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Complex potential equations

I. A technique for solution

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Abstract. This work amalgamates and solves certain problems arising in differential equation theory and in classical differential geometry.

It describes a novel technique for solving systems of non-linear partial differential equations of the form

$$(\nabla\gamma)^2 = f(\gamma),$$
$$\nabla^2\gamma = g(\gamma),$$

where $f(\gamma)$ and $g(\gamma)$ are arbitrarily assigned functions. The circumstances are determined under which compatible solutions exist, not only when γ is real, but also when γ is complex, and all of the corresponding solutions are found. This is done by using a geometric technique that incorporates the equipotential surfaces of constant γ. In general, these surfaces are imaginary, and a fairly extensive treatment of such surfaces in (complexified) 3-dimensional Euclidean space is included. A close association is discovered between the set of equipotential surfaces and the class of surfaces of constant radii of principal normal curvature.

1. *Introduction*

Both the eikonal equation of geometrical optics

$$(\nabla\gamma)^2 = 1$$

and Laplace's equation of classical physics

$$\nabla^2\gamma = 0,$$

where γ is real, have been extensively investigated, and much is known about their solutions. Certain problems arise in mathematical physics in which it becomes necessary to solve their generalizations

$$(\nabla\gamma)^2 = f(\gamma) \tag{1·1a}$$

and

$$\nabla^2\gamma = g(\gamma), \tag{1·1b}$$

where γ is real or complex, $f(\gamma)$ and $g(\gamma)$ are given functions, and the operators ∇ and ∇^2 refer to 3-dimensional Euclidean space. In the present paper we study equations (1·1) as a combined system, by examining the circumstances under which compatible

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real or complex solutions exist, and determining what those solutions are. This question has a bearing on certain problems in classical applied mathematics (in connexion with the wave equation) and in general relativity (with reference to solutions of Einstein's vacuum field equations). Such applications will be considered at greater length (see section 3 of this paper, and a subsequent article).

In the absence of systematic methods of solving general non-linear partial differential equations, certain *ad hoc* procedures are frequently employed (e.g. (9), and references cited). Some nonlinear partial differential equations may be solved in a systematic way by using group-theoretic techniques (1). While such techniques can be fairly powerful, they do not usually produce *general* solutions. Since the present aim is to determine *all* solutions to system (1·1) we shall not therefore approach the problem from the point of view of group theory. Instead, we shall devise a method which yields all solutions to the system (1·1).

By taking real and imaginary parts, this system is seen to consist essentially of four equations for only two unknowns, without boundary conditions. It is therefore heavily over-determined, and the number of independent solutions is consequently very restricted. We determine the conditions under which compatible solutions exist (i.e. the forms of the functions $f(\gamma)$ and $g(\gamma)$ for which there are common solutions), and we solve the system completely in each case. This is done in Sections 3 and 4 by adapting a geometric technique of Friedlander (3), who solved a system of non-linear partial differential equations associated with the classical wave equation by considering, in effect, (real) 'equipotential surfaces'. A major distinction in the present case is that the equipotential surfaces (surfaces of constant γ) may well be imaginary. The geometry of complex surfaces is therefore developed in section 2, and the results are then employed in section 4 to adapt Friedlander's technique to the complex case. It is shown that a necessary and sufficient condition for compatible solutions of (1·1) is that the equipotentials must be surfaces of constant principal radii of normal curvature.

One special type of surface that arises in this discussion is an 'isotropic developable' (2), for which the normals are always null, (i.e. a surface $\{f(x,y,z) = \text{constant}\}$ such that $(\nabla \mathbf{f})^2 \equiv 0$). We shall call such surfaces 'null surfaces', in keeping with the terminology employed in general relativity and in pseudo-Riemannian geometry. An example of a null surface is provided by the plane: $f(x,y,z) \equiv x+iy$ whose normal is in the direction $(1,i,0)$, and is thus null. Other examples include general 'null planes': $f(x,y,z) \equiv lx+my+nz$ (l,m,n all constants, not all zero, satisfying $l^2+m^2+n^2=0$), and 'null cones': surfaces of the form $(x-a)^2+(y-b)^2+(z-c)^2=0$ (a, b, c all constants), which are also known variously as 'isotropic cones', 'absolute cones' and 'point spheres' (8). Eisenhart (2) has presented a general prescription for the set of all null surfaces, but his formulation is rather unwieldy. We propose in section 2 an alternative prescription which is considerably simpler and more amenable to application. In fact, solutions to the system (1·1) for which the equipotentials are null surfaces can exist only in the case $f(\gamma) \equiv g(\gamma) \equiv 0$, and when this is so the equipotentials are null planes.

The results of this paper are summarized in Table 1. In the case where the equipotentials are not null ($f(\gamma) \not\equiv 0$) we can assume without loss of generality that

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Table 1. *A summary of compatible solutions to the system* $(\nabla\gamma)^2 = f(\gamma)$, $\nabla^2\gamma = g(\gamma)$

The quantity $1/r$ is a non-zero constant, and ρ and R are defined by

$$\rho = [(1-l^2)(x-x_1)^2 + (1-m^2)(y-y_1)^2 + (1-n^2)(z-z_1)^2 - 2mn(y-y_1)(z-z_1) - 2nl(z-z_1)(x-x_1) - 2lm(x-x_1)(y-y_1)]^{\frac{1}{2}}$$

and

$$R = [(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{\frac{1}{2}}$$

respectively. The entry $0 \not\equiv f(\gamma) \to 1$ means that $f(\gamma)$ is not identically zero and that it has been transformed to unity at a change of variable.

$(\nabla\gamma)^2 = f(\gamma)$	$\nabla^2\gamma = g(\gamma)$	Solution, γ	(Generalized) principal radii of normal curvature of equipotentials	Conditions
0	0	$\gamma = \gamma_0$	—	γ_0 constant
		$\psi(\gamma) = l(\gamma)x + m(\gamma)y + n(\gamma)$	0, 0	ψ, l, m, n arbitrary; l, m, n not all zero, and $l^2+m^2+n^2 = 0$
$f(\gamma) \not\equiv 0$	$g(\gamma)$	$\gamma = \gamma_0$	—	γ_0 constant, $f(\gamma_0) = g(\gamma_0) = 0$
$0 \not\equiv f(\gamma) \to 1$	$N/(\gamma - A)$	$lx+my+nz+A$ $\mu_1 x+\mu_2 y+\mu_3 z+\psi(\lambda_1 x+\lambda_2 y+\lambda_3 z)$	0, 0	$N = 0$: l, m, n, A constant; $l^2+m^2+n^2 = 1$ ψ arbitrary, $\psi'' \not\equiv 0$; $\mu_1, \mu_2, \mu_3, \lambda_1, \lambda_2, \lambda_3$ constant; $\mu_1^2+\mu_2^2+\mu_3^3 = 1$; $\lambda_1, \lambda_2, \lambda_3$ not all zero and $\lambda_1^2+\lambda_2^2+\lambda_3^2 = 0$; $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3 = 0$
$0 \not\equiv f(\gamma) \to 1$	$N/(\gamma - A)$	$\pm\rho + A$	$0, 1/r$	$N = 1$ $l, m, n, x_1, y_1, z_1, A$ constant; $l^2+m^2+n^2 = 1$
		$\pm R + A$ $\partial F/\partial\beta + A$	$1/r, 1/r$	$N = 2$: x_1, y_1, z_1, A constant $F(\beta, x, y, z) \equiv \phi(\beta) + \frac{1}{2}[(1-\beta^2)x + i(1+\beta^2)y - 2\beta z]$ with ϕ arbitrary; $\phi''' \not\equiv 0$

$f(\gamma) = 1$; this is achieved by a change of dependent variable. Compatible solutions will exist only if $g(\gamma) = 0$, $(\gamma - A)^{-1}$ or $2(\gamma - A)^{-1}$, where A is a constant. In the real case, the equipotentials are planes, right-circular cylinders and spheres respectively. These are the only real surfaces which have constant principal radii of normal curvature. In the complex case the equipotentials are generalizations of these surfaces, together with a more complicated set of surfaces which is not admitted for real γ. The fundamental reason for there being a larger class of complex surfaces is that in the real case the shear tensor of the normal congruence *must* have two linearly independent eigenvectors, whereas in the complex case the number of linearly independent eigenvectors may be either 1 or 2. When there is only one eigenvector the associated eigenvalue is zero for $g(\gamma) = 0$ and non-zero for $g(\gamma) = 2(\gamma - A)^{-1}$. If the equipotential surfaces are null, then as mentioned above, they must be null planes. This exhausts all possibilities for the equipotentials of solutions to the system (1·1) and the only other cases admitted are those for which there are no equipotentials at all, viz. γ is identically constant.

2. *The geometry of complex surfaces*

Before discussing complex surfaces imbedded in Euclidean space, it will be convenient to consider first some algebraic operations on complex vectors. By a *complex vector* we shall mean a triple $\mathbf{u} = (u^1, u^2, u^3)$ of complex numbers u^1, u^2 and u^3. We can define the *scalar product* $(\cdot)$ of two complex vectors $\mathbf{u} = (u^1, u^2, u^3)$ and $\mathbf{v} = (v^1, v^2, v^3)$ by $\mathbf{u} \cdot \mathbf{v} \equiv u^1 v^1 + u^2 v^2 + u^3 v^3$. In general the quantity $\mathbf{u} \cdot \mathbf{v}$ will be complex, and so the scalar product is not an inner product in the usual sense. If $\mathbf{u} \cdot \mathbf{v} = 0$ for two vectors $\mathbf{u}$ and $\mathbf{v}$, we shall say that $\mathbf{u}$ and $\mathbf{v}$ are '*orthogonal*'.

If a non-zero vector $\mathbf{u}$ is such that $\mathbf{u} \cdot \mathbf{u} \equiv \mathbf{u}^2 = 0$, we shall call it *null* or *isotropic*, in keeping with terminology employed in general relativity and in real pseudo-Riemannian geometry.

The *vector product*, $\mathbf{u} \wedge \mathbf{v}$, of vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by $(\mathbf{u} \wedge \mathbf{v})^i = \epsilon_{ijk} u^j v^k$, numerically. Here ϵ_{ijk} is the completely antisymmetric symbol, with $\epsilon_{123} = 1$.

LEMMA 2·1. *A null vector* $\mathbf{u}$ *is orthogonal to a null vector* $\mathbf{v}$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel.*

Proof. (i) Let $\mathbf{u}$ and $\mathbf{v}$ be parallel. Then there exists a non-zero scalar λ such that $\mathbf{u} = \lambda \mathbf{v}$, whence $\mathbf{u} \cdot \mathbf{v} = \lambda \mathbf{v} \cdot \mathbf{v} = 0$, and so u and v are orthogonal.

(ii) Let u and v be orthogonal. Since $\mathbf{u} \neq \mathbf{0}$, at least one component of $\mathbf{u}$ is non-zero. Without loss of generality, we may let this component be u^3. Then we may write $u^1 = iu^3 \cos\theta$ and $u^2 = iu^3 \sin\theta$ for some scalar θ. The condition $\mathbf{u} \cdot \mathbf{v} = 0$ is then equivalent to $v^3 = -i(v^1 \cos\theta + v^2 \sin\theta)$, and combining this with the condition $v^2 = 0$ we obtain $v^1 \sin\theta = v^2 \cos\theta$. Direct calculation then shows that

$$v^1 u^2 - v^2 u^1 = v^2 u^3 - v^3 u^2 = v^3 u^1 - v^1 u^3 = 0,$$

whence $\mathbf{u}$ and $\mathbf{v}$ are parallel. |

Remark. This Lemma does not hold in higher dimensions. For instance, $(1, i, 0, 0)$ and $(0, 0, 1, i)$ can be regarded as null vectors which are orthogonal and yet linearly independent.

An important result which will be of use in Section 4 is provided by

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THEOREM 2·2. *Every complex symmetric* 3×3 *matrix,* A, *has* either 3 *linearly independent non-null eigenvectors* or *one null eigenvector and at most one linearly independent non-null eigenvector. In both cases the eigenvectors are orthogonal, or may be chosen as such.*

Proof. Let $\mathbf{C}$ be the vector space of all complex vectors. The general theory of linear equations guarantees that there is at least one eigenvector, $\mathbf{x}_1$, being a non-trivial solution to the equation $A\mathbf{x} = \lambda_1 \mathbf{x}$. Suppose that $\mathbf{x}$ is not null. Let $\mathbf{C}_1$ be the subspace of all vectors orthogonal to $\mathbf{x}_1$: $\mathbf{C}_1 = \{\mathbf{y} \in \mathbf{C},\ \mathbf{y} \cdot \mathbf{x}_1 = 0\}$. Then by symmetry of A, $(A\mathbf{y}) \cdot \mathbf{x}_1 = \mathbf{y} \cdot (A\mathbf{x}_1) = \lambda_1 \mathbf{y} \cdot \mathbf{x}_1 = 0$, and so A maps $\mathbf{C}_1$ into itself; thus $\mathbf{C}_1$ is an invariant subspace of $\mathbf{C}$. Choose a basis such that the eigenvector $\mathbf{x}_1$ is in the 1-direction. Then, if $A = (a_{ij})$, $1 \leqslant i,j \leqslant 3$, it follows that $a_{11} = \lambda_1$ and $a_{1j} = a_{j1} = 0$ $(2 \leqslant j \leqslant 3)$. We now repeat the argument, using the 2×2 matrix $A_1 = (a_{ij})$, $2 \leqslant i,j \leqslant 3$, in the subspace $\mathbf{C}_1$. If A_1 possesses a non-null eigenvector then it possesses two such eigenvectors; in this case the matrix A has three non-null eigenvectors. Now suppose that, on the other hand, k $(0 \leqslant k \leqslant 1)$ non-null eigenvectors have been determined, but that there are no further non-null eigenvectors. Then there is a null eigenvector, $\mathbf{w}$, of the $(3-k) \times (3-k)$ matrix A_k. Suppose that $A_k \mathbf{w} = \mu \mathbf{w}$ and that a second linearly independent eigenvector $\mathbf{w}_1$ exists, with associated eigenvalue μ_1. Then

$$(A_k \mathbf{w}) \cdot \mathbf{w}_1 = \mu(\mathbf{w} \cdot \mathbf{w}_1) = \mathbf{w} \cdot A_k \mathbf{w}_1 = \mu_1(\mathbf{w} \cdot \mathbf{w}_1).$$

Since $\mathbf{w}$ and $\mathbf{w}_1$ are linearly independent, it follows from Lemma 2·1 that $\mathbf{w} \cdot \mathbf{w}_1 \neq 0$, and hence $\mu_1 = \mu$. But then $\mathbf{w} + \mathbf{w}_1$ is a non-null eigenvector of A_k, which contradicts our assumption. Consequently $\mathbf{w}$ is the unique null eigenvector of A_k. In this case the matrix A has one null eigenvector, and at most one linearly independent non-null eigenvector.

The orthogonality result follows by construction. ∎

Corollary. Any complex symmetric 2×2 matrix B either has 2 linearly independent non-null eigenvectors, or has form

$$\begin{bmatrix} a & \pm \frac{1}{2}i(a-b) \\ \pm \frac{1}{2}i(a-b) & b \end{bmatrix}$$

where $a \neq b$, with associated eigenvalue $\frac{1}{2}(a+b)$ and associated eigenvector $(1, \pm i)$. In the latter case the eigenvector is unique up to arbitrary rescaling, and is null.

Proof. Proceeding as in the proof of the theorem, we see that if the matrix B does not have 2 linearly independent eigenvectors, then it must possess a unique null eigenvector. This requires the characteristic equation to have equal roots, i.e.

$$(trB)^2 = 4 \det B.$$

If $B = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ this occurs if and only if $(a-b)^2 = 4c^2$, $c = \pm \frac{1}{2}i(a-b)$. Direct calculation then shows that the eigenvector is $\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$, unless $a = b$, in which case $c = 0$ and the matrix is isotropic, i.e. *every* vector is an eigenvector, which is impossible. ∎

Remark. In view of the fact that Lemma 2·1 is not valid in dimensions higher than 3, this theorem does not appear to generalize to higher dimensions. Thus the numbers of

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linearly independent non-null and null eigenvectors of the following 4×4 matrices are respectively 4 and 0, 2 and 1, 1 and 1 and 0 and 2:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & i & 2 \\ 0 & i & -1 & 2i \\ 0 & 2 & 2i & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & i & 0 & 0 \\ i & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \end{bmatrix}.$$

A *complex curve* can be defined parametrically by a relationship

$$\mathbf{x} = \mathbf{x}(\alpha) \Leftrightarrow x = x(\alpha), \quad y = y(\alpha), \quad z = z(\alpha), \tag{2·1}$$

where α is a (generally complex) coordinate along the curve. In the case† where the derivatives

$$x'(\alpha) \equiv y'(\alpha) \equiv z'(\alpha) \equiv 0 \tag{2·2}$$

equations (2·1) locally define a point. We shall henceforth assume that (2·2) is false. The *tangent vector*, $\mathbf{t}$, to a curve $\mathbf{x} = \mathbf{x}(\alpha)$ is defined to be a vector in the direction $\mathbf{x}_\alpha$, where the suffix denotes differentiation. We can construct a unit vector parallel to $\mathbf{t}$ provided $\mathbf{t}$ is not null. Any curve whose tangent vector is null at a point p will be called *null at* p, and a curve whose tangent vector is null everywhere (sufficiently locally) is called a *null curve*, a *curve of length zero* or a *minimal curve* (2).

In a similar way, we can define a *complex surface* parametrically by a relationship

$$\mathbf{x} = \mathbf{x}(\alpha, \beta) \Leftrightarrow x = x(\alpha, \beta), \quad y = y(\alpha, \beta), \quad z = z(\alpha, \beta) \tag{2·3}$$

where α and β are (generally complex) coordinates in the surface. In the case where

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} \equiv \frac{\partial(y, z)}{\partial(\alpha, \beta)} \equiv \frac{\partial(z, x)}{\partial(\alpha, \beta)} \equiv 0 \tag{2·4}$$

the parameters α and β are not independent, and then equations (2·3) locally define a curve or a point. We shall henceforth assume that (2·4) is false. The *normal vector* $\mathbf{n}$, to a surface is defined to be a vector in the direction $\mathbf{x}_\alpha \wedge \mathbf{x}_\beta$ (here suffices denote partial derivatives). When $\mathbf{n}$ is not null we can normalize it by requiring $\mathbf{n}^2 = 1$. A surface whose normal is null at a point p will be called *null at* p, and a surface whose normal is null everywhere (sufficiently locally) will be called a *null surface*, or *isotropic developable* (2). A vector $\mathbf{v}$ which is spanned by the pair $\{\mathbf{x}_\alpha, \mathbf{x}_\beta\}$ at a point p of a surface is orthogonal to $\mathbf{n}$ and is called a *tangent vector to the surface at* p.

LEMMA 2·3. *A surface S is null at a point p if and only if the normal at p is tangent to S at p.*

Proof. Let $\mathbf{n} = \mathbf{x}_\alpha \wedge \mathbf{x}_\beta$. Then $\mathbf{n}^2 = 0 \Leftrightarrow \mathbf{x}_\alpha^2 \mathbf{x}_\beta^2 = (\mathbf{x}_\alpha \cdot \mathbf{x}_\beta)^2$. This is the necessary and sufficient condition that there should exist a non-trivial solution for scalars λ and μ to the equations

$$\mathbf{x}_\alpha \cdot (\lambda \mathbf{x}_\alpha + \mu \mathbf{x}_\beta) = 0 \quad \text{and} \quad \mathbf{x}_\beta \cdot (\lambda \mathbf{x}_\alpha + \mu \mathbf{x}_\beta) = 0.$$

Hence if $\mathbf{n}^2 = 0$ there is a non-zero vector $\lambda \mathbf{x}_\alpha + \mu \mathbf{x}_\beta$ which is orthogonal to $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$. It is therefore self-orthogonal, and hence null. Since $\mathbf{n} \cdot \mathbf{x}_\alpha = \mathbf{n} \cdot \mathbf{x}_\beta = 0$, it follows that

† The purpose of the present article is expository, and here and throughout functions will be assumed to possess the requisite differentiability properties, at least locally.

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$\lambda \mathbf{x}_\alpha + \mu \mathbf{x}_\beta$ is a null vector orthogonal to $\mathbf{n}$, and so by Lemma 2·1, $\mathbf{n}$ is parallel to $\lambda \mathbf{x}_\alpha + \mu \mathbf{x}_\beta$. Conversely, if there are scalars λ and μ such that $\mathbf{n} = \lambda \mathbf{x}_\alpha + \mu \mathbf{x}_\beta$, then, since $\mathbf{n} \cdot \mathbf{x}_\alpha = \mathbf{n} \cdot \mathbf{x}_\beta = 0$, it follows that $\mathbf{n}^2 = 0$. |

The *first fundamental form* of a complex surface $\mathbf{x} = \mathbf{x}(\alpha, \beta)$ is the induced metric on the surface, viz.

$$ds^2 = \mathbf{x}_\alpha^2 \, d\alpha^2 + 2\mathbf{x}_\alpha \cdot \mathbf{x}_\beta \, d\alpha \, d\beta + \mathbf{x}_\beta^2 \, d\beta^2. \tag{2·5}$$

By equating this expression to zero we define in general two distinct families of null curves in the surface. One exceptional case arises, in which the two families coincide, and as the following theorem shows, this corresponds to the surface being null.

THEOREM 2·4. (Cf. (2).) *A surface is null if and only if the set of null curves in the surface forms a* 1*-parameter family.*

Proof. The condition for there to be only a 1-parameter family of null curves is that, when equated to zero, the expression (2·5) should have equal roots. This occurs if and only if $(\mathbf{x}_\alpha \cdot \mathbf{x}_\beta)^2 = \mathbf{x}_\alpha^2 \, \mathbf{x}_\beta^2$. The proof of Lemma 2·3 shows that this occurs if and only if $\mathbf{n}$ is null, and the theorem is thus proved. |

Simple examples of null surfaces are provided by (i) *null planes*: $lx + my + nz = p$, where l, m, n and p are constants and (l, m, n) is a null vector, and (ii) *null cones*: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 0$, where x_0, y_0 and z_0 are constants. Null cones are sometimes called *isotropic cones*, *absolute cones* or *point spheres* (8). It will be useful to obtain formulas for the set of all null surfaces, and this is done in the following theorem.

THEOREM 2·5. *The set, $\mathscr{N}$, of all null surfaces consists of the set, $\mathscr{P}$, of all null planes, together with a set, $\mathscr{S}$, of surfaces described by the equations*

$$x = i\beta \sin \alpha + \mu(\alpha),$$

$$y = i\beta \cos \alpha - \int^{\alpha} \mu_\alpha(a) \tan a \, da + \alpha_0$$

and

$$z = \beta,$$

where $\mu(\alpha)$ is an arbitrary function and α_0 is an arbitrary constant. No member of $\mathscr{S}$ is a null plane. Symbolically, $\mathscr{N} = \mathscr{P} \cup \mathscr{S}$ with $\mathscr{P} \cap \mathscr{S} = \varnothing$.

Proof. It follows from Theorem 2·4 that the set of null curves in a null surface $\mathbf{x} = \mathbf{x}(\alpha, \beta)$ forms a 1-parameter family. Consequently, the first fundamental form is (cf. (2·5))

$$ds^2 = E \, d\alpha^2 + 2F \, d\alpha \, d\beta + G \, d\beta^2 \tag{2·6}$$

where $E = \mathbf{x}_\alpha^2$, $F = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta$, $G = \mathbf{x}_\beta^2$ and $EG - F^2 = 0$. We show first that, without loss of generality, we can assume that $F \equiv G \equiv 0$ (2). We may write (2·6) in the form

$$ds = (E^{\frac{1}{2}} \, d\alpha + G^{\frac{1}{2}} \, d\beta)^2.$$

There exists an integrating factor, $I = I(\alpha, \beta) \not\equiv 0$, of $E^{\frac{1}{2}} \, d\alpha + G^{\frac{1}{2}} \, d\beta$, and so there is a function A such that $dA = I(E^{\frac{1}{2}} \, d\alpha + G^{\frac{1}{2}} \, d\beta)$. Thus $ds^2 = I^{-2} \, dA^2$. Comparing this expression with (2·6) it follows (reverting to the former notation) that $F \equiv G \equiv 0$, and $E \not\equiv 0$.

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We next consider the system

$$F = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta = x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta \equiv 0 \tag{2·7}$$

and

$$G = \mathbf{x}_\beta^2 = x_\beta^2 + y_\beta^2 + z_\beta^2 \equiv 0. \tag{2·8}$$

At most one of x_β, y_β and z_β can be identically zero, for if at least two are zero, then so is the third, by (2·8), and in that case (2·4) is satisfied, contrary to our assumptions. Two cases therefore arise, one for which exactly one of x_β, y_β and z_β is zero, and the second for which $x_\beta y_\beta z_\beta \not\equiv 0$.

Case 1. Without loss of generality we may suppose that $z_\beta \equiv 0$ and $x_\beta y_\beta \not\equiv 0$. It follows from (2·8) that $y_\beta = \pm i x_\beta$, and substitution of this into (2·7) shows that $y_\alpha = \pm i x_\alpha$. Integrating, it follows that

$$y = \pm ix + p, \tag{2·9}$$

where p is constant, and hence the surface is a null plane containing the z-axis. The other cases may be obtained by permuting the symbols x, y and z.

Case 2. $x_\beta y_\beta z_\beta \not\equiv 0$. We show first that we can choose $z = \beta$. Equations (2·7) and (2·8) are then simplified to the required extent.

The coordinates (α, β) such that (2·7) and (2·8) hold are fixed up to a transformation $\alpha = \alpha(A)$, $\beta = \beta(A, B)$, where the Jacobian of the transformation is finite and not identically zero: $\alpha_A \beta_B \not\equiv 0 \Leftrightarrow A_\alpha B_\beta \not\equiv 0$. Thus the coordinate β is perfectly arbitrary, provided of course that it is functionally independent of α. Choose any function B satisfying $B_\beta = z_\beta$ (which is always possible: the Jacobian of the transformation is finite, and not zero since $z_\beta \not\equiv 0$). Then $z_\beta = z_B B_\beta = z_B z_\beta \Rightarrow z_B = 1$. Reverting to the original notation, it follows that $z_\beta = 1$, the coordinate freedom now being given by $\alpha = \alpha(A)$, $\beta = B + \phi(\alpha)$, where $\phi(\alpha)$ is an arbitrary function. Since $z = \beta + \zeta(\alpha)$ for some function $\zeta(\alpha)$, we can perform the transformation $\beta = B - \zeta(\alpha)$, and then $z = B$. Reverting once again to the original notation, we now have $z = \beta$, with associated coordinate freedom $\alpha = \alpha(A)$, $\beta = B$, where $\alpha_A A_\alpha \not\equiv 0$.

The defining equations (2·7) and (2·8) now become

$$x_\alpha x_\beta + y_\alpha y_\beta = 0 \tag{2·7′}$$

and

$$x_\beta^2 + y_\beta^2 + 1 = 0. \tag{2·8′}$$

From (2·8′) it follows that there exists a function $\lambda(\alpha, \beta)$ such that $x_\beta = i \sin\lambda$, $y_\beta = i \cos\lambda$, and, since $x_\beta y_\beta \not\equiv 0$, $\sin 2\lambda \not\equiv 0$. Differentiating (2·8′) with respect to α, we see that

$$\mathbf{x}_{\beta\alpha} \cdot \mathbf{x}_\beta \equiv x_{\alpha\beta} x_\beta + y_{\alpha\beta} y_\beta = 0, \tag{2·10}$$

while differentiating (2·7′) with respect to β and using (2·10) shows that

$$\mathbf{x}_\alpha \cdot \mathbf{x}_{\beta\beta} \equiv x_\alpha x_{\beta\beta} + y_\alpha y_{\beta\beta} = 0. \tag{2·11}$$

Equations (2·7′) and (2·11) together imply that either $x_\alpha \equiv y_\alpha \equiv 0$ or $x_\beta y_{\beta\beta} - y_\beta x_{\beta\beta} = 0$. In the former case, (2·4) is satisfied, contrary to our assumptions, while in the latter case $\lambda_\beta = 0$, i.e. $\lambda = \lambda(\alpha)$. The expressions $x_\beta = i \sin\lambda(\alpha)$ and $y_\beta = i \cos\lambda(\alpha)$ now imply that $x = i\beta \sin\lambda(\alpha) + \mu(\alpha)$ and $y = i\beta \cos\lambda(\alpha) + \nu(\alpha)$ where $\mu(\alpha)$ and $\nu(\alpha)$ are functions

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of α only. Substitution into (2·7′) implies that μ and ν are related by $\nu_\alpha = -\mu_\alpha \tan\lambda$. In the case where $\lambda_\alpha \equiv 0$, i.e. λ is constant, it follows that

$$x\sin\lambda + y\cos\lambda - iz = p \tag{2·12}$$

for some constant p and for $\sin 2\lambda \not\equiv 0$; the surface is a null plane. On the other hand, if $\lambda_\alpha \not\equiv 0$, we can use the remaining freedom $\alpha \to A(\alpha)$, where $A_\alpha \not\equiv 0$, to write $A = \lambda(\alpha)$. Reverting back to the former notation, it then follows that

$$\left.\begin{aligned} x &= i\beta\sin\alpha + \mu(\alpha), \\ y &= i\beta\cos\alpha - \int^{\alpha} \mu_\alpha(a)\tan a\, da + \alpha_0 \end{aligned}\right\} \tag{2·13}$$

and

$$z = \beta,$$

where $\mu(\alpha)$ is an arbitrary function, and α_0 is an arbitrary constant. In this case $\mathbf{x}_\alpha = (i\beta\cos\alpha + \mu_\alpha)(1, -\tan\alpha, 0)$ and $\mathbf{x}_\beta = (i\sin\alpha, i\cos\alpha, 1)$, so the normal $\mathbf{n}$ is parallel to the null vector $\mathbf{x}_\beta$. Since the direction of this vector is not constant on the surface, it follows that no surface of form (2·13) is a null plane. In addition, *every* null plane is encompassed by the equations of type (2·9) and (2·12). Thus, in the terminology of the theorem, $\mathscr{N} = \mathscr{P} \cup \mathscr{S}$, where $\mathscr{P} \cap \mathscr{S} = \varnothing$. ∎

Remarks. It is instructive to determine the function $\mu(\alpha)$ and constant α_0 that generate the set of null cones. This is done by substituting (2·13) into the expression

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = 0$$

to obtain an identity in α and β. Thus

$$i(\mu(\alpha)-x_0)\sin\alpha - i\left[\int^{\alpha} \mu_\alpha(a)\tan a\, da - \alpha_0 + y_0\right]\cos\alpha - z_0 = 0 \tag{2·14}$$

and

$$(\mu(\alpha)-x_0)^2 + \left[\int^{\alpha} \mu_\alpha(a)\tan a\, da - \alpha_0 + y_0\right]^2 + z_0^2 = 0. \tag{2·15}$$

Division of (2·14) by $\cos\alpha$, and subsequent differentiation with respect to α shows that $\mu(\alpha) = x_0 - iz_0\sin\alpha$, and substitution of this into either (2·14) or (2·15) yields $\alpha_0 = y_0$.

It is felt that the prescription for null surfaces provided by Theorem 2·5 is preferable to that of Eisenhart (2), in that it is simpler and more amenable to application.

3. *Real solutions of the system* (1·1)

In this section we examine the system of equations

$$(\nabla\gamma)^2 = f(\gamma) \tag{1·1a}$$

and

$$\nabla^2\gamma = g(\gamma), \tag{1·1b}$$

determining the functions $f(\gamma)$ and $g(\gamma)$ for which real solutions exist, and the solutions themselves in each case. The method employed here is based on the analysis of Friedlander (3), who sought to determine all simple progressive real solutions of the classical wave equation

$$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = u_{tt}.$$

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These solutions have the special form $u = w(x,y,z)\,F(t-h(x,y,z))$ where F denotes an arbitrary function, (x,y,z) are ordinary Cartesian coordinates, and t is the Newtonian time variable.

It was shown that this problem is equivalent to determining common solutions of the system

$$(\nabla h)^2 = 1, \tag{3·1}$$

$$2\nabla w\cdot\nabla h + w\nabla^2 h = 0 \tag{3·2}$$

and

$$\nabla^2 w = 0. \tag{3·3}$$

Equation (3·1) means that the equipotential surfaces of constant h are orthogonal to a ray congruence of straight lines. This fact forms the basis of Friedlander's development of the geometrical technique.

Friedlander's methods are now adapted in order to study the problem of determining all real solutions of the system (1·1). Our general result is

THEOREM 3·1. *Real solutions to the system*

$$(\nabla\gamma)^2 = f(\gamma) \quad \textit{and} \quad \nabla^2\gamma = g(\gamma)$$

in 3-dimensional Euclidean space can exist only when one of the following holds:

(i) $f(\gamma)\equiv g(\gamma)\equiv 0$. *Then* $\gamma\equiv\gamma_0$, *where* γ_0 *is a constant, or*

(ii) $f(\gamma)\not\equiv 0, f(\gamma)\geqslant 0$. *Then* $\gamma\equiv\gamma_0$ *is a solution if and only if* γ_0 *is a common zero of* $f(\gamma)$ *and* $g(\gamma)$, *or*

(iii) $f(\gamma)\not\equiv 0, f(\gamma)\geqslant 0$, *and, if* ν *is any solution to the equation* $\nu' = [f(\gamma)]^{-\frac{1}{2}}$, *where a prime* $(')$ *denotes differentiation with respect to* γ, *then*

$$g(\gamma(\nu)) - \tfrac{1}{2}f'(\gamma(\nu)) \equiv \frac{N[f(\gamma(\nu))]^{\frac{1}{2}}}{\nu - A}$$

where A *is an arbitrary real constant and* $N = 0, 1$ *or* 2. *The equipotentials are planes* $(N=0)$, *right-circular cylinders* $(N=1)$ *or spheres* $(N=2)$, *and the general solution for* γ *is* $\gamma = \gamma(\nu)$, *where* $\nu' = [f(\gamma)]^{-\frac{1}{2}}$ *and*

$$\nu = \begin{cases} lx+my+nz+A, & l,m,n \text{ constants}, \quad l^2+m^2+n^2 = 1 \qquad (N=0), \\ \pm\rho + A, & \rho \equiv [(1-l^2)(x-x_1)^2 + (1-m^2)(y-y_1)^2 + (1-n^2)(z-z_1)^2 \\ & \qquad -2mn(y-y_1)(z-z_1) - 2nl(z-z_1)(x-x_1) \\ & \qquad -2lm(x-x_1)(y-y_1)]^{\frac{1}{2}}, \\ & l,m,n,x_1,y_1,z_1 \text{ constants}, \quad l^2+m^2+n^2 = 1 \qquad (N=1), \\ \pm R + A, & R \equiv [(x-x_1)^2+(y-y_1)^2+(z-z_1)^2]^{\frac{1}{2}}, \quad x_1,y_1,z_1 \text{ constants}, \quad (N=2). \end{cases}$$

The constants l, m, n, x_1, y_1, z_1, γ_0 *and* A *are all real.*

Proof. If $f(\gamma)\equiv 0$ then the only real solution is clearly $\gamma = \gamma_0$, where γ_0 is an arbitrary constant. In this case, $g(\gamma)$ must be identically zero.

If $f(\gamma)\not\equiv 0$, then $\gamma = \gamma_0$ is a solution if and only if $f(\gamma)$ and $g(\gamma)$ possess a common zero, γ_0.

Real solutions with γ not identically constant will exist only when $f(\gamma)\not\equiv 0$ and $f(\gamma)\geqslant 0$, and we shall now focus attention on this remaining case. Defining a new

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variable, ν, by $\dfrac{d\nu}{d\gamma} \equiv \nu' = [f(\gamma)]^{-\frac{1}{2}}$, it follows that $(\nabla\nu)^2 = 1$. This means† that the equipotential surfaces are orthogonal to a congruence of straight lines (called 'rays'). For sufficiently local regions, there will be a unique ray passing through each point. We now introduce a 'Gaussian' coordinate system (α, β, ν) by choosing some initial surface‡

$$\nu(x, y, z) = 0 \tag{3·4}$$

and specifying any point p by means of its normal distance ν to this surface, and the position (α, β) of the point, q, of intersection of the ray through p with the initial surface. The equipotential surface $\{\nu = \text{constant}\}$ is then parallel to the surface (3·4). This means that the equipotential surfaces share the same normal congruence (2).

The coordinates (α, β) in the surface are specified most conveniently by using orthogonal curvature line parameters, thus making the coordinate system (α, β, ν) orthogonal. If, therefore, the surface (3·4) is specified by

$$\mathbf{x} = \mathbf{a}(\alpha, \beta)$$

then the coordinates of an arbitrary point are

$$\mathbf{x} = \mathbf{a}(\alpha, \beta) + \nu\mathbf{n}(\alpha, \beta),$$

where $\mathbf{n}$ is a unit normal to the surface (3·4). From this expression we find that

$$\begin{aligned} d\mathbf{x} &= \mathbf{a}_\alpha d\alpha + \mathbf{a}_\beta d\beta + \mathbf{n}\, d\nu + \nu\mathbf{n}_\alpha d\alpha + \nu\mathbf{n}_\beta d\beta \\ &= \mathbf{a}_\alpha \left(1 - \frac{\nu}{r}\right) d\alpha + \mathbf{a}_\beta \left(1 - \frac{\nu}{s}\right) d\beta + \mathbf{n}\, d\nu, \end{aligned}$$

since by Rodrigues' formula

$$\mathbf{a}_\alpha = -r\mathbf{n}_\alpha \quad \text{and} \quad \mathbf{a}_\beta = -s\mathbf{n}_\beta, \tag{3·5}$$

where $r = r(\alpha, \beta)$ and $s = s(\alpha, \beta)$ denote the principal radii of curvature (which are non-zero but may be infinite) of the initial surface. The metric therefore becomes

$$d\mathbf{x}^2 = H^2 d\alpha^2 + K^2 d\beta^2 + d\nu^2,$$

where $H \equiv (\mathbf{a}_\alpha \cdot \mathbf{a}_\alpha)^{\frac{1}{2}}(1 - \nu/r)$ and $K \equiv (\mathbf{a}_\beta \cdot \mathbf{a}_\beta)^{\frac{1}{2}}(1 - \nu/s)$. In this coordinate system, equation (1·1a) is seen to follow automatically, while equation (1·1b) gives

$$\nabla^2\nu = \frac{1}{HK}\frac{\partial}{\partial\nu}(HK) = \frac{g(\gamma(\nu)) - \frac{1}{2}f'(\gamma(\nu))}{[f(\gamma(\nu))]^{\frac{1}{2}}} \tag{3·6}$$

and thus

$$g(\gamma(\nu)) - \tfrac{1}{2}f'(\gamma(\nu)) = [f(\gamma(\nu))]^{\frac{1}{2}}\psi(\nu),$$

where

$$\psi(\nu) \equiv \begin{cases} 0 & (r = s = \infty) \qquad (3·7a) \\ \dfrac{1}{\nu - r} & (s = \infty,\ r \text{ finite}) \qquad (3·7b) \\ \dfrac{1}{\nu - r} + \dfrac{1}{\nu - s} & (r, s \text{ finite}). \qquad (3·7c) \end{cases}$$

† A proof is given in the remarks following the theorem.

‡ Without loss of generality this will exist, because ν is specified only up to a constant of integration.

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By considering the functional dependence on the independent variables α, β and ν, of both sides of equation (3·6), it is easy to verify that r and s must be constants whenever they are finite. In case (3·7b) this follows immediately. In case (3·7c), differentiate both sides with respect to α and β to obtain

$$\frac{r_\alpha}{(\nu-r)^2}+\frac{s_\alpha}{(\nu-s)^2}=\frac{r_\beta}{(\nu-r)^2}+\frac{s_\beta}{(\nu-s)^2}=0.$$

By equating coefficients of ν we obtain

$$r_\alpha+s_\alpha\equiv sr_\alpha+rs_\alpha\equiv s^2r_\alpha+r^2s_\alpha\equiv r_\beta+s_\beta\equiv sr_\beta+rs_\beta\equiv s^2r_\beta+r^2s_\beta\equiv 0.$$

These equations are compatible if and only if r and s are constants.

We consider the three cases (3·7a), (3·7b) and (3·7c) separately.

Case (3·7a) $r=s=\infty$. In this case it follows from (3·5) that $\mathbf{n}$ is independent of α and β. Consequently, the normal is fixed in direction, and if $\mathbf{x}$ is a vector to the surface $\{\nu=\text{constant}\}$, $(\mathbf{x}\cdot\mathbf{n})_\alpha=\mathbf{x}_\alpha\cdot\mathbf{n}=(\mathbf{a}_\alpha+\nu\mathbf{n}_\alpha)\cdot\mathbf{n}=0$ and similarly $(\mathbf{x}\cdot\mathbf{n})_\beta=0$. Also $(\mathbf{x}\cdot\mathbf{n})_\nu=1$ and thus $\mathbf{x}\cdot\mathbf{n}=\nu+\text{constant}$. This means that the equipotentials are planes, and that the general solution to our equation is $\nu=lx+my+nz+A$, where $l^2+m^2+n^2=1$, and l, m, n and A are (real) constants.

Case (3·7b) $s=\infty$, r finite. From equation (3·5) we have $\mathbf{x}_\alpha=(\nu-r)\,\mathbf{n}_\alpha$ and $\mathbf{n}_\beta=\mathbf{0}$. Thus $\mathbf{x}(\alpha,\beta)=(\nu-r)\,\mathbf{n}(\alpha)+\mathbf{f}(\beta)$, for some vector function $\mathbf{f}(\beta)$. Since at each point $\mathbf{x}$ on the surface $\{\nu=\text{constant}\}$, $\mathbf{x}_\alpha$, $\mathbf{x}_\beta$ and $\mathbf{n}$ constitute an orthogonal triad, there exists a scalar $k=k(\alpha,\beta)\not\equiv 0$ such that $\mathbf{x}_\beta=k(\mathbf{x}_\alpha\wedge\mathbf{n})$. Because $\mathbf{x}_{\alpha\beta}=0$, we obtain

$$\mathbf{f}''(\beta)=\mathbf{x}_{\beta\beta}=k_\beta(\mathbf{x}_\alpha\wedge\mathbf{n})=\frac{k_\beta}{k}\mathbf{x}_\beta=\frac{k_\beta}{k}\mathbf{f}'(\beta).$$

This means that there exists a constant unit vector $\mathbf{u}$ such that $\mathbf{f}'(\beta)=\psi(\beta)\,\mathbf{u}$ for some scalar $\psi(\beta)$. Hence there exists a scalar function $\phi(\beta)$ and a constant vector $\mathbf{v}$ such that $\mathbf{f}(\beta)=\phi(\beta)\,\mathbf{u}+\mathbf{v}$. Moreover, the condition $(\mathbf{x}\cdot\mathbf{n})_\beta=0$ requires $\phi'(\beta)\,\mathbf{u}\cdot\mathbf{n}\equiv 0$. The case $\phi'(\beta)\equiv 0$ requires $\mathbf{a}_\beta\equiv 0$, which means that (2·4) is satisfied, a contradiction. So $\phi'(\beta)\not\equiv 0$, and thus $\mathbf{u}\cdot\mathbf{n}=0$. In this case $(\mathbf{x}-\mathbf{v})-[(\mathbf{x}-\mathbf{v})\cdot\mathbf{u}]\,\mathbf{u}=(\nu-r)\,\mathbf{n}$, i.e. the equipotentials are right-circular cylinders, with axis parallel to $\mathbf{u}=(l,m,n)$ and passing through $\mathbf{v}=(x_1,y_1,z_1)$. Thus

$$\nu=r\pm\rho,$$

where

$$\rho\equiv[(1-l^2)(x-x_1)^2+(1-m^2)(y-y_1)^2+(1-n^2)(z-z_1)^2 - 2mn(y-y_1)(z-z_1)-2nl(z-z_1)(x-x_1)-2lm(x-x_1)(y-y_1)]^{\frac{1}{2}}.$$

Case (3·7c) r and s finite. We shall first show that $r=s$. Suppose that $r\neq s$. Then $\mathbf{a}_{\alpha\beta}=\mathbf{a}_{\beta\alpha}\Rightarrow r\mathbf{n}_{\alpha\beta}=s\mathbf{n}_{\alpha\beta}\Rightarrow\mathbf{n}_{\alpha\beta}=\mathbf{0}$. Hence $\mathbf{n}$ is of the form $\mathbf{n}=\mathbf{A}(\alpha)+\mathbf{B}(\beta)$ where $\mathbf{a}_\alpha=-r\mathbf{A}_\alpha$, $\mathbf{a}_\beta=-s\mathbf{B}_\beta$ and $\mathbf{a}_\alpha\cdot\mathbf{a}_\beta=rs\mathbf{A}_\alpha\cdot\mathbf{B}_\beta=0$. A detailed algebraic argument, given in the Appendix, proves that in these circumstances $\mathbf{a}_\alpha$ and $\mathbf{a}_\beta$ are parallel null vectors, which requires that $\mathbf{n}=\mathbf{0}$, a contradiction. It follows that $r=s$, in which

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case if $\mathbf{x}$ is a vector to the surface $\{\nu = \text{constant}\}$, $\mathbf{x} = (\nu - r)\,\mathbf{n} + \mathbf{v}$ where $\mathbf{v} = (x_1, y_1, z_1)$ is a constant vector. Thus $\nu = r \pm R$, where

$$R \equiv [(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{\frac{1}{2}}. \quad \blacksquare$$

Remarks. It may be observed that the substitution $h = \nu$, $w = -(Ah+B)^{-1}$ into Friedlander's system (3·1)–(3·3) gives $(\nabla\nu)^2 = 1$ and $\nabla^2\nu = 2A(A\nu+B)^{-1}$. When $A = 0$ this corresponds to case (3·7*a*) ($N = 0$) of the present theorem, and when $A \neq 0$, $\nabla^2\nu = \dfrac{2}{\nu + B/A}$, which corresponds to case (3·7*c*) ($N = 2$). These results are in agreement with (3).

Another way of describing the principal curvature directions is by means of the eigendirections of the shear tensor of the ray congruence, in a manner similar to that employed in the pseudo-Riemannian manifolds of general relativistic cosmology. Resolving the quantity $\nu_{;\lambda;\mu}$ into parts perpendicular and parallel to the normal†:

$$\nu_{;\lambda;\mu} = v_{\lambda\mu} + (\nu_{;\lambda})^{\cdot}\,\nu_{;\mu}, \tag{3·8}$$

where $v_{\lambda\mu}\nu^{;\mu} = 0$, $v_{\lambda\mu}\nu^{;\lambda} = 0$, and the notation $(\ )^{\cdot}$ denotes derivative along the normal congruence, viz. $(\ )^{\cdot} \equiv (\ )_{;\mu}\nu^{;\mu}$. The second term disappears, since the fact that $(\nabla\nu)^2 = \nu^{;\alpha}\nu_{;\alpha} = 1$ means that $(\nu^{;\alpha})^{\cdot} = \nu^{;\alpha}{}_{;\beta}\,\nu^{;\beta} = \nu_{;\beta}{}^{;\alpha}\nu^{;\beta} = \frac{1}{2}(\nu_{;\beta}\nu^{;\beta})^{;\alpha} = 0$, and so the curves of the congruence normal to the equipotential surfaces $\{\nu = \text{constant}\}$ are geodesics (straight lines, in this case). Splitting the term $v_{\lambda\mu}$ into symmetric and antisymmetric parts: $v_{\lambda\mu} = \theta_{\lambda\mu} + \omega_{\lambda\mu}$, where $\theta_{\lambda\mu} = \theta_{\mu\lambda}$, $\omega_{\lambda\mu} = -\omega_{\mu\lambda}$, $\theta_{\lambda\mu}\nu^{;\mu} = \omega_{\lambda\mu}\nu^{;\mu} = 0$, the term $\omega_{\lambda\mu} = 0$, since the normal congruence is by definition surface-orthogonal. Finally, we can express $\theta_{\lambda\mu}$ in terms of its trace and trace-free parts: $\theta_{\lambda\mu} = \sigma_{\lambda\mu} + \frac{1}{2}\theta h_{\lambda\mu}$, where $\sigma^{\lambda}{}_{\lambda} = 0$, $\theta \equiv \theta^{\mu}{}_{\mu}$, $\sigma_{\lambda\mu}\nu^{;\mu} = 0$, $\sigma_{\lambda\mu} = \sigma_{\mu\lambda}$, and $h_{\lambda\mu}$ is the 'projection operator' into the tangent plane at each point on a surface, i.e. $h_{\lambda\mu} = \delta_{\lambda\mu} - \nu_{;\lambda}\nu_{;\mu}$, $h_{\lambda\mu} = h_{\mu\lambda}$, $h^{\lambda}{}_{\lambda} = 2$, and $h_{\mu\lambda}h^{\mu\tau} = h_{\lambda}{}^{\tau}$. Thus (3·8) becomes

$$\nu_{;\lambda;\mu} = \theta_{\lambda\mu} = \sigma_{\lambda\mu} + \tfrac{1}{2}\theta h_{\lambda\mu}. \tag{3·8′}$$

The interpretations of $\theta_{\lambda\mu}, \sigma_{\lambda\mu}$ and θ are that they are respectively the 'expansion tensor', the 'shear tensor', and the 'area expansion' of the normal congruence. The symmetric tensor $n_{\lambda;\mu}$ will possess three linearly independent eigenvectors, one of which is $\mathbf{n}$, the other two, $\mathbf{t}_1$ and $\mathbf{t}_2$, being orthogonal to $\mathbf{n}$ and, without loss of generality, to each other. Thus we have

$$n_{\lambda;\mu}t_1{}^{\mu} = T_1 t_{1\lambda} \quad \text{and} \quad n_{\lambda;\mu}t_2{}^{\mu} = T_2 t_{2\lambda},$$

for some scalar eigenvalues T_1 and T_2. These scalars will, apart from a sign, be the principal curvatures of the surface, and the eigendirections will be the principal directions of curvature. Writing the position vector to the initial surface $\{\nu = 0\}$ in the form $\mathbf{a} = \mathbf{a}(\pi_1, \pi_2)$, where π_1 and π_2 are curvature line parameters, we have $\mathbf{a}_{,\pi_i} \equiv \mathbf{t}_i$

† A semicolon denotes covariant derivative, and as usual indices are raised and lowered using the metric tensor. Greek indices λ, μ and τ denote component indices, and $\delta_{\lambda}{}^{\mu}$ is the Kronecker delta tensor. The summation notation for repeated indices is employed.

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and $n_{\nu,\pi_i} \equiv n_{\nu;\mu} t_i^{\mu}$ $(i = 1, 2)$. The above relations then become $\mathbf{a}_{,\pi_i} = \frac{1}{T_i}\mathbf{n}_{,\pi_i}$ (no sum over i), which is Rodrigues' formula.

4. *Complex solutions of the system* (1·1)

In this section we extend the geometric formalism of sections 2 and 3 in order to generalize our results to the case of complex solutions of the system (1·1). The main result is contained in the following

THEOREM 4·1. *Complex solutions to the system*

$$(\nabla\gamma)^2 = f(\gamma) \quad \text{and} \quad \nabla^2\gamma = g(\gamma)$$

in complexified 3-dimensional Euclidean space can exist only when one of the following holds:

(i) $f(\gamma) \equiv g(\gamma) \equiv 0$. *Two possibilities arise:*

either (*a*) $\gamma \equiv \gamma_0$, *where* γ_0 *is constant*

or (*b*) γ *satisfies an equation of form* $k(\gamma) = l(\gamma)x + m(\gamma)y + n(\gamma)z$, *where* $k(\gamma)$, $l(\gamma)$, $m(\gamma)$ *and* $n(\gamma)$ *are arbitrary, except that* $l^2 + m^2 + n^2 \equiv 0$ *and not all of* l, m, n *are zero; the equipotentials are null planes.*

or (ii) $f(\gamma) \not\equiv 0$. *Then* $\gamma = \gamma_0$ *is a solution if and only if* γ_0 *is a common zero of* $f(\gamma)$ *and* $g(\gamma)$.

or (iii) $f(\gamma) \not\equiv 0$, *and if* ν *is any solution to the equation* $\nu' = [f(\gamma)]^{-\frac{1}{2}}$ *where a prime* (′) *denotes differentiation with respect to* γ, *then*

$$g(\gamma(\nu)) - \tfrac{1}{2}f'(\gamma(\nu)) \equiv \frac{N[f(\gamma(\nu))]^{\frac{1}{2}}}{\nu - A},$$

where A *is an arbitrary complex constant, and* $N = 0$, 1 *or* 2. *Three possibilities occur*

either (*a*) *the equipotentials are planes* ($N = 0$), *right-circular cylinders* ($N = 1$) *or spheres* ($N = 2$), *and the general solution for* γ *is* $\gamma = \gamma(\nu)$, *where* $\nu' = [f(\gamma)]^{-\frac{1}{2}}$ *and*

$$\nu = \begin{cases} lx + my + nz + A, & l, m, n \text{ constants}, \quad l^2 + m^2 + n^2 = 1, & (N = 0), \\ \pm\rho + A, & \rho \equiv [(1-l^2)(x-x_1)^2 + (1-m^2)(y-y_1)^2 + (1-n^2)(z-z_1)^2 \\ & \quad - 2mn(y-y_1)(z-z_1) - 2nl(z-z_1)(x-x_1) \\ & \quad - 2lm(x-x_1)(y-y_1)]^{\frac{1}{2}}, \\ & l, m, n, x_1, y_1, z_1 \text{ constants}, \quad l^2 + m^2 + n^2 = 1, & (N = 1), \\ \pm R + A, & R \equiv [(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{\frac{1}{2}}, \quad x_1, y_1, z_1 \text{ constants}, & (N = 2). \end{cases}$$

The constants l, m, n, x_1, y_1, z_1, γ_0 *and* A *are complex.*

or (*b*) $N = 0$ *and* $\nu = \mu_1 x + \mu_2 y + \mu_3 z + \psi(\lambda_1 x + \lambda_2 y + \lambda_3 z)$ *where* ψ *is an arbitrary function with* $\psi'' \not\equiv 0$, *and* μ_1, μ_2, μ_3, λ_1, λ_2, λ_3 *are arbitrary complex constants satisfying* $\mu_1^2 + \mu_2^2 + \mu_3^2 = 1$, $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$, $\mu_1\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_3 = 0$. *Not all of* λ_1, λ_2 *and* λ_3 *are zero.*

or (*c*) $N = 2$ *and* $\nu = A + \dfrac{\partial F}{\partial \beta}$, *where* F *and* β *are given by*

$$F(\beta, x, y, z) \equiv \phi(\beta) + \tfrac{1}{2}[(1-\beta^2)x + i(1+\beta^2)y - 2\beta z] = 0.$$

$\phi(\beta)$ *is an arbitrary analytic function of* β *with* $\phi''' \not\equiv 0$.

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Proof. As in the proof of Theorem 3·1 it is clear that γ is identically (complex) constant, $\gamma \equiv \gamma_0$, if and only if $f(\gamma_0) = g(\gamma_0) = 0$, i.e. γ_0 is a common zero of $f(\gamma)$ and $g(\gamma)$. This proves case (ii), and the first part of case (i).

If γ is not identically constant, we observe from (1·1a) that the equipotential surfaces for γ have null normals if and only if $f(\gamma) = 0$, and hence if the normal to an equipotential is null at one point, it is null everywhere. Consequently either $f(\gamma) \equiv 0$ (the second part of case (i)), or there are equipotential surfaces whose normals at each point are not null (case (iii)).

Suppose that the normals to all (sufficiently local) equipotential surfaces are null; we show that the only case that can arise is that for which the equipotentials are null planes. For if the equipotentials $\{\gamma = \text{constant}\}$ are not null planes, then by Theorem 2·5, they may be described by the equations

$$x = i\beta \sin\alpha + \mu(\alpha, \gamma),$$

$$y = i\beta \cos\alpha - \int^{\alpha} \mu_{\alpha}(a, \gamma) \tan a \, da + \alpha_0(\gamma)$$

and

$$z = \beta.$$

In these coordinates the metric is of the form

$$ds^2 = A\, d\alpha^2 + 2B\, d\alpha\, d\gamma + 2C\, d\beta\, d\gamma + D\, d\gamma^2,$$

where $A \equiv (i\beta + \mu_{\alpha} \sec\alpha)^2$ and $C = C(\alpha, \gamma) \not\equiv 0$. It is easily seen that this form guarantees that $(\nabla\gamma)^2 = 0$. Moreover,

$$\nabla^2\gamma = \frac{1}{2C}(\ln A)_{\beta} = [C(\alpha, \gamma)(\beta - i\mu_{\alpha} \sec\alpha)]^{-1},$$

which cannot be a function of γ only, by virtue of the β-dependence of the right side. Consequently, the set of equipotentials (if it exists at all) is a 1-parameter family of null planes. In this case we must have $k(\gamma) = l(\gamma)x + m(\gamma)y + n(\gamma)z$ for some function $k(\gamma)$, and for functions $l(\gamma)$, $m(\gamma)$ and $n(\gamma)$ satisfying $l^2 + m^2 + n^2 = 0$. One can readily check that $(\nabla\gamma)^2 = \nabla^2\gamma = 0$ for arbitrary $k(\gamma)$. In other words, when $f(\gamma) \equiv 0$, we obtain non-constant solutions to the system (1·1) only in the case $g(\gamma) \equiv 0$, and the equipotentials are then null planes. This completes case (i).

It remains to investigate case (iii). It is clear that whenever the shear tensor of the normal congruence possesses three linearly independent non-null eigenvectors we can directly generalize the arguments of the proof of Theorem 3·1, which concerned the real case. We can reconstruct equipotential surfaces parallel to the initial surface $\{\nu(x, y, z) = 0\}$, where $(\nabla\nu)^2 = 1$, and the entire geometric formulation upon which the proof of Theorem 3·1 was based can be applied to the present case. The only change that must be made is that the constants l, m, n, x_1, y_1, z_1, and A may now be complex.

In the case where the shear tensor possesses only one non-null eigenvector (i.e. $\mathbf{n}$) then there exists a single null eigenvector $\mathbf{a}_{\alpha}$, which is orthogonal to $\mathbf{n}$. There is therefore only one curvature line parameter, α. Since the equipotential surfaces are *not* null, it follows from Theorem 2·4 that there are *two* families of null curves in the surfaces. One of these families is the set of α-lines (tangent to $\mathbf{a}_{\alpha}$). Let β be a parameter for the second family, i.e. the β-lines (tangent to $\mathbf{a}_{\beta}$) are null, and $\mathbf{a}_{\alpha} \cdot \mathbf{a}_{\beta} \neq 0$.

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The symmetric tensor $n_{\mu;\nu}$ is required to have a unique eigenvector, $\mathbf{t}$. Let $\boldsymbol{\tau}$ be a null vector with the property $\boldsymbol{\tau}\cdot\mathbf{n}=0$, $\boldsymbol{\tau}\cdot\mathbf{t}\neq 0$. Then writing

$$n_{\mu;\nu}=At_\mu t_\nu+B(t_\mu\tau_\nu+\tau_\mu t_\nu)+C\tau_\mu\tau_\nu$$

we have

$$n_{\mu;\nu}t^\nu=B(\boldsymbol{\tau}\cdot\mathbf{t})\,t_\mu+C(\boldsymbol{\tau}\cdot\mathbf{t})\,\tau_\mu$$

and

$$n_{\mu;\nu}\tau^\nu=A(\boldsymbol{\tau}\cdot\mathbf{t})\,t_\mu+B(\boldsymbol{\tau}\cdot\mathbf{t})\,\tau_\mu.$$

Since t^ν is an eigenvector of $n_{\mu;\nu}$ it follows that $C=0$. That part of the matrix $n_{\mu;\nu}$ orthogonal to $\mathbf{n}$ may be expressed in the form given in the corollary to Theorem 2·2, viz.

$$n_{\mu;\nu}=\begin{bmatrix} a & \pm\frac{1}{2}i(a-b)\\ \pm\frac{1}{2}i(a-b) & b\end{bmatrix},\quad a\neq b,$$

by putting $t^\nu=t\begin{pmatrix}1\\ \pm i\end{pmatrix}$, $\tau^\nu=\frac{1}{2}\tau\begin{pmatrix}1\\ \mp i\end{pmatrix}$ with $t\tau\neq 0$. Thus we obtain $A=\dfrac{1}{2t^2}(a-b)\neq 0$, $B=\dfrac{1}{2t\tau}(a+b)$, $a=t(At+B\tau)$ and $b=-t(At-B\tau)$. In terms of the parameters α and β, and a position vector $\mathbf{a}(\alpha,\beta)$ to the initial surface, we may thus write

$$\mathbf{n}_\alpha=-\frac{1}{r}\mathbf{a}_\alpha \tag{4·1}$$

and

$$\mathbf{n}_\beta=-\frac{1}{s}\mathbf{a}_\alpha-\frac{1}{r}\mathbf{a}_\beta, \tag{4·2}$$

where $s=s(\alpha,\beta)\neq 0$ is finite and $r=r(\alpha,\beta)\neq 0$ is either finite or infinite.

We may now proceed along the same lines as in section 3. A general position vector $\mathbf{x}$ can be expressed (sufficiently locally) in the form $\mathbf{x}=\mathbf{a}(\alpha,\beta)+\nu\mathbf{n}(\alpha,\beta)$, where $\mathbf{n}$ is the unit normal to the initial surface, and we therefore obtain the line element

$$d\mathbf{x}^2=2(\mathbf{a}_\alpha\cdot\mathbf{a}_\beta)\left(1-\frac{\nu}{r}\right)^2 d\alpha\,d\beta-\frac{2\nu}{s}(\mathbf{a}_\alpha\cdot\mathbf{a}_\beta)\left(1-\frac{\nu}{r}\right)d\beta^2+d\nu^2, \tag{4·3}$$

where use is made of the relations

$$\mathbf{n}\cdot\mathbf{a}_\alpha=\mathbf{n}\cdot\mathbf{a}_\beta=\mathbf{n}\cdot\mathbf{n}_\alpha=\mathbf{n}\cdot\mathbf{n}_\beta=\mathbf{a}_\alpha\cdot\mathbf{a}_\alpha=\mathbf{a}_\beta\cdot\mathbf{a}_\beta=0.$$

It follows from (4·3) that $(\nabla\nu)^2=1$ and that $\nabla^2\nu=\dfrac{2}{\nu-r}$. By demanding that $\nabla^2\nu$ is a function of ν only (i.e. equation (1·1b)), we see that r must be constant (possibly infinite). Note that at this stage there is the freedom

$$\alpha\to\tilde{\alpha}(\alpha),\quad \beta\to\tilde{\beta}(\beta),\quad \tilde{\alpha}'(\alpha)\tilde{\beta}'(\beta)\not\equiv 0,$$

in the choice of the parameters α and β.

We consider first the case where $r=\infty$. This means that the expansion, $n^\mu{}_{;\mu}$, of the normals is zero. Equations (4·1) and (4·2) become

$$\mathbf{n}_\alpha=\mathbf{0} \tag{4·1′}$$

and

$$\mathbf{n}_\beta=-\frac{1}{s}\mathbf{a}_\alpha, \tag{4·2′}$$

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where $s = s(\alpha, \beta)$ is finite. It follows from (4·2′) that $\mathbf{n}_\beta \not\equiv 0$, since we are assuming (2·4) to be false. We show first that $\mathbf{a}_{\alpha\beta} = \mathbf{0}$. By expanding $\mathbf{a}_{\alpha\beta}$ in terms of the basis $\{\mathbf{a}_\alpha, \mathbf{a}_\beta, \mathbf{n}\}$ we have $\mathbf{a}_{\alpha\beta} = C_1\mathbf{a}_\alpha + C_2\mathbf{a}_\beta + C_3\mathbf{n}$, for scalars C_1, C_2 and C_3. Since $\mathbf{a}_\alpha \cdot \mathbf{a}_\alpha = 0$, it follows by differentiation that $\mathbf{a}_\alpha \cdot \mathbf{a}_{\alpha\beta} = 0$ and so $C_2 = 0$. It follows similarly that $C_1 = 0$. Also, $\mathbf{a}_\alpha \cdot \mathbf{n} = 0$, and by differentiating and using (4·2′) we obtain $\mathbf{a}_{\alpha\beta} \cdot \mathbf{n} = C_3 = 0$. Thus $\mathbf{a}_{\alpha\beta} = \mathbf{0}$, and so there exist vector functions $\mathbf{h}(\alpha)$ and $\mathbf{k}(\beta)$ such that $\mathbf{a} = \mathbf{h}(\alpha) + \mathbf{k}(\beta)$, where $\mathbf{h}'(\alpha)$ and $\mathbf{k}'(\beta)$ are linearly independent null vectors. The next step is to calculate and simplify the unit normal vector $\mathbf{n}$. Using (4·1′) and (4·2′) we obtain

$$\mathbf{n}'(\beta) = -\frac{1}{s(\alpha, \beta)}\mathbf{h}'(\alpha).$$

The parameter dependence on each side of this equation requires that

$$\mathbf{n}'(\beta) = N(\beta)\boldsymbol{\lambda} \not\equiv \mathbf{0} \quad \text{and} \quad \mathbf{h}'(\alpha) = H(\alpha)\boldsymbol{\lambda} \not\equiv \mathbf{0},$$

where λ is a constant null vector, and H and N are scalar functions of α and β respectively. Thus $\mathbf{n} = \left[\int^\beta N(\beta)\, d\beta\right]\boldsymbol{\lambda} + \boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is a constant vector. Now use the freedom $\beta \to \tilde{\beta}(\beta) = \int^\beta N(\beta)\, d\beta$ to put $\mathbf{n} = \beta\boldsymbol{\lambda} + \boldsymbol{\mu}$, without loss of generality. Similarly, we may write $\mathbf{h}(\alpha) = \alpha\boldsymbol{\lambda}$, absorbing any constant of integration into the function $\mathbf{k}(\beta)$. The requirement that $\mathbf{n} \cdot \mathbf{a}_\alpha = 0$ means that $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} = 0$, whilst $\mathbf{n}^2 = 1 \Rightarrow \boldsymbol{\mu}^2 = 1$. Thus $\mathbf{a} \cdot \boldsymbol{\lambda} = \mathbf{k}(\beta) \cdot \boldsymbol{\lambda}$ and $\mathbf{a} \cdot \boldsymbol{\mu} = \mathbf{k}(\beta) \cdot \boldsymbol{\mu}$. Also, since $\mathbf{a}_\alpha \cdot \mathbf{a}_\beta \not\equiv 0$, we have $\mathbf{k}'(\beta) \cdot \boldsymbol{\lambda} \not\equiv 0$, and hence $\mathbf{a} \cdot \boldsymbol{\lambda}$ cannot be constant. Consequently, $\mathbf{a} \cdot \boldsymbol{\mu} + \psi(\mathbf{a} \cdot \boldsymbol{\lambda}) = 0$, for some function ψ. Since $\mathbf{x} = \mathbf{a} + \nu\mathbf{n}$, we have $\mathbf{x} \cdot \boldsymbol{\mu} = \mathbf{a} \cdot \boldsymbol{\mu} + \nu$ and $\mathbf{x} \cdot \boldsymbol{\lambda} = \mathbf{a} \cdot \boldsymbol{\lambda}$, so

$$\nu = \mathbf{x} \cdot \boldsymbol{\mu} + \psi(\mathbf{x} \cdot \boldsymbol{\lambda})$$

for some function ψ. We observe that with the restrictions $\boldsymbol{\mu}^2 = 1$, $\boldsymbol{\lambda}^2 = 0$, $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} = 0$, ν satisfies $(\nabla\nu)^2 = 1$ and $\nabla^2\nu = 0$ for *arbitrary* ψ. However, the requirement that (4·1′) and (4·2′) hold does place restrictions on ψ. This is seen by writing the initial surface $\{\nu = 0\}$ in the form $\mathbf{a} \cdot \boldsymbol{\mu} + \psi(\mathbf{a} \cdot \boldsymbol{\lambda}) = 0$ and calculating the unit normal directly by taking the gradient of the left side. Thus $\mathbf{n} = \psi'\boldsymbol{\lambda} + \boldsymbol{\mu}$, whence $\mathbf{n}_\alpha = \mathbf{0}$ and

$$\mathbf{n}_\beta = \psi''.(\mathbf{a}_\beta \cdot \boldsymbol{\lambda})\boldsymbol{\lambda} = \psi''.(\mathbf{k}'(\beta) \cdot \boldsymbol{\lambda})\,\mathbf{a}_\alpha,$$

which is compatible with (4·1′) and (4·2′) if and only if $\psi'' \not\equiv 0$. This completes case (iii *b*).

There now remains only case (iii *c*), for which $(\nabla\nu)^2 = 1$, $\nabla^2\nu = \dfrac{2}{\nu - r}$, and r is a finite non-zero constant. We focus attention on the parameter β, rather than ν. We observe from (4·3) that

$$(\nabla\beta)^2 = 0 \quad \text{and} \quad \nabla^2\beta = 0,$$

and so from our previous results (case (i) of the present theorem) we have

$$\psi(\beta) = l(\beta)x + m(\beta)y + n(\beta)z \tag{4·4}$$

for an arbitrary function ψ, and for l, m and n not all zero and satisfying

$$l^2 + m^2 + n^2 = 0.$$

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We first dismiss two possibilities, one for which $l-im \equiv 0$, and the other for which $l-im \not\equiv 0$ and $\frac{n}{l-im} = \text{constant}$. A straightforward calculation shows that in these cases $\beta = \beta(\tilde{l}x+\tilde{m}y+\tilde{n}z)$ for *constant* $\tilde{l}$, $\tilde{m}$ and $\tilde{n}$ not all zero and satisfying

$$\tilde{l}^2+\tilde{m}^2+\tilde{n}^2 = 0.$$

Write $\boldsymbol{\lambda} = (\tilde{l},\tilde{m},\tilde{n})$. Then from (4·3), $\nabla\beta\cdot\nabla\nu = 0 \Rightarrow \nabla\nu\cdot\boldsymbol{\lambda} = 0$. Let $\boldsymbol{\mu}$ be a constant vector satisfying $\boldsymbol{\mu}\cdot\boldsymbol{\lambda} = 0, \boldsymbol{\mu}\cdot\boldsymbol{\mu} = 1$. Then $\nabla\nu = D\boldsymbol{\lambda}\pm\boldsymbol{\mu}$, for some scalar D, and thus $\nu = \pi \pm \mathbf{x}\cdot\boldsymbol{\mu}$, where $\nabla\pi = D\boldsymbol{\lambda}$, whence $\pi = \pi(\mathbf{x}\cdot\boldsymbol{\lambda})$. Consequently $\nabla^2\nu = \nabla^2\pi = \nabla D\cdot\boldsymbol{\lambda} = \pi''\boldsymbol{\lambda}^2 = \mathbf{0}$, which contradicts our assumption that $\nabla^2\nu = \frac{2}{\nu-r}$ with r finite. It follows that in (4·4), $l-im \not\equiv 0$ and $\frac{n}{l-im} \not\equiv$ constant. Thus reparametrizing $\beta \to \tilde{\beta}(\beta) = -\frac{n(\beta)}{l(\beta)-im(\beta)}$ we obtain $\tilde{\beta}^2 = -\left(\frac{l+im}{l-im}\right)$. Define $\phi(\tilde{\beta}) \equiv -\frac{\psi(\beta)}{l(\beta)-im(\beta)}$. Then (4·4) becomes

$$F(\tilde{\beta},x,y,z) \equiv \phi(\tilde{\beta})+\tfrac{1}{2}(1-\tilde{\beta}^2)x+\tfrac{1}{2}i(1+\beta^2)y-\tilde{\beta}z = 0. \tag{4·5}$$

From now on we shall write β instead of $\tilde{\beta}$. We observe† from (4·5) that

$$\nabla\beta = \frac{1}{F_\beta}\{-\tfrac{1}{2}(1-\beta^2), -\tfrac{1}{2}i(1+\beta^2), \beta\},$$

and that $\boldsymbol{\mu} \equiv (-\beta, i\beta, -1)$ satisfies $\boldsymbol{\mu}^2 = 1$ and $\boldsymbol{\mu}\cdot\nabla\beta = 0$. Since $\nabla\nu$ also satisfies these conditions, viz. $(\nabla\nu)^2 = 1$ and $\nabla\nu\cdot\nabla\beta = 0$, there is a scalar function, D, such that $\nabla\nu = D\nabla\beta\pm\boldsymbol{\mu}$. Now observe from (4·5) that $\nabla F_\beta = F_{\beta\beta}\nabla\beta+\boldsymbol{\mu}$. This suggests the substitution $\nu = \pm F_\beta+\Omega$, which, for compatibility, requires $\nabla\Omega = (D\mp F_{\beta\beta})\nabla\beta$. Thus $\Omega = \Omega(\beta)$, and consequently $\Omega'(\beta) = D\mp F_{\beta\beta}$ with $(\nabla\Omega)^2 = 0$ and $\nabla^2\Omega = 0$. Finally, we must satisfy the equation $\nabla^2\nu = \frac{2}{\nu-r}$, which implies

$$\pm[\phi'''(\beta)\nabla\beta+(-1,i,0)]\cdot\nabla\beta\pm\operatorname{div}\boldsymbol{\mu} = \frac{2}{\nu-r},$$

or $\pm(-\beta_x+i\beta_y) = \pm\frac{1}{F_\beta} = \frac{1}{\nu-r}$, and so $\nu = r\pm F_\beta$. Both equations (1·1*a*) and (1·1*b*) are now satisfied, for arbitrary $\phi(\beta)$ in (4·5). However, as in the previous case (iii*b*), we must check the validity of (4·1) and (4·2). Consider the initial surface $\nu = 0$:

$$r\pm[\phi'(\beta)-\beta x+i\beta y-z] = 0. \tag{4·6}$$

A direct calculation using the above formulae shows that the unit normal $\mathbf{n}$ to this surface is

$$\begin{aligned}\mathbf{n} &= \nabla F_\beta = F_{\beta\beta}\nabla\beta+\boldsymbol{\mu}\\ &= \frac{F_{\beta\beta}}{F_\beta}(-\tfrac{1}{2}(1-\beta^2), -\tfrac{1}{2}i(1+\beta^2), \beta)+(-\beta, i\beta, -1)\\ &\equiv \pm\frac{1}{r}\mathbf{G}(\beta)\mp\frac{\mathbf{a}}{r},\end{aligned}$$

† In the following a β-suffix indicates partial differentiation with respect to β, holding x, y and z constant.

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where $\mathbf{G}(\beta) \equiv (\frac{1}{2}\phi''.(1-\beta^2)+\beta\phi'-\phi,\ i[\frac{1}{2}\phi''.(1+\beta^2)-\beta\phi'+\phi],\ -\beta\phi''+\phi')$.

Thus
$$\mathbf{n}_\alpha = \mp\frac{1}{r}\mathbf{a}_\alpha$$

and
$$\mathbf{n}_\beta = \pm\frac{1}{r}\mathbf{G}'(\beta) \mp \frac{1}{r}\mathbf{a}_\beta.$$

Comparison with (4·1) and (4·2) shows† that we must identify the term $(1/r)\,\mathbf{G}'(\beta)$ with $-(1/s)\,\mathbf{a}_\alpha$. Thus $-(1/s)\,\mathbf{a}_\alpha \equiv \phi'''\nabla\beta$. This must be non-zero, since s is supposed finite, and thus we require $\phi''' \not\equiv 0$. Thus ϕ is arbitrary, except that it must not be of form $\phi(\beta) = a\beta^2+b\beta+c$ for constants a, b and c. It is interesting to observe that this function is generated from $\phi(\beta) \equiv 0$ in (4·5) by the shift of origin

$$x \to x-(a-c), \quad y \to y-i(a+c), \quad z \to z-b.$$

This completes the proof of the theorem. |

Remark. Some of the results of Theorems 3·1 and 4·1 can be checked with more familiar results in 2 dimensions. In this case the equation $\nabla^2\gamma = 0$ has general solution $\gamma = \phi(x+iy)+\psi(x-iy)$, where ϕ and ψ are arbitrary functions and (x, y) are Cartesian coordinates. The additional requirement $(\nabla\gamma)^2 = f(\gamma)$ means that

$$4\phi_\zeta\psi_{\bar\zeta} = f(\phi(\zeta)+\psi(\bar\zeta)), \tag{4·7}$$

where $\zeta = x+iy$ and $\bar\zeta = x-iy$.

If $f(\gamma) \equiv 0$, it follows from (4·7) that either $\phi_\zeta = 0$ or $\psi_{\bar\zeta} = 0$, and thus $\gamma = \gamma(x \pm iy)$. This agrees with the result of case (i) of Theorem 4·1 when there is no explicit z dependence.

If $f(\gamma) \not\equiv 0$, take logarithms and differentiate (4·7), successively with respect to ζ and $\bar\zeta$, to obtain $\left(\frac{f'}{f}\right)'\phi_\zeta\psi_{\bar\zeta} \equiv 0$. Since $f \not\equiv 0$, it follows that $\phi_\zeta\psi_{\bar\zeta} \not\equiv 0$, and so $\frac{f'}{f} = B$, a constant. Thus $f(\gamma) = ke^{B\gamma}$ for some non-zero constant k. If $B = 0$, we have

$$4\phi_\zeta\psi_{\bar\zeta} = k,$$

and since ζ and $\bar\zeta$ are independent variables, this requires

$$\phi(\zeta) = a\zeta+b \quad \text{and} \quad \psi(\bar\zeta) = c\bar\zeta+d,$$

where a, b, c and d are constants such that $4ac = k$. Consequently in this case

$$\gamma = (a+c)\,x+i(a-c)\,y+b+d,$$

arising from planar equipotentials. Referring to the statement of the theorem, we have $f(\gamma) \equiv k$, $\nu = \gamma k^{-\frac{1}{2}}$, $g(\gamma) \equiv 0$, and so $N = 0$, and this corresponds to case (iii a) with planar equipotential surfaces. If $B \neq 0$, $\phi_\zeta e^{-B\phi(\zeta)} = \frac{1}{4}k/[\psi_{\bar\zeta}e^{-B\psi(\zeta)}]$ and by the independence of ζ and $\bar\zeta$ this requires $\phi(\zeta) = -\frac{1}{B}\ln(a\zeta+b)$, $\psi(\bar\zeta) = -\frac{1}{B}\ln(c\bar\zeta+d)$, where a, b, c and d are constants such that $4ac = kB^2$. In this case

$$\gamma = -\frac{1}{B}\ln\{ac[(x-\lambda)^2+(y-\mu)^2]\}, \quad \text{where} \quad \lambda = -\frac{1}{2ac}(ad+bc)$$

† We can without loss of generality take the top sign (+) in (4·6), since with the minus sign in (4·6) we merely reverse $\mathbf{n}$ in the argument.

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and $\mu = -\dfrac{i}{2ac}(ad-bc)$, arising from cylindrical equipotentials. In the theorem, we have $f(\gamma) \equiv ke^{B\gamma}$, $\nu = -\dfrac{2k^{\frac{1}{2}}}{B}e^{-\frac{1}{2}B\gamma}$, $g(\gamma) \equiv 0$, and so $A = 0$ and $N = 1$, corresponding to case (iii a) with cylindrical equipotential surfaces.

These cases are the only ones for which $g(\gamma) \equiv 0$ and there is no z-dependence.

When the equipotential surfaces are *not* null, they have *constant* principal radii of normal curvature. In the cases where the equipotential surfaces *are* null, these radii are not so well-defined. However, we can obtain a useful extension of the idea, as follows. In general, the normal congruence of geodesics will satisfy $n_{\mu;\nu}n^{\nu} = \lambda n_{\mu}$, where λ is a function of position. The vector $\mathbf{N} = \kappa\mathbf{n}$ will satisfy $N_{\mu;\nu}N^{\nu} = 0$, if and only if κ satisfies $(\ln\kappa)_{;\nu}n^{\nu} = -\lambda$. In this case, $\mathbf{N}$ is fixed up to a factor which is constant along each curve of the null congruence, and, having chosen $\mathbf{N}$, we can *define* the principal radii of normal curvature, ρ_1 and ρ_2, by expressions of the form

$$\mathbf{N}_{\alpha} = A\mathbf{a}_{\alpha} + B\mathbf{a}_{\beta},$$
$$\mathbf{N}_{\beta} = C\mathbf{a}_{\alpha} + D\mathbf{a}_{\beta}$$

with $\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2} \equiv -(A+D)$, and $\dfrac{1}{\rho_1\rho_2} \equiv AD-BC$. We then find that when the equipotential surfaces are null, they have *constant* principal radii of normal curvature. This, and a converse result, is proved in the following

THEOREM 4·2. *Equipotential surfaces for (non-constant) solutions of the system*

$$(\nabla\gamma)^2 = f(\gamma) \tag{1·1 a}$$

and

$$\nabla^2\gamma = g(\gamma) \tag{1·1 b}$$

have constant (possibly infinite) principal radii of normal curvature. Conversely given any (sufficiently smooth) family of surfaces $\{\gamma = constant\}$ with $(\nabla\gamma)^2 = f(\gamma)$, and with constant (possibly infinite) principal radii of normal curvature, it is a family of equipotential surfaces for the system (1·1).

Proof. We observe from our proofs of Theorems 3·1 and 4·1 that when the equipotentials are not null, and the shear tensor of the normal congruence possesses three linearly independent non-null eigenvectors (cases Th. 3·1 (iii) and Th. 4·1 (iii a)), the principal radii are constants if and only if $\nabla^2\gamma$ is independent of α and β. If the equipotential surfaces are not null and there is a null shear eigenvector (cases Th. 4·1 (iii b) and (iii c)), then the principal radii of normal curvature are again constants if and only if $\nabla^2\gamma$ is independent of α and β.

There remains the case of null equipotentials, where $(\nabla\gamma)^2 = 0$. We have shown in Theorem 4·1 (i) that the only possible solution is given by the set of null planes: $k(\gamma) = l(\gamma)x + m(\gamma)y + n(\gamma)z$, with k arbitrary, and l, m, n not all zero and satisfying $l^2+m^2+n^2 = 0$. The normal, $\mathbf{n}$, is parallel to (l, m, n), so writing

$$\mathbf{n} = \lambda(\alpha,\beta,\gamma)\,(l(\gamma), m(\gamma), n(\gamma))$$

we have

$$\mathbf{n}_{\alpha} = \lambda_{\alpha}\mathbf{n} = \lambda_{\alpha}(A\mathbf{a}_{\alpha} + B\mathbf{a}_{\beta})$$

and

$$\mathbf{n}_{\beta} = \lambda_{\beta}\mathbf{n} = \lambda_{\beta}(A\mathbf{a}_{\alpha} + B\mathbf{a}_{\beta}),$$

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for some coefficients A and B. Thus $\frac{1}{\rho_1}+\frac{1}{\rho_2} = -(\lambda_\alpha A+\lambda_\beta B)$ and $\frac{1}{\rho_1\rho_2} = 0$. Choosing $\mathbf{n}$ to be the normal $\mathbf{N}$ ($N_{\mu;\nu}N^\nu = 0$), we have

$$0 = \nabla_{\mathbf{n}}\mathbf{n} = A\mathbf{n}_\alpha + B\mathbf{n}_\beta = (\lambda_\alpha A+\lambda_\beta B)\mathbf{n} \Rightarrow \frac{1}{\rho_1} = \frac{1}{\rho_2} = 0,$$

and so the principal radii of curvature are infinite.

Conversely, given a family of surfaces $\{\gamma = \text{constant}\}$ with $(\nabla\gamma)^2 = 0$, either they are null planes (in which case $\nabla^2\gamma = 0$) or they are expressible in the form $\mathbf{a} = (x, y, z)$ where

$$x = i\beta\sin\alpha+\mu(\alpha,\gamma),$$

$$y = i\beta\cos\alpha-\int^\alpha \mu_\alpha(a,\gamma)\tan a\,da+\alpha_0(\gamma)$$

and

$$z = \beta$$

(Theorem 2·5); direct calculation then shows that $\mathbf{n} = \kappa\mathbf{a}_\beta$ for some $\kappa = \kappa(\alpha,\beta,\gamma) \not\equiv 0$, and also

$$\mathbf{n}_\alpha = \frac{i\kappa\cos\alpha}{i\beta\cos\alpha+\mu_\alpha}\mathbf{a}_\alpha+\kappa_\alpha\mathbf{a}_\beta \quad \text{and} \quad \mathbf{n}_\beta = \kappa_\beta\mathbf{a}_\beta.$$

Choosing $\mathbf{n}$ to be the normal, $\mathbf{N}$, we have

$$0 = \nabla_{\mathbf{n}}\mathbf{n} = \kappa\mathbf{n}_\beta = \kappa\kappa_\beta\mathbf{a}_\beta$$

and so $\kappa = \kappa(\alpha,\gamma)$. The principal radii of curvature, ρ_1 and ρ_2, then satisfy

$$\frac{1}{\rho_1}+\frac{1}{\rho_2} = \frac{i\kappa\cos\alpha}{i\beta\cos\alpha+\mu_\alpha}, \quad \frac{1}{\rho_1\rho_2} = 0.$$

Since, by virtue of the β dependence, the expression for $\frac{1}{\rho_1}+\frac{1}{\rho_2}$ cannot be constant, it follows that in this case there are no null surfaces except null planes with constant principal radii of curvature. This completes the proof. ∎

5. *Conclusions.* We have exhibited all compatible real or complex solutions of the system

$$(\nabla\gamma)^2 = f(\gamma), \tag{1·1a}$$

$$\nabla^2\gamma = g(\gamma). \tag{1·1b}$$

An interesting relationship was discovered between the non-trivial solutions ($\gamma \not\equiv$ constant) and the set of surfaces that possess constant principal radii of normal curvature. It was found that non-trivial solutions of (1·1) exist only for special functions $f(\gamma)$ and $g(\gamma)$, and that if $f(\gamma) \not\equiv 0$, a change of variable can be made which reduces $f(\gamma)$ to unity. This being done, we can take real and imaginary parts in (1·1) and can conclude that we have obtained all solutions to the real set of equations:

$$(\nabla X)^2-(\nabla Y)^2 = \epsilon,$$

$$\nabla X\cdot\nabla Y = 0,$$

$$\nabla^2 X = \frac{N(X-a)}{(X-a)^2+(Y-b)^2}$$

and

$$\nabla^2 Y = \frac{-N(Y-b)}{(X-a)^2+(Y-b)^2},$$

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where $\epsilon = 0$ or 1, $N = 0, 1$ or 2, and a and b are real constants. These results are contained in Theorems 3·1, 4·1 and 4·2, and summarised in Table 1 of Section 1.

The method employed was based on a consideration of the geometry of the equipotential surfaces $\{\gamma = \text{constant}\}$, embedded in flat (complexified, 3-dimensional) Euclidean space. It would be of interest to explore how this method could be generalized (a) to higher dimensional spaces, i.e. more independent variables, and (b) to embeddings in non-flat manifolds, i.e. non-Euclidean operators ∇ and ∇^2. Generalization (b) is to some extent encompassed by (a), since it is a standard result of Riemannian geometry that an n-dimensional (pseudo-)Riemannian manifold may be locally isometrically imbedded in a flat (pseudo-)Euclidean manifold of dimension no greater than $\frac{1}{2}n(n+1)$. Some difficulties might be encountered in generalization (a), in view of the remark following Lemma 2·1.

For certain geometrical problems it may be helpful to consider the complexification and then to take appropriate submanifolds when specializing to the real case. This could be done in the present article to recover the results of Theorem 3·1 from those of Theorem 4·1, by considering the submanifold given by $\{(x, y, z), \mathscr{I}m\, x = \mathscr{I}m\, y = \mathscr{I}m\, z = 0\}$. Similarly, one could obtain results pertaining, for instance, to three dimensional Lorentzian geometry by means of the isomorphic submanifold given by

$$\{(x, y, z), \mathscr{I}m\, x = \mathscr{I}m\, y = \mathscr{R}e\, z = 0\}.$$

Complex manifolds can be treated at a more sophisticated level (4), and an investigation of the role that they might play in the understanding of problems in general relativity is already under way, notably by Newman (5), (6) and Penrose (7).

The present author has applied the results of the present paper to a discussion of Kerr–Schild metrics in general relativity. This discussion will appear in a subsequent article.

Appendix. The case (3·7b)

We show here that the equations $\mathbf{n} = \mathbf{A}(\alpha) + \mathbf{B}(\beta)$, $\mathbf{n}^2 = 1$, $\mathbf{a}_\alpha = -r\mathbf{n}_\alpha \neq \mathbf{0}$, $\mathbf{a}_\beta = -s\mathbf{n}_\beta \neq \mathbf{0}$, $\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = 0$, with r and s constant, imply that $\mathbf{a}_\alpha$ and $\mathbf{a}_\beta$ are null vectors, and deduce that $\mathbf{n} = \mathbf{0}$, a contradiction, as required in section 3.

Write $\mathbf{A}(\alpha) = (A_1(\alpha), A_2(\alpha), A_3(\alpha))$ and $\mathbf{B}(\beta) = (B_1(\beta), B_2(\beta), B_3(\beta))$. Then $\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = 0 \Rightarrow A_1' B_1' + A_2' B_2' + A_3' B_3' = 0$, where a prime denotes derivative with respect to argument (α or β). If $A_1' = A_2' = A_3' = 0$, then $\mathbf{a}_\alpha = \mathbf{0}$ which is impossible, since (2·4) is false. So there is some component, A_1', say, which is non-zero. If $B_2' = B_3' = 0$ then $\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = 0 \Rightarrow B_1' = 0$, and hence $\mathbf{a}_\beta = \mathbf{0}$, which again is impossible. So there is some component, B_3', say, which is non-zero. Thus $\frac{B_1'}{B_3'} + \frac{A_2'}{A_1'}\frac{B_2'}{B_3'} + \frac{A_3'}{A_1'} = 0$. By differentiating this successively with respect to α and β it follows that $\left(\frac{A_2'}{A_1'}\right)' \left(\frac{B_2'}{B_3'}\right)' \equiv 0$. Without loss of generality (relabelling if necessary) we may assume $\left(\frac{B_2'}{B_3'}\right)' \equiv 0$. Then $\frac{B_1'}{B_3'} = \lambda_1$ and $\frac{B_2'}{B_3'} = \lambda_2$, where λ_1 and λ_2 are constants, and so

$$B_1(\beta) = \lambda_1 B_3(\beta) + \mu_1 \quad \text{and} \quad B_2(\beta) = \lambda_2 B_3(\beta) + \mu_2,$$

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where μ_1 and μ_2 are constants. However, the constant vector $\boldsymbol{\mu} \equiv (\mu_1, \mu_2, 0)$ can be absorbed into $\mathbf{A}(\alpha)$. The requirement $\mathbf{n}^2 = 1$ is now a quadratic equation for $B_3(\beta)$ in terms of certain coefficients that are functions of α only:

$$\boldsymbol{\lambda}^2 B_3^2(\beta) + (\mathbf{A}(\alpha)\cdot\boldsymbol{\lambda})\, B_3(\beta) + (\mathbf{A}^2(\alpha) - 1) = 0,$$

where we have written $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, 1)$. Since by assumption $B_3'(\beta) \not\equiv 0$, we must have $\boldsymbol{\lambda}^2 = 0$, $\mathbf{A}(\alpha)\cdot\boldsymbol{\lambda} = 0$ and $\mathbf{A}^2(\alpha) = 1$. Thus $\mathbf{B} = B_3(\beta)\boldsymbol{\lambda}$ is a null vector in a fixed direction orthogonal to $\mathbf{A}(\alpha)$. Writing $\lambda_1 = i\cos\theta$, $\lambda_2 = i\sin\theta$, for some scalar constant θ, we define $\mathbf{u} = (\sin\theta, -\cos\theta, 0)$ and $\mathbf{p} = (0, 0, 1)$. The set of vectors $\{\boldsymbol{\lambda}, \mathbf{u}, \mathbf{p}\}$ is linearly independent, so we may expand $\mathbf{A}$ in terms of these vectors; with the above restrictions on $\mathbf{A}$ we find that $\mathbf{A} = \nu(\alpha)\boldsymbol{\lambda} \pm \mathbf{u}$ for some function $\nu(\alpha)$. Thus $\mathbf{a}_\alpha = -r\nu'(\alpha)\boldsymbol{\lambda}$ and $\mathbf{a}_\beta = -sB_3'(\beta)\boldsymbol{\lambda}$ are parallel null vectors. This means that $\mathbf{n} = \mathbf{0}$, a contradiction.

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