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NASH-WILLIAMS' THEOREM ON DECOMPOSING GRAPHS INTO FORESTS

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Abstract. We give a simple graph-theoretic proof of a classical result due to Nash-Williams on covering graphs by forests. Moreover, we derive a slight generalization of this statement where some edges are preassigned to distinct forests.

§1. *Introduction.* All graphs considered in this paper are finite. The sets of vertices and edges of a graph $\mathcal{G}$ are denoted by $\mathsf{V}(\mathcal{G})$ and $\mathsf{E}(\mathcal{G})$, respectively. The *restriction* of a graph $\mathcal{G} = (V, E)$ to a subset X of V, i.e. the graph on X whose edges are precisely those members of E both of whose ends belong to X, is indicated by $\mathcal{G}|X$. We write $\|X\|$ for the number $|\mathsf{E}(\mathcal{G}|X)|$ of edges of that graph. For any integer $r \geqslant 0$ we set $[r] = \{1, 2, \dots, r\}$.

The following result is due to Nash-Williams (see [3], and the related articles [2] and [4], as well as [1] for another simple proof).

THEOREM 1 [3]. *If $\mathcal{G} = (V, E)$ is a graph, and $r \geqslant 0$ is an integer such that for all non-empty subsets X of V one has $\|X\| \leqslant r(|X| - 1)$, then there exists a partition $E = E_1 \cup E_2 \cup \cdots \cup E_r$ such that (V, E_i) is a forest for $i \in [r]$.*

It is plain that the sufficient condition for such a partition to exist given here is also necessary. Also, the cases $r = 0$ and $r = 1$ of this statement are immediate, and the case $r = 2$ was recently posed at the All-Russian Mathematical Olympiad, [5]. That case can be dealt with by some peculiar tricks not discussed here, that do not straightforwardly generalize to $r > 2$, but which nevertheless motivated us to reprove the general case independently. In fact, it turned out that our arguments for the case $r = 2$ yielded slightly more, namely, that for any two distinct edges of $\mathcal{G}$ there exists such a partition in which one of the edges belongs to E_1 while the other one belongs to E_2. Hence one might guess the following corollary.

COROLLARY 2. *Given a graph $\mathcal{G} = (V, E)$, an integer $r \geqslant 0$ such that for all non-empty subsets X of V one has $\|X\| \leqslant r(|X| - 1)$, and moreover a sequence $e_1, e_2, \dots, e_r$ of distinct edges of $\mathcal{G}$, there exists a partition $E = E_1 \cup E_2 \cup \cdots \cup E_r$ such that $e_i \in E_i$ for $i \in [r]$ and (V, E_i) is a forest for all $i \in [r]$.*

As we shall see in §3, this can in fact be derived from Theorem 1.

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All statements and arguments contained in this paper are valid irrespective of whether multiple edges are allowed to occur in our graphs or not.

§2. *Proving Theorem 1.* In this section we give a simple proof of Theorem 1 that is, to the best of our knowledge, new. For this purpose we need some preparation. Let $\mathcal{T}$ be a forest, $k \geqslant 2$ an integer and $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ mutually vertex-disjoint, connected subgraphs of $\mathcal{T}$. Obviously, $\mathcal{T}_1, \mathcal{T}_2, \ldots \mathcal{T}_k$ are trees and each of them is contained in exactly one component of $\mathcal{T}$. We call $\mathcal{T}_i$ *isolated* in $\mathcal{T}$ if there is no $\mathcal{T}_j$ with $j \neq i$ that is contained in the same component of $\mathcal{T}$ as $\mathcal{T}_i$. Furthermore, we call $\mathcal{T}_i$ *peculiar* in $\mathcal{T}$ if there is an edge $e_i \in E(\mathcal{T}) - (E(\mathcal{T}_1) \cup \cdots \cup E(\mathcal{T}_k))$ incident with a vertex of $\mathcal{T}_i$, such that $\mathcal{T}_i$ is isolated in $\mathcal{T} - e_i$ (the reader may note that one gets an equivalent notion without demanding e_i to be incident with a vertex of $\mathcal{T}_i$).

FACT 3. *At least two of the subgraphs $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ are isolated or peculiar in $\mathcal{T}$.*

Proof. Otherwise take a counterexample where $\mathcal{T}$ has as few vertices as possible. If no component of $\mathcal{T}$ contains two or more of the $\mathcal{T}_i$, then each them is isolated and we are done. In the remaining cases $\mathcal{T}$ is a tree. Consider any leaf x of $\mathcal{T}$. As we cannot delete x thus getting a smaller counterexample, there has to be some $\mathcal{T}_i$ consisting solely of x, and the edge of $\mathcal{T}$ incident with x witnesses that this $\mathcal{T}_i$ is peculiar. Because of $k \geqslant 2$ the tree $\mathcal{T}$ has at least two vertices and consequently at least two leaves. Applying the foregoing argument to any two leaves of $\mathcal{T}$ we see that at least two of the trees $\mathcal{T}_1, \mathcal{T}_2, \ldots,$ and $\mathcal{T}_k$ are peculiar. □

Proof of Theorem 1. Arguing indirectly we choose a graph $\mathcal{G} = (V, E)$ and an integer $r \geqslant 0$ contradicting Theorem 1 with $|E|$ minimal. Then $|E| \neq 0$. Let $e = ab$ be an arbitrary edge of $\mathcal{G}$.

Because of the choice of $\mathcal{G}$ there is at least one partition $E - \{e\} = E_1' \cup E_2' \cup \cdots \cup E_r'$ such that (V, E_i') is a forest for $i = 1, 2, \ldots, r$. For each of these partitions we consider the component (C, E_C) of (V, E_1') containing the vertex a. Let $E - \{e\} = E_1' \cup E_2' \cup \cdots \cup E_r'$ be one of these partitions with $|C|$ minimum. Let $\overline{C} = \mathsf{E}(\mathcal{G}|C)$.

If $b \notin C$ the partition $E = (E_1' \cup \{e\}) \cup E_2' \cup \cdots \cup E_r'$ would satisfy all conditions of Theorem 1, consequently $b \in C$ and $e \in \overline{C}$. Therefore,

$$|E_1' \cap \overline{C}| + \cdots + |E_r' \cap \overline{C}| < |\overline{C}| = \|C\| \leqslant r(|C| - 1).$$

Thus, there is an $i \in [r]$ with $|E_i' \cap \overline{C}| < |C| - 1$. This implies that $(C, E_i' \cap \overline{C})$ is not connected. Because of the definition of (C, E_C) we have $i \neq 1$, so we can without loss of generality (w.l.o.g.) assume $i = 2$.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ be the connected components of $(C, E_2' \cap \overline{C})$, where obviously $k \geqslant 2$. Thus, $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are mutually vertex-disjoint, connected subgraphs of the forest (V, E_2'). We define isolation and peculiarity in (V, E_2') as applying to these subgraphs. By Fact 3 at least two of the subgraphs $\mathcal{D}_1, \ldots, \mathcal{D}_k$

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are isolated or peculiar in (V, E_2'), and at most one of them contains a. Let w.l.o.g. $\mathcal{D}_1$ be isolated or peculiar in (V, E_2') and $a \notin \mathsf{V}(\mathcal{D}_1)$.

If $\mathcal{D}_1$ is peculiar in (V, E_2'), there is an edge

$$e_1 \in E_2' - (\mathsf{E}(\mathcal{D}_1) \cup \cdots \cup \mathsf{E}(\mathcal{D}_k))$$

incident with a vertex v_1 of $\mathcal{D}_1$, such that $\mathcal{D}_1$ is isolated in $(V, E_2' - \{e_1\})$. Note that

$$\mathsf{E}(\mathcal{D}_1) \cup \cdots \cup \mathsf{E}(\mathcal{D}_k) = \overline{C} \cap E_2'$$

yields $e_1 \notin \overline{C}$, wherefore e_1 connects v_1 and a vertex not in C.

If $\mathcal{D}_1$ is isolated in (V, E_2'), let v_1 be an arbitrary vertex of $\mathcal{D}_1$.

We consider the uniquely determined path from a to v_1 in the tree (C, E_C). Since $a \notin \mathsf{V}(\mathcal{D}_1)$ and $v_1 \in \mathsf{V}(\mathcal{D}_1)$, this path contains an edge e_d connecting a vertex of $\mathcal{D}_1$ with a vertex of some D_i with $i \neq 1$.

First, we consider the case when $\mathcal{D}_1$ is isolated in (V, E_2'). Then the graph $(V, E_2' \cup \{e_d\})$ is a forest, because e_d connects different components of (V, E_2'). Obviously, the graph $(V, E_1' - \{e_d\})$ is also a forest and its component including a is a subgraph of (C, E_C). But this subgraph does not contain v_1 and has therefore a number of vertices smaller than $|C|$. This contradicts

$$E - \{e\} = (E_1' - \{e_d\}) \cup (E_2' \cup \{e_d\}) \cup \cdots \cup E_r'$$

being one of the partitions considered at the beginning.

For the second case let $\mathcal{D}_1$ now be peculiar in (V, E_2'). Then $\mathcal{D}_1$ is isolated in $(V, E_2' - \{e_1\})$ and the graph $(V, (E_2' - \{e_1\}) \cup \{e_d\})$ is therefore a forest. On the other hand, the graph $(V, E_1' - \{e_d\})$ is also a forest and a subgraph of the forest (V, E_1'). Because e_d belongs to the component (C, E_C) of (V, E_1'), the graph $(V, E_1' - \{e_d\})$ has two components being subgraphs of (C, E_C), one of which contains a and the other one v_1. The edge e_1 connects v_1 and a vertex not in C, consequently the graph $(V, (E_1' - \{e_d\}) \cup \{e_1\})$ is a forest and its component including a is equal to the component of $(V, E_1' - \{e_d\})$ including a. Thus, its number of vertices is smaller than $|C|$. This contradicts

$$E - \{e\} = ((E_1' - \{e_d\}) \cup \{e_1\}) \cup ((E_2' - \{e_1\}) \cup \{e_d\}) \cup \cdots \cup E_r'$$

being one of the partitions considered at the beginning.

Since we have obtained a contradiction in each of the two cases, our assumption must have been wrong and Theorem 1 is true. □

§3. *Deducing Corollary 2.* The strengthening given by Corollary 2 will now be deduced from Theorem 1 by means of a short argument.

Proof of Corollary 2. Let $\mathcal{G} = (V, E)$, $r \geqslant 0$ and $e_1, \ldots, e_r$ be as in Corollary 2. We call an integer $0 \leqslant r' \leqslant r$ *restrained* if there is a partition $E = E_1 \cup E_2 \cup \cdots \cup E_r$ such that $e_i \in E_i$ for $i \in [r']$ and (V, E_i) is acyclic for all $i \in [r]$. By Theorem 1 the integer 0 is restrained. Corollary 2 is equivalent to r being restrained. It is therefore sufficient to prove the following: if an integer k with $0 \leqslant k \leqslant r - 1$ is restrained, then the integer $k + 1$ is also restrained.

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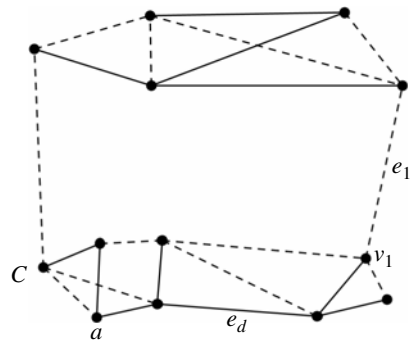


Figure 1: $\mathcal{D}_1$ is peculiar.

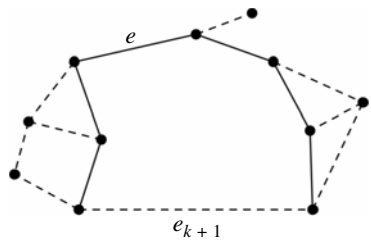


Figure 2: Switching e and e_{k+1}.

Let $E = E_1 \cup E_2 \cup \cdots \cup E_r$ be a partition such that $e_i \in E_i$ for $i \in [k]$ and (V, E_i) is acyclic for all $i \in [r]$. If $e_{k+1} \in E_{k+1}$, we are done. We can therefore assume $e_{k+1} \in E_l$ with $l \neq k+1$. Obviously, we can assume that the two vertices of e_{k+1} belong to the same component of (V, E_{k+1}).

The two vertices of e_{k+1} belong to different components of the forest $(V, E_l - \{e_{k+1}\})$. Therefore, the uniquely determined path between these two vertices in the forest (V, E_{k+1}) contains vertices of different components of $(V, E_l - \{e_{k+1}\})$. Hence, there is an edge $e \in E_{k+1}$ of this path connecting vertices of different components of $(V, E_l - \{e_{k+1}\})$. Then the graph $(V, (E_l - \{e_{k+1}\}) \cup \{e\})$ is a forest. On the other hand, the graph $(V, (E_{k+1} - \{e\}) \cup \{e_{k+1}\})$ is by the definition of e also a forest.

Thus, the partition gained from $E = E_1 \cup E_2 \cup \cdots \cup E_r$ by substituting E_l by $(E_l - \{e_{k+1}\}) \cup \{e\}$ and E_{k+1} by $(E_{k+1} - \{e\}) \cup \{e_{k+1}\}$ fulfills all conditions for $k+1$ being restrained. □

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