^p(X, \mu)$ as $\mathcal{A}$, the algebra $L^\infty(X, \mu)$ as $\mathcal{C}$, and an arbitrary positive linear operator on $L^p(X, \mu)$ as E. We call the support of the phase algebra $\mathcal{C}$ the phase space of the positive process and usually denote it by X. The positive process differs from the positive flow only by the absence of the shift homomorphism $\delta : \mathcal{C} \to \mathcal{C}$ and the mapping $\alpha : X \to X$ associated with it. Nevertheless, we shall see soon that one can assign in a standard way a certain positive flow to any positive process.

A *trajectory* of a positive process $(\mathcal{A}, \mathcal{C}, E)$ is an arbitrary sequence $\bar{x} = (x_1, x_2, x_3, \dots)$ of points of the phase space X. We denote the set of all trajectories supplied with the cylinder topology by $\bar{X}$. By Tychonoff's theorem, $\bar{X}$ is compact.

For every natural n define the subalgebra $C(X^n) \subset C(\bar{X})$ as the set of continuous functions of the form $F(x_1, \dots, x_n)$, which depend only on the first n coordinates of the point $x \in \bar{X}$. The subalgebras $C(X^n)$ are naturally embedded into each other: $\mathbb{R} = C(X^0) \subset C(X^1) \subset C(X^2) \subset \cdots$. By Stone's theorem the union $\bigcup_n C(X^n)$ is dense in $C(\bar{X})$.

Consider the left shift $\sigma : (x_1, x_2, x_3, \dots) \longmapsto (x_2, x_3, \dots)$ on $\bar{X}$. It generates the homomorphism of $C(\bar{X})$ taking each function F to $F \circ \sigma$. Obviously, $F \circ \sigma^m(x_1, x_2, \dots) = F(x_{m+1}, x_{m+2}, \dots)$. Also, if $F \in C(X^n)$, then $F \circ \sigma^m \in C(X^{n+m})$.

The following object to be defined is the *crossed product* of the algebra $C(\bar{X})$ and the semigroup $\{E^n \mid n \geq 0\}$ generated by the evolution operator E. We use the notation $\mathcal{B} = C(\bar{X}) \times_\sigma \{E^n\}$ for this crossed product. By definition, the crossed algebra $\mathcal{B}$ consists of formal sums $\sum_{i=0}^n F_i E^i$, where $F_i \in C(\bar{X})$, which are multiplied by the rule

$$\left(\sum_{i=0}^n F_i E^i\right)\left(\sum_{j=0}^m G_j E^j\right) = \sum_{i=0}^n \sum_{j=0}^m (F_i \cdot (G_j \circ \sigma^i)) E^{i+j}. \tag{8.1}$$

We call the elements of $\mathcal{B}$ the *crossed operators*. We say that the crossed operator $\sum_{i=0}^n F_i E^i$ has *degree* n. From (8.1) it follows that the degree of the product of two crossed operators is equal to the sum of their degrees. Evidently, the set of all crossed operators of zero degree forms a subalgebra in $\mathcal{B}$ isomorphic to $C(\bar{X})$. Denote by $\mathcal{B}_+$ the set of crossed operators of the form $\sum_{i=0}^n F_i E^i$, where all of the coefficients $F_i \in C(\bar{X})$ are non-negative. The cone $\mathcal{B}_+$ induces a semiorder in $\mathcal{B}$. A crossed operator $\sum_{i=0}^n F_i E^i$ will be called *proper* if $F_i \in C(X^i)$ for all $i = 1, \dots, n$. We denote the set of all proper crossed operators by $\mathcal{B}_p$. Formula (8.1) shows that $\mathcal{B}_p$ is a subalgebra in $\mathcal{B}$.

THEOREM 8.1. *Let $(\mathcal{A}, \mathcal{C}, E)$ be a positive process with phase space X, $\mathcal{B}$ be the corresponding crossed algebra with cone $\mathcal{B}_+$, and $\mathcal{B}_p \subset \mathcal{B}$ be the subalgebra of proper operators. Then there exists a unique algebraic homomorphism $\pi : \mathcal{B}_p \to \mathcal{A}$ that is positive (maps the cone $\mathcal{B}_p \cap \mathcal{B}_+$ to $\mathcal{A}_+$) and takes each crossed operator of the form $f_1(x_1) \cdots f_n(x_n) E^n$, where $f_i \in C(X)$ and $(x_1, \dots, x_n) \in X^n$, to the ordinary product $f_1 E f_2 E \cdots f_n E \in \mathcal{A}$. Moreover, for each function $F \in C(X^n)$ the estimate $\|\pi(FE^n)\| \leq 2\|E^n\| \sup F$ holds.*

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Proof. At the beginning let us assume that such a homomorphism does exist and prove its uniqueness. Consider an arbitrary function $F \in C(X^n)$. Set $t = \max|F|$. Then $0 \le t \pm F \le 2t$. By the positiveness of π and the monotonicity of the norm on $\mathcal{A}$ we have

$$\begin{aligned} 2\|\pi(FE^n)\| &= \|\pi((t+F)E^n) - \pi((t-F)E^n)\| \\ &\le \|\pi((t+F)E^n)\| + \|\pi((t-F)E^n)\| \le 2\|\pi(2tE^n)\| = 4t\|E^n\|. \end{aligned} \tag{8.2}$$

Therefore, $\pi(FE^n)$ depends continuously on $F \in C(X^n)$. Denote by $\mathcal{C}_n$ the algebra of functions of the form $F = \sum_{i=1}^m f_1^i(x_1)\cdots f_n^i(x_n)$, where $f_j^i \in C(X)$ and $(x_1, \dots, x_n) \in X^n$. By Stone's theorem, $\mathcal{C}_n$ is dense in $C(X^n)$. By the condition of the theorem, $\pi(FE^n)$ is defined uniquely for all $F \in \mathcal{C}_n$. This argument implies the required uniqueness of π.

Now let us prove the existence of π. For all $F \in \mathcal{C}_n$, $\pi(FE^n)$ is already defined by the condition of the theorem. Suppose that a function $F \in \mathcal{C}_n$ is non-negative. Then by Theorem 3.4 for any $\varepsilon > 0$ the sum $F + \varepsilon$ can be represented as $F + \varepsilon = \sum_{i=1}^m f_1^i(x_1)\cdots f_n^i(x_n)$, where all of the functions $f_j^i \in C(X)$ are non-negative. Hence, $\pi((F+\varepsilon)E^n) = \sum_{i=1}^m f_1^i E \cdots f_n^i E \in \mathcal{A}_+$ and so $\pi(FE^n) \in \mathcal{A}_+$ as well. Now calculation (8.2) becomes valid for all $F \in \mathcal{C}_n$. It shows that the linear mapping $F \mapsto \pi(FE^n)$ is continuous over $\mathcal{C}_n$. By the continuity it extends to $C(X^n)$ preserving the positiveness. Thus, we have constructed the linear mapping $\pi : \mathcal{B}_p \to \mathcal{A}$. It is easy to see that this π is an algebraic homomorphism. Indeed, if $A = f_1(x_1)\cdots f_n(x_n)E^n$ and $B = g_1(x_1)\cdots g_m(x_m)E^m$, then $AB = f_1(x_1)\cdots f_n(x_n)g_1(x_{n+1})\cdots g_m(x_{n+m})E^{n+m}$ and so $\pi(AB) = f_1 E \cdots f_n E g_1 E \cdots g_m E = \pi(A)\pi(B)$. □

On the crossed algebra $\mathcal{B} = C(\bar{X}) \times_\sigma \{E^n\}$ one may define various monotone norms and seminorms (a seminorm on $\mathcal{B}$ is called monotone if $\|A\| \le \|A+B\|$, provided that $A, B \in \mathcal{B}_+$). We say that a monotone seminorm $\|\cdot\|$ on $\mathcal{B}$ is *compatible* with the norm on the initial algebra $\mathcal{A}$ if $\|A\| = \|\pi(A)\|$ for any proper positive operator $A \in \mathcal{B}_p \cap \mathcal{B}_+$. It turns out that compatible seminorms do exist and there is a maximal one among them, which can be defined in the following way. For a positive crossed operator $A \in \mathcal{B}_+$ set

$$\|A\| = \inf\{\|\pi(A')\| \mid A' \in \mathcal{B}_p \text{ and } A' - A \in \mathcal{B}_+\}, \tag{8.3}$$

and for the rest $A \in \mathcal{B}$ let

$$\|A\| = \inf\{\|A'\| + \|A''\| \mid A', A'' \in \mathcal{B}_+ \text{ and } A' - A'' = A\}. \tag{8.4}$$

THEOREM 8.2. *Suppose that $(\mathcal{A}, \mathcal{C}, E)$ is a positive process and $\mathcal{B}$ is the corresponding crossed algebra with cone $\mathcal{B}_+$ and $\mathcal{B}_p \subset \mathcal{B}$ is the subalgebra of proper operators. Then formulas (8.3) and (8.4) determine a monotone seminorm on $\mathcal{B}$ compatible with the norm on $\mathcal{A}$. This seminorm is the maximum of all monotone compatible seminorms on $\mathcal{B}$.*

Proof. Consider a pair of positive crossed operators $A, B \in \mathcal{B}_+$. Suppose that for some proper operators $A', B' \in \mathcal{B}_p$ we have $A' - A \in \mathcal{B}_+$ and $B' - B \in \mathcal{B}_+$. Then $A', B' \in \mathcal{B}_+$ and $A'B' - AB = (A'-A)B' + A(B'-B) \in \mathcal{B}_+$. Therefore, (8.3) implies $\|AB\| \le \|\pi(A'B')\| \le \|\pi(A')\|\|\pi(B')\|$. Passing here to the infimum over A', B' we obtain $\|AB\| \le \|A\|\|B\|$. Similarly (and even a little more easily) it follows from (8.3) that $\|A\| + \|B\| \ge \|A+B\|$ and $\|A+B\| \ge \|A\|$. Further, if $A \in \mathcal{B}_+$, then the infimum

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in (8.4) is attained at $A' = A$ and $A'' = 0$. Therefore, formulas (8.3) and (8.4) give the same value of $\|A\|$ for $A \in \mathcal{B}_+$.

Now consider a pair of arbitrary operators $A, B \in \mathcal{B}$. Suppose that some operators $A', A'', B', B'' \in \mathcal{B}_+$ satisfy $A' - A'' = A$ and $B' - B'' = B$. Then

$$A + B = (A' + B') - (A'' + B''),$$
$$AB = (A' - A'')(B' - B'') = (A'B' + A''B'') - (A'B'' + A''B'),$$

and by virtue of (8.4),

$$\|A + B\| \le \|A' + B'\| + \|A'' + B''\| \le \|A'\| + \|A''\| + \|B'\| + \|B''\|, \tag{8.5}$$
$$\|AB\| \le \|A'B' + A''B''\| + \|A'B'' + A''B'\| \le (\|A'\| + \|A''\|)(\|B'\| + \|B''\|) \tag{8.6}$$

(here we use the inequalities $\|C + D\| \le \|C\| + \|D\|$ and $\|CD\| \le \|C\|\|D\|$, which are already proved for positive operators). Passing in (8.5) and (8.6) to the infimums over A', A'', B', B'' we get the triangle inequality $\|A + B\| \le \|A\| + \|B\|$ and the estimate for the product $\|AB\| \le \|A\|\|B\|$. Eventually we have proved that (8.3) and (8.4) determine a monotone seminorm on $\mathcal{B}$. The compatibility of it is evident because in the case $A \in \mathcal{B}_p \cap \mathcal{B}_+$ the infimum in (8.3) is attained at $A' = A$.

Let us prove that this seminorm is maximal. Consider any other compatible monotone seminorm $|\cdot|$ on $\mathcal{B}$. If $A \in \mathcal{B}_+$, then, in view of (8.3),

$$|A| \le \inf\{|A'| \mid A' \in \mathcal{B}_p \text{ and } A' - A \in \mathcal{B}_+\} = \|A\|, \tag{8.7}$$

and for an arbitrary $A \in \mathcal{B}$, in view of (8.4) and (8.7), we have

$$|A| \le \inf\{|A'| + |A''| \mid A', A'' \in \mathcal{B}_+ \text{ and } A' - A'' = A\} \le \|A\|. \qquad \square$$

PROPOSITION 8.3. *Suppose that $\mathcal{B}$ is the crossed algebra corresponding to a positive process $(\mathcal{A}, \mathcal{C}, E)$ with phase space X. Then for any function $F \in C(\bar{X})$ we have $\|F\| \le 2 \max |F| \le 2\|F\|$, where $\|\cdot\|$ is the seminorm (8.3) and (8.4).*

Proof. The algebra $C(\bar{X})$ is naturally embedded in $\mathcal{B}$ (being the subalgebra of crossed operators of zero degree). By definition, the intersection $C(\bar{X}) \cap \mathcal{B}_p$ contains only constant functions. Therefore, for non-negative F, formula (8.3) gives $\|F\| = \max |F|$. For an arbitrary $F \in C(\bar{X})$ let $F^+ = \max\{0, F\}$ and $F^- = \max\{0, -F\}$. Obviously, $F = F^+ - F^-$. Hence, (8.4) gives $\|F\| = \|F^+\| + \|F^-\| \le 2 \max |F|$. On the other hand, $\max |F| \le \|F^+\| + \|F^-\| = \|F\|$. $\square$

In the sequel we always assume that the crossed algebra $\mathcal{B} = C(\bar{X}) \times_\sigma \{E^n\}$ corresponding to the positive process $(\mathcal{A}, \mathcal{C}, E)$ is equipped with the maximal compatible monotone seminorm (8.3) and (8.4). In view of Proposition 8.3 the natural embedding $C(\bar{X}) \hookrightarrow \mathcal{B}$ is continuous. Let us form the quotient of $\mathcal{B}$ by the crossed operators with zero seminorm (i.e. we regard all of the operators with zero seminorm as zeros). On this quotient algebra the maximal compatible seminorm becomes a norm. Complete the quotient algebra in the norm. We call the resulting Banach algebra, as before, the crossed product and denote it by the same letter $\mathcal{B}$. Obviously, this new $\mathcal{B}$ is semiordered (by the closure of the cone $\mathcal{B}_+$), positive, and includes the whole of $C(\bar{X})$.

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Formula (8.1) yields the homological identity $EF = (1E)(FE^0) = (F \circ \sigma)E$ for all $F \in C(\bar{X})$. Denote by σ^* the homomorphism $\sigma^*(F) = F \circ \sigma$, where $F \in C(\bar{X})$. It follows from the above that the quadruple $(\mathcal{B}, C(\bar{X}), E, \sigma^*)$ forms a positive flow associated with the dynamical system $(\bar{X}, \sigma)$. We call it the *suspension* over the positive process $(\mathcal{A}, \mathcal{C}, E)$.

For the suspension $(\mathcal{B}, C(\bar{X}), E, \sigma^*)$, just as for any positive flow, we can define the spectral potential, the equilibrium measures on X, the eigen- and sensitive states. For them, of course, all of the probabilistic results proved in Sections 6 and 7 are valid.

In conclusion we note two more useful properties of the spectral potential $\lambda(\varphi)$ of the positive flow $(\mathcal{B}, C(\bar{X}), E, \sigma^*)$. First, its restriction to the subalgebra $C(X) \subset C(\bar{X})$ coincides with the logarithm of the spectral radius of the operator $e^{\varphi}E$ in the initial algebra $\mathcal{A}$. This follows from the compatibility of the norms on $\mathcal{B}$ and on $\mathcal{A}$. Second, the value of $\lambda(\varphi)$ in (5.2) does not at all depend on the choice of a monotone seminorm on the crossed algebra $\mathcal{B}$, provided that this seminorm is compatible with the norm on $\mathcal{A}$. We leave this statement without proof because it will not be used below.

9. *Supports of positive flows*

Let $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ be a covariant positive flow with phase space X. It can be considered as the positive process $(\mathcal{A}, \mathcal{C}, E)$. However, the latter, in turn, is identified with the positive flow $(\mathcal{B}, C(\bar{X}), E, \sigma^*)$ (suspension). In the current section we investigate relations between the spectral potentials, equilibrium measures, and eigen-states of the initial positive flow and the suspension over it.

A function $f \in C(X)$ will be called *unessential* for the positive flow $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ if $fE^n = 0$ for a certain $n \in \mathbb{N}$. Evidently, the set of all unessential functions forms an ideal in $C(X)$. Denote it by $\mathcal{I}$. Note that by the homological identity $EfE^n = (f \circ \alpha)E^{n+1}$. Hence, the ideal $\mathcal{I}$ is α-invariant.

Define the *support* of the positive flow as the annihilator of $\mathcal{I}$. In other words, the support is $\bigcap_{f \in \mathcal{I}} f^{-1}(0)$. The α-invariance of $\mathcal{I}$ implies α-invariance of the support.

PROPOSITION 9.1. *The value of the spectral potential $\lambda(\varphi)$ depends only on the restriction of φ to the support of the positive flow.*

Proof. A function $\psi \in C(X)$ vanishes on the support if and only if it belongs to the closure of $\mathcal{I}$. Therefore, it is sufficient to prove the identity $\lambda(\varphi + \psi) = \lambda(\varphi)$ for all $\psi \in \mathcal{I}$. Suppose that $\psi E^m = 0$. Consider the operator $E^n_{\varphi+\psi}E^m - E^n_{\varphi}E^m = (e^{S_n(\varphi+\psi)} - e^{S_n\varphi})E^{n+m}$. The difference $e^{S_n(\varphi+\psi)} - e^{S_n\varphi}$ can be represented as $f_n S_n \psi$, where $f_n \in C(X)$. From the homological identity it follows that $S_n \psi E^{n+m} = \sum_{i=0}^{n-1} E^i \psi E^{n+m-i} = 0$. Hence, $E^n_{\varphi+\psi}E^m = E^n_{\varphi}E^m$ for all n. Put $t = \max|\varphi|$. Since the norm on $\mathcal{A}$ is monotone, $\|E^n_{\varphi}E^m\|e^{-mt} \le \|E^{n+m}_{\varphi}\| \le \|E^n_{\varphi}E^m\|e^{mt}$. These inequalities show that $\lambda(\varphi) = \lim_{n\to\infty} n^{-1} \ln \|E^n_{\varphi}E^m\|$. Absolutely in the same way, $\lambda(\varphi + \psi) = \lim_{n\to\infty} n^{-1} \ln \|E^n_{\varphi+\psi}E^m\|$. Therefore, $\lambda(\varphi) = \lambda(\varphi + \psi)$. □

PROPOSITION 9.2. *Equilibrium measures of a positive flow are concentrated on its support.*

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Proof. Each equilibrium measure μ is a subgradient of the spectral potential $\lambda(\varphi)$ at a point $\varphi \in C(X)$. It is sufficient to show that $\mu(\psi) = 0$ for all functions ψ that vanish on the support of the flow. By the definition of the subgradient, $\lambda(\varphi + t\psi) - \lambda(\varphi) \geq \mu(t\psi)$ for all $t \in \mathbb{R}$. In addition, by Proposition 9.1, we have $\lambda(\varphi + t\psi) = \lambda(\varphi)$. Hence, $\mu(\psi) = 0$. □

We study the positive flow $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$, which generates a dynamical system (X, α). Set $\bar{X} = X^{\mathbb{N}}$ and define the algebraic homomorphism $\pi : C(\bar{X}) \to C(X)$ that takes each function $F(x_1, x_2, x_3, \dots)$ to $\pi F(x) = F(x, \alpha(x), \alpha^2(x), \dots)$. Consider the crossed product $\mathcal{B} = C(\bar{X}) \times_\sigma \{E^n\}$. Extend π up to a mapping $\pi : \mathcal{B} \to \mathcal{A}$ by means of the equality $\pi\left(\sum_{i=0}^n F_i E^i\right) = \sum_{i=0}^n \pi(F_i) E^i$. It follows from the homological identity that this extended π is also a homomorphism. Which, of course, is positive. On crossed operators of the form $f_1 E f_2 E \cdots f_n E$, where $f_i \in C(X)$, our extended π coincides with the homomorphism that was defined in Theorem 8.1 and was denoted there by the same letter π. Since linear combinations of operators having the indicated form are dense in the subalgebra of proper crossed operators $\mathcal{B}_p \subset \mathcal{B}$, these two homomorphisms coincide on $\mathcal{B}_p$. Thus, we see that the just defined π is an extension (from $\mathcal{B}_p$ to $\mathcal{B}$) of the homomorphism from Theorem 8.1.

PROPOSITION 9.3. *If a positive flow generates a dynamical system (X, α), then the set of trajectories of the dynamical system contains the support of the suspension over the flow.*

Proof. Suppose that a sequence $\bar{a} = (a_1, a_2, a_3, \dots)$ is not a trajectory of the dynamical system (X, α). It is sufficient to show that $\bar{a}$ lies out of the support of the suspension over the initial flow. And to prove this it is sufficient to find an unessential (with respect to the suspension) function $F \in C(\bar{X})$ such that $F(\bar{a}) \neq 0$. Fix a number n such that $\alpha(a_{n-1}) \neq a_n$. Choose neighborhoods $O(a_{n-1})$ and $O(a_n)$ in X such that $\alpha(O(a_{n-1})) \cap O(a_n) = \emptyset$. There exist non-negative functions $f_i \in C(X)$, $i = n-1, n$, vanishing out of $O(a_i)$ and equal to one at a_i. Evidently, $f_{n-1} \cdot (f_n \circ \alpha) \equiv 0$. Consider the function $F(\bar{x}) = f_{n-1}(x_{n-1}) f_n(x_n)$ on $\bar{X}$. By construction, it takes the unit value at $\bar{a}$. In addition, by the compatibility of norms we have $\|FE^n\|_{\mathcal{B}} = \|\pi(FE^n)\|_{\mathcal{A}} = \|E^{n-1} f_{n-1} \cdot (f_n \circ \alpha) E\|_{\mathcal{A}} = 0$. Thus, F is unessential. □

Now let $\lambda(\varphi)$ be the spectral potential of a positive flow $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ with phase space X and $\Lambda(\varphi)$ be the spectral potential of the corresponding suspension $(\mathcal{B}, C(\bar{X}), E, \sigma^*)$. Define the embedding $i_\alpha : X \to \bar{X}$ that assigns to each point $x \in X$ its trajectory $i_\alpha(x) = (x, \alpha(x), \alpha^2(x), \dots)$. This embedding is associated with the homomorphism π (that is $\pi F = F \circ i_\alpha$).

PROPOSITION 9.4. *The embedding i_α establishes a one-to-one correspondence between the support of the initial positive flow and the support of the suspension over it. In addition, $\Lambda(\varphi) \equiv \lambda(\varphi \circ i_\alpha)$.*

Proof. Recall that $C(X)$ is naturally embedded into $C(\bar{X})$. For any $\varphi \in C(\bar{X})$ the difference $\varphi - \varphi \circ i_\alpha$ vanishes on the set of trajectories $i_\alpha(X)$ and, hence, on the support of the suspension as well. By Proposition 9.1, we have $\Lambda(\varphi) \equiv \Lambda(\varphi \circ i_\alpha)$. In view of the compatibility of norms, the spectral radius of the operator $e^{\varphi \circ i_\alpha} E$ in $\mathcal{B}$ is equal to the spectral radius of the same operator in $\mathcal{A}$. Therefore, $\lambda(\varphi \circ i_\alpha) = \Lambda(\varphi \circ i_\alpha) = \Lambda(\varphi)$.

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Suppose that a trajectory $i_\alpha(a) = (a, \alpha(a), \alpha^2(a), \dots)$ lies in the support of the suspension and that a function $f \in C(X)$ is unessential with respect to the initial flow. Then f^2 is unessential as well. Hence, for a certain n the operator $f^2E^n \in \mathcal{A}$ is zero. By the compatibility of norms the operator $f^2E^n \in \mathcal{B}_+$ is zero as well. Therefore, f^2 is unessential with respect to the suspension and so $f^2(i_\alpha(a)) = f^2(a) = 0$. From this it follows that a belongs to the support of the initial flow.

Conversely, suppose that $a \in X$ belongs to the support of the initial flow. We have to prove that if a function $F \in C(\bar{X})$ is unessential with respect to the suspension, then it vanishes at $i_\alpha(a)$. Let $FE^n = 0$. Then, of course, $F^2E^n = 0$. Consider the function $G(x_1, \dots, x_n) = \max\{F^2(x_1, \dots, x_n, x_{n+1}, \dots) \mid x_{n+1}, x_{n+2}, \dots \in X\}$. It follows from the definition of the maximal compatible seminorm on the crossed algebra that $\|F^2E^n\|_{\mathcal{B}} = \|\pi(G)E^n\|_{\mathcal{A}}$. Therefore, $\|\pi(G)E^n\|_{\mathcal{A}} = 0$ and, hence, the function $\pi(G) = G \circ i_\alpha$ is unessential with respect to the initial flow. Thus, $G(i_\alpha(a)) = 0$ and so $F^2(i_\alpha(a)) = 0$. □

Proposition 9.4 shows that the embedding $i_\alpha : X \to \bar{X}$ not only establishes a homeomorphism between the support of the initial positive flow and the support of the suspension over it, but also establishes a one-to-one correspondence between the sets of equilibrium measures on both of the supports. Moreover, it is easy to see that the embedding i_α conjugates the mapping $\alpha : X \to X$ with the shift $\sigma : \bar{X} \to \bar{X}$, i.e. $i_\alpha \circ \alpha = \sigma \circ i_\alpha$.

Finally, let us look at how eigen-states of the initial positive flow and those of the corresponding suspension are related to each other. Obviously, each state ν on $\mathcal{A}$ induces the state $\mu = \nu \circ \pi$ on the crossed algebra $\mathcal{B}$. Suppose that ν is a right eigen-state corresponding to the function $\pi\varphi = \varphi \circ i_\alpha$, where $\varphi \in C(\bar{X})$. Then $\mu = \nu \circ \pi$ is a right eigen-state corresponding to φ because

$$\mu(\cdot\, e^{\varphi}E) = \nu(\pi(\cdot)e^{\pi\varphi}E) = e^{\lambda(\pi\varphi)}\nu(\pi(\cdot)) = e^{\Lambda(\varphi)}\mu(\cdot).$$

On the other hand, the question of whether any right eigen-state on the crossed algebra $\mathcal{B}$ can be represented as $\mu = \nu \circ \pi$, where ν is a right eigen-state on $\mathcal{A}$, still remains open. In brief, the difficulty of this question lies in the fact that the equality $\mu = \nu \circ \pi$ uniquely determines the restriction of ν to the subalgebra $\pi(\mathcal{B}) \subset \mathcal{A}$ but it is not at all evident that this restriction is positive (in the sense of the cone $\mathcal{A}_+$) or at least continuous with respect to the norm on $\mathcal{A}$.

10. *Examples*

In conclusion we present several examples illustrating the notions of positive flow, positive process, spectral potential, dual entropy, and eigen-state.

Example 1. Suppose that $\alpha : X \to X$ is a continuous mapping of a Hausdorff compact set X. Denote by $\mathcal{A}$ the algebra of all continuous linear operators on $C(X)$. Take $C(X)$ as a phase algebra $\mathcal{C}$. Let an evolution operator E and a homomorphism $\delta : \mathcal{C} \to \mathcal{C}$ be given by $Ef = \delta(f) = f \circ \alpha$. For this positive flow, the spectral potential $\lambda(\varphi)$ is the logarithm of the spectral radius of the weighted shift operator $E_\varphi = e^{\varphi}E$ that assigns to each $f \in C(X)$ the function $e^{\varphi} \cdot f \circ \alpha$. It is well known (see [**1, 13, 15, 16**]) that in this case

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the spectral potential has the form $\lambda(\varphi) = \sup_{\mu \in M_\alpha(X)} \int_X \varphi \, d\mu$, where $M_\alpha(X)$ is the set of all α-invariant Borel probability measures on X. From the remark after Proposition 1.4 it follows that this spectral potential is maximal among all dynamical potentials with fixed value $\lambda(0) = 0$.

Let us consider separately the case of zero weight function φ. From the form of the functional $\lambda(\varphi)$ we see that it is Gâteaux differentiable at the point $\varphi = 0$ if and only if the dynamical system (X, α) has a unique invariant probability measure μ. In this case $\lambda(\varphi) \equiv \int_X \varphi \, d\mu$. Dynamical systems with unique invariant probability measure μ are called *unique ergodic*. In the unique ergodic dynamical systems all empirical measures converge to μ (in the *-weak topology) uniformly with respect to initial point (see [**12**]).

Any probability measure μ on X generates the state ν on $\mathcal{A}$ that is given by $\nu(A) = \int_X A(1) \, d\mu$. Obviously, this state satisfies $[E\nu](A) = \nu(AE) = \nu(A)$. Hence ν is a right eigen-state corresponding to $\varphi \equiv 0$. If, in addition, the measure μ is invariant, then the ν is a two-sided eigen-state because $\nu(EA) = \int_X A(1) \circ \alpha \, d\mu = \int_X A(1) \, d\mu = \nu(A)$.

Now consider an arbitrary function $\varphi \in C(X)$. In the proof of Theorem 4.3 a way was given to construct a right eigen-state corresponding to φ. Let us try to apply this construction to our particular case. Evidently, if a linear operator $A : C(X) \to C(X)$ is positive, then $\|A\| = \max A(1)$. For any $\theta < e^{-\lambda(\varphi)}$ consider the operator $R_\theta = \sum_{n=0}^{\infty} \theta^n E_\varphi^n$. Its norm is equal to $\|R_\theta\| = \max R_\theta(1) = \max \sum_{n=0}^{\infty} \theta^n e^{S_n \varphi}$. Let δ_θ be a unit measure concentrated at one of points where the function $R_\theta(1)$ attains its maximum. Define the state ν_θ as $\nu_\theta(A) = \delta_\theta(A(1))$. It has the unit norm and $\nu_\theta(R_\theta) = \|R_\theta\|$. In the proof of Theorem 4.3 the right eigen-state corresponding to φ was defined as a limit point of the family of states $\|R_\theta\|^{-1} \nu_\theta(\cdot \, R_\theta)$ as $\theta \nearrow e^{-\lambda(\varphi)}$. The restriction of the state $\|R_\theta\|^{-1} \nu_\theta(\cdot \, R_\theta)$ to the algebra $C(X)$ coincides with δ_θ. Therefore, the restriction of the right eigen-state to $C(X)$ is a limit point of the family of measures δ_θ as $\theta \nearrow e^{-\lambda(\varphi)}$. So this restriction is itself an atomic measure. Eventually we have proved that for any function $\varphi \in C(X)$ there exists a right eigen-state such that its restriction to $C(X)$ is an atomic probability measure.

Of course, the law of large numbers with respect to an atomic measure has little significance. However, we can still use formula (6.7), which gives us a sequence of distributed probabilities $\mathbf{P}_n$ such that the law of large numbers is valid with respect to these $\mathbf{P}_n$. For the sake of simplicity let us consider the case of separable algebra $C(X)$. Suppose that $\{\varphi_1, \varphi_2, \dots\}$ is a countable dense subset in $C(X)$. For each φ_i there exists a right eigen-state ν_i whose restriction to $C(X)$ is concentrated at one point x_i and has the mass 2^{-i}. Then according to (6.7) the measures $\mathbf{P}_n$ have the form

$$\mathbf{P}_n(f) = \frac{\sum_{i=1}^{\infty} \nu_i(f e^{S_n(\varphi - \varphi_i)}) e^{n\lambda(\varphi_i)}}{\sum_{i=1}^{\infty} \nu_i(e^{S_n(\varphi - \varphi_i)}) e^{n\lambda(\varphi_i)}} = \frac{\sum_{i=1}^{\infty} 2^{-i} f(x_i) e^{S_n\varphi(x_i) - S_n\varphi_i(x_i) + n\lambda(\varphi_i)}}{\sum_{i=1}^{\infty} 2^{-i} e^{S_n\varphi(x_i) - S_n\varphi_i(x_i) + n\lambda(\varphi_i)}}.$$

They depend on the weight function $\varphi \in C(X)$. If there exists the Gâteaux derivative $\lambda'(\varphi) = \mu$, then by the law of large numbers we have

$$\lim_{n \to \infty} \mathbf{P}_n\{x \in X \mid |\delta_{x,n}(f) - \mu(f)| > \varepsilon\} = 0 \quad \text{for all } \varepsilon > 0.$$

Example 2. Let $\alpha : X \to X$ be a measurable mapping of a measure space (X, μ). Consider the shift $E = \alpha^*$ that takes each function f on X to $Ef = f \circ \alpha$.

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Suppose that there exists a constant C so large that $\mu(\alpha^{-1}(G)) \leq C\mu(G)$ for any measurable subset $G \subset X$. Then the shift E is continuous on the space $L^1(X, \mu)$ and $\|E\| \leq C$. Denote by $\mathcal{A}$ the algebra of bounded linear operators on $L^1(X, \mu)$ and by $\mathcal{C}$ the algebra of essentially bounded functions $L^\infty(X, \mu)$. Then the quadruple $(\mathcal{A}, \mathcal{C}, E, \alpha^*)$ forms a positive flow.

Note that the phase space of this flow (that is, the support of $\mathcal{C} = L^\infty(X, \mu)$) is not X, but a certain compactification of X. Finite Borel measures on the phase space are identified with continuous linear functionals on the phase algebra $\mathcal{C} = L^\infty(X, \mu)$ and these functionals, in turn, are identified with finitely additive measures on X that are absolutely continuous with respect to μ. For any finitely additive measure m absolutely continuous with respect to μ we can define the dual entropy $S(m) = \inf_{\varphi \in \mathcal{C}}(\lambda(\varphi) - m(\varphi))$, where $\lambda(\varphi)$ is the logarithm of the spectral radius of the weighted shift operator $E_\varphi = e^\varphi E$. It turns out that the restriction of $S(m)$ to the set of equilibrium measures (subgradients of the spectral potential $\lambda(\varphi)$) can be computed in a different way avoiding the use of the spectral radius. Denote by $\mathfrak{M}_\alpha$ the set of all α-invariant finitely additive probability measures on X that are absolutely continuous with respect to μ. We consider all possible finite measurable partitions $D = \{G_1, \ldots, G_k\}$ of the space X. Define the functional

$$\tau(m) = \inf_{n, D} \sup_{\|f\|=1} \sum_{G \in D} m(G) \ln \frac{\int_G f \circ \alpha^n \, d\mu}{m(G)}, \quad m \in \mathfrak{M}_\alpha, \tag{10.1}$$

where the supremum is taken over all non-negative functions $f \in L^1(X, \mu)$ with unit norm. In [**2, 3**] the equality $\lambda(\varphi) = \max_{m \in \mathfrak{M}_\alpha}(m(\varphi) + \tau(m))$ was announced. It was called the 'variational principle' for the spectral radius of the weighted shift operator. Since the Legendre transform is involutive (see [**9**]), this principle implies that the functionals $S(m)$ and $\tau(m)$ coincide on $\mathfrak{M}_\alpha$.

Special cases of Example 2 are all stationary discrete time stochastic processes. To verify this fact it is sufficient to define X as the space of trajectories of a stochastic process, μ as the probability distribution on the X, and α as the shift taking each trajectory $x(t)$ to the trajectory $x(t + 1)$. In this case we can define the state ν_0 on the algebra $\mathcal{A}$ of all bounded linear operators on $L^1(X, \mu)$ by means of the formula $\nu_0(A) = \int_X A(1) \, d\mu$. The restriction of ν_0 to the phase algebra $\mathcal{C} = L^\infty(X, \mu)$ coincides with μ. Our stochastic process is stationary if and only if the measure μ is α-invariant. On the other hand, the α-invariance of μ yields that ν_0 is a two-sided eigen-state for the shift $Ef = f \circ \alpha$ on $L^1(X, \mu)$.

Example 3. Suppose that $\alpha : X \to X$ is a continuous and locally expanding mapping of a metric compact set X. Denote by $H^\gamma(X)$ the set of real-valued functions on X satisfying the Hölder condition with exponent $\gamma > 0$. For each function $\varphi \in H^\gamma(X)$ define the following operator E_φ on $C(X)$:

$$E_\varphi f(x) = \sum_{y \in \alpha^{-1}(x)} e^{\varphi(y)} f(y).$$

Usually it is called the Perron–Frobenius operator or the transfer operator. It is easy to see that it is positive and satisfies the contravariant homological identity $E_\varphi((f \circ \alpha)g) = fE_\varphi g$.

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The operator E_φ has a unique positive eigenfunction $h_\varphi \in H^\gamma(X)$ and a unique positive eigenfunctional $\mu_\varphi : C(X) \to \mathbb{R}$, and the corresponding eigenvalue e^{λ_φ} coincides with the spectral radius of E_φ (see [**11, 18**]). These eigen-elements can be chosen in such a way that they depend analytically on $\varphi \in H^\gamma(X)$ and satisfy

$$E_\varphi h_\varphi = e^{\lambda_\varphi} h_\varphi, \quad \mu_\varphi \circ E_\varphi = e^{\lambda_\varphi} \mu_\varphi, \quad \mu_\varphi(h_\varphi) = 1, \quad \mu_\varphi(1) = 1.$$

Let us define $\mathcal{A}$ as the algebra of bounded linear operators on $C(X)$ with the transposed multiplication $A * B = B \circ A$, take $\mathcal{C} = H^\gamma(X)$, and define δ as the shift $\delta(f) = f \circ \alpha$. Then for any functions $f \in \mathcal{C}$ and $g \in C(X)$ we have $(E_\varphi * f)g = fE_\varphi g = E_\varphi(\delta(f)g) = (\delta(f) * E_\varphi)g$. So we get the covariant homological identity $E_\varphi * f = \delta(f) * E_\varphi$. Hence, the quadruple $(\mathcal{A}, \mathcal{C}, E_\varphi, \delta)$ forms a covariant positive flow. The number λ_φ defined above is its spectral potential. In our example the evolution operator E_φ has a two-sided eigen-state ν_φ on the algebra $\mathcal{A}$. It is given by $\nu_\varphi(A) = \mu_\varphi(Ah_\varphi)$. The restriction of ν_φ to $C(X)$ induces an α-invariant ergodic probability measure on X. Any function $f \in C(X)$ can be considered as a random variable with respect to the measure just defined. In [**5**] it was shown that if $f \in H^\gamma(X)$, then the sequence of the random variables $S_n f = f + f \circ \alpha + \cdots + f \circ \alpha^{n-1}$ satisfies the central limit theorem and, moreover, Cramer's type asymptotics for the probabilities of large deviations is valid as well.

Example 4. Let Ω be an arbitrary measurable space and $B(\Omega)$ be the set of all bounded measurable functions on Ω supplied with the supremum norm. Suppose that E is a positive linear operator on $B(\Omega)$ such that $E1 \equiv 1$. Then E is continuous. Consider the positive process $(\mathcal{A}, B(\Omega), E)$, where $\mathcal{A}$ is the algebra of bounded linear operators on $B(\Omega)$. Denote by X the support of the phase algebra $B(\Omega)$. Since $B(\Omega)$ is complete, it is naturally isomorphic to $C(X)$. The crossed algebra $\mathcal{B}$ corresponding to the positive process $(\mathcal{A}, B(\Omega), E)$ consists of formal sums $\sum_{i=0}^n F_i E^i$, where the coefficients F_i are continuous functions on the trajectory space $\bar{X} = X^{\mathbb{N}}$.

Consider the linear span of all crossed operators of the form FE^n, where $F \in C(\bar{X})$ is a product $F(x) = f_1(x_1) \cdots f_m(x_m)$, $f_i \in C(X)$. This linear span is dense in the crossed algebra $\mathcal{B}$. Hence, any state on $\mathcal{B}$ is uniquely determined by its values on the operators indicated above. Let μ be a positive linear functional on $B(\Omega)$ with $\mu(1) = 1$. Define the state ν on $\mathcal{B}$ by means of the formula

$$\nu(FE^n) = \mu(f_1 E(f_2 E(f_3 \ldots E(f_m E(1)) \ldots))). \tag{10.2}$$

It is easy to see that ν is a right eigen-state for the evolution operator E. The restriction of ν to the subalgebra $C(\bar{X}) \subset \mathcal{B}$ generates a Borel probability measure $\mathbf{P}$ on $\bar{X}$. In its turn, the linear functional μ on $B(\Omega) \cong C(X)$ is identified with a Borel probability measure on X. Further, the evolution operator E is naturally transferred from $B(\Omega)$ to $C(X)$. For each point $x \in X$ consider the positive linear functional $\mathbf{P}_x(f) = Ef(x)$ on $C(X)$. It is also identified with a probability measure on X. Thus, we have constructed the Markov chain with phase space X, transition probabilities $\mathbf{P}_x$, and initial probability distribution μ on X. From (10.2) it follows that the corresponding probability measure on the space of trajectories of this Markov chain coincides with $\mathbf{P}$. If we wish, we can assume that the phase space of this Markov chain is not X, but Ω, although in this case all of the probability measures will be only finitely additive.

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