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CONDUCTIVITY OF RANDOM TREES

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We prove that the effective resistances of spherically symmetric random trees dominate in mean the effective resistances of random trees corresponding branching processes in varying environments and having the same growth law of spherically symmetric trees. We conjecture that the statement does not necessarily hold true in the case of stochastic domination and give an idea of constructing a counterexample.

1. INTRODUCTION

Consider a finite graph (electrical network) G and apply a voltage v between two vertices (nodes) a and b such that the voltage at a is v_a and at b is $v_b = 0$. The effective resistance, R_{eff}, of G is defined as

$$R_{\text{eff}} = \frac{v_a}{i_a},$$

where i_a is the current flowing into the circuit at node a. A flow j on G from a to b is a function defined on the set E of edges of G such that if $xy \in E$, then the following hold:

1. $j_{xy} = -j_{yx}$.
2. $\sum_y j_{xy} = 0$ if $x \neq a, b$.
3. $j_{kl} = 0$ if $kl \notin E$.

The energy dissipation of a flow j is defined to be

$$\frac{1}{2} \sum_{x,\, y} j_{xy}^2 r_{xy},$$

where r_{xy} is the resistance assigned to the edge xy.

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The principle of conservation of energy [1, p. 61] states the following. If w is any function defined on the vertices of a graph G and j is a flow from a vertex a to a vertex b, then

$$(w_a - w_b)j_a = \frac{1}{2}\sum_{x,y}(w_x - w_y)j_{xy}.$$

If a voltage v_a is imposed between a and b with $v_b = 0$, we obtain a voltage v_x and currents i_{xy}. The current i gives a flow from a to b, and by the above principle, we conclude that

$$v_a i_a = \frac{1}{2}\sum_{x,y}(v_x - v_y)i_{xy}$$

and from Ohm's law, we get

$$v_a i_a = \frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy}.$$

Since $R_{\text{eff}} = v_a/i_a$,

$$i_a^2 R_{\text{eff}} = \frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy}.$$

If a unit resistance is assigned to every edge $xy \in E$ and a voltage is applied between a and b such that $i_a = 1$, then

$$R_{\text{eff}} = \frac{1}{2}\sum_{x,y} i_{xy}^2;$$

that is, the effective resistance is the energy dissipation of the unit current flow when unit resistances are assigned to the edges.

We now consider an infinite tree T and we assign, from now on, a unit resistance to every edge of T. Let T_n denote the portion of T from its root r to the vertices having distance n from the root r. The effective resistance of the infinite tree T is defined to be the limit, as n tends to infinity, of the effective resistance of T_n. This limit does exist (probably infinity) because R_{eff} of a finite tree T_n does not decrease as n gets larger. The energy dissipation of a flow j on infinite trees is defined similarly.

The following two laws are useful in determining an upper and lower bound for the effective resistances.

Thomson's Principle [1, p. 63]: If i is the unit current flow between two vertices a and b, then the energy dissipation $\frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy}$ minimizes the energy dissipation $\frac{1}{2}\sum_{x,y} j_{xy}^2 r_{xy}$ among all unit flows from a to b.

One way to modify the network whose resistance we are interested in so as to get a simpler network is by making shorting. Shorting involves connecting a given set of nodes together with perfectly conducting wires so that the current can pass freely between them. All nodes that were shorted together behave as if they were a

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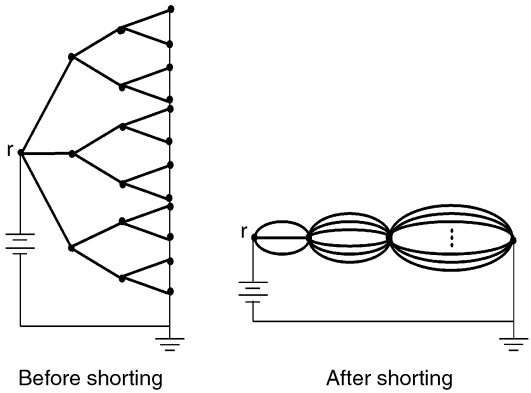


FIGURE 1. Shorting levels of a tree.

single node (see Fig. 1). The usefulness of the shorting procedure stems from the following principle.

Shorting Principle [1, p. 100]: Shorting certain sets of nodes together can only decrease the effective resistance of the network between two given nodes.

The degree of a vertex x is the number of edges meeting at x. The level n of the tree is the set of all vertices having distance n from the root. A tree $\breve{T}$ is called spherically symmetric if the degree of any vertex depends only on its distance from the root. For such a tree, let d_n denote the degree of each vertex of the level n. Assume that d_n is a random variable having distribution F_n. Let $\{d_{nk}: n = 0,1,2,\ldots; k = 1,2,\ldots\}$ be a doubly-indexed family of independent random variables and, for fixed n, they are identically distributed according to the distribution F_n. If d_{nk} denotes the degree of the kth vertex of the nth level, then we have a tree T^* corresponding a branching process in varying environments. We call a vertex v a branching vertex if its degree is greater than 2. It is useful to note that the unit current flow in a spherically symmetric tree splits equally at every branching vertex and shorting together all nodes of each level does not affect either the current flow or the effective resistance. Thus, for $\breve{T}$, the energy dissipation is

$$\sum_{n=1}^{\infty} \frac{1}{Z_n}$$

where Z_n is the size of the nth level. See [2].

All infinite trees we consider are assumed to be leafless; that is, the degree of any vertex is greater than 1.

2. THE CONDUCTIVITIES OF $\breve{T}$ AND T^*

Let $\breve{T}$ and T^* denote, as mentioned earlier, the infinite random spherically symmetric tree and the infinite tree, respectively, corresponding to a branching process in vary-

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ing environments that grows following the growth distribution of $\breve{T}$. We assign a unit resistance to each edge of $\breve{T}$ and T^*. We show in the following theorem that the effective resistance of $\breve{T}$ dominates in mean that of T^*.

THEOREM 1: *For $\breve{T}$ and T^*, we have*

$$E(R^*_{\text{eff}}) \leq E(\breve{R}_{\text{eff}}),$$

where R^*_{eff} and $\breve{R}_{\text{eff}}$ denote the effective resistances of T^* and $\breve{T}$, respectively.

PROOF: Let j be the unit flow, applied at the root r of T^*, that splits equally at every branching vertex and D stands for its energy dissipation. We assume that there are Z vertices and each is connected to the root r by one edge (Fig. 2). At each of these vertices, we apply a unit flow divided equally at each branching vertex. Let D_i stand for the energy dissipation of that flow applied at the vertex i. We can easily verify that

$$D = \frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i.$$

It follows from Thomson's Principle that

$$R^*_{\text{eff}} \leq \frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i$$

and, hence,

$$\begin{aligned} E(R^*_{\text{eff}}) &\leq E\left(\frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i\right) \\ &= E\left(E\left(\frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i | Z\right)\right) \\ &= E\left(\frac{1}{Z} + \frac{1}{Z}ED_1\right). \end{aligned}$$

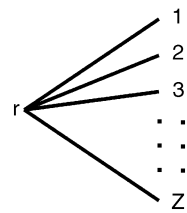


FIGURE 2. T^*.

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The last equation follows because D_i, $i = 1,2,\ldots,Z$, are identically distributed random variables. The quantity $(1/Z) + (1/Z)D_1$ is the energy dissipation of the unit current flow when it is applied at the root of $\breve{T}$, which is, in turn, the effective resistance of $\breve{T}$ and the theorem is proved. ■

Consider the two finite trees $\breve{S}_n$ and S_n^*. Suppose that each tree begins with two edges coming from the root and then each continues with one more edge up to level $n+1$ (see Figs. 3 and 4). Now, let each of the two edges coming from the root of $\breve{S}_n$ have a unit resistance and each of the two following edges have a random resistance R_1 and each of the two subsequent edges have a random resistance R_2, and so forth. See Figure 3. On S_n^*, let each of the two edges coming from the root have a unit resistance and the two following edges have random resistances R_{11} and R_{12} which are independent copies of R_1 and the two subsequent edges have resistances R_{21} and R_{22} which are independent copies of R_2, and so forth. See Figure 4.

We assume that the random variables R_i and R_{ij}, where $i = 1,2,..,n$ and $j = 1,2$, are independent random variables, and for fixed i, R_i, R_{i1}, and R_{i2} have the same distribution.

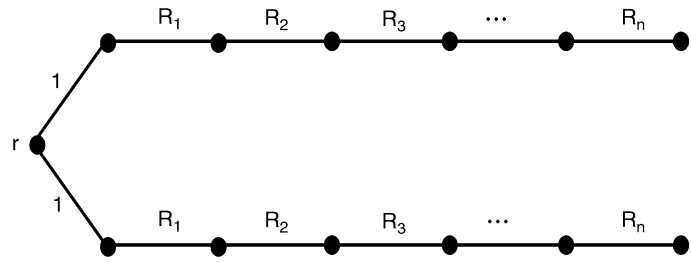


FIGURE 3. $\breve{S}_n$.

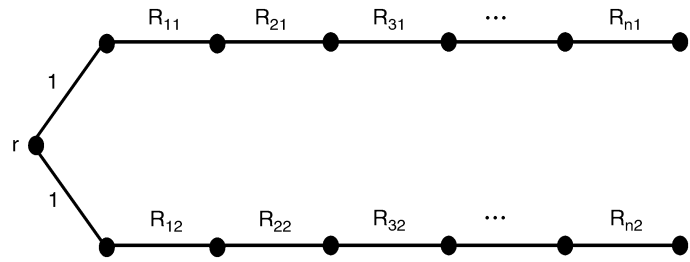


FIGURE 4. S_n^*.

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Consider R_i to have the following distribution:

$$R_i := \begin{cases} k & \text{with probability } 1 - p_i \\ r_i & \text{with probability } p_i, \end{cases}$$

where $r_{i+1} \leq r_i$ and $r_i < k$ for some constant $k \leq 1$.

Let $R_{\text{eff}}(\check{S}_n)$ and $R_{\text{eff}}(S_n^*)$ be the effective resistances of $\check{S}_n$ and S_n^*, respectively, and let $R_{\text{eff}}(\check{S})$ and $R_{\text{eff}}(S^*)$ stand for their limits as $n \to \infty$.

In what follows, we give an idea of an example that shows that the stochastic domination may not hold between R_{eff}^* and $\check{R}_{\text{eff}}$.

LEMMA 2: *If the sequences* $\{r_i\}$ *and* $\{p_i\}$ *are chosen such that* $0 < p_i < 1$, $0 < r_i < 1$, $\sum(1 - p_i) < \infty$, *and* $\sum r_i < \infty$, *then*

$$E(R_{\text{eff}}(S^*)) \leq E(R_{\text{eff}}(\check{S})) < \infty.$$

PROOF: We first show that $E(R_{\text{eff}}(\check{S})) < \infty$. Since $\check{S}_n$ is symmetric and the resistances are assigned symmetrically to its edges, shorting together nodes of the same level will not affect the flow of the current and, hence, has no effect on the effective resistance of $\check{S}_n$. From this, the effective resistance $R_{\text{eff}}(\check{S})$ of $\check{S}$ is

$$R_{\text{eff}}(\check{S}) = \frac{1}{2}\left[1 + \sum_{i=1}^{\infty} R_i\right]$$

and, hence,

$$\begin{aligned} E(R_{\text{eff}}(\check{S})) &= \frac{1}{2}\left[1 + \sum_{i=1}^{\infty} E(R_i)\right] \\ &\leq \frac{1}{2} + \frac{1}{2}\sum_i [1 - p_i + r_i p_i] \\ &= \frac{1}{2} + \frac{1}{2}\sum_i [1 - (1 - r_i)p_i]. \end{aligned}$$

Consequently, $E(R_{\text{eff}}(\check{S})) < \infty$ if and only if $\prod(1 - \mathrm{r_i})\mathrm{p_i}$ converges to a nonzero value if and only if $\sum r_i < \infty$ and $\sum(1 - p_i) < \infty$. Thus, $R_{\text{eff}}(\check{S}) < \infty$ a.s.

It remains to show that $E(R_{\text{eff}}(S^*)) \leq E(R_{\text{eff}}(\check{S}))$. The mean energy dissipation of the unit flow that divides equally at the root of S^* is

$$\begin{aligned} E\left[\frac{1}{2}\sum_{x,y} j_{xy}^2 R_{xy}\right] &= \frac{1}{8}\left(2 + 2E(R_{11}) + 2E(R_{12}) + \cdots\right) \\ &= \frac{1}{8}\left(2 + 2E(R_1) + 2E(R_2) + \cdots\right). \end{aligned}$$

The last quantity is the mean effective resistance of $\check{S}$. Applying Thomson's Principle assures that this quantity is greater than the mean effective resistance of S^*. Hence, we are done. ■

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The following is another lemma that compares the effective resistances of $\breve{S}_n$ and S_n^*.

LEMMA 3:

$$p(R_{\text{eff}}(S_n^*) > R_{\text{eff}}(\breve{S}_n)) > 0.$$

PROOF:

$$\begin{aligned}
&p(R_{\text{eff}}(S_n^*) > R_{\text{eff}}(\breve{S}_n)) \\
&\quad \geq p\left(R(\breve{S}_n) = \frac{1}{2}(1 + r_1 + \cdots + r_n), R_{11} = k, R_{12} = k\right) \\
&\quad = p\left(R(\breve{S}_n) = \frac{1}{2}(1 + r_1 + \cdots + r_n)\right)(p(R_{11} = k))^2 \\
&\quad \geq p(R_1 = r_1, \ldots, R_n = r_n)(1 - p_1)^2 \\
&\quad = \left(\prod_{i=1}^{n} P_i\right)(1 - p_1)^2 > 0.
\end{aligned}$$

■

Subject to the conditions of Lemma 2 on the two sequences $\{r_i\}$ and $\{p_i\}$, we prove the following proposition.

PROPOSITION 4:

$$p(R_{\text{eff}}(S^*) > R_{\text{eff}}(\breve{S})) > 0.$$

PROOF:

$$\begin{aligned}
&p(R_{\text{eff}}(S^*) > R_{\text{eff}}(\breve{S})) \\
&\quad \geq p(R_{\text{eff}}(S_n^*) > R_{\text{eff}}(\breve{S}_n), R_{n+1,1} \geq R_{n+1}, R_{n+1,2} \geq R_{n+1}, \cdots) \\
&\quad = p(R_{\text{eff}}(S_n^*) > R_{\text{eff}}(\breve{S}_n)) \prod_{m=n+1}^{\infty} [p(R_{m,1} \geq R_m)]^2 \\
&\quad \geq \prod_{i=1}^{n} p_i(1 - p_1)^2 \prod_{m=n+1}^{\infty} [1 - p(R_{m,1} < R_m)]^2 \\
&\quad = \prod_{i=1}^{n} p_i(1 - p_1)^2 \prod_{m=n+1}^{\infty} [1 - p_m(1 - p_m)]^2 \\
&\quad \geq \prod_{i=1}^{n} p_i(1 - p_1)^2 \prod_{m=n+1}^{\infty} [1 - (1 - p_m)]^2 \\
&\quad \geq (1 - p_1)^2 \prod_{i=1}^{\infty} p_i^2.
\end{aligned}$$

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Since $\sum_i (1 - p_i) < \infty$, $\prod_i p_i$ converges to a nonzero limit and so does $\prod p_i^2$. Thus,

$$p(R_{\text{eff}}(S^*) > R_{\text{eff}}(\check{S})) > 0.$$

■

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