Forum of Mathematics, Sigma (2026), **Vol. 14**:e59 1–41
doi:10.1017/fms.2026.10191

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Tropicalization of ψ classes

Andreas Gross[1] and Renzo Cavalieri[2]

[1]Department of Mathematics, Goethe-Universität Frankfurt, Germany; E-mail: gross@math.uni-frankfurt.de.
[2]Department of Mathematics, Colorado State University, Fort Collins CO, USA;
E-mail: renzo@colostate.edu (Corresponding author).

Received: 5 May 2025; **Revised:** 21 November 2025; **Accepted:** 21 November 2025

2020 Mathematics Subject Classification: *Primary* – 14T20; *Secondary* – 14A21

Abstract
Under suitable conditions on a family of logarithmic curves, we endow the tropicalization of the family with an affine structure in a neighborhood of the sections in such a way that the tropical ψ classes from [6] arise as tropicalizations of algebraic ψ classes.

1. Introduction

The main goal of this paper is to address the following question, posed by the authors and Hannah Markwig in [6]:

Are tropical ψ classes the tropicalization of algebraic ψ classes?

The answer provided in this work makes a rigorous connection between the combinatorial theory of [6] and algebraic geometry via a new notion of tropicalization, and explores the reach that tropical geometry has in retaining algebraic information. In Section 1.1 we summarize the results and give a streamlined account of the *story* this work is telling. In Section 1.2 we provide context, motivation, and a discussion of the ideas informing our constructions. We allowed for significant overlap between the sections: readers may choose the order in which to read them based on their level of familiarity with the field.

1.1. Results

The first step in the journey is to define a notion of tropicalization that takes value in the category of tropical spaces, that is, spaces that are locally modeled on abstract rational polyhedral complexes, and in addition are endowed with a sheaf of affine linear functions.

Definition A (Tropicalization). Let X be a toroidal embedding with no self-intersections. The **tropicalization** of X in the category of tropical spaces is obtained by endowing the boundary (extended) cone complex $\overline{\Sigma}_X$ with the sheaf of affine functions from Definition 3.1.

Informally, if $\sigma \in \Sigma_X$ corresponds to a (closed) stratum $V(\sigma) \subseteq X$, a piecewise linear function ϕ defined on a neighborhood Σ_X^σ of σ is declared affine when the corresponding line bundle (defined on the open set X_σ obtained by removing all boundary divisors that do not meet $V(\sigma)$) trivializes on $V(\sigma)$:

$$\mathcal{O}_{X_\sigma}(\phi)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}. \tag{1.1}$$

© The Author(s), 2026. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

When σ/τ is a cone at infinity, one makes the additional requirement that ϕ is constant on τ, meaning that the support of the divisor associated to ϕ does not contain any boundary divisor containing $V(\tau)$. This notion of tropicalization is functorial, and it is invariant under log modifications of the toroidal variety X.

Theorem B (Proposition 3.5, Proposition 3.8). *A morphism of toroidal varieties $f : X \to Y$ induces a morphism of tropical spaces:*

$$F := \mathtt{Trop}(f) \colon |\Sigma_X| \to |\Sigma_Y|. \tag{1.2}$$

If f is a log modification, then F is an isomorphism.

Since the main application of this technology is to families of curves, an important check of the soundness of the definitions is that in genus zero this notion of tropicalization recovers the previous ones (via torus embedding [20] or cross ratios [6]).

Theorem C (Theorem 3.17). *The tropicalization of the toroidal variety $\overline{\mathcal{M}}_{0,n}$ is the tropical space $\mathcal{M}_{0,n}^{\mathtt{trop}}$.*

The situation is more subtle for curves of positive genus. A family $\pi : \mathcal{C} \to \mathcal{B}$ of logarithmic curves naturally produces a map of extended cone complexes $\Pi : \overline{\mathsf{C}} \to \overline{\mathsf{B}}$; we call the family **tropicalizable** if its tropicalization gives rise to the expected (set theoretic) moduli map. As soon as a family of curves is tropicalizable, it is *almost* a family of tropical curves in the sense of [6, Definition 3.10].

Theorem D (Proposition 3.21, Proposition 3.22). *Given $\pi : \mathcal{C} \to \mathcal{B}$ a tropicalizable family of stable, n-marked logarithmic curves, we obtain a map of tropical spaces $\Pi : \overline{\mathsf{C}} \to \overline{\mathsf{B}}$ such that:*

- *if x is a genus-0 point of $\overline{\mathsf{C}}$, we have the exact sequence*

$$0 \to \mathrm{Aff}_{\overline{\mathsf{B}},\Pi(x)} \to \mathrm{Aff}_{\overline{\mathsf{C}},x} \to \Omega^1_{\overline{\mathsf{C}}_{\Pi(x)},x} \to 0. \tag{1.3}$$

- *if x is a rational vertex of $\overline{\mathsf{C}}$ or a point on an edge adjacent to a rational vertex, then, near x, the affine functions on its fiber consist of all harmonic functions.*

A necessary step towards tropicalizing ψ classes is to define the notion of tropicalization of line bundles. In the next result we discuss when such an object is a tropical line bundle.

Theorem E (Proposition 4.4, Proposition 4.6). *Denote by $\mathcal{L}$ the invertible sheaf of a line bundle $L \to X$; let $\mathtt{Trop}(L)$ be the tropicalization of the total space of L and $\mathtt{U} \subseteq \overline{\Sigma}_X$ an open subcomplex of the tropicalization of the base space.*

1. *$\mathtt{Trop}(L)$ is a tropical line bundle on $\mathtt{U}$ if and only if for every $\sigma/\tau \in U$ there exists a strictly piecewise linear function $\phi_{\sigma/\tau}$ on a neighborhood of τ that is constant on τ such that*

$$\big(\mathcal{O}_{X_\sigma}(\phi_{\sigma/\tau}) \otimes \mathcal{L}\big)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}. \tag{1.4}$$

2. *if $\mathcal{L} = \mathcal{O}_X(\phi)$, for ϕ a strict piecewise linear function on Σ_X, then condition (1.4) is equivalent to the existence of an affine function $\chi_{\sigma/\tau}$ on a neighborhood of σ such that*

$$\chi|_\tau = \phi|_\tau. \tag{1.5}$$

With this technology in place, we turn our attention to the i-th cotangent line bundle of a family of curves $\mathcal{C} \to \mathcal{B}$. The main result is that if the tropicalization $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$ has enough local affine functions near the i-th section at infinity, then the tropicalization of the cotangent line bundle agrees with the definition of the tropical cotangent line bundle from [6, Definition 6.16].

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Theorem F (Theorem 5.5, Corollary 5.11). *Let $\mathcal{C} \to \mathcal{B}$ be a family of n-marked stable curves with tropicalization $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$, and let ϕ_i be the strict piecewise linear function on C having slope one along the ray dual to the i-th section of the family, and 0 on all other rays. If $\mathrm{Aff}_{\overline{\mathsf{C}}}(\phi_i)$ is a tropical line bundle, then*

$$\mathtt{Trop}(\mathbb{L}_i) \cong \mathbb{L}_i^{\mathtt{trop}}. \tag{1.6}$$

By taking first Chern classes, we obtain

$$\mathtt{Trop}(\psi_i) = \psi_i^{\mathtt{trop}}. \tag{1.7}$$

In genus greater than 1, the ψ classes are *not* boundary classes, but supported on the interior $\mathcal{M}_{g,n}$. Tropical geometry usually only captures information about the boundary. So Theorem 5.5 can be paraphrased in elementary terms as follows: if we are able to sufficiently degenerate a family of curves such that it becomes tropicalizable, a condition which can be checked easily in applications, tropical geometry captures all numerical information about the ψ classes, even though they are not boundary.

We conclude the manuscript with an extended example. We compute the tropical ψ classes for two two-dimensional families of genus-one, tropical curves, obtained by stabilization from a space of tropical admissible covers. We show that the tropical ψ classes of [6] are the tropicalization of the algebraic ψ classes, and that they agree with the operational perspective of tropicalization from [17].

1.2. Context and commentary

Tropical geometry and moduli spaces of curves have enjoyed fruitful interactions ever since Mikhalkin's seminal paper [21] interpreted the space of phylogenetic trees of [2] as the tropicalization of $\mathcal{M}_{0,n}$. With Hannah Markwig, we gave our perspective and account on the evolution of the subject in the introduction to [6], to which we refer the interested reader. The one line (cheeky) summary is, when it comes to tropical geometry and moduli spaces of curves, anything you may wish comes true for rational curves, while it appears to crash and burn for positive genus. This article is part of a collection of works (including, but not limited to [1, 5, 8, 6]) arguing that tropical geometry can still provide a meaningful and powerful approach to the study of the geometry and intersection theory of moduli spaces of positive genus curves.

The notion of tropical ψ classes for families of tropical curves of arbitrary genus was introduced in [6]. The key step there was to define the notion of family of tropical curves by requiring an *affine structure*, that is, distinguished subsheaves of the sheaves of piecewise linear functions on both the base and the total space of the family satisfying some natural requirements: for example that the restriction of affine functions to fibers are harmonic (see Section 2.1 for a more complete review). Borrowing intuition from the algebraic identification of the ψ class with the negative self-intersection of the corresponding section, the tropical cotangent line bundle $\mathbb{L}_i$ of a family of tropical curves $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$ is defined to be the pullback via the i-th section of the tropical counterpart of the co-normal bundle to the section: the $\mathrm{Aff}_{\mathsf{C}}$ torsor whose local sections are affine functions on the finite part of $\overline{\mathsf{C}}$, going to infinity towards the section s_i with slope 1 along the fibers of the family.

The theory thus obtained is a combinatorial theory where the affine structures are canonically determined in certain cases: for example, for families of explicit tropical curves (that is when all vertices are rational); in this case we have natural correspondence statements with the corresponding algebraic families. In general, the theory of tropical ψ classes is strictly broader than the algebraic one: in [6, Example 6.17], the authors exhibited one-dimensional families of tropical curves where the degree of the tropical ψ class cannot agree with the degree of ψ on any corresponding algebraic family.

In order to make a meaningful connection with algebraic geometry, we define a notion of tropicalization which is less restrictive than torus embedded tropicalization, yet still contains information about affine structures. While ultimately we wish to apply our constructions to logarithmic families of log smooth curves, we choose to develop the main definitions in the simpler set-up and language of toroidal embeddings with no self-intersections (which we call toroidal varieties for simplicity): varieties with a

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

distinguished divisor called boundary, which are locally isomorphic to toric varieties. A toroidal variety is assigned, in a functorial way, a cone complex agreeing with the (cone over the topological) boundary complex, where each cone is additionally endowed with an integral lattice, see [18].

In order to define a natural notion of affine structure on the cone complex of a toroidal variety, we use the following intuition from toric varieties and tropicalization of subvarieties of tori: piecewise linear functions on the fan of a toric variety correspond to Cartier divisors, and therefore line bundles; the main role of the torus in these theories is that its characters are a class of functions that are invertible in the interior of the space, which can be thought of as sections of bundles that trivialize in the interior. We thus choose to define a piecewise linear function on (an open set of) the cone complex of a toroidal variety to be affine when the corresponding line bundle trivializes (on some collection of orbits depending on the chosen open set)[1]. The resulting affine structure makes the cone complex into a tropical space, which we call the **tropicalization** of the toroidal variety.

In the broader generality of logarithmic schemes with a divisorial log structure, a functorial assignment of a generalized cone complex is given in [25], where it is called logarithmic tropicalization; the fact that multiple faces of a single cone can be identified makes it a bit delicate to immediately define an affine structure on the logarithmic tropicalization. We circumvent this problem by observing that the sheaf of affine functions defined is invariant under logarithmic modifications (roughly speaking sequences of blow-ups and blow-downs of strata and their transforms); a generalized cone complex can be refined to an honest cone complex, which is the cone complex of an appropriate log modification of the original space. Hence one may define the affine structure on the logarithmic tropicalization via any suitable log modification making the space toroidal.

There are two classes of objects whose tropicalization we are especially interested in: families of curves and line bundles. We begin discussing the latter.

Besides special cases like toric varieties, Mumford curves, or totally degenerating Abelian varieties, there is no known procedure to tropicalize (algebraic) line bundles to tropical line bundles in the sense of [22, Definition 4.4](see Section 2.4). Of course, there is a naïve approach: a line bundle L over a toroidal variety X may be given a natural toroidal boundary, obtained as the union of the pull-back of the boundary on the base, plus the zero section. We define the (naïve) tropicalization of L essentially to be the tropicalization of its total space, see Remark 4.2. The result is a tropical space, but it may not have enough affine functions to actually be a tropical line bundle. We find that there is a concise condition for the tropicalization of L to be a tropical line bundle on an open neighborhood of a cone $\sigma \in \Sigma_X$: it amounts to the invertible sheaf $\mathcal{L}$ of L restricted to the orbit $V(\sigma)$ being the restriction of a sheaf of **boundary type**, i.e.,

$$\mathcal{L}|_{V(\sigma)} \cong \mathcal{O}_{X_\sigma}\Big(\sum a_\rho D_\rho\Big)\Big|_{V(\sigma)}, \tag{1.8}$$

where the D_ρ's are the boundary divisors in X_σ. For a cone at infinity σ/τ the situation is analogous: we need $\mathcal{L}|_{V(\sigma)}$ to be of the form $\mathcal{O}_{V(\tau)_\sigma}(\sum a_\rho(D_\rho \cap V(\tau)))|_{V(\sigma)}$, where the D_ρ's are boundary divisors which do not contain the orbit $V(\tau)$: this means that on $V(\sigma)$, the line bundle is the restriction of a line bundle of boundary type on the orbit $V(\tau)$. It follows that a line bundle of boundary type, which we associate to a piecewise linear function ϕ on Σ_X, tropicalizes to a tropical line bundle on the cone complex Σ_X, but not necessarily on the whole extended cone complex $\overline{\Sigma}_X$. We identify a combinatorial condition, which we call being **combinatorially principal**, that makes it a tropical line bundle on a cone at infinity of the form σ/τ: there must exist an affine function ϕ_τ which agrees with ϕ on the cone τ.

Turning our attention to families of curves, given a family $\pi\colon \mathcal{C} \to \mathcal{B}$, the tropicalization gives a map of tropical spaces $\Pi\colon \overline{\mathsf{C}} \to \overline{\mathsf{B}}$. A paraphrase of $\mathcal{C} \to \mathcal{B}$ being tropicalizable is that, once appropriately endowing the points of $\overline{\mathsf{C}}$ with a genus function, the fibers of Π are the dual graphs of the corresponding fibers of π. This is some sort of minimal requirement, that due to automorphisms and monodromy issues

[1]The idea of connecting functions on the tropicalization with Picard theory of the variety also appears in [4, 12].

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

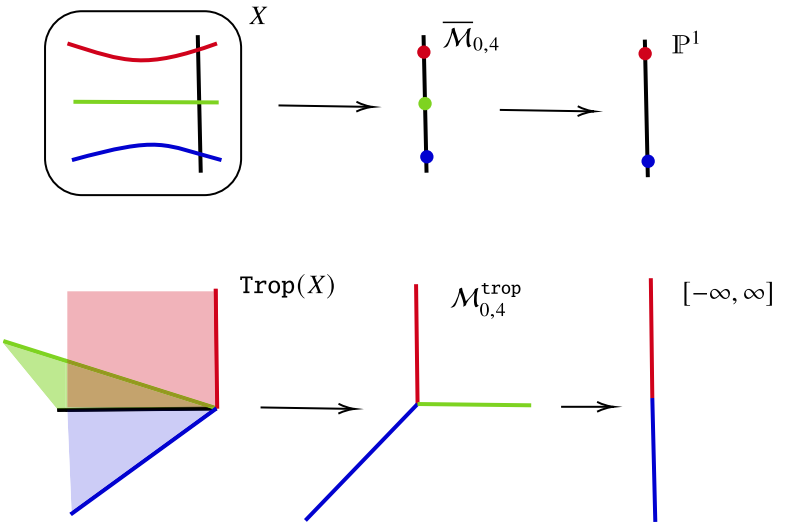


Figure 1. *An illustration that tropical cross ratios are the tropicalization of algebraic cross ratios. The top of the figure shows a cross ratio as factoring through a morphism to $\overline{\mathcal{M}}_{0,4}$, which is then identified with $\mathbb{P}^1$ (the difference between the two spaces is the boundary structure). Tropical cross ratios are similarly obtained in the row below. Functoriality of tropicalization shows the bottom part of the figure to be the tropicalization of the top.*

may not always be guaranteed: in Example 3.20 we see an instance of what may go wrong and how it can be remedied.

By functoriality, tropical cross ratios are tropicalizations of algebraic cross ratios, see Figure 1. This fact guarantees that this notion of tropicalization agrees with the *classical* notion (coming from embedding into a torus) for families of rational pointed curves and gives yet another equivalent way to get to $\mathcal{M}_{0,n}^{\mathrm{trop}}$. The tropicalization $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$ is in general not a family of tropical curves in the sense of [6, Definition 3.10]: one may not have enough (germs of) affine functions on the fibers, but this failure is restricted to points of positive genus, or points on edges that are not adjacent to any rational vertex. In particular, we recover the result from [6, Proposition 4.24] that $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$ is a family of tropical curves when the image of B via the tropical moduli map lies in the *good* locus $\mathcal{V}_{g,n}^{\mathrm{good}}$ ([6, Definition 4.22]).

For a family of marked tropical curves $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$, the tropical cotangent line bundle to the i-th section is defined as

$$\mathbb{L}_i^{\mathrm{trop}} := \mathsf{s}_i^*(\mathrm{Aff}_{\overline{\mathsf{C}}}(-\mathsf{s}_i)), \tag{1.9}$$

see [6, Definition 6.16]. In this paper, we define $\mathrm{Aff}_{\overline{\mathsf{C}}}(-\mathsf{s}_i)$ when $\overline{\mathsf{C}}$ is a tropical space, and observe that in general it is a pseudo $\mathrm{Aff}_{\overline{\mathsf{C}}}$ torsor: its local sections either give a torsor or are empty. An important observation from [6, Propositions 3.24, 3.25] is that $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$ being a family of tropical curves ensures that $\mathrm{Aff}_{\overline{\mathsf{C}}}(-\mathsf{s}_i)$ really is an $\mathrm{Aff}_{\overline{\mathsf{C}}}$-torsor. However, this condition is stronger than strictly necessary: for $\mathrm{Aff}_{\overline{\mathsf{C}}}(-\mathsf{s}_i)$ to be a tropical line bundle, the affine structure of $\overline{\mathsf{C}}$ only matters locally near s_i, so one only needs $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$ to satisfy the conditions on the affine structure of families of tropical curves in a neighborhood of $\mathsf{s}_i(\overline{\mathsf{B}})$. When this happens, it is then possible to check that the tropicalization of the cotangent line bundle is a tropical line bundle, and it agrees with $\mathsf{s}_i^*(\mathrm{Aff}_{\overline{\mathsf{C}}}(-\mathsf{s}_i))$.

Pulling all the loose strings together, one finally can answer the motivating question we began this section with.

Given a tropicalizable family of stable marked curves $\mathcal{C} \to \mathcal{B}$ with tropicalization $\overline{\mathsf{C}} \to \overline{\mathsf{B}}$, we have

$$\mathtt{Trop}(\psi_i) = \psi_i^{\mathrm{trop}} \tag{1.10}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

whenever $\mathrm{Aff}_{\overline{C}}(-\mathtt{s}_i)$ *is a tropical line bundle; further, this notion admits the following combinatorial characterization: the piecewise linear function* ϕ_{ρ_i} *on* $\overline{\mathtt{C}}$ *having slope one on the ray* ρ_i *dual to* s_i *and zero on all other rays is combinatorially principal.*

One might interpret the above conclusion as follows: tropical geometry does not see the algebraic information contained in ψ classes on all families of curves. In order to capture this information tropically, the family must degenerate *sufficiently* near the section. However, when tropical geometry does see ψ classes, it indeed sees the tropical ψ classes of [6].

1.3. Computations

We conclude this paper with an extended example of a computation of tropical ψ classes for two two-dimensional families of tropical curves of genus 1 with two marked points. The families are obtained from tropical admissible covers by forgetting some of the marked ends. In both cases we show that the tropical ψ class is the tropicalization of the algebraic ψ class, and exhibit it as a one-dimensional tropical cycle, that is a Minkowski weight on the base of the family. The tropical cycle obtained agrees with the operational tropicalization of the ψ class, that is with the Minkowski weight decorating each cone with the intersection number of ψ with the corresponding stratum. Besides illustrating a concrete instance of the theoretical story told, there are some interesting lessons one can observe from this extended example. In order to prove that the tropicalization of the families curves have enough affine functions to define the tropical ψ classes it is more efficient to rely on functoriality, rather than on the direct definition of tropicalization. Then one is essentially able to use the affine functions from the genus zero theory to obtain affine functions near the section. The second observation is that the combinatorics of tropical intersection theory is manageable, which is encouraging that this technology might prove a useful tool for the study of tautological rings of moduli spaces of curves. There has been recent interest in approaching the study of the logarithmic tautological ring of moduli spaces of curves [13, 23, 24] as the image of a ring homomorphism from the ring of piecewise polynomial functions on $\mathcal{M}_{g,n}^{\mathtt{trop}}$. The affine linear functions defined here generate an ideal that lies in the kernel of that ring homomorphism.

1.4. Notations and Conventions

We assume throughout working on an algebraically closed field of characteristic 0; we expect all constructions to go through for perfect fields, but care should be taken to what objects the constructions are applied to. In this paper, the tropical semiring is $\mathbb{T} = \{(-\infty, \infty], \min, +\}$; the choice of this convention is to make the affine structure near the section at infinity parallel the affine structure of the tropicalization of the normal bundle to the section near its "zero" section. We mostly use calligraphic fonts for algebraic objects, and typewriter ones for tropical objects. The superscript `trop` is used for certain tropical objects where the use of an identifying font was considered not viable, whereas the tropicalization functor is denoted by `Trop`.

2. Preliminaries

We recall the relevant definitions and results about tropical moduli spaces of curves, tropical cycles, and tropical line bundles in the generality introduced in [6]. We refer to that manuscript for a comprehensive list of references to the relevant preexisting literature.

2.1. Tropical curves and their moduli

In [6], moduli spaces of tropical curves are introduced as stacks over the category of tropical spaces. We review the relevant definitions.

A *TPL-space* is a pair (X, PL_X) consisting of a topological space X and a sheaf PL_X of continuous $\mathbb{T}$-valued functions on X that is locally isomorphic to (U, PL_U), the sheaf of continuous, integral,

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

piecewise linear functions of some open subset U of a rational polyhedral set P in $\mathbb{T}^n$. Here, two pairs $(Y, \mathcal{F}_Y)$ and $(Z, \mathcal{F}_Z)$ of topological spaces and sheaves of functions on them are isomorphic if there exists a homeomorphism $f\colon Y \to Z$ so that for a function ϕ on an open subset of Z we have that $\phi \circ f$ is a section of $\mathcal{F}_Y$ if and only if ϕ is a section of $\mathcal{F}_Z$. For a TPL-space X we denote by $\mathrm{PL}_X^{\mathrm{fin}}$ the subsheaf of $\mathbb{R}$-valued piecewise linear functions. A morphism of TPL-spaces is a continuous map that pulls piecewise linear functions (section of PL) back to piecewise linear functions.

A *tropical space* is a pair $(\mathtt{X}, \mathrm{Aff}_{\mathtt{X}})$ consisting of a TPL-space $\mathtt{X}$ and a subsheaf $\mathrm{Aff}_{\mathtt{X}} \subseteq \mathrm{PL}_{\mathtt{X}}^{\mathrm{fin}}$, the sheaf of *affine functions*, that contains the constant sheaf $\mathbb{R}_{\mathtt{X}}$. A morphism of tropical spaces is morphism of TPL-spaces that pulls affine functions back to affine functions. In that case, we also say that the underlying morphism of TPL-spaces is *linear*. The *tropical cotangent bundle* is the quotient sheaf $\Omega^1_{\mathtt{X}} := \mathrm{Aff}_{\mathtt{X}}/\mathbb{R}_{\mathtt{X}}$.

A *tropical curve* is a one-dimensional tropical space $\mathtt{C}$, together with a finitely supported genus function $\gamma : \mathtt{C} \to \mathbb{Z}_{\geq 0}$. The genus of a tropical curve is $\sum_{x\in\mathtt{C}} \gamma(x) + \mathrm{b}_1(\mathtt{C})$. A *cycle rigidification* of a genus-g tropical curve $\mathtt{C}$ is a g-tuple of elements spanning $H_1(\mathtt{C}, \mathbb{Z})$ and such that each element is either 0 or coming from a circuit in $\mathtt{C}$. We denote by $\overline{\mathcal{V}}_{g,n}$ the set of isomorphism classes of cycle-rigidified stable[2] n-marked tropical curves of genus g. Forgetting the metric structure (that is, only retaining the topological graph and the cycle rigidification, the so-called combinatorial type) stratifies $\overline{\mathcal{V}}_{g,n}$ into locally closed sets that can be naturally identified with open cones. For a combinatorial type Γ we denote by σ_Γ° the corresponding open cone and by σ_Γ its closure. The natural face morphisms among the closed cones gives $\overline{\mathcal{V}}_{g,n}$ the structure of an extended cone complex. Given an $n \sqcup \{\star\}$-marked curve, one can forget the $\star$-mark and stabilize. As stabilization is a contraction, a cycle rigidification on the original graph induces a cycle rigidification on the stabilized graph. We thus obtain a map

$$\mu_\star\colon \overline{\mathcal{V}}_{g,n\sqcup\{\star\}} \to \overline{\mathcal{V}}_{g,n}, \tag{2.1}$$

the forgetful map, which is a morphism of extended cone complexes. The fiber of $\mu_\star$ over a point $\overline{\mathcal{V}}_{g,n}$ corresponding to a cycle-rigidified tropical curve $\mathtt{C}$ is isomorphic to $\mathtt{C}$. In particular, there are n natural sections of $\mu_\star$ and a global genus function $\overline{\mathcal{V}}_{g,n\sqcup\{\star\}} \to \mathbb{Z}_{\geq 0}$.

A *family of genus- g, n-marked stable TPL-curves* over a TPL-space X is a morphism $\pi\colon C \to X$ of TPL-spaces, together with n sections $s_i\colon X \to C$, $1 \leq i \leq n$, and a genus function $C \to \mathbb{Z}_{\geq 0}$, such that every $x \in X$ has an open neighborhood U for which there exists a map $f\colon U \to \overline{\mathcal{V}}_{g,n}$ and an isomorphism $\pi^{-1}U \cong \overline{\mathcal{V}}_{g,n\sqcup\{\star\}} \times_{\mu_\star,\overline{\mathcal{V}}_{g,n},f} U$ over U that respects the sections and the genus function.

A *family of genus- g, n-marked stable tropical curves* over a tropical space $\mathtt{X}$ is a family of genus-g, n-marked TPL-curves $\Pi\colon \mathtt{C} \to \mathtt{X}$ satisfying the following conditions:

1. for each $x \in \mathtt{X}$, the fiber $\mathtt{C}_x := \Pi^{-1}\{x\}$, equipped with the induced affine structure, is a stable tropical curve,
2. for every $y \in \mathtt{C}$, the sequence

$$0 \to \Omega^1_{\mathtt{X},\Pi(y)} \to \Omega^1_{\mathtt{C},y} \to \Omega^1_{\mathtt{C}_{\Pi(y)},y} \to 0 \tag{2.2}$$

induced by pull-back from $\mathtt{X}$ and restriction to $\mathtt{C}_{\Pi(y)}$, is exact. This can be phrased equivalently as saying that affine functions that are constant on fibers are pull-backs of affine functions on the base.

There is a natural affine structure on the space $\overline{\mathcal{V}}_{g,n}$ induced by *cross ratio functions*. We describe these functions, starting with $\overline{\mathcal{V}}_{0,4} = \overline{\mathcal{M}}^{\mathrm{trop}}_{0,4}$. Choosing an ordering, for example, $((p_1,p_2),(p_3,p_4))$, of the four markings defines two unique minimal paths in a curve $[\mathtt{C}] \in \overline{\mathcal{M}}^{\mathrm{trop}}_{0,4}$, one from p_1 to p_2 and one from p_3 to p_4. The value of the cross ratio function $\xi_{((p_1,p_2),(p_3,p_4))}$ at $[\mathtt{C}]$ is the signed length of the intersection of these two paths. Let Γ be a combinatorial type of cycle-rigidified n-marked

[2] see [6, Definitions 3.2, 3.4] for the notion of stability of a tropical curve. Such notion parallels the algebraic one, that allows only nodal singularities and finitely many automorphisms; it takes a little bit of care in describing the tropical version of a nodal singularity.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

genus-g curves, let $\widetilde{\Gamma}$ denote the universal covering space of Γ, and let $T \subset \widetilde{\Gamma}$ be a connected open subset all of whose vertices have genus 0. Note that T is automatically a tree, that is $h^1(T) = 0$, and its leaves are open edge segments. For every combinatorial type Γ' specializing to Γ, the preimage T' of T under the induced retraction $\widetilde{\Gamma}' \to \widetilde{\Gamma}$ of universal covers is a connected open subset of $\widetilde{\Gamma}'$ not containing higher-genus vertices, and the retraction induces a bijection between the leaves of T' and the leaves of T. Forgetting the rest of the curve, this produces for every Γ' specializing to Γ a map $\sigma^\circ_{\Gamma'} \to \overline{\mathcal{M}}^{\mathrm{trop}}_{0,m}$, where m is the number of leaves of T. Given a quadruple $((p_1, p_2), (p_3, p_4))$ of distinct leaves of T, we can postcompose with the forgetful morphism $\overline{\mathcal{M}}^{\mathrm{trop}}_{0,m} \to \overline{\mathcal{M}}^{\mathrm{trop}}_{0,4}$ that forgets all but the marks $p_1, \dots, p_4$. From $\overline{\mathcal{M}}^{\mathrm{trop}}_{0,4}$ we can then pull-back the cross ratio function $\xi_{((p_1,p_2),(p_3,p_4))}$ to obtain a function on $\sigma_{\Gamma'}$. For different $\Gamma' \to \Gamma$, these functions glue together and define a cross ratio function $\xi_{(T,(p_1,p_2),(p_3,p_4))}$ on a neighborhood of σ°_Γ. This is the cross ratio function ξ_c associated to the cross ratio datum $c = (T, (p_1, p_2), (p_3, p_4))$ (the tuple c is not literally a cross ratio datum in the sense of [6, Definition 4.8], but there is a natural induced cross ratio datum). The affine structure of $\overline{\mathcal{V}}_{g,n}$ is the sheaf generated by cross ratio functions and constants. It is in fact generated by constants and the cross ratio functions associated to *primitive* cross ratio data: these correspond to the cases where T is a neighborhood of a single vertex v or of a single edge e. The four markings $p_1, \dots, p_4$ are induced by two pairs of flags $((f_1, f_2), (f_3, f_4))$ at v (resp. at the end points of e) and we write $c_{((f_1,f_2),(f_3,f_4))}$ (resp. $c_{(e,(f_1,f_2),(f_3,f_4))}$) for the cross ratio datum induced by $(T, (p_1, p_2), (p_3, p_4))$ in the two respective cases and $\xi_{((f_1,f_2),(f_3,f_4))}$ (resp. $\xi_{(e,(f_1,f_2),(f_3,f_4))}$) for the associated cross ratio functions.

With these affine structures, the forgetful map $\mu_\star \colon \overline{\mathcal{V}}_{g,n\sqcup\{\star\}} \to \overline{\mathcal{V}}_{g,n}$ is a morphism of tropical spaces, but not a family of tropical curves. What fails is the condition on the fibers: it is true that all functions on $\overline{\mathcal{V}}_{g,n\sqcup\{\star\}}$ are harmonic on the fibers, but not every harmonic function on a fiber is the restriction of an affine function.

2.2. Cone complexes and toroidal embeddings

We recall how to associate a cone complex to a toroidal embedding without self intersections. The original reference is [18], where what we call cone complex was called conical polyhedral complex with integral structures, and we refer there and to [1, 25, 16, 5] for a more detailed treatment.

A *(abstract) cone* is a pair (σ, M^σ) consisting of a topological space σ and a lattice M^σ of real-valued continuous functions on σ with the property that the natural map $\sigma \to \mathrm{Hom}(M^\sigma, \mathbb{R})$ maps σ homeomorphically onto its image, a strictly convex rational polyhedral cone in $\mathrm{Hom}(M^\sigma, \mathbb{R})$. If we denote $M^\sigma_+ = \{m \in M^\sigma : m \geq 0\}$, then the image of σ in $\mathrm{Hom}(M^\sigma, \mathbb{R})$ is precisely $\mathrm{Hom}(M^\sigma_+, \mathbb{R}_{\geq 0})$. The faces of (σ, M^σ) are the pairs $(\tau, \{m|_\tau : m \in M^\sigma\})$, where τ is a face of σ. The *relative interior* of a cone σ is the complement of all its proper faces; we denote it by σ°. A *cone complex* is pair $(|\Sigma|, \Sigma)$ consisting of a topological space $|\Sigma|$ and a collection Σ of cones whose underlying sets are closed subsets of $|\Sigma|$ and such that $|\Sigma| = \bigsqcup_{\sigma\in\Sigma} \sigma^\circ$. Moreover, the set Σ is closed under taking faces and intersections. A continuous function ϕ on a subset $U \subseteq \Sigma$ is *strict piecewise linear* if for all $\sigma \in \Sigma$ there exists $m \in M^\sigma$ and $c \in \mathbb{R}$ with $\phi|_{\sigma\cap U} = m|_{\sigma\cap U} + c$. We denote the group of strict piecewise linear functions on U by $\mathrm{sPL}_\Sigma(U)$.

For a cone σ, the *extended cone* $\overline{\sigma}$ is given by $\overline{\sigma} = \mathrm{Hom}(M^\sigma_+, \mathbb{R}_{\geq 0} \cup \{\infty\})$. The extended cones of the cones in a cone complex Σ can be glued to an extended cone complex $\overline{\Sigma}$. The extended cone complex Σ has a natural stratification $\overline{\Sigma} = \bigsqcup_{\tau<\sigma\in\Sigma} \sigma/\tau$. Any choice of generators of M^σ_+ induces an embedding of an extended cone $\overline{\sigma}$ into $\mathbb{T}^n$ for some $n \in \mathbb{Z}_{\geq 0}$ and thus $\overline{\sigma}$ obtains the structure of a TPL-space. Declaring a function on an extended cone complex to be piecewise linear if and only if its restrictions to all extended cones of the complex are piecewise linear defines a TPL-structure on any extended cone complex.

A toroidal embedding without self-intersection is a pair (X_0, X) consisting of a variety X and an open subset $X_0 \subseteq X$ that locally looks like the inclusion of the big torus into a toric variety. More precisely, for every point $x \in X$ there exists a toric chart, that is, an open neighborhood U of x, a toric

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

variety Y with big open torus T, and an étale morphism $f\colon U \to Y$ with $f^{-1}T = U \cap X_0$.[3] The toroidal embedding X has an associated cone complex Σ_X, with each $\sigma \in \Sigma_X$ corresponding to a *stratum* $O(\sigma)$ in the stratification of X induced by the toric charts. If we denote by $V(\sigma)$ the closure of $O(\sigma)$, the strata closures $V(\rho)$ for rays $\rho \in \Sigma_X(1)$ are the irreducible components of the *boundary* $X \setminus X_0$, and all other strata are connected components of $\bigcap_{\rho\in I} V(\rho) \setminus \bigcup_{\rho\notin I} V(\rho)$ for subsets $I \subseteq \Sigma_X(1)$. For a cone $\sigma \in \Sigma_X$, the monoid M^σ is naturally identified with the set of effective boundary Cartier divisors near $O(\sigma)$. In particular, the functions in $\mathrm{sPL}(\Sigma)$ with trivial constant part are in natural bijection with boundary Cartier divisors on X.

A dominant toroidal morphism of toroidal embeddings X and Y is a dominant morphism $X \to Y$ of schemes that can be expressed as a morphism of toric varieties in toric charts. A toroidal morphism from X to Y is any morphism of schemes $X \to Y$ that factors through a dominant toroidal morphism $X \to V(\sigma)$ for some $\sigma \in \Sigma_Y$.[4] A dominant toroidal morphism $f\colon X \to Y$ induces a morphism $\mathtt{Trop}(f)\colon \Sigma_X \to \Sigma_Y$, that is a continuous map $|\Sigma_X| \to |\Sigma_Y|$ mapping the cones of Σ_X linearly into cones of Σ_Y. The map $\mathtt{Trop}(f)$ extends to a morphism on the extended cone complexes. For $\tau \in \Sigma_Y$, we have a natural identification of $\Sigma_{V(\tau)}$ with the subcomplex $\bigcup_{\tau\subseteq\sigma\in\Sigma_Y} \sigma/\tau$ of $\overline{\Sigma}_Y$. In particular, for a not necessarily dominant toroidal morphism $f\colon X \to Y$ there is an induced morphism $\mathtt{Trop}(f)\colon \overline{\Sigma}_X \to \overline{\Sigma}_Y$.

2.3. Tropical cycles

We present the notion of tropical cycles and their intersection pairing with tropical divisors in the generality developed in [6].

A *k-weight* on a cone complex Σ is an equivalence class of pairs (Δ, c), where Δ is a proper subdivision of Σ and $c\colon \Delta(k) \to \mathbb{Z}$ is a map. Here, $\Delta(k)$ denote the set of k-dimensional cones of Δ, and the equivalence relation is induced by compatibility under refinements of Δ. Given an affine structure Aff_Σ, which we assume to satisfy the condition that Ω^1_Σ is constant on all cones, we define the balancing condition for weights as follows: a 1-weight represented by (Δ, c) is balanced with respect to an affine function $\phi \in \mathrm{Aff}_\Sigma(\Sigma)$ with $\phi(0_\Sigma) = 0$ (here, 0_Σ refers to the cone point of Σ), if we have $\sum_{\rho\in\Delta(1)} \phi(u_\rho)\cdot c(\rho) = 0$, where u_ρ denotes the primitive lattice generator of a ray ρ. We say that c is *balanced* if it is balanced with respect to every $\phi \in \mathrm{Aff}_\Sigma(\Sigma)$ with $\phi(0_\Sigma) = 0$. More generally, a k-weight represented by (Δ, c) is balanced if for every $(k-1)$-dimensional cone $\tau \in \Delta$ and every affine function $\phi \in \mathrm{Aff}_\Sigma(\Delta^\tau)$ with $\phi|_\tau = 0$, the induced 1-weight $\overline{c}$ on Σ^τ/τ is balanced with respect to the induced function $\overline{\phi}$. Note that the balancing condition is independent of the choice of representative (Δ, c). A balanced k-weight is called a *tropical k-cycle*.

We say a function $\phi \in \mathrm{sPL}_\Sigma(\Sigma)$ is *combinatorially principal at the cone* σ, and write $\phi \in \mathrm{CP}(\sigma)$ if there exists $\chi \in \mathrm{Aff}_\Sigma(\Sigma^\sigma)$ with $\chi|_\sigma = \phi|_\sigma$. We say ϕ is *combinatorially principal* if it is combinatorially principal at every cone $\sigma \in \Sigma$, and denote the group of combinatorially principal functions on Σ by $\mathrm{CP}(\Sigma)$. For a function $\phi \in \mathrm{sPL}_\Sigma(\Sigma)$ and a tropical 1-cycle A represented by (Δ, c), the intersection product $\phi \cdot A$ is the unique 0-cycle whose weight at the origin is given by

$$-\sum_{\rho\in\Delta(1)} \mathrm{slope}_\rho(\phi)c(\rho), \tag{2.3}$$

where $-\mathrm{slope}_\rho(\phi) = -\phi(u_\rho)$ is the incoming slope of ϕ at the origin, as required by the min convention. This only depends on the class of $\phi \in \mathrm{sPL}_\Sigma(\Sigma)/\mathrm{Aff}_\Sigma(\Sigma)$. Let $\phi \in \mathrm{CP}(\Sigma)$ and let $A = [(\Delta, c)]$ be a tropical k-cycle on Σ. The intersection product $\phi \cdot A$ is the tropical cycle represented by the $(k-1)$-weight whose weight on $\tau \in \Delta(k-1)$ is given by the weight at the origin of $\overline{\phi}\cdot\overline{c}$, where $\overline{c}$ is the tropical 1-cycle on Δ^τ/τ induced by c and $\overline{\phi}$ is the function on Δ^τ/τ induced by $\phi - \chi$ for some $\chi \in \mathrm{Aff}_\Sigma(\Sigma)$ with $\chi|_\tau = \phi|_\tau$. By construction, $\phi \cdot A$ depends only on the class of ϕ in $\mathrm{CP}(\Sigma)/\mathrm{Aff}_\Sigma(\Sigma)$.

[3] In the language of logarithmic geometry, toroidal embeddings without self-intersections are log smooth log schemes over a point with trivial log structure, where the log structure is defined on the Zariski site.

[4] Dominant toroidal morphisms are precisely the log smooth morphisms between toroidal varieties. Nondominant toroidal morphisms do not define morphisms of log schemes.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

We just presented the local theory of tropical cycles, which is all we need in this paper. These concepts can be globalized: there exists a sheaf Z_k of tropical k-cycles on tropical spaces, a sheaf CP (denoted by Rat in [6, Definition 6.3]) of combinatorially principal functions, and an intersection-pairing

$$\mathrm{CP}/\mathrm{Aff} \otimes_{\mathbb{Z}} Z_k \to Z_{k-1} \tag{2.4}$$

that locally looks like the pairing in the case of cone complexes with affine structure. The sheaf CP/Aff is the sheaf of combinatorially principal tropical Cartier divisors, sitting inside the sheaf $\mathrm{PL}^{\mathrm{fin}}/\mathrm{Aff}$ of *tropical Cartier divisors*. The *k-th tropical Chow group* $A_k(\mathtt{X})$ *of a tropical space* $\mathtt{X}$ is given by

$$A_k(\mathtt{X}) \coloneqq Z_k(\mathtt{X})/(\mathrm{CP}(\mathtt{X}) \cdot Z_{k+1}(\mathtt{X})). \tag{2.5}$$

2.4. Tropical line bundles and psi classes

A tropical line bundle on a tropical space $\mathtt{X}$ is an $\mathrm{Aff}_{\mathtt{X}}$-torsor. Equivalently, it is a morphism $\mathtt{L} \to \mathtt{X}$ whose fibers are $\mathbb{T}$-torsors and which locally trivializes: it is locally isomorphic to $\mathtt{X} \times \mathbb{T}$ as tropical spaces, and the isomorphism is an isomorphism of $\mathbb{T}$-torsors on each fiber. One can translate the two notions into each other by associating to a line bundle $\mathtt{L} \to \mathtt{X}$ the sheaf $\underline{\mathrm{Isom}}(\mathbb{T} \times \mathtt{X}, \mathtt{L})$ of isomorphisms with the trivial line bundle, which is an $\mathrm{Aff}_{\mathtt{X}}$-torsor because $\underline{\mathrm{Aut}}(\mathbb{T} \times \mathtt{X}) \cong \mathrm{Aff}_{\mathtt{X}}$.

By [15, Lemma 4.5], every tropical line bundle $\mathtt{L}$ has a piecewise linear section. Any such section defines a tropical Cartier divisor $\mathtt{D}$ whose associated tropical line bundle is isomorphic to $\mathtt{L}$. In the language of torsors, the existence of a piecewise linear section means that given any $\mathrm{Aff}_{\mathtt{X}}$-torsor, the associated PL-torsor is trivial. However, the associated CP-torsor is not necessarily trivial, that is the line bundle does not necessarily have a combinatorially principal section (i.e., the tropical Cartier divisor $\mathtt{D}$ from above is not necessarily combinatorially principal). If a tropical line bundle $\mathtt{L}$ on a tropical space $\mathtt{X}$ has a combinatorially principal section, any choice of such section defines the same element in $(\mathrm{CP}/\mathrm{Aff})(\mathtt{X})/\mathrm{CP}(\mathtt{X})$ (a combinatorially principal tropical Cartier divisor modulo linear equivalence), and this element defines a map

$$c_1(\mathtt{L})\colon A_*(\mathtt{X}) \to A_{*-1}(\mathtt{X}) \tag{2.6}$$

via the intersection pairing. The map $c_1(\mathtt{L})$ is called the *first Chern class of* $\mathtt{L}$. Given $\alpha \in A_*(\mathtt{X})$, we denote $c_1(\mathtt{L})(\alpha)$ by $c_1(\mathtt{L}) \frown \alpha$ to avoid proliferation of nested parentheses.

We recall the construction of tropical ψ classes from [6]. Given a family of n-marked, stable tropical curves $\overline{\mathtt{C}} \to \overline{\mathtt{B}}$, for every $1 \le i \le n$ and $k \in \mathbb{Z}$ the i-th section $\mathtt{s}_i\colon \overline{\mathtt{B}} \to \overline{\mathtt{C}}$ defines a tropical line bundle $\mathrm{Aff}_{\overline{\mathtt{C}}}(k\mathtt{s}_i)$ in the form of an $\mathrm{Aff}_{\overline{\mathtt{C}}}$-torsor on the total space $\overline{\mathtt{C}}$. Namely, every point in the image of $\mathtt{s}_i$ is the infinite point of a leg in its fiber. Therefore, we can define $\mathrm{Aff}_{\overline{\mathtt{C}}}(k\mathtt{s}_i)$ as the subsheaf of $\iota_* \mathrm{Aff}_{\overline{\mathtt{C}}\setminus \mathtt{s}_i(\overline{\mathtt{B}})}$, where $\iota\colon \overline{\mathtt{C}} \setminus \mathtt{s}_i(\mathtt{B}) \to \mathtt{C}$ is the inclusion, of all functions whose slope on the leg approaching $\mathtt{s}_i$ is $-k$. The *i-th tropical cotangent* line bundle $\mathbb{L}_i^{\mathtt{trop}}$ is the line bundle on $\overline{\mathtt{B}}$ defined as $\mathtt{s}_i^* \mathrm{Aff}_{\overline{\mathtt{C}}}(-\mathtt{s}_i)$.

If $\mathtt{L} = \mathbb{L}_i^{\mathtt{trop}}$ is the i-the cotangent bundle of a marked family of tropical curves, we call the first Chern class of $\mathtt{L}$ the *i-th* ψ *class* and denote it by

$$\psi_i \coloneqq c_1(\mathbb{L}_i^{\mathtt{trop}}). \tag{2.7}$$

3. Affine Structures for Tropicalizations

This section contains the core technical definitions and constructions. Given a toroidal variety X, we endow its cone complex with an affine structure, calling the result the tropicalization of X. Tropicalization becomes a functor valued in the category of tropical spaces. To simplify the exposition, we initially make the following seemingly strong assumption:

(∗) there are no self-intersections among the toroidal strata or multiple intersections among pairs of strata.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

We prove that the affine structure induced by tropicalization is invariant under log modifications and explain in Remark 3.9 how this allows us to remove such hypothesis without loss of generality.

3.1. Affine structures on cone complexes

Let Σ be a cone complex. A *subcomplex* S of Σ is a union

$$S = \bigcup_{\sigma \in I} \sigma^\diamond, \tag{3.1}$$

where I is a subset of cones of Σ. We say that S is an open (resp. closed) subcomplex of Σ if it defines an open (resp. closed) set.

For a cone $\sigma \in \Sigma$, we define the *open star* Σ^σ of σ to be the open subcomplex

$$\Sigma^\sigma = \bigcup_{\sigma \subseteq \tau} \tau^\diamond, \tag{3.2}$$

given by the union of the relative interiors of all cones containing σ. The usual *star* of σ, consisting of the union of all cones containing σ, is the closure of the open star, and we denote it by $\overline{\Sigma^\sigma}$.[5]

Let X be a toroidal variety satisfying $(*)$, and Σ_X its cone complex as described in Section 2.2. Every $\sigma \in \Sigma_X$ corresponds to a stratum $O(\sigma)$.

For a cone $\sigma \in \Sigma_X$ we denote by X_σ the open neighborhood of $V(\sigma)$ given by

$$X_\sigma = \bigcup_{\tau \in \overline{\Sigma_X^\sigma}} O(\tau). \tag{3.3}$$

A strict piecewise linear function ϕ on Σ_X^σ determines a Cartier divisor, and consequently an invertible sheaf on X_σ: extend ϕ to $\overline{\Sigma_X^\sigma}$ by linearity, and for any ray $\rho \in \overline{\Sigma_X^\sigma}$ denote the slope of $\phi|_\rho$ by $\mathrm{slope}_\rho(\phi)$. We define:

$$D_{X_\sigma}(\phi) = \sum_{\rho \in \overline{\Sigma_X^\sigma}} \mathrm{slope}_\rho(\phi) V(\rho), \qquad \mathcal{O}_{X_\sigma}(\phi) = \mathcal{O}_{X_\sigma}\big(D_{X_\sigma}(\phi)\big). \tag{3.4}$$

Definition 3.1. Let Σ_X be the cone complex of a toroidal variety X, and $\sigma \in \Sigma_X$.
An **affine function** at σ is a strict piecewise linear function ϕ on Σ^σ such that $\mathcal{O}_{X_\sigma}(\phi)|_{V(\sigma)}$ is trivial.
The **affine structure** Aff_{Σ_X} on Σ_X induced by X is defined to be the subsheaf of $\mathrm{PL}^{\mathrm{fin}}_{\Sigma_X}$ generated by affine functions at cones $\sigma \in \Sigma_X$.

As shown in the following lemma, affine functions in the sense of Definition 3.1 are closed under restriction, hence we are not generating any new sections of Aff_Σ by restriction.

Lemma 3.2. *Let $\sigma \in \Sigma$, let τ be a face of σ, and let ϕ be an affine function at τ. Then $\phi|_{\Sigma^\sigma}$ is an affine function at σ.*

Proof. Since $V(\sigma)$ is contained in $V(\tau)$, $\mathcal{O}_{X_\tau}(\phi)|_{V(\tau)}$ being trivial implies that the bundle $\mathcal{O}_{X_\tau}(\phi)|_{V(\sigma)}$ is trivial as well. The assertion now follows from the fact that

$$\mathcal{O}_{X_\tau}(\phi)|_{X_\sigma} = \mathcal{O}_{X_\sigma}(\phi|_{\Sigma_\sigma}). \tag{3.5}$$

□

[5] We could just as well have defined the open star as the interior of the star, and adapted the notation accordingly. We chose this order because the open star, which provides a natural open neighborhood of the generic point of a cone, plays a more prominent role in the definition of sheaves of affine functions.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Corollary 3.3. *Let $\sigma \in \Sigma$. Then $\mathrm{Aff}_\Sigma(\Sigma^\sigma)$ consists precisely of affine functions at σ.*

Proof. This is an immediate consequence of Lemma 3.2. □

Remark 3.4. This notion of affine functions is related to the linear functions on Cartwright's tropical complexes [4]. To compare the two notions, consider the situation where we are given a proper morphism $\pi\colon X \to \mathbb{A}^1$ such that X is smooth and connected and $\pi^{-1}\{0\}$ is a reduced simple normal crossings divisor. In particular, the pair $(\pi^{-1}(\mathbb{A}^1 \setminus \{0\}), X)$ defines a toroidal embedding and π is a dominant toroidal morphism. We obtain an induced morphism $\Pi\colon \Sigma_X \to \mathbb{R}_{\geq 0}$ of cone complexes and the linear functions of [4] are defined on the fiber $\Delta := \Pi^{-1}\{1\}$. To be linear in the sense of Cartwright, a function χ on Δ has to satisfy the condition that for each $\sigma \in \Sigma$ with $V(\sigma)$ a curve, there exists a function $\phi \in \mathrm{sPL}(\Sigma^\sigma)$ with

$$\phi|_{\Delta\cap\Sigma^\sigma} = \chi|_{\Delta\cap\Sigma^\sigma}, \qquad \text{and} \tag{3.6}$$

$$\deg(\mathcal{O}_{X_\sigma}(\phi)|_{V(\sigma)}) = 0. \tag{3.7}$$

So Cartwright's linearity is only checked locally around $\sigma \in \Sigma$ with $V(\sigma)$ a curve, and our condition that $\mathcal{O}_{X_\sigma}(\phi)|_{V(\sigma)}$ be trivial is weakened to it having degree 0. In other words, our notion of affine functions agrees with Cartwright's locally around $\sigma \in \Sigma$ with $V(\sigma) \cong \mathbb{P}^1$, and differs from it otherwise.

We now show that the assignment of tropical spaces Σ_X to toroidal varieties X is functorial.

Proposition 3.5. *Let $f : X \to Y$ be a dominant morphism of toroidal varieties. Then the natural piecewise linear map*

$$F := \mathtt{Trop}(f)\colon |\Sigma_X| \to |\Sigma_Y| \tag{3.8}$$

is a morphism of tropical spaces.

Proof. Let $\tau \in \Sigma_Y$ and let $\phi \in \mathrm{Aff}_{\Sigma_Y}(\Sigma_Y^\tau)$ be an affine function at τ. Let σ be a cone of $F^{-1}\Sigma_Y^\tau$. We want to show that $F^*(\phi)|_{\Sigma_X^\sigma}$ is an affine function at σ, that is, $F^*(\phi)|_{\Sigma_X^\sigma} \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\sigma)$. After potentially restricting ϕ to the star of a cone containing τ, we may assume by Lemma 3.2 that $F(\sigma^\diamond) \subseteq \tau^\diamond$. Then the morphism f maps $V(\sigma)$ into $V(\tau)$ and thus induces a morphism $f|_{V(\sigma)}\colon V(\sigma) \to V(\tau)$. We have

$$\mathcal{O}_{X_\sigma}(F^*(\phi))|_{V(\sigma)} \cong \Big(f|_{X_\sigma}^*\mathcal{O}_{Y_\tau}(\phi)\Big)|_{V(\sigma)} \cong f|_{V(\sigma)}^*\big(\mathcal{O}_{Y_\tau}(\phi)|_{V(\tau)}\big) \cong f|_{V(\sigma)}^*\big(\mathcal{O}_{V(\tau)}\big) \cong \mathcal{O}_{V(\sigma)}, \tag{3.9}$$

where the second to last isomorphism follows from ϕ being affine. Hence $F^*(\phi)$ is affine in a neighborhood of σ. □

Remark 3.6. In Proposition 3.5, the assumption that f is a dominant toroidal morphism is too strong. All that is needed is that f defines a morphism of log schemes, that is that $f(X_0) \subseteq Y_0$.

In the proof of the invariance of the affine structure under modifications, we use the following lemma.

Lemma 3.7. *Let $f\colon X \to Y$ be a log modification, let $\tau \in \Sigma_X$, and let $\sigma \in \Sigma_Y$ be the unique cone with $\tau^\diamond \subseteq \sigma^\diamond$. If $\phi \in \mathrm{sPL}_{\Sigma_Y}(\Sigma_Y^\sigma)$ is a piecewise linear function which restricts to an affine function $\phi|_{\Sigma_X^\tau} \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\tau)$, then we have $\phi \in \mathrm{Aff}_{\Sigma_Y}(\Sigma_Y^\sigma)$.*

Proof. Let $\theta \in \Sigma_X$ be an inclusion-maximal cone with $\tau \subseteq \theta \subseteq \sigma$, note that θ and σ have the same dimension. The induced morphism $\Sigma_X^\theta/\theta \to \Sigma_Y^\sigma/\sigma$ is a subdivision, and the corresponding logarithmic modification agrees with the restriction $f_\theta := f|_{V(\theta)}\colon V(\theta) \to V(\sigma)$. By Corollary 3.3, the function $\phi|_{\Sigma_X^\theta}$ is affine at θ and hence

$$\mathcal{O}_{V(\theta)} = \mathcal{O}_{X_\theta}(\phi|_{\Sigma_X^\theta})|_{V(\theta)} \cong f_\theta^*\big(\mathcal{O}_{Y_\sigma}(\phi)|_{V(\sigma)}\big) \tag{3.10}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

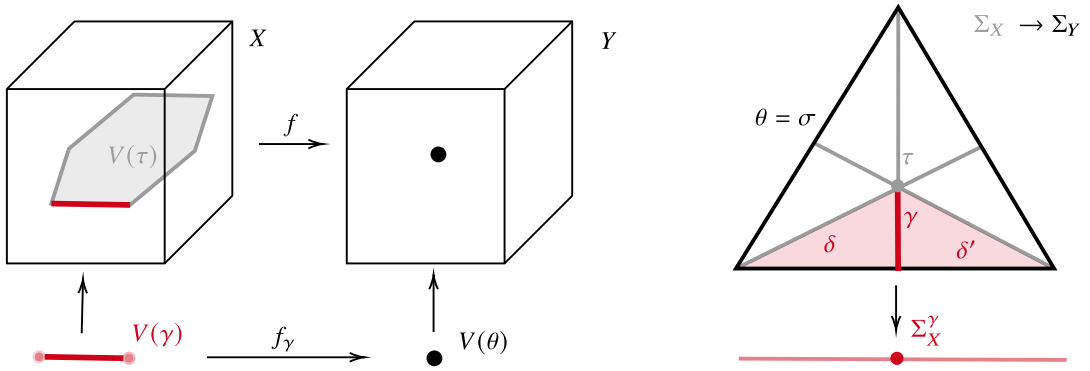


Figure 2. *The setup used in showing that a function $\phi \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\tau)$ is the restriction of a function in $\mathrm{Aff}_{\Sigma_Y}(\Sigma_Y^\sigma)$. The key point is that the map f_γ is a $\mathbb{P}^1$ bundle. In the illustration we draw $\theta = \sigma$ for lack of dimensions: σ is the minimal cone of Σ_Y containing τ, and θ a maximal cone with such property.*

Taking first Chern classes and using the projection formula, we conclude that

$$\mathcal{O}_{Y_\sigma}(\phi)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}, \tag{3.11}$$

which is equivalent to $\phi \in \mathrm{Aff}_{\Sigma_Y}(\Sigma_Y^\sigma)$. □

Proposition 3.8. *If $f : X \to Y$ is a log modification then the induced morphism*

$$F\colon |\Sigma_X| \to |\Sigma_Y| \tag{3.12}$$

is an isomorphism of tropical spaces.

Proof. Since f is a log modification, we know F is a bijective function, and by Proposition 3.5 affine functions pull-back to affine functions. It remains to show that any affine function on Σ_X can be obtained by pull-back from an affine function of Σ_Y. Let $\tau \in \Sigma_X$ and let $\sigma \in \Sigma_Y$ be the unique cone with $\tau^\circ \subseteq \sigma^\circ$. We must prove that every function $\phi \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\tau)$ is the restriction of a function in $\mathrm{Aff}_{\Sigma_Y}(\Sigma_Y^\sigma)$.

First we show that ϕ is the restriction of a function in $\mathrm{sPL}_{\Sigma_Y}(\Sigma_Y^\sigma)$, see Figure 2 for the setup. Let $\theta \in \Sigma_Y$ be a maximal cone containing σ and let $\delta \in \Sigma_X$ be a maximal cone of Σ_X with $\tau \subseteq \delta \subseteq \theta$. Since the cones δ and θ have the same dimension, the restriction $\phi|_\delta$ extends uniquely to a linear function m_θ^δ on the cone θ. We now show that this function does not depend on the choice of δ. Let $\delta' \in \Sigma_X$ be a second maximal cone with $\tau \subseteq \delta' \subseteq \theta$. Because δ can be reached from δ' by passing through codimension-1 cones of Σ_X that contain τ and are contained in θ, we may assume that δ and δ' are adjacent cones, and $\gamma := \delta \cap \delta'$ has codimension 1. Let $\varphi = \phi|_{\Sigma_X^\gamma} - m_\theta^\delta|_{\Sigma_X^\gamma}$. Then

$$\mathcal{O}_{X_\gamma}(\phi|_{\Sigma_X^\gamma}) \cong \mathcal{O}_{X_\gamma}(\varphi) \otimes f|_{X_\gamma}^* \mathcal{O}_{Y_\theta}(m_\theta^\delta). \tag{3.13}$$

Denote by $f_\gamma\colon V(\gamma) \to V(\theta)$ the restriction of f. Restricting (3.13) to $V(\gamma)$ and using the hypothesis of ϕ being affine we obtain

$$\mathcal{O}_{V(\gamma)} \cong \mathcal{O}_{X_\gamma}(\varphi)|_{V(\gamma)} \otimes f_\gamma^* \mathcal{O}_{Y_\theta}(m_\theta^\delta)|_{V(\theta)}. \tag{3.14}$$

The function φ vanishes on γ and therefore induces a function $\overline{\varphi}$ on the star $\Sigma_X^\gamma/\gamma = \Sigma_{V(\gamma)}$. Since γ has codimension 1 in θ, the star Σ_X^γ/γ is a union of two rays, corresponding to the divisors $V(\delta)$ and $V(\delta')$ in $V(\gamma)$, which both map isomorphically onto $V(\theta)$. That makes f_γ a $\mathbb{P}^1$-bundle on $V(\theta)$ with two distinct sections. The function $\overline{\varphi}$ has slope 0 on the ray corresponding to $V(\delta)$, we show that it must have slope 0 along the other ray as well. If k is the slope of $\overline{\varphi}$ on the ray corresponding to $V(\delta')$, then

$$\mathcal{O}_{X_\gamma}(\varphi)|_{V(\gamma)} \cong \mathcal{O}_{V(\gamma)}(\overline{\varphi}) \cong \mathcal{O}_{V(\gamma)}(V(\delta'))^{\otimes k} \tag{3.15}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

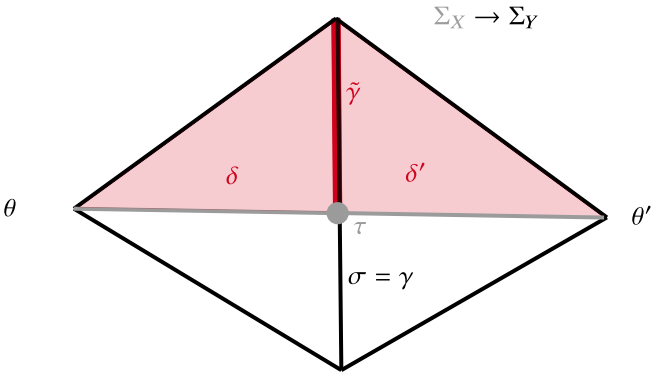


Figure 3. *Illustration for the second part of the argument in Proposition 3.8. Here by lack of dimensions we have drawn $\sigma = \gamma$. Observe that $\tilde{\gamma} \subset \gamma$, and the cones θ and θ' are the two-dimensional triangles bounded by the black edges.*

and hence (3.14) gives

$$\mathcal{O}_{V(\gamma)} \cong \mathcal{O}_{V(\gamma)}(V(\delta'))^{\otimes k} \otimes f_\gamma^* \mathcal{O}_{Y_\theta}(m_\theta^\delta)|_{V(\theta)}. \tag{3.16}$$

Since $\mathrm{Pic}(V(\gamma)) = \mathbb{Z} \cdot \mathcal{O}_{V(\gamma)}(V(\delta')) \oplus \mathrm{Pic}(V(\theta))$ by virtue of the projective bundle formula, it follows that $k = 0$, that is, $\varphi|_{\delta'} = 0$. From this it follows immediately that $m_\theta^\delta = m_\theta^{\delta'}$.

For $\theta \in \Sigma_Y$ a maximal cone containing σ we can now define $m_\theta = m_\theta^\delta$ for some maximal cone $\delta \in \Sigma_X$ with $\tau \subseteq \delta \subseteq \theta$; this is independent of the choice of δ. Given a second maximal cone $\theta' \in \Sigma_Y$ containing σ intersecting θ in $\gamma := \theta \cap \theta'$, we claim that

$$m_\theta|_\gamma = m'_\theta|_\gamma. \tag{3.17}$$

Referring to Figure 3, if $\widetilde{\gamma} \in \Sigma_X$ is a maximal cone in γ containing τ, then there exist maximal cones $\delta, \delta' \in \Sigma_X$ containing $\widetilde{\gamma}$ with $\delta \subseteq \theta$ and $\delta' \subseteq \theta'$. Then we have

$$m_\theta|_{\widetilde{\gamma}} = m_\theta^\delta|_{\widetilde{\gamma}} = \phi|_{\widetilde{\gamma}} = m_{\theta'}^{\delta'}|_{\widetilde{\gamma}} = m_{\theta'}|_{\widetilde{\gamma}}. \tag{3.18}$$

As $\widetilde{\gamma}$ is full-dimensional in γ, this shows the desired equality. We conclude that the functions m_θ glue to give a function $\widetilde{\phi} \in \mathrm{sPL}_{\Sigma_Y}(\Sigma_Y^\sigma)$, and by construction we have $\widetilde{\phi}|_{\Sigma_X^\tau} = \phi$. By Lemma 3.7, we have $\widetilde{\phi} \in \mathrm{Aff}_{\Sigma_Y}(\Sigma_Y^\sigma)$, completing the proof. □

Remark 3.9. Proposition 3.8 allows us to remove the hypothesis that the boundary of X does not have strata with self intersections or pairs of strata with multiple intersections. Hypothesis $(*)$ may always be achieved via some log modification, for example, by subdividing barycentrically the generalized cone complex Σ_X of the original space. We then define the affine structure on the tropicalization as the affine structure on any refinement where $(*)$ holds; by Proposition 3.8, it is a well defined affine structure on Σ_X.

3.2. The affine structure at infinity

In this section, we extend the previous constructions to the context of extended cone complexes. Given a cone complex Σ, its extended cone complex is set-theoretically described as a disjoint union of quotients of cone complexes indexed by the cones of Σ:

$$\overline{\Sigma} = \bigsqcup_{\sigma \in \Sigma} \Sigma^\sigma / \sigma. \tag{3.19}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Observe that if 0_Σ is the cone point of Σ, then $\Sigma^{0_\Sigma}/0_\Sigma = \Sigma$. We call sets of the form $\sigma/\tau \subseteq \overline{\Sigma}$ for some $\sigma \supseteq \tau \in \Sigma$ the *cells* of the extended cone complex. The *open star* of a cell of an extended cone complex is

$$\overline{\Sigma}^{\sigma/\tau} = \bigcup_{\substack{\tilde{\sigma} \supseteq \sigma \\ \tilde{\tau} \subseteq \tau}} (\tilde{\sigma}/\tilde{\tau})^\circ. \tag{3.20}$$

Taking $\tau = 0_\Sigma$, (3.20) agrees with the earlier notion of open star of a cone in a cone complex (3.2). We will denote by $\mathrm{sPL}(\overline{\Sigma}^{\sigma/\tau})$ the subgroup of $\mathrm{sPL}(\Sigma^\sigma)$ consisting of those functions whose continuous extensions to $\overline{\Sigma^\sigma}$ are constant on τ. Equivalently, the functions in $\mathrm{sPL}(\overline{\Sigma}^{\sigma/\tau})$ are precisely those functions in $\mathrm{sPL}(\Sigma^\sigma)$ that extend continuously to $\overline{\Sigma}^{\sigma/\tau}$.

Definition 3.10. Let X be a toroidal variety, and let $\overline{\Sigma}_X$ denote the extended cone complex associated to X. We extend the affine structure on Σ_X to $\overline{\Sigma}_X$ by defining

$$\mathrm{Aff}_{\overline{\Sigma}_X} = \iota_* \mathrm{Aff}_{\Sigma_X} \cap \mathrm{PL}^{\mathrm{fin}}_{\overline{\Sigma}_X}, \tag{3.21}$$

where $\iota\colon \Sigma_X \to \overline{\Sigma}_X$ denotes the inclusion, and the intersection is taken in $\iota_*\mathrm{PL}^{\mathrm{fin}}_{\Sigma_X}$. In other words, the affine functions on $\overline{\Sigma}_X$ are precisely the continuous extension of the affine functions on Σ_X.

An important feature of extended tropicalization is that *at infinity* one can witness a stratification by the Σ^σ/σ's, which is parallel (rather than opposite) to the original toroidal stratification. The idea is that the geometry at infinity in the direction of τ should reflect the geometry of $V(\tau)$. For a cell at infinity of the the form σ/τ, we define $V(\sigma/\tau) = V(\sigma)$, which we think of as a closed stratum inside $V(\tau)$. We show that affine structures are compatible with this perspective.

Lemma 3.11. *Let $\sigma \in \Sigma_X$, let τ be a face of σ, and let $\phi \in \mathrm{sPL}(\Sigma_X^\sigma)$ such that $\phi|_\tau$ is constant. Denote by $\pi_\tau\colon \Sigma_X^\tau \to \Sigma_X^\tau/\tau$ the projection and by χ the unique piecewise linear function on $(\Sigma_X^\tau/\tau)^{\sigma/\tau}$ such that $\phi = \pi_\tau^*\chi$. Then ϕ is affine at σ if and only if χ is affine at σ/τ in the affine structure on Σ_X^τ/τ defined using the natural identification $\Sigma_X^\tau/\tau \cong \Sigma_{V(\tau)}$.*

Proof. As ϕ is constant on τ, the support of the Cartier divisor $D_{X_\sigma}(\phi)$ is does not contain $V(\tau)$, hence its restriction to $V(\tau)$ is transversal:

$$D_{X_\sigma}(\phi)|_{V(\tau)_{\sigma/\tau}} = D_{V(\tau)_{\sigma/\tau}}(\chi). \tag{3.22}$$

It follows that $\mathcal{O}_{X_\sigma}(\phi)|_{V(\tau)_{\sigma/\tau}}$ agrees with $\mathcal{O}_{V(\tau)_{\sigma/\tau}}(\chi)$, and, in particular, their restriction to $V(\sigma)$ agree, from which the assertion follows immediately. □

Proposition 3.12. *Let X be a toroidal variety with tropicalization Σ_X, and let $\tau \in \Sigma_X$. With the natural identification $\Sigma_X^\tau/\tau = \Sigma_{V(\tau)}$, we have an isomorphism*

$$\mathrm{Aff}_{\overline{\Sigma}_X}|_{\Sigma_X^\tau/\tau} \cong \mathrm{Aff}_{\Sigma_{V(\tau)}} \tag{3.23}$$

Proof. By definition, the affine structure Aff_{Σ_X} is generated by strict piecewise linear functions. Such a function extends to a finite piecewise linear function in a neighborhood of a cell $\sigma/\tau \subseteq \overline{\Sigma}_X$ if and only if it is constant on τ. The assertion now follows from Lemma 3.11. □

Corollary 3.13. *Let $f : X \to Y$ be a (not necessarily dominant) toroidal morphism of toroidal varieties. Then the natural piecewise linear map*

$$\mathtt{Trop}(f)\colon |\overline{\Sigma}_X| \to |\overline{\Sigma}_Y| \tag{3.24}$$

is a morphism of tropical spaces.

Proof. Combine Proposition 3.5 with Proposition 3.12. □

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

3.3. *Affine structures on families of curves*

We apply the constructions in Section 3.1 to families of curves and their tropicalizations. We first observe, in what seems like a *groundhog day* moment but is a necessary verification, that in genus zero the affine structures we obtain from tropicalization agree with those from [6] and with those obtained from any prior incarnation of $\mathcal{M}_{0,n}^{\mathrm{trop}}$. Next we study the tropicalization of families of curves of arbitrary genus, and observe that while tropicalization gives a map of TPL-spaces with affine structures, this is not always a family of tropical curves with affine structures in the sense of [6]. However we identify combinatorial conditions describing a locus where the affine structures agree. We begin with some technical definitions that are useful in developing these ideas.

Definition 3.14. Let Σ be a pure-dimensional cone complex, let $\sigma \in \Sigma$, and let H^σ be a subgroup of $\mathrm{sPL}(\Sigma^\sigma)$ such that $[\Sigma^\sigma]$ (the weight on Σ^σ which is 1 on all top-dimensional cones) is balanced in the affine structure induced by H^σ. Let $\mathrm{CP}_{H^\sigma}(\Sigma^\sigma)$ denote the subgroup of $\mathrm{sPL}(\Sigma^\sigma)$ consisting of those piecewise linear functions ϕ for which on every cone δ of Σ^σ there exists a function $h \in H^\sigma$ with $\phi|_\delta = h|_\delta$. We define the closure $\overline{H^\sigma}$ of H^σ as

$$\overline{H^\sigma} = \{\phi \in \mathrm{CP}_{H^\sigma}(\Sigma^\sigma) \mid \phi \cdot [\Sigma^\sigma] = 0\} \tag{3.25}$$

and say that H^σ is **normal** if $H^\sigma = \overline{H^\sigma}$. When no superscript is added, we assume that we are choosing σ to be the cone point 0_Σ.

Example 3.15. Consider a cone complex Σ with one vertex and four rays denoted ρ_i, for $i = 1, \dots, 4$. Denote by φ_i the piecewise linear function with slope one along ρ_i and zero along all other rays. For σ equal to the vertex 0_Σ, if we choose

$$H = \langle \varphi_1 - \varphi_3, \varphi_2 - \varphi_4 \rangle_{\mathbb{Z}}, \tag{3.26}$$

then $\Sigma^\sigma = \Sigma$ is balanced at σ in the affine structure induced by H. In this case $\mathrm{CP}_H(\Sigma) = \mathrm{sPL}(\Sigma)$, as one can find on each ray a function in H with any given integral slope.

The piecewise linear function $\varphi_3 - \varphi_4$ belongs to $\overline{H} \setminus H$, thus showing that H is not normal. One can verify that adding this function one spans all of $\overline{H}$, thus

$$\overline{H} = \langle \varphi_1 - \varphi_4, \varphi_2 - \varphi_4, \varphi_3 - \varphi_4 \rangle_{\mathbb{Z}}. \tag{3.27}$$

See Figure 4 for a geometric illustration of this example.

In the next proposition, we show that $\overline{H^\sigma}$ is the largest subgroup of $\mathrm{CP}_{H^\sigma}(\Sigma^\sigma)$ containing H^σ in which Σ^σ is balanced at σ.

Proposition 3.16. *Let X be a toroidal variety with associated cone complex Σ_X, let $\sigma \in \Sigma_X$, and let $H^\sigma \subseteq \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\sigma)$ be a subgroup such that $[\Sigma_X^\sigma]$ is balanced. Then we have*

$$\mathrm{Aff}_{\Sigma_X}(\Sigma_X^\sigma) \cap \mathrm{CP}_{H^\sigma}(\Sigma_X^\sigma) \subseteq \overline{H^\sigma}. \tag{3.28}$$

In particular, if $\mathrm{CP}_{H^\sigma}(\Sigma_X^\sigma) = \mathrm{sPL}(\Sigma_X^\sigma)$, we have $\mathrm{Aff}_{\Sigma_X}(\Sigma_X^\sigma) \subseteq \overline{H^\sigma}$.

Proof. Let $\phi \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\sigma) \cap \mathrm{CP}_{H^\sigma}(\Sigma_X^\sigma)$. We will show in Corollary 5.8 that with the affine structure Aff_{Σ_X}, the intersection product $\phi \cdot [\Sigma_X^\sigma]$ vanishes. Since $H^\sigma \subseteq \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\sigma)$, we have $\phi \cdot [\Sigma_X^\sigma] = 0$ even with respect to the affine structure defined by H^σ, that is $\phi \in \overline{H^\sigma}$. □

Theorem 3.17. *Via the canonical identification of the underlying cone complexes, the affine structure on $\overline{\Sigma}_{\overline{\mathcal{M}}_{0,n}}$ agrees with the affine structure on $\overline{\mathcal{M}}_{0,n}^{\mathrm{trop}}$ (as defined in [6, Section 4]).*

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

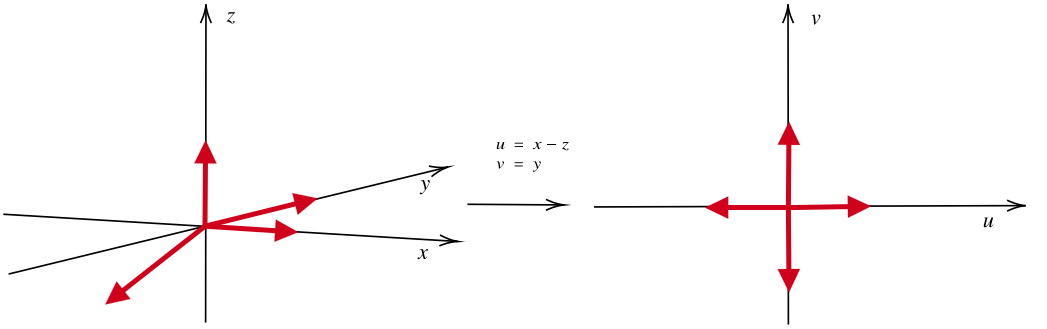


Figure 4. *The primitive ray generators for the cone complex Σ, depicted in red, are embedded into $\mathbb{R}^2$ via the chosen generators of H, and into $\mathbb{R}^3$ via the generators of $\overline{H}$. The two balanced fans are related by a linear function. The failure of normality of H detects the fact that the embedding of Σ as a plane balanced fan is not as linearly independent as possible.*

Proof. By Definition 3.10, we have

$$\mathrm{Aff}_{\overline{\Sigma}_{\overline{\mathcal{M}}_{0,n}}} = \iota_* \mathrm{Aff}_{\Sigma_{\overline{\mathcal{M}}_{0,n}}} \cap \mathrm{PL}^{\mathrm{fin}}_{\overline{\Sigma}_{\overline{\mathcal{M}}_{0,n}}}, \tag{3.29}$$

where $\iota\colon \Sigma_{\overline{\mathcal{M}}_{0,n}} \to \overline{\Sigma}_{\overline{\mathcal{M}}_{0,n}}$ is the inclusion. By [6, Lemma 4.16] we also have

$$\mathrm{Aff}_{\overline{\mathcal{M}}^{\mathrm{trop}}_{0,n}} = \kappa_* \mathrm{Aff}_{\mathcal{M}^{\mathrm{trop}}_{0,n}} \cap \mathrm{PL}^{\mathrm{fin}}_{\overline{\mathcal{M}}^{\mathrm{trop}}_{0,n}}, \tag{3.30}$$

where $\kappa\colon \mathcal{M}^{\mathrm{trop}}_{0,n} \to \overline{\mathcal{M}}^{\mathrm{trop}}_{0,n}$ denotes the inclusion. Since $\mathrm{PL}^{\mathrm{fin}}_{\overline{\Sigma}_{\overline{\mathcal{M}}_{0,n}}} = \mathrm{PL}^{\mathrm{fin}}_{\overline{\mathcal{M}}^{\mathrm{trop}}_{0,n}}$, it suffices to show that the affine structures on $\Sigma_{\overline{\mathcal{M}}_{0,n}}$ and $\mathcal{M}^{\mathrm{trop}}_{0,n}$ coincide.

By [6, Theorem 4.32], affine functions on $\mathcal{M}^{\mathrm{trop}}_{0,n}$ are given by cross ratios [6, Definition 4.8]. We first show that the tropicalization of a cross ratio on $\overline{\mathcal{M}}_{0,n}$ is a tropical cross ratio. All cross ratios are pull-backs of cross ratios on $\mathcal{M}_{0,4}$ via the forgetful map, both algebraically and tropically, and tropicalization respects the S_4-action on $\mathcal{M}_{0,4}$. Thus, by the functoriality of taking cone complexes of Proposition 3.5, it suffices to show that the cross ratio $(p_1, p_2; p_3, p_4)$ on $\mathcal{M}_{0,4}$ tropicalizes to a cross ratio. We have

$$(\infty, 0; 1, z) = z, \tag{3.31}$$

whose associated divisor is $(0) - (\infty)$. The point 0 corresponds to the nodal curve $(p_2p_4|p_1p_3)$, whereas the point ∞ corresponds to the curve $(p_1p_4|p_2p_3)$. Therefore, the tropicalization of the cross ratio $(p_1, p_2; p_3, p_4)$ is the function with slope 1 on the ray $\rho_{(13|24)}$, slope -1 on the ray $\rho_{(14|23)}$, and slope 0 on the ray $\rho_{(12|34)}$. This is precisely the tropical cross ratio function $\xi_{((1,2),(3,4))}$ corresponding to the primitive vertex cross ratio datum $c_{(((1,2),(3,4))}$ at the unique vertex of the tropical curve in the minimal cone of $\mathcal{M}^{\mathrm{trop}}_{0,4}$.

We have shown that the natural identification

$$\Sigma_{\overline{\mathcal{M}}_{0,n}} \to \mathcal{M}^{\mathrm{trop}}_{0,n} \tag{3.32}$$

is linear. It remains to show that there are not more functions on $\Sigma_{\overline{\mathcal{M}}_{0,n}}$. The subgroup $H = \mathrm{Aff}_{\mathcal{M}^{\mathrm{trop}}_{0,n}}(\Sigma_{\overline{\mathcal{M}}_{0,n}})$ is naturally identified with the dual to the integral lattice in $\mathbb{R}^{\binom{n-1}{2}-1}$. Since $\mathcal{M}^{\mathrm{trop}}_{0,n}$ is a tropical linear space [20, Theorem 6.4.12], we have that $H = \overline{H}$ by [10, Proposition 4.5]. In addition

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

$\mathrm{CP}_H(\mathcal{M}_{0,n}^{\mathrm{trop}}) = \mathrm{sPL}(\mathcal{M}_{0,n}^{\mathrm{trop}})$, because $\mathcal{M}_{0,n}^{\mathrm{trop}}$ is embedded in $\mathbb{R}^{\binom{n-1}{2}-1}$ by its global linear functions. We can thus apply Proposition 3.16 to conclude that $\mathrm{Aff}_{\Sigma_{\overline{\mathcal{M}}_{0,n}}}(\Sigma_{\overline{\mathcal{M}}_{0,n}}) \subseteq \overline{H} = \mathrm{Aff}_{\mathcal{M}_{0,n}^{\mathrm{trop}}}(\Sigma_{\overline{\mathcal{M}}_{0,n}})$.

Given any cone $\sigma \in \mathcal{M}_{0,n}^{\mathrm{trop}}$, the same argument applies to $\Sigma^{\sigma}_{\overline{\mathcal{M}}_{0,n}}$, because $\Sigma^{\sigma}_{\overline{\mathcal{M}}_{0,n}} - \sigma$ is again a tropical linear space [9, Proposition 2.5]. This shows that $\mathrm{Aff}_{\Sigma_{\overline{\mathcal{M}}_{0,n}}}(\Sigma^{\sigma}_{\overline{\mathcal{M}}_{0,n}}) = \mathrm{Aff}_{\mathcal{M}_{0,n}^{\mathrm{trop}}}(\Sigma^{\sigma}_{\overline{\mathcal{M}}_{0,n}})$, thus concluding the proof. □

We turn our attention to more general families of curves and their tropicalization.

Definition 3.18. A **tropicalizable** family of n-marked stable curves of genus g is a toroidal morphism $\pi\colon \mathcal{C} \to \mathcal{B}$ of toroidal varieties, together with n toroidal sections $(\sigma_i)_{1\le i\le n}$, satisfying the following three properties:

1. The family π together with the sections $(\sigma_i)_i$ is a family of n-marked genus-g stable curves.
2. The family $\mathtt{Trop}(\pi)\colon \overline{\mathsf{C}} \to \overline{\mathsf{B}}$, together with the sections $\mathtt{Trop}(\sigma_i)$ and the genus function that assigns on σ° for a cone $\sigma \in \mathsf{C}$ the arithmetic genus of the generic fiber of $V(\sigma) \to \pi(V(\sigma))$, is an n-marked family of TPL-curves; that is, there exists a cartesian diagram

$$\begin{array}{ccc} \overline{\mathsf{C}} & \xrightarrow{f_\star} & \overline{\mathcal{V}}_{g,n\sqcup\{\star\}} \\ \downarrow{\scriptstyle \mathtt{Trop}(\pi)} & & \downarrow{\scriptstyle \mu_\star} \\ \overline{\mathsf{B}} & \xrightarrow{f} & \overline{\mathcal{V}}_{g,n}\,. \end{array} \tag{3.33}$$

3. For every $\sigma \in \mathsf{C}$ the generic fiber of the induced map $V(\sigma) \to \pi(V(\sigma))$ is irreducible.

Remark 3.19. To allow for greater flexibility, we allow the total space $\mathcal{C}$ (but not $\mathcal{B}$, for simplicity of the exposition) to have self-intersections, that is to be a toroidal embedding without self-intersections only étale locally. There still is an associated cone complex C, albeit one in which some cones have some of their faces glued together. One can remove the self-intersections by blowing up $\mathcal{C}$ according to a suitable subdivision of C (essentially, the only thing that can go wrong is the existence of loops in the fibers; these need to be subdivided), but this comes at the cost of trading a family of stable curves for one of semistable curves.

Example 3.20. While the strata $O(\sigma)$ and their closures $V(\sigma)$ are always irreducible, condition (3) in Definition 3.18 is not superfluous. It can happen that the fibers of $V(\sigma) \to \pi(V(\sigma))$ are reducible, but because of the monodromy action on the irreducible components, the domain $V(\sigma)$ is still irreducible. As an example consider the family of plane curves over $\mathrm{Spec}\,\mathbb{C}[t^{\pm 1}]$ given by a union of three lines as

$$\begin{aligned} &\{(t-1)x + (t-\zeta)y + (t-\zeta^2)z = 0\} \quad \cup \\ &\qquad \cup \quad \{(t-\zeta)x + (t-\zeta^2)y + (t-1)z = 0\} \quad \cup \\ &\qquad\qquad \cup \quad \{(t-\zeta^2)x + (t-1)y + (t-\zeta)z = 0\}, \end{aligned} \tag{3.34}$$

where $\zeta \in \mathbb{C}^*$ denotes a primitive third root of unity. Traveling from $t = 1$ to $t = \zeta$ permutes the three lines cyclically, and in particular the family descends to an irreducible family over $\mathrm{Spec}\,\mathbb{C}[t^{\pm 3}]$ whose fibers are unions of three lines in a plane that do not meet in a point. The construction of this example already shows how to fix families in which condition (3) is violated: one has to pass to an étale cover that trivializes the monodromy action on the components of the fibers.

Proposition 3.21. *Let $\pi : \mathcal{C} \to \mathcal{B}$ be a tropicalizable family of stable, n-marked logarithmic curves, and let $\Pi := \mathtt{Trop}(\pi)\colon \overline{\mathsf{C}} \to \overline{\mathsf{B}}$ be its tropicalization. Then the maps f and $f_\star$ from diagram (3.33) are linear.*

Proof. We first show the linearity of f on the interior B of $\overline{\mathsf{B}}$. Let $b_0 \in \mathsf{B}$ and let σ be the unique cone of B with $b_0 \in \sigma^\circ$. Let G be the cycle-rigidified, n-marked, genus-g stable graph such that $f(b_0) \in \sigma_G^\circ$. As the combinatorial type of the fibers of Π remains constant on σ°, we have $f(\sigma^\circ) \subseteq \sigma_G^\circ$ and f and $f_\star$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

induce an identification of the combinatorial type of $\overline{\mathsf{C}}_b$ with G for every $b \in \sigma^\diamond$. Since linear functions on $\overline{\mathcal{V}}_{g,n}$ are generated by vertex and edge primitive cross ratios ([6, Proposition 4.13]), we show linearity of f by showing that the pullback of a primitive cross ratio is linear on $\mathtt{B}$.

Let $c = c_{((f_s,f_e),(f_{-1},f_1))}$ be a primitive cross ratio of vertex type at a rational vertex v of G with associated function ξ_c on $(\overline{\mathcal{V}}_{g,n})^{\sigma_G}$. We show that $f^*\xi_c$ is linear on $\mathtt{B}^\sigma$.

We may assume, possibly after subdividing some cones of $\mathtt{B}$, that $\mathtt{B}^\sigma$ is simply connected. We may also assume, possibly after subdividing some cones of $\overline{\mathsf{C}}$, that the four flags f_s, f_e, f_{-1}, f_1 belong to distinct edges in each fiber. Then the vertex v defines a component $V(v)$ of $\mathcal{C}_{V(\sigma)}$. The four tangent directions f_s, f_e, f_{-1}, and f_1 determine four sections of $V(v)$ and thus a morphism

$$V(\sigma) \to \overline{\mathcal{M}}_{0,4}; \tag{3.35}$$

precomposing its tropicalization with the map quotienting σ, we obtain $\mathtt{B}^\sigma \to \mathcal{M}_{0,4}^{\mathtt{trop}}$. Since $f^*\xi_c$ agrees with the pullback of the appropriate cross ratio from $\mathcal{M}_{0,4}^{\mathtt{trop}}$, linearity follows from Theorem 3.17.

Now let $c = c_{(e,(f_s,f_e),(f_{-1},f_1))}$ be a primitive cross ratio of edge type on G at an edge e. First assume that there exists a face τ of σ such that $f(\tau^\diamond) \subseteq \sigma_{\overline{G}}$ for some edge contraction $\overline{G}$ of G in which e got contracted to a genus-0 vertex v. Then f_s, f_e, f_{-1}, and f_1 induce flags at v and in particular a primitive cross ratio $\overline{c} = c_{((f_s,f_e),(f_1,f_1))}$ of vertex type on $\overline{G}$. By the vertex type case above, the function $f^*\xi_{\overline{c}}$ is linear on $\mathtt{B}^\tau$, and by Lemma 3.2 the restriction $f^*\xi_{\overline{c}}|_{\mathtt{B}^\sigma}$ is linear on $\mathtt{B}^\sigma$. But $f^*\xi_{\overline{c}}|_{\mathtt{B}^\sigma}$ coincides with $f^*\xi_c$ on $\mathtt{B}^\sigma$, concluding the proof in this case.

Next, assume that for any face of σ where the edge e gets contracted, a vertex of positive genus is formed. We can view $\mathcal{C}$ as the base of a family of $(n \sqcup \{\star\})$-marked stable curves

$$\pi_\star \colon \mathcal{C}_\star \to \mathcal{C}. \tag{3.36}$$

The cycle rigidification of Π induced by the maps f and $f_\star$ from diagram (3.33) produces a cycle rigidification of the family

$$\overline{\mathsf{C}}_\star := \Sigma_{\mathcal{C}_\star} \xrightarrow{\mathtt{Trop}(\pi_\star)} \overline{\mathsf{C}}, \tag{3.37}$$

that is a Cartesian TPL-diagram

$$\begin{array}{ccc} \overline{\mathsf{C}}_\star & \xrightarrow{f_{\star,\bullet}} & \overline{\mathcal{V}}_{g,n\sqcup\{\star,\bullet\}} \\ \downarrow & & \downarrow \\ \overline{\mathsf{C}} & \xrightarrow{f_\star} & \overline{\mathcal{V}}_{g,n\sqcup\{\star\}}\,. \end{array} \tag{3.38}$$

Let $\sigma_e \in \Sigma$ be the cone over σ whose fibers corresponds to e. The stratum $V(\sigma_e)$ is the node corresponding to e in every fiber of $\mathcal{C}_{V(\sigma)} \to V(\sigma)$ and thus the restriction of π defines an isomorphism $V(\sigma_e) \to V(\sigma)$. The pull-back $\mu_\star^*\xi_c$ is a cross ratio on $\overline{\mathcal{V}}_{g,n\sqcup\{\star\}}$, although not a primitive one. More precisely, if $G_\star$ denotes the rigidified $(n \sqcup \{\star\})$-marked graph, obtained by introducing a vertex v on e in G and attaching a $\star-$marked leg $l_\star$ at v, then $f_\star(\sigma_e) \subseteq \sigma_{G_\star}$ and c induces a cross ratio datum on $G_\star$ by ignoring the fact that e has been subdivided. If e_1 and e_2 denote the two edges adjacent to v in the order seen when traversing e according to c, and g_1 and g_2 are the corresponding half-edges starting at v, as shown in Figure 5, then we have

$$\mu_\star^*\xi_c = \xi_{(e_1,(f_s,l_\star),(f_{-1},g_2))} + \xi_{(e_2,(l_\star,f_e),(g_2,f_1))} \tag{3.39}$$

The two edges e_1 and e_2 are contracted on the two faces of σ_e that map isomorphically onto σ and correspond to the two vertices of e. Therefore, for each of the two primitive cross ratio summands in

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

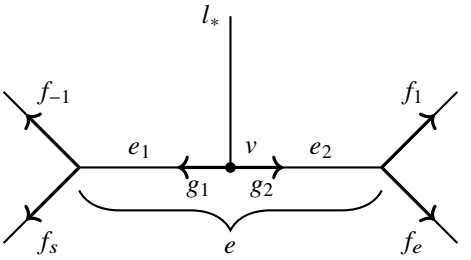


Figure 5. *The subdivision of the edge e in $G_\star$.*

(3.39) we are in the special situation from before: the edge of the edge-type primitive cross ratio is contracted to a genus zero vertex. Algebraically, this yields

$$\begin{aligned}\mathcal{O}_{V(\sigma_e)} &\cong \left(\mathcal{O}_{\mathcal{C}_{\sigma_e}}(f_\star^*\xi_{(e_1,(f_s,l_\star),(f_{-1},g_2))}) \otimes \mathcal{O}_{\mathcal{C}_{\sigma_e}}(f_\star^*\xi_{(e_2,(l_\star,f_e),(g_2,f_1))})\right)|_{V(\sigma_e)} \cong \\ &\cong \mathcal{O}_{\mathcal{C}_{\sigma_e}}(f_\star^*\mu_\star^*\xi_c)|_{V(\sigma_e)} \cong \mathcal{O}_{\mathcal{C}_{\sigma_e}}(\Pi^* f^*\xi_c|_{\sigma_e})|_{V(\sigma_e)} \cong \left(\pi^*\mathcal{O}_{\mathcal{B}_\sigma}(f^*\xi_c)\right)\Big|_{V(\sigma_e)}. \end{aligned} \tag{3.40}$$

As we have noticed before, π induces an isomorphism $V(\sigma_e) \to V(\sigma)$, so we conclude that

$$\mathcal{O}_{\mathcal{B}_\sigma}(f^*\xi_c)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}, \tag{3.41}$$

which is equivalent to $f^*\xi_c$ being linear on $\mathtt{B}^\sigma$.

To show that f is also linear on the boundary of $\mathtt{B}$ it suffices to note that on both source (by Definition 3.10) and target (by [6, Lemma 4.16]), the affine structure near the boundary is induced by the affine structure on the interior in the sense that the affine functions at the boundary are precisely the continuous extensions of affine functions on the interior.

To show that $f_\star$ is linear one applies the same arguments to the map $\pi_\star$ and the rigidification displayed in (3.38). □

Proposition 3.22. *With the set-up and notation of Proposition 3.21, let $x \in \overline{\mathtt{C}}$. If x has genus 0, then all functions in $\mathrm{Aff}_{\mathtt{C},x}$ are linear on the fibers of Π close to x. Moreover, we have an exact sequence*

$$0 \to \mathrm{Aff}_{\overline{\mathtt{B}},\Pi(x)} \to \mathrm{Aff}_{\overline{\mathtt{C}},x} \to \Omega^1_{\overline{\mathtt{C}}_{\Pi(x)},x} \to 0. \tag{3.42}$$

If x is a vertex (of genus 0) in its fiber, or is on an edge adjacent to a vertex of genus 0, then $\Omega^1_{\overline{\mathtt{C}}_{\Pi(x)},x}$ consists of the differentials of all harmonic functions.

Proof. The injectivity of the pull-back is obvious, so is the surjectivity of restriction to the fiber (we consider the fiber with the induced affine structure). We show the harmonicity of affine functions on fibers and the exactness in the middle simultaneously.

We first prove the proposition in the case when $x \in \mathtt{C}$. Suppose x is on an edge in its fiber. In that case a function $\phi \in \mathrm{Aff}_{\mathtt{C},x}$ is linear, and hence harmonic, on all fibers in a neighborhood of x. If $\phi|_{\Pi^{-1}\{\Pi(x)\}}$ is constant near x, then ϕ is constant on fibers on a small neighborhood of x. Let $\sigma \in \mathtt{C}$ be the unique cone with $x \in \sigma^\diamond$ and let $\tau = \Pi(\sigma)$. By definition of the affine structure on $\mathtt{C}$, ϕ defines a piecewise linear function on $\mathtt{C}^\sigma$, and ϕ is constant on all the fibers of the induced map $\mathtt{C}^\sigma \to \mathtt{B}^\tau$. Taking any section $\mathtt{B}^\tau \to \mathtt{C}^\sigma$ of this map (for example a section corresponding to one of the two vertices of the edge on which x sits), we obtain a piecewise linear function χ on $\mathtt{B}^\tau$ with $\Pi^*\chi = \phi$. The stratum $V(\sigma)$ is a node in each fiber, and the projection $\pi|_{V(\sigma)} : V(\sigma) \to V(\tau)$ is an isomorphism. We have

$$\pi|^*_{V(\sigma)}\mathcal{O}_{\mathcal{B}_\tau}(\chi)|_{V(\tau)} \cong \mathcal{O}_{\mathcal{C}_\sigma}(\Pi^*\chi)|_{V(\sigma)} \cong \mathcal{O}_{\mathcal{C}_\sigma}(\phi)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}, \tag{3.43}$$

where the last isomorphism follows because ϕ is affine; we conclude that χ is affine on $\mathtt{B}$.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Now suppose x is a vertex of genus 0, let σ and τ be as above, and let $\phi \in \mathrm{Aff}_{\mathtt{C},x}$. Let e be an edge in the fiber of x that is adjacent to x. After subtracting a suitable cross ratio, which is harmonic on fibers and affine by Proposition 3.21, we may assume that ϕ is constant on e. So by the case where x was on an edge, there exists an affine function $\chi \in \mathrm{Aff}_{\mathtt{B},\Pi(x)}$ such that $\phi - \Pi^*\chi$ vanishes on e. In particular, by replacing ϕ with this difference of affine functions, we may assume that $\phi|_\sigma = 0$. By Proposition 3.12, ϕ is induced by an affine function $\overline{\phi}$ on $\mathtt{C}^\sigma/\sigma = \mathtt{C}_{V(\sigma)}$. Let $T_x(\mathtt{C}_{\Pi(x)})$ denote the set of half-edges starting at x in its fiber. For each $h \in T_x(\mathtt{C}_{\Pi(x)})$ let m_h be the slope of ϕ in the direction of h, and let σ_h be the cone of $\mathtt{C}$ spanned by σ and h. Then the normalization $\widetilde{V}(\sigma)$ of the stratum $V(\sigma)$ is a family of $T_x(\mathtt{C}_{\Pi(x)})$-marked rational stable curves over $V(\tau)$, and $V(\sigma_h)$ is the section of this family corresponding to $h \in T_x(\mathtt{C}_{\Pi(x)})$. The fact that ϕ is affine means that the divisor

$$\sum_{h \in T_x(\mathtt{C}_{\Pi(x)})} m_h V(\sigma_h) \tag{3.44}$$

is rationally equivalent to 0 on $\widetilde{V}(\sigma)$. Since the fiber of $\widetilde{V}(\sigma) \to V(\tau)$ over any $y \in O(\tau)$ is isomorphic to $\mathbb{P}^1$, we conclude that $\sum_h m_h = 0$, because every divisor on $\mathbb{P}^1$ rationally equivalent to 0 has degree 0. This shows that ϕ is harmonic at x. Onto exactness in the middle of (3.43), if x is a vertex and $\phi|_{\Pi^{-1}\{\Pi(x)\}}$ is constant near x, then it is constant on an edge adjacent to x; by the case where x was on an edge we obtain an affine function $\chi \in \mathrm{Aff}_{\mathtt{B},\Pi(x)}$ with $\phi = \Pi^*\chi$.

Let x be either a rational vertex or adjacent to a genus 0 vertex in its fiber. By the linearity of $f_\star$ (Proposition 3.21), all (pull-backs of) cross ratios are linear on $\mathtt{C}$. By the condition on x, by [6, Proposition 4.24] there are enough cross ratio functions defined at x to realize all possible differentials of harmonic functions on the fiber.

It remains to analyze the case where $x \in \overline{\mathtt{C}} \setminus \mathtt{C}$. Let δ be the unique cone of $\mathtt{C}$ with $x \in \mathtt{C}^\delta/\delta$ and let $\theta = \Pi(\delta)$, which is a cone in $\mathtt{B}$. We can either have $\dim(\delta) = \dim(\theta) + 1$ or $\dim(\delta) = \dim(\theta)$. In the first case, the point x is either the endpoint of a leg or a node at infinity in its fiber (see [6, p. 14]). It follows that $\Omega^1_{\Pi(x),x} = 0$ and we need to show that the pull-back $\mathrm{Aff}_{\overline{\mathtt{B}},\Pi(X)} \to \mathrm{Aff}_{\overline{\mathtt{C}},x}$ is an isomorphism. The morphism $V(\delta) \to V(\theta)$ induced by π is an isomorphism because $V(\delta)$ either consists of a single marked point or a single node in each fiber over $V(\theta)$. So π also induces an isomorphism $\Sigma_{V(\delta)} \to \Sigma_{V(\theta)}$ of tropical spaces and we are done by Proposition 3.12. In the second case, that is, when δ and θ have the same dimension, a neighborhood of x in $\overline{\mathtt{C}}_{\Pi(x)}$ is contained in $\mathtt{C}^\delta/\delta$. It therefore suffices to prove the assertion for the family

$$\Sigma_{V(\delta)} \cong \mathtt{C}^\delta/\delta \to \mathtt{B}^\theta/\theta \cong \Sigma_{V(\theta)},$$

where the canonical isomorphisms come from Proposition 3.12. This family is the tropicalization of the induced family of curves $V(\delta) \to V(\theta)$ and x is in the interior of $\Sigma_{V(\delta)}$, so we have reduced to the case that we already proved. □

Remark 3.23. Consider the set $\mathtt{U} \subseteq \mathtt{C}$ of all genus-0 points that are either vertices or on an edge adjacent to a genus-0 vertex. The locus $\mathtt{U}$ is closely related to the locus $\mathcal{V}^{\mathrm{good}}_{g,n}$ introduced in [6, Definition 4.22]. More precisely, we have

$$f^{-1}\mathcal{V}^{\mathrm{good}}_{g,n} = \{y \mid \mathtt{C}_y \subseteq \mathtt{U}\} \tag{3.45}$$

where $\mathtt{C}_y$ denotes the fiber over y. One can view $\mathtt{U}$ as the locus where $\mathtt{C} \to \mathtt{B}$ is a family of tropical curves in the sense of [6]. The affine structure on $\mathtt{U}$ is completely determined by that on $\mathtt{B}$. Equivalently, the inclusion

$$\mathtt{U} \to \mathtt{B} \times_{\overline{\mathcal{V}}_{g,n}} \overline{\mathcal{V}}_{g,n \sqcup \{\star\}}, \tag{3.46}$$

where the fiber product is taken in the category of tropical spaces, is an open immersion.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

4. Tropicalizations of line bundles

Let X be a toroidal variety and let $\pi\colon L \to X$ be a line bundle on X. We equip the total space of L with the toroidal structure whose boundary is $Z \cup \pi^{-1}\partial X$, where Z denotes the zero-section of the bundle. The cone complex Σ_L is naturally identified with $\Sigma_X \times \rho_Z$, where ρ_Z is the ray of Σ_L corresponding to the boundary divisor Z. Via the construction from Section 3, we obtain an affine structure on $\overline{\Sigma}_X \times \overline{\rho}_Z$.

Definition 4.1. Extend the affine structure on $\overline{\Sigma}_X \times (\overline{\rho}_Z \setminus \{0\})$ to $\overline{\Sigma}_X \times \mathbb{T}$ by linearity, where we identify $\mathbb{T}_{>0}$ with $\overline{\rho}_Z \setminus \{0\}$. We call the morphism

$$\overline{\Sigma}_X \times \mathbb{T} \to \overline{\Sigma}_X \tag{4.1}$$

of spaces with affine structures the **tropicalization** of L and denote it by $\mathtt{Trop}(L)$.

Remark 4.2. We describe an equivalent way of defining $\mathtt{Trop}(L)$: first compactify the total space L by adding a section at infinity, that is consider the projective bundle $P_L = \mathbb{P}(L \oplus \mathcal{O}_X)$; the boundary consists of the union of the pull-back of ∂X with the two sections. If ρ_∞ is the ray of Σ_{P_L} corresponding to the section at infinity, then $\mathtt{Trop}(L)$ is the complement of $\Sigma_{P_L}^{\rho_\infty}/\rho_\infty$ in $\overline{\Sigma}_{P_L}$. More concisely, we add a section at infinity, tropicalize, and then remove the tropical section at infinity.

The tropicalization of L is not necessarily a tropical line bundle (see §2.4), as we illustrate in the following example.

Example 4.3. We present here three examples of tropicalization of a line bundle $\pi\colon L \to X$ on a toroidal variety X with exactly one boundary divisor H. The extended tropicalization $\overline{\Sigma}_X$ of X is the closed segment $[0,\infty]$ and we denote by $\{V_0, V_\infty\}$ the open cover $\{[0,\infty), (0,\infty]\}$. The map of extended cone complexes giving the tropicalization of L is depicted in Figure 6; in all three cases the combinatorics of the tropicalization is the same, what differs is the affine structure on the total space $\overline{\Sigma}_X \times \mathbb{T}$. We describe the affine structure by giving the affine functions on the six types of open sets $\mathtt{U}_{ij}$ shaded in Figure 6. In each case we will denote $\widetilde{H} := \pi^* H$.

Case 1: $(X, H) = (\mathbb{P}^1, pt. = c_1(\mathcal{O}_{\mathbb{P}^1}(1)))$, $L = \mathcal{O}_{\mathbb{P}^1}(m)$.

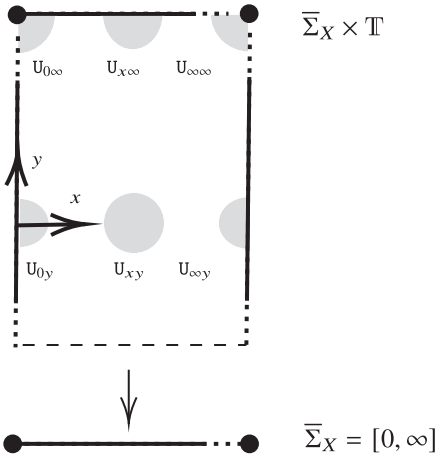


Figure 6. *The tropicalization of a line bundle on a toroidal space with a unique boundary divisor H. As suggested by the orientation of the picture, we call x a linear coordinate for the ray ρ_H and y the coordinate for the ray ρ_Z dual to the zero section $Z \cong X$.*

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

The local affine functions are the following:

(a) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{0\infty}) = \mathbb{R}$
(b) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{x\infty}) = \langle x\rangle_{\mathbb{Z}} \oplus \mathbb{R}$
(c) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{\infty\infty}) = \mathbb{R}$
(d) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{0y}) = \langle y - mx\rangle_{\mathbb{Z}} \oplus \mathbb{R}$
(e) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{xy}) = \langle x, y\rangle_{\mathbb{Z}} \oplus \mathbb{R}$
(f) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{\infty y}) = \langle y\rangle_{\mathbb{Z}} \oplus \mathbb{R}$

(a) since the open set $\mathtt{U}_{0\infty}$ contains the cell at infinity ρ_Z/ρ_Z, any affine function must be constant in y. The local function $\alpha x + r$ is affine if and only if $\mathcal{O}_L(\alpha\widetilde{H})|_Z \cong \mathcal{O}_Z = \mathcal{O}_{\mathbb{P}^1}$. We have

$$\mathcal{O}_L(\alpha\widetilde{H})|_Z \cong \mathcal{O}_Z(\widetilde{H}\cdot Z\alpha) = \mathcal{O}_{\mathbb{P}^1}(\alpha); \tag{4.2}$$

it follows that $\alpha = 0$;
(b) affine functions need to be constant in y. We want to see when $\mathcal{O}_L(\alpha\widetilde{H})|_{Z\cap\widetilde{H}} \cong \mathcal{O}_{Z\cap\widetilde{H}}$, but $Z \cap \widetilde{H}$ consists of a single point, and therefore any line bundle restricted to it must be trivial. There is no condition on α;
(c) the set $\mathtt{U}_{\infty\infty}$ contains the point (∞, ∞) and therefore affine functions must be constant in x and y and therefore constant.
(d) we are looking for functions of the form $\alpha x + \beta y + r$ such that $\mathcal{O}_L(\alpha\widetilde{H} + \beta Z)|_Z \cong \mathcal{O}_{\mathbb{P}^1}$

$$\mathcal{O}_L(\alpha\widetilde{H} + \beta Z)|_Z \cong \mathcal{O}_Z((Z\cdot\widetilde{H})\alpha + c_1(L)\beta) = \mathcal{O}_{\mathbb{P}^1}(\alpha + m\beta); \tag{4.3}$$

setting $\alpha + m\beta = 0$ shows that affine functions have the form $\beta(y - mx) + r$;
(e) in this case we are again restricting a line bundle on the total space of L to the point $Z \cap \widetilde{H}$, therefore always obtaining a trivial bundle. There is no restriction on the integral slopes of affine functions.
(f) affine functions must be constant in x. The function $\beta y + r$ is affine if $\mathcal{O}_L(\beta\widetilde{H})|_{\widetilde{H}}$ is trivial. But since $\widetilde{H} \cong \mathbb{C}$, any line bundle may be trivialized and the result follows.

We see that $\mathtt{Trop}(L)$ is a tropical line bundle on $\overline{\Sigma}_X$ with Cartier data $\{(V_0, -mx), (V_\infty, 0)\}$.

Case 2: $(X, H) = (\mathbb{P}^2, H = c_1(\mathcal{O}_{\mathbb{P}^2}(1)))$, $L = \mathcal{O}_{\mathbb{P}^2}(m)$.

The local affine functions are the following:

(a) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{0\infty}) = \mathbb{R}$
(b) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{x\infty}) = \mathbb{R}$
(c) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{\infty\infty}) = \mathbb{R}$
(d) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{0y}) = \langle y - mx\rangle_{\mathbb{Z}} \oplus \mathbb{R}$
(e) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{xy}) = \langle y - mx\rangle_{\mathbb{Z}} \oplus \mathbb{R}$
(f) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{\infty y}) = \mathbb{R}$

We don't go through the computations in detail, since they are very similar to those in the first case. To highlight what the difference is, we just look at what is different in computing the affine sections for the open set $\mathtt{U}_{xy}$: we are now looking for functions of the form $\alpha x + \beta y + r$ such that $\mathcal{O}_L(\alpha\widetilde{H} + \beta Z)|_{Z\cap\widetilde{H}} \cong \mathcal{O}_{Z\cap\widetilde{H}}$. Notice that $Z \cap \widetilde{H}$ is now a hyperplane in $\mathbb{P}^2$, and it is therefore isomorphic to $\mathbb{P}^1$. We have

$$\mathcal{O}_L(\alpha\widetilde{H} + \beta Z)|_{Z\cap\widetilde{H}} \cong \mathcal{O}_Z((Z\cdot\widetilde{H})\alpha + c_1(L)\beta)|_{\widetilde{H}} = \mathcal{O}_{\mathbb{P}^1}(\alpha + m\beta); \tag{4.4}$$

there must be a nontrivial relation between α and β in order for the line bundle to trivialize, yielding the result from (e). In this case $\mathtt{Trop}(L)$ is a (trivial) tropical line bundle on the cone complex Σ_X, but not on the extended space $\overline{\Sigma}_X$.

Case 3: $(X, H) = (\mathbb{P}^1 \times \mathbb{P}^1, H = c_1(\mathcal{O}_{\mathbb{P}^1}(1, 0)))$, $L = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0, 1)$. In this case the local affine functions are:

(a) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{0\infty}) = \mathbb{R}$
(b) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{x\infty}) = \langle x\rangle_{\mathbb{Z}} \oplus \mathbb{R}$
(c) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{\infty\infty}) = \mathbb{R}$
(d) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{0y}) = \mathbb{R}$
(e) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{xy}) = \langle x\rangle_{\mathbb{Z}} \oplus \mathbb{R}$
(f) $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{\infty y}) = \mathbb{R}$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Again, the computations are similar to the first case, so we just show the most interesting example. We compute the local sections $\mathrm{Aff}_{\mathtt{Trop}(L)}(\mathtt{U}_{0y})$. These are functions of the form $\alpha x + \beta y + r$ such that $\mathcal{O}_L(\alpha\widetilde{H} + \beta Z)|_Z \cong \mathcal{O}_Z = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$. In this case:

$$\mathcal{O}_L(\alpha\widetilde{H} + \beta Z)|_Z = \mathcal{O}_Z((Z\cdot\widetilde{H})\alpha + c_1(L)\beta) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\alpha\widetilde{H} + \beta V), \tag{4.5}$$

where $V = c_1(L)$ denotes the class of the vertical fiber in $\mathbb{P}^1\times\mathbb{P}^1$. For the bundle to be trivial we must have $\alpha = \beta = 0$. We thus see that $\mathtt{Trop}(L)$ is not a tropical line bundle on Σ_X.

We investigate when the tropicalization of a line bundle is indeed a tropical line bundle. We will say that a line bundle is *tropicalizable* when this happens.

Proposition 4.4. *Let L be a line bundle on X with associated invertible sheaf $\mathcal{L}$ and let $\mathtt{U}\subseteq\overline{\Sigma}_X$ be an open subcomplex. Then $\mathtt{Trop}(L)$ is a tropical line bundle on $\mathtt{U}$ if and only if for every cell σ/τ of $\mathtt{U}$ there exists a function $\phi_{\sigma/\tau}\in\mathrm{sPL}(\overline{\Sigma}_X^{\sigma/\tau})$ (see page 17) with*

$$\big(\mathcal{O}_{X_\sigma}(\phi_{\sigma/\tau})\otimes\mathcal{L}\big)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}. \tag{4.6}$$

In that case, $\mathtt{Trop}(L)$ is the tropical line bundle associated to the tropical Cartier divisor $(\overline{\Sigma}_X^{\sigma/\tau},\phi_{\sigma/\tau})_{\sigma/\tau\subseteq\mathtt{U}}$.

Proof. The total space $\mathtt{Trop}(L)$ is a tropical line bundle on $\mathtt{U}$ if and only if for every cell σ/τ of $\mathtt{U}$ there exists exactly one element in $\mathrm{Aff}_{\Sigma_L}(\overline{\Sigma}_L^{(\sigma/\tau)\times\rho_Z})/\mathrm{Aff}_{\overline{\Sigma}_X}(\overline{\Sigma}_X^{\sigma/\tau})$ with slope 1 along the ray ρ_Z dual to the zero section of L. Let I denote the set of functions in $\mathrm{sPL}(\overline{\Sigma}_L^{(\sigma/\tau)\times\rho_Z})$ with slope 1 along ρ_Z. There is a bijection

$$\alpha\colon \mathrm{sPL}(\overline{\Sigma}_X^{\sigma/\tau}) \to I,\quad \phi\mapsto \mathrm{pr}^*_{\overline{\Sigma}_X}\phi + \mathrm{pr}^*_{\rho_Z}\mathrm{id}, \tag{4.7}$$

where we identify ρ_Z with $\mathbb{R}_{\geq 0}$. The structure map $L\to X$ restricts to an isomorphism $V(\sigma\times\rho_Z)\to V(\sigma)$. Since $\mathcal{O}_L(\alpha(0))$ is the line bundle associated to the zero-section, and the normal bundle to the zero-section of L is L itself, we obtain an identification of

$$\mathcal{O}_L(\alpha(\phi))|_{V((\sigma/\tau)\times\rho_Z)} = (\mathcal{O}_{X_\sigma}(\phi)\otimes\mathcal{L})|_{V(\sigma)} \tag{4.8}$$

for every $\phi\in\mathrm{sPL}(\overline{\Sigma}_X^{\sigma/\tau})$. It follows that $I\cap\mathrm{Aff}_{\overline{\Sigma}_L}(\overline{\Sigma}_L^{(\sigma/\tau)\times\rho_Z})$ is either empty or an $\mathrm{Aff}_{\overline{\Sigma}_X}(\overline{\Sigma}_X^{\sigma/\tau})$-torsor; the nonemptiness is equivalent to the existence of a function $\phi_{\sigma/\tau}\in\mathrm{sPL}(\overline{\Sigma}_X^{\sigma/\tau})$ with

$$\big(\mathcal{O}_{X_\sigma}(\phi_{\sigma/\tau})\otimes\mathcal{L}\big)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}. \tag{4.9}$$

For the last statement, given two cells σ_1/τ_1 and σ_2/τ_2 of $\mathtt{U}$, it is immediate from (4.9) that the difference $\phi_{\sigma_2/\tau_2}-\phi_{\sigma_1/\tau_1}$ is affine on $\overline{\Sigma}_X^{\sigma_1/\tau_1}\cap\overline{\Sigma}_X^{\sigma_2/\tau_2}$ and hence $(\overline{\Sigma}_X^{\sigma/\tau},\phi_{\sigma/\tau})_{\sigma/\tau\subseteq\mathtt{U}}$ represents a tropical Cartier divisor. It is the Cartier divisor associated to the canonical piecewise linear section

$$\overline{\Sigma}_X \xrightarrow{\cong} \overline{\Sigma}_X\times\{0\}\subseteq\overline{\Sigma}_X\times\overline{\rho}_Z\subseteq\mathtt{Trop}(L) \tag{4.10}$$

restricted to $\mathtt{U}$, thus concluding the proof. □

Proposition 4.5. *Let $f\colon X\to Y$ be a morphism of toroidal varieties, and denote $\mathtt{F}:=\mathtt{Trop}(f)$. Let L be a line bundle on Y, and let $\mathtt{U}\subseteq\overline{\Sigma}_Y$ such that $\mathtt{Trop}(L)$ is a tropical line bundle on $\mathtt{U}$. Then $\mathtt{Trop}(f^*L)$ is a line bundle on $\mathtt{F}^{-1}\mathtt{U}$ and we have*

$$\mathtt{Trop}(f^*L)|_{\mathtt{F}^{-1}\mathtt{U}} \cong \mathtt{F}^*(\mathtt{Trop}(L)|_{\mathtt{U}}). \tag{4.11}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Proof. Let $\mathcal{L}$ denote the invertible sheaf associated to L. By Proposition 4.4, for every σ/τ of $\mathtt{U}$ there exists a function $\phi_{\sigma/\tau} \in \mathrm{sPL}(\overline{\Sigma}_Y^{\sigma/\tau})$ such that

$$\big(\mathcal{O}_{Y_\sigma}(\phi_{\sigma/\tau}) \otimes \mathcal{L}\big)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}, \tag{4.12}$$

and $\mathtt{Trop}(L)|_{\mathtt{U}}$ corresponds to the Cartier divisor $D = (\overline{\Sigma}_Y^{\sigma/\tau}, \phi_{\sigma/\tau})_{\sigma/\tau \subseteq \mathtt{U}}$. For every cell $\tilde{\sigma}/\tilde{\tau}$ of $\mathtt{F}^{-1}\mathtt{U}$, we have $\mathtt{F}(\overline{\Sigma}_X^{\tilde{\sigma}/\tilde{\tau}}) \subseteq \overline{\Sigma}_Y^{\sigma/\tau} \subseteq \mathtt{U}$ for the inclusion-minimal $\sigma/\tau \subseteq \overline{\Sigma}_Y$ with $\mathtt{F}(\tilde{\sigma}/\tilde{\tau}) \subseteq \sigma/\tau$. On the algebraic side, f induces a morphism $f|_{V(\tilde{\sigma})} \colon V(\tilde{\sigma}) \to V(\sigma)$ and we have

$$\begin{aligned}\mathcal{O}_{V(\tilde{\sigma})} \cong f|_{V(\tilde{\sigma})}^*\big(\big(\mathcal{O}_{Y_\sigma}(\phi_{\sigma/\tau}) \otimes \mathcal{L}\big)|_{V(\sigma)}\big) \cong \\ \big(f^*\mathcal{O}_{Y_\sigma}(\phi_{\sigma/\tau}) \otimes f^*\mathcal{L}\big)|_{V(\tilde{\sigma})} \cong \Big(\mathcal{O}_{X_{\tilde{\sigma}}}\big((\mathtt{F}^*\phi_{\sigma/\tau})|_{\overline{\Sigma}_X^{\tilde{\sigma}/\tilde{\tau}}}\big) \otimes f^*\mathcal{L}\Big)|_{V(\tilde{\sigma})}.\end{aligned} \tag{4.13}$$

Using Proposition 4.4, we see that $\mathtt{Trop}(f^*L)$ is a tropical line bundle on $\mathtt{F}^{-1}\mathtt{U}$ and that it is the line bundle associated to the tropical Cartier divisor

$$E \coloneqq \Big(\overline{\Sigma}_X^{\tilde{\sigma}/\tilde{\tau}}, (\mathtt{F}^*\phi_{\sigma/\tau})|_{\overline{\Sigma}_X^{\tilde{\sigma}/\tilde{\tau}}}\Big)_{\tilde{\sigma}/\tilde{\tau} \subseteq \mathtt{F}^{-1}\mathtt{U}}. \tag{4.14}$$

Noting that $E = \mathtt{F}^*D$ concludes the proof. □

Proposition 4.6. *Let $\phi \in \mathrm{sPL}(\Sigma_X)$. Then $\mathtt{Trop}(\mathcal{O}_X(\phi))$ is a tropical line bundle on $\overline{\Sigma}_X^{\sigma/\sigma}$ if and only if $\phi \in \mathrm{CP}(\sigma)$.*

Proof. Assume $\mathtt{Trop}(\mathcal{O}_X(\phi))$ is a tropical line bundle on $\overline{\Sigma}_X^{\sigma/\sigma}$. By Proposition 4.4, there is a function $\phi_\sigma \in \mathrm{sPL}(\overline{\Sigma}_X^{\sigma/\sigma})$ such that

$$\mathcal{O}_{X_\sigma}(\phi_\sigma + \phi)|_{V(\sigma)} \cong (\mathcal{O}_{X_\sigma}(\phi_\sigma) \otimes \mathcal{O}_X(\phi))|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}. \tag{4.15}$$

By definition of Aff_{Σ_X}, this means that

$$\phi_\sigma + \phi \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X^\sigma), \tag{4.16}$$

and after subtracting a constant we may assume that ϕ_σ vanishes on σ; that means $\phi_\sigma + \phi$ is an affine function on Σ_X^σ that restricts to $\phi|_\sigma$ on σ, which shows that $\phi \in \mathrm{CP}(\sigma)$.

Conversely, suppose that $\phi \in \mathrm{CP}(\sigma)$, and let $\xi \in \mathrm{Aff}_{\overline{\Sigma}_X}(\Sigma_X^\sigma)$ such that $\phi|_\sigma = \xi|_\sigma$. Then for every cell δ/τ of $\overline{\Sigma}_X^{\sigma/\sigma}$ (that means for all cones δ and τ with $\tau \subseteq \sigma \subseteq \delta$), the function $\xi - \phi$ vanishes on τ and thus defines an element $(\xi - \phi)|_{\Sigma_X^\delta} \in \mathrm{sPL}(\overline{\Sigma}_X^{\delta/\tau})$. Moreover, we have

$$(\mathcal{O}_{X_\delta}((\xi - \phi)|_{\Sigma_X^\delta}) \otimes \mathcal{O}_X(\phi))|_{V(\delta)} \cong \mathcal{O}_{X_\delta}(\xi|_{\Sigma_X^\delta})|_{V(\delta)} \cong \mathcal{O}_{V(\delta)}, \tag{4.17}$$

where the second isomorphism follows from the fact that $\xi|_{\Sigma_X^\delta}$ is affine at δ by Lemma 3.2. Applying Proposition 4.4 with $\phi_{\delta/\tau} = (\xi - \phi)|_{\Sigma_X^\delta}$ concludes the proof. □

Corollary 4.7. *Let $\mathcal{L}$ be an invertible sheaf on a toroidal variety X. Then $\mathtt{Trop}(\mathcal{L})$ is a tropical line bundle on $\overline{\Sigma}_X$ if and only if $\mathcal{L} \cong \mathcal{O}_X(\phi)$ for some $\phi \in \mathrm{CP}(\Sigma_X)$. In particular, if $\chi \in \mathrm{sPL}(\Sigma_X)$, then $\mathtt{Trop}(\mathcal{O}_X(\chi))$ is a tropical line bundle on $\overline{\Sigma}_X$ if and only if $\chi \in \mathrm{CP}(\Sigma_X)$.*

Proof. If $\mathtt{Trop}(\mathcal{L})$ is a tropical line bundle on $\overline{\Sigma}_X$, then $\mathcal{L} \cong \mathcal{O}_X(\phi)$ for some $\phi \in \mathrm{sPL}(\Sigma)$ by Proposition 4.4 (take $\phi = -\phi_{0_\Sigma/0_\Sigma}$ in the notation of the proposition). By Proposition 4.6, we have $\phi \in \mathrm{CP}(\Sigma_X)$.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Conversely, suppose $\phi \in \mathrm{CP}(\Sigma_X)$. Then $\mathtt{Trop}(\mathcal{O}_X(\phi))$ is a tropical line bundle on $\overline{\Sigma}_X^{\sigma/\sigma}$ for all $\sigma \in \Sigma$ by Proposition 4.6. Since $\overline{\Sigma}_X = \bigcup_{\sigma\in\Sigma} \overline{\Sigma}_X^{\sigma/\sigma}$, it follows that $\mathtt{Trop}(\mathcal{O}_X(\phi))$ is a tropical line bundle on all of $\overline{\Sigma}_X$.

Finally, if $\mathtt{Trop}(\mathcal{O}_X(\chi))$ is a tropical line bundle, then by what we have just shown we have $\mathcal{O}_X(\chi) \cong \mathcal{O}_X(\phi)$ for some $\phi \in \mathrm{CP}(\Sigma_X)$. This implies $\chi - \phi \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X)$ and hence $\chi \in \mathrm{CP}(\Sigma_X)$ because every affine function is combinatorially principal. □

5. Tropicalization of ψ classes

We turn our attention to the tautological cotangent line bundle to a section of a family of curves and study its tropicalization.

5.1. Tropicalization of the cotangent line bundle

In [6, Definition 6.16], tropical ψ classes are defined as first Chern classes of certain torsors over the sheaf of affine functions of the base of a family of tropical curves. In this section we develop language and describe suitable conditions that allow us to compare the tropicalization of the i-th cotangent line bundle with the torsors above.

Definition 5.1. Let Σ be a cone complex with a sheaf of affine functions Aff_Σ, let $\mathtt{U} \subseteq \overline{\Sigma}$ be an open subcomplex, and let $\phi \in \mathrm{sPL}(\Sigma \cap \mathtt{U})$. We define the sheaf $\mathrm{Aff}_\mathtt{U}(\phi)$ by

$$\mathrm{Aff}_\mathtt{U}(\phi)(\mathtt{V}) := \{\chi \in \mathrm{Aff}_\Sigma(\mathtt{V} \cap \Sigma) \mid \chi + \phi \text{ extends to a section of } \mathrm{PL}^{\mathrm{fin}}(\mathtt{V})\} \tag{5.1}$$

for open subsets $\mathtt{V} \subseteq \mathtt{U}$. Note that $\mathrm{Aff}_\mathtt{U}(\phi)$ is a pseudo-torsor over $\mathrm{Aff}_\mathtt{U}$, that is it is a torsor whenever it is nonempty. If $\mathrm{Aff}_\mathtt{U}(\phi)$ is a torsor, it defines a tropical line bundle, which we also denote by $\mathrm{Aff}_\mathtt{U}(\phi)$ by abuse of notation.

Proposition 5.2. *Let X be a toroidal variety, let $\phi \in \mathrm{sPL}(\Sigma_X)$ and let $\mathtt{U} \subseteq \overline{\Sigma}_X$. If $\mathrm{Aff}_\mathtt{U}(\phi)$ is a tropical line bundle, then $\mathtt{Trop}(\mathcal{O}_X(\phi))|_\mathtt{U}$ is a tropical line bundle as well and*

$$\mathtt{Trop}(\mathcal{O}_X(\phi))|_\mathtt{U} \cong \mathrm{Aff}_\mathtt{U}(\phi) \tag{5.2}$$

Proof. By Proposition 4.4, $\mathtt{Trop}(\mathcal{O}_X(\phi))|_\mathtt{U}$ is a tropical line bundle if and only if for every cell σ/τ of $\mathtt{U}$ there exists $\phi_{\sigma/\tau} \in \mathrm{sPL}(\overline{\Sigma}_X^{\sigma/\tau})$ with

$$\mathcal{O}_{X_\sigma}(\phi_{\sigma/\tau} + \phi)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)} \tag{5.3}$$

the collection of all the $-\phi_{\sigma/\tau}$'s generates the pseudo-torsor $\mathtt{Trop}(\mathcal{O}_X(\phi))|_\mathtt{U}(\overline{\Sigma}_X^{\sigma/\tau})$ of local affine sections. By definition of $\mathrm{Aff}_{\overline{\Sigma}_X}$, equation (5.3) means that $\phi_{\sigma/\tau} + \phi \in \mathrm{Aff}_{\overline{\Sigma}_X}(\overline{\Sigma}_X^{\sigma/\tau} \cap \Sigma_X)$, which is equivalent to $-(\phi_{\sigma/\tau} + \phi) \in \mathrm{Aff}_\mathtt{U}(\phi)(\overline{\Sigma}_X^{\sigma/\tau})$. The $\mathrm{Aff}_\mathtt{U}(\overline{\Sigma}_X^{\sigma/\tau})$-equivariant map

$$\mathtt{Trop}(\mathcal{O}_X(\phi))|_\mathtt{U}(\overline{\Sigma}_X^{\sigma/\tau}) \to \mathrm{Aff}_\mathtt{U}(\phi)(\overline{\Sigma}_X^{\sigma/\tau}), \quad -\phi_{\sigma/\tau} \mapsto \chi = -\phi_{\sigma/\tau} - \phi \tag{5.4}$$

is an isomorphism of pseudo-torsors. Therefore, $\mathtt{Trop}(\mathcal{O}_X(\phi))|_\mathtt{U}$ being a line bundle is equivalent to $\mathrm{Aff}_\mathtt{U}(\phi)$ being a torsor, and the collection of maps (5.4) for $\sigma/\tau \subseteq \mathtt{U}$ defines an isomorphism between $\mathtt{Trop}(\mathcal{O}_X(\phi))|_\mathtt{U}$ and $\mathrm{Aff}_\mathtt{U}(\phi)$. □

We now apply these constructions to our main objects of interest: families of pointed curves and their tropicalizations.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Proposition 5.3. *Let $\mathcal{C} \to \mathcal{B}$ be a tropicalizable family of n-marked stable curves with associated family of tropical curves $\overline{\mathtt{C}} \to \overline{\mathtt{B}}$, and let $1 \le i \le n$. Let ϕ be the piecewise linear function that has slope 1 on the ray ρ_i corresponding to the i-th section s_i and is 0 on all cones not containing ρ_i. Then there exists an isomorphism of pseudo $\mathrm{Aff}_{\overline{\mathtt{C}}}$-torsors*

$$\mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{s}_i) \cong \mathrm{Aff}_{\overline{\mathtt{C}}}(\phi). \tag{5.5}$$

Proof. Let $\mathtt{U} \subseteq \overline{\mathtt{C}}$ be open. The sections of $\mathrm{Aff}_{\overline{\mathtt{C}}}(\phi)(\mathtt{U})$ are affine functions χ on $\mathtt{U} \cap \mathtt{C}$ for which $\chi + \phi$ can be extended continuously to $\mathtt{U}$. Since ϕ has slope 1 on ρ_i and vanishes on cones that do not contain ρ_i, this is equivalent to saying that χ extends continuously to $\mathtt{U} \setminus \mathtt{s}_i$ and has slope -1 in the direction of ρ_i near $\mathtt{s}_i$. By Definition 3.10, the continuous extension of χ to $\mathtt{U} \setminus \mathtt{s}_i$ is automatically affine on $\mathtt{U} \setminus \mathtt{s}_i$. We conclude that $\mathrm{Aff}_{\overline{\mathtt{C}}}(\phi)(\mathtt{U})$ is in natural bijection with the subset of $\mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{U} \setminus \mathtt{s}_i)$ of affine functions that approach $\mathtt{s}_i$ with slope -1. But this set equals $\mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{s}_i)(\mathtt{U})$ by definition. □

Definition 5.4. Let $\mathcal{C} \to \mathcal{B}$ be a tropicalizable family of n-marked stable curves with tropicalization $\overline{\mathtt{C}} \to \overline{\mathtt{B}}$. If $\mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{s}_i)$ is a tropical line bundle, we define the i-th tropical cotangent bundle analogously to [6, Definition 6.16]:

$$\mathbb{L}_i^{\mathtt{trop}} := (\mathtt{s}_i)^* \mathrm{Aff}_{\overline{\mathtt{C}}}(-\mathtt{s}_i), \tag{5.6}$$

where $\mathtt{s}_i = \mathtt{Trop}(s_i)$ is the tropicalization of the i-th section

Theorem 5.5. *With setup and notation as in Definition 5.4, assume $\mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{s}_i)$ is a tropical line bundle. Then $\mathbb{L}_i$ tropicalizes to a tropical line bundle and we have*

$$\mathtt{Trop}(\mathbb{L}_i) \cong \mathbb{L}_i^{\mathtt{trop}}. \tag{5.7}$$

Proof. Let ϕ be the function on $\mathtt{C}$ that has slope 1 on the ray ρ_i corresponding to s_i and is 0 on all cones of $\mathtt{C}$ that do not contain ρ_i. By Proposition 5.3, we have $\mathrm{Aff}_{\overline{\mathtt{C}}}(\phi) \cong \mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{s}_i)$. Using the assumption that $\mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{s}_i)$ is a tropical line bundle, we conclude that $\mathrm{Aff}_{\overline{\mathtt{C}}}(\phi)$ is a tropical line bundle as well. Applying Proposition 5.2, we see that the bundle $\mathcal{O}_{\mathcal{C}}(s_i) \cong \mathcal{O}_{\mathcal{C}}(\phi)$ is tropicalizable and that the second isomorphism in the chain of isomorphisms

$$\mathtt{Trop}(\mathcal{O}_{\mathcal{C}}(s_i)) \cong \mathtt{Trop}(\mathcal{O}_{\mathcal{C}}(\phi)) \cong \mathrm{Aff}_{\overline{\mathtt{C}}}(\phi) \cong \mathrm{Aff}_{\overline{\mathtt{C}}}(\mathtt{s}_i) \tag{5.8}$$

exists, the other ones being evident. Dualizing and applying Proposition 4.5 to the section s_i then shows that $\mathtt{Trop}(\mathbb{L}_i)$ is defined and that we have a sequence of isomorphisms

$$\begin{aligned} \mathbb{L}_i^{\mathtt{trop}} &= \mathtt{s}_i^* \mathrm{Aff}_{\overline{\mathtt{C}}}(-\mathtt{s}_i) \cong \\ &\cong (\mathtt{s}_i)^* \mathtt{Trop}(\mathcal{O}_{\mathcal{C}}(-s_i)) \cong \mathtt{Trop}(s_i^* \mathcal{O}_{\mathcal{C}}(-s_i)) \cong \mathtt{Trop}(\mathbb{L}_i). \end{aligned} \tag{5.9}$$

□

5.2. Tropicalizations of Cycles and Tropical Cycles

In order to translate Theorem 5.5 into a statement about ψ classes, we need to take the first Chern class of a tropical line bundle. In this section we present the necessary bits of tropical intersection theory for this task. Similar definitions and constructions appeared in [6, Section 6]; here we present them in the slightly larger generality needed for the current context.

Definition 5.6. Let X be a complete toroidal variety with associated complex Σ_X, and let $c \in A^*(X)$. Then the tropicalization of c is given by the weight

$$\mathtt{Trop}_X(c) \colon \Sigma_X \to \mathbb{Z}, \quad \sigma \mapsto \int_X c \cdot [V(\sigma)] \tag{5.10}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Proposition 5.7. *Let X be an n-dimensional complete toroidal variety and let $c \in A^k(X)$. Then the $(n-k)$-weight $\mathtt{Trop}(c)$ is balanced.*

Proof. Balancing can be checked locally around each cone $\sigma \in \Sigma_X$ of dimension $n-k-1$. The weight $\mathtt{Trop}_X(c)$ induces a weight on Σ_X^σ/σ and this weight coincides with $\mathtt{Trop}(\iota^* c)$, where $\iota\colon V(\sigma) \to X$ is the inclusion. Because the functions near σ that vanish on σ are pulled back from $\Sigma_X^\sigma/\sigma \cong \Sigma_{V(\sigma)}$ by Lemma 3.11, it suffices to prove that $\mathtt{Trop}(\iota^* c)$ is balanced on $\Sigma_{V(\sigma)}$. We have reduced to the case where $k = n-1$ and $\mathtt{Trop}(c)$ is 1-dimensional. Let $\phi \in \mathrm{Aff}_{\Sigma_X}(\Sigma_X)$. We have

$$\sum_{\rho \in \Sigma_X(1)} \mathrm{slope}_\rho(\phi) c(\rho) = \int_X c \cdot \left(\sum_{\rho \in \Sigma_X(1)} \mathrm{slope}_\rho(\phi)[V(\rho)] \right) = \int_X c \cdot c_1(\mathcal{O}_X(\phi)) = 0, \tag{5.11}$$

where the last equality follows from the fact that $\mathcal{O}_X(\phi)$ is trivial because ϕ is affine. The vanishing of the first term in (5.11) shows balancing, thus concluding the proof. □

Applying Proposition 5.7 to the unity of the Chow ring, we obtain a tropical fundamental class.

Corollary 5.8. *Let X be an n-dimensional complete toroidal variety. The constant weight with value 1 on the n-dimensional cones of Σ_X is balanced. We denote by $[\Sigma_X]$ the corresponding tropical cycle.*

Proof. The constant weight 1 on n-dimensional cones is equal to $\mathtt{Trop}_X(1)$, where $1 \in A^0(X)$ is the multiplicative identity of the Chow ring. The assertion follows from Proposition 5.7. □

Definition 5.9. For a complete toroidal variety X of dimension n we call the tropical cycle represented by the balanced n-weight with value 1 on all n-dimensional cones of Σ_X the **fundamental class** of Σ_X and denote it by $[\Sigma_X]$

The final missing ingredient to show that tropicalizations of ψ classes coincide with tropical ψ classes operationally is the compatibility of tropicalization with intersections with first Chern classes of line bundles. This is the content of the following proposition, which generalizes [11, Proposition 4.13].

Proposition 5.10. *Let X be a complete toroidal variety with cone complex Σ_X, let $c \in A^k(X)$, and let $\mathcal{L}$ be a tropicalizable line bundle on X with tropicalization $\mathtt{L}$ on $\overline{\Sigma}_X$. Then we have*

$$[\mathtt{Trop}_X(c_1(\mathcal{L}) \cdot c)] = c_1(\mathtt{L}) \frown [\mathtt{Trop}_X(c)] \tag{5.12}$$

Proof. By Corollary 4.7, there exists a function $\phi \in \mathrm{CP}(\Sigma_X)$ with $\mathcal{L} \cong \mathcal{O}_X(\phi)$. Let $\tau \in \Sigma_X$ be a cone of dimension $n-k-1$, where $n = \dim(X)$. We need to show that the weights on both sides of (5.12) have the same value on τ. Since $\phi \in \mathrm{CP}(\Sigma_X)$, we may assume that $\phi|_\tau = 0$. We denote by ϕ^τ the induced function on Σ_X^τ/τ. Then we have $\mathcal{O}_X(\phi)|_{V(\tau)} \cong \mathcal{O}_{V(\tau)}(\phi^\tau)$ and thus

$$c_1(\mathcal{L}) \cdot [V(\tau)] = \sum_\sigma \mathrm{slope}_{\sigma/\tau}(\phi^\tau)[V(\sigma)] \quad \text{in } A^{k+1}(X), \tag{5.13}$$

where the sum is taken over all $(n-k)$-dimensional cones of Σ_X containing τ. Therefore, we have

$$\mathtt{Trop}_X(c_1(\mathcal{L}) \cdot c)(\tau) = \int_{V(\tau)} c_1(\mathcal{L}) \cdot c = \sum_\sigma \mathrm{slope}_{\sigma/\tau}(\phi^\tau) \mathtt{Trop}_X(c)(\sigma). \tag{5.14}$$

By Proposition 4.4, the function $-\phi^\tau$ is the local equation on $\overline{\Sigma}^{\tau/\tau}$ for a tropical Cartier divisor whose associated line bundle is $\mathtt{L}$. Therefore, the last term in (5.14) equals the local expression for the weight at τ of $c_1(\mathtt{L}) \frown [\mathtt{Trop}_X(c)]$ (compare with (2.3)). □

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Corollary 5.11. *Let $\mathcal{C} \to \mathcal{B}$ be a tropicalizable family of n-marked stable curves with tropicalization $\mathtt{C} \to \mathtt{B}$, and let $1 \le i \le n$. Assume $\mathrm{Aff}_{\overline{\mathtt{C}}}(-\mathtt{s}_i)$ is a tropical line bundle. Then*

$$\mathtt{Trop}(\psi_i) = \psi_i^{\mathtt{trop}}. \tag{5.15}$$

Proof. Combine Theorem 5.5 and Proposition 5.10. □

6. An extended example in genus one

This section is dedicated to an extended example of the theory set-up in the previous part of the paper. We study two 2-dimensional families of genus-one curves with two marked points. They arise from a single family of admissible covers by forgetting different subsets of the marked ramification. We exhibit the tropical ψ class as a 1-dimensional cycle on the base and show it is the tropicalization of the corresponding algebraic cycle.

6.1. Algebraic set-up

We consider the (algebraic) moduli space of admissible covers

$$\overline{Adm}_{1\to 0}((3),(2,1)^4). \tag{6.1}$$

Its closed points parameterize degree-3 admissible covers of a rational curve, where one branch point is the image of exactly one ramification point (of order 3), whereas the other four branch points are simple: their inverse image consists of two points, one unramified and one simply ramified. There is a natural branch morphism:

$$\mathrm{br} : \overline{Adm}_{1\to 0}((3),(2,1)^4) \to \overline{\mathcal{M}}_{0,5}, \tag{6.2}$$

recording the target of the cover together with the marked branch points. It is a finite morphism of degree 9. We adopt the convention that all inverse images of branch points are marked, and therefore we have a natural source morphism:

$$\mathrm{src} : \overline{Adm}_{1\to 0}((3),(2,1)^4) \to \overline{\mathcal{M}}_{1,9}. \tag{6.3}$$

Let $r_1 : \overline{\mathcal{M}}_{1,9} \to \overline{\mathcal{M}}_{1,2}$ (resp. r_2) be the morphism that forgets all marks except (the one corresponding to) the point of triple ramification and one of the unramified (resp. simply ramified) marked points. We adopt the notation that the first mark in $\overline{\mathcal{M}}_{1,2}$ is the point of triple ramification of the cover.

One may compose $r_i \circ \mathrm{src}$, and pull-back the universal family of $\overline{\mathcal{M}}_{1,2}$ to obtain a family of genus-one, two-marked curves:

$$\begin{array}{ccc} \mathcal{C}_i & \longrightarrow & \mathcal{U}_{1,2} \\ \downarrow & & \downarrow \\ \overline{Adm}_{1\to 0}((3),(2,1)^4) = \mathcal{B}_i & \xrightarrow{\varphi_i = r_i \circ \mathrm{src}} & \overline{\mathcal{M}}_{1,2}. \end{array} \tag{6.4}$$

We denote the space of admissible covers by $\mathcal{B}_i$ both to shorten the notation and to emphasize that we think of it as the base of a family of curves, determined by the map $\varphi_i : \mathcal{B}_i \to \overline{\mathcal{M}}_{1,2}$.

Lemma 6.1. *We determine the degree of the maps φ_1, φ_2.*

1. *The degree of $\varphi_1 : \mathcal{B}_1 \to \overline{\mathcal{M}}_{1,2}$ is equal to 24, that is,*

$$\varphi_{1,*}([\mathcal{B}_1]) = 24[\overline{\mathcal{M}}_{1,2}]. \tag{6.5}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

2. *The degree of* $\varphi_2 : \mathcal{B}_2 \to \overline{\mathcal{M}}_{1,2}$ *is equal to* 6*, that is,*

$$\varphi_{2,*}([\mathcal{B}_2]) = 6[\overline{\mathcal{M}}_{1,2}]. \tag{6.6}$$

Proof. Consider a general point $[(E, p_1, p_2)] \in \overline{\mathcal{M}}_{1,2}$, that is, a smooth genus-one curve with two marked points. A map $f : E \to \mathbb{P}^1$ in the inverse image of $[(E, p_1, p_2)]$ via φ_1 corresponds to a principal divisor of the form $3p_1 - p_2 - 2x$, where x is a point of E to be determined. One can solve for x in $Jac(E) \cong E$, and find that there are four solutions (pick any particular solution and then one can add any one of the four two-torsion points of $Jac(E)$). There are therefore four distinct maps that have triple ramification at p_1 and are unramified at p_2 (but where p_2 is the inverse image of a branch point). Generically, for each of these maps there are 3! ways to label the remaining simple ramification points, and once that is done the remaining marks are determined. There is therefore a total of 24 inverse images, which proves (1).

Similarly, to determine the inverse images via φ_2 of $[(E, p_1, p_2)]$ one must solve the equation $3p_1 - 2p_2 - x = 0 \in Jac(E)$, which has a unique solution. There is only one map with triple ramification at p_1 and simple ramification at p_2 and 3! = 6 ways to label the remaining simple ramification points, for a total of six inverse images. This proves (2). □

We conclude the description of the algebraic set up by spelling out the relation between the class ψ_1 on $\overline{\mathcal{M}}_{1,2}$ and the pushforward of the class ψ_1 on $\mathcal{B}_i$ via the morphism φ_i.

Lemma 6.2. *Denote by W the class of the Weierstrass divisor in* $\overline{\mathcal{M}}_{1,2}$ *(parameterizing curves of genus one where both marks are Weierstrass points of the same* g^1_2*). We have the following equalities in* $A^1(\overline{\mathcal{M}}_{1,2})$*:*

$$\varphi_{1,*}(\psi_1) = 24\psi_1, \tag{6.7}$$
$$\varphi_{2,*}(\psi_1) = 6(\psi_1 + W). \tag{6.8}$$

Proof. Since we need to pull-back and push-forward classes on different spaces that are all called ψ_1 on their respective space, we adopt the convention of just letting the context indicate which class is which. There are two fundamental pull-back relations that we will be using:

$$\psi_1 = r_i^*(\psi_1) + D_i, \tag{6.9}$$

where D_i denotes the sum of all rational tails boundary divisors of $\overline{\mathcal{M}}_{1,9}$ parameterizing nodal curves where the mark 1 lies on the rational tail, and the mark 2 lies on the genus-one component of the curve. This relation follows from iterated application of the standard pull-back relation for ψ classes along forgetful morphisms (see, e.g., [19, Lemma 1.3.1]). With respect to the morphism src,

$$\psi_1 = \mathrm{src}^*(\psi_1), \tag{6.10}$$

follows from the fact that the first mark is a point of full ramification for the cover, that can therefore not live in an unstable component of the source curve of the admissible cover.

Next we analyze the divisor $\mathrm{src}^*(D_i)$. When $i = 1$, that is, when the second mark corresponds to a nonramified point of the cover, then one may observe that the image of src is disjoint from D_1, and therefore $\mathrm{src}^*(D_1) = 0$.

In the other case, $\mathrm{src}^*(D_2)$ is the sum of three irreducible boundary divisors in the space of admissible covers, depicted in Figure 7. They parameterize the same topological type of covers, and differ only in the labeling of the inverse images of the branch points.

The final step is to apply the projection formula. In the case $i = 1$, we have that $\psi_1 = \varphi_1^*(\psi_1)$, and we have shown that φ_1 is a finite morphism of degree 24, from which (6.7) follows.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

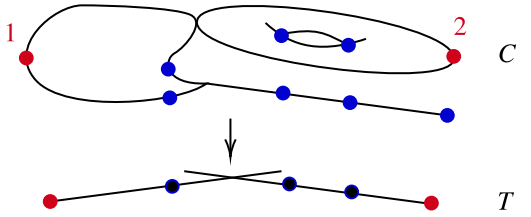


Figure 7. *A topological cartoon of the admissible covers parameterized by the general points of* $\mathrm{src}^*(D_2)$. *The blue points are the marked points that get forgotten by the map* r_2, *which we did not label to avoid cluttering the figure. There are three possible ways to order the pairs of blue points, giving rise to three irreducible components.*

In the second case, we have

$$\varphi_{2,*}(\psi_1) = 6\psi_1 + \varphi_{2,*}(\mathrm{src}^*(D_2)). \tag{6.11}$$

One finally observes that each of the three boundary divisors comprising $\mathrm{src}^*(D_2)$ pushes forward to $2W$ (the factor of 2 coming from the gluing ghost automorphisms of the node), thus yielding (6.8). □

6.2. Tropical family

We consider the moduli space of tropical admissible covers as introduced in [3, 7]

$$\overline{Adm}^{\mathtt{trop}}_{1\to 0}((3),(2,1)^4). \tag{6.12}$$

It is a two-dimensional cone complex, with 20 rays and 45 two-dimensional cones which we call faces for simplicity. Forgetting the labeling of the five points on the target trees, one can group the rays into 4 distinct types; similarly, the faces are grouped into 5 types. The topological types of tropical covers corresponding to these types are depicted in Figure 8. We describe the various incidence properties of rays and faces.

Rays:

type ρ_{a} there are six rays of type ρ_a; each is adjacent to two faces of type σ_1, a face of type σ_4 and one of type σ_5;
type ρ_{b} there are six rays of type ρ_b; each is adjacent to two faces of type σ_2, two faces of type σ_3 and one of type σ_5;
type ρ_{c} there are four rays of type ρ_c; each is adjacent to three faces of type σ_1, and three faces of type σ_2;
type ρ_{d} there are four rays of type ρ_d; each is adjacent to three faces of type σ_3.

Faces:

type σ_1 there are twelve faces of type σ_1, each bounded by a ray of type ρ_a and one of type ρ_c;
type σ_2 there are twelve faces of type σ_2, each bounded by a ray of type ρ_b and one of type ρ_c;
type σ_3 there are twelve faces of type σ_3, each bounded by a ray of type ρ_b and one of type ρ_d;
type σ_4 there are three faces of type σ_4, each bounded by two rays of type ρ_a;
type σ_5 there are six faces of type σ_5, each bounded by a ray of type ρ_a and one of type ρ_b.

6.3. Tropicalization and the fundamentalish cycle.

The tropicalization of the Hurwitz space (compactified in the space of admissible covers) is a cone complex which may be naturally identified with the Berkovich skeleton of the analytic Hurwitz space. In [7], the authors exhibit a natural tropicalization map $\mathtt{T}$ from the Berkovich analytic skeleton Σ of

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

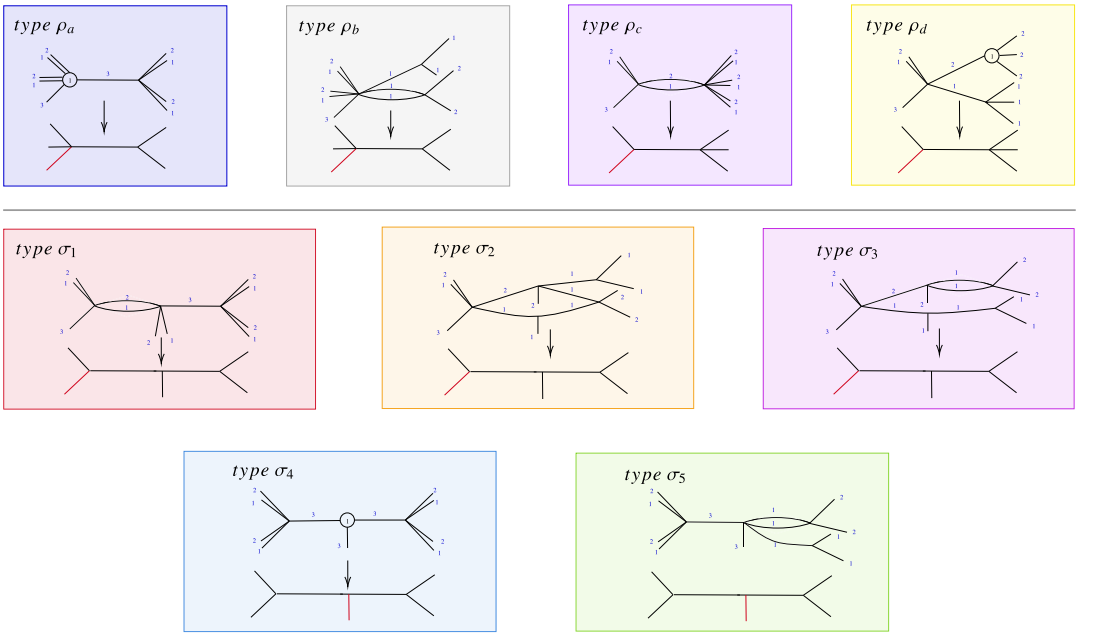


Figure 8. *The topological types of tropical covers corresponding to the rays and two-dimensional cones of* $\overline{Adm}^{\mathrm{trop}}_{1\to 0}((3),(2,1)^4)$. *Types are labeled as in their description in Section 6.2. The color coding is meant to facilitate the understanding of the structure of the cone complex of tropical admissible covers, represented in Figure 9.*

the Hurwitz space (Σ agrees with the cone complex associated to the toroidal compactification given by the space of admissible covers) to the cone complex of tropical admissible covers. This map is a strict morphism of cone complexes, but in general it is not an isomorphism. Surjectivity fails if some tropical admissible cover is not dual to any algebraic one. This situation is detected by the vanishing of some local Hurwitz number associated to a vertex of some tropical admissible cover. Injectivity may fail for two reasons: multiple strata may give rise to the same topological type of tropical cover if they arise from the normalization of a singular locus in the space of admissible covers: this happens when in a tropical admissible cover, there are multiple edges mapping to a given edge of the target graph, and their expansion factors are not pairwise coprime. Alternatively, it may be that there are multiple algebraic covers of a component of the target curve: this situation is a bit more delicate and it is assessed by observing the local Hurwitz numbers and the automorphism factors around the vertices of a tropical admissible covers.

The tropical source and target morphisms factor via the tropicalization map $\mathtt{T}$; in the specific case we are studying they give rise to the following commutative diagram:

$$\begin{array}{ccccc} \Sigma = \Sigma_{\mathcal{B}_i} = \Sigma_{\overline{Adm}_{1\to 0}((3),(2,1)^4)} & & & & \\ & \searrow^{\mathtt{Src}} & & & \\ & \swarrow_{\mathtt{T}} & & & \\ & & \overline{Adm}^{\mathrm{trop}}_{1\to 0}((3),(2,1)^4) & \xrightarrow{\mathtt{src}} & \overline{\mathcal{M}}^{\mathrm{trop}}_{1,9} \\ & \swarrow_{\mathtt{Br}} & \downarrow_{\mathtt{br}} & & \\ & & \overline{\mathcal{M}}^{\mathrm{trop}}_{0,5} & & \end{array} \tag{6.13}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Table 1. *The factors computing the weights of two-dimensional cones in* $\overline{Adm}^{\mathtt{trop}}_{1\to 0}((3),(2,1)^4)$.

Face type	W1	W2	W3	$\omega(\sigma)$
σ_1	1	1	1	1
σ_2	1	1	1	1
σ_3	$\frac{1}{2}$	1	1	$\frac{1}{2}$
σ_4	1	$\frac{1}{3}$	1	$\frac{1}{3}$
σ_5	$\frac{1}{2}$	2	1	1

By the functoriality of tropicalization (Proposition 3.5), any (local, affine) linear function on $\overline{\mathcal{M}}^{\mathtt{trop}}_{1,9}$ (resp. $\overline{\mathcal{M}}^{\mathtt{trop}}_{0,5}$) pulls-back via $\mathtt{Src}$ (resp. $\mathtt{Br}$) to a linear function on the cone complex Σ of $\overline{Adm}_{1\to 0}((3),(2,1)^4)$.

We declare that linear functions pulled back via $\mathtt{src}$ and $\mathtt{br}$ are linear functions on the tropical moduli space $\overline{Adm}^{\mathtt{trop}}_{1\to 0}((3),(2,1)^4)$. By the commutativity of (6.13) such a declaration is compatible with further pulling back via $\mathtt{T}$. As a further consequence, any intersection-theoretic computation on Σ involving cycles pulled back from source or target morphism may be reproduced, by the projection formula, on the moduli space $\overline{Adm}^{\mathtt{trop}}_{1\to 0}((3),(2,1)^4)$.

It is then necessary to study the push-forward of the fundamental class of Σ via $\mathtt{T}$. We have

$$\mathtt{T}_*([\Sigma]) = \sum_{\sigma} \omega(\sigma)\sigma, \tag{6.14}$$

where the sum runs over all top-dimensional cones of $\overline{Adm}^{\mathtt{trop}}_{1\to 0}((3),(2,1)^4)$. The weight $\omega(\sigma)$, for a cone σ parameterizing tropical admissible covers $\Gamma \to \Delta$ of combinatorial type Θ is as in [7, Definition 22] the product of the following factors:

(W1) a factor of $\frac{1}{|\mathrm{Aut}_0(\Theta)|}$, with $\mathrm{Aut}_0(\Theta)$ denoting automorphisms of the tropical cover fixing the source curve;
(W2) a factor of local Hurwitz numbers $\prod_{v\in V(\Gamma)} H(v)$;
(W3) a factor of $\prod_{e\in E(\Delta)} w_e$, where w_e is the product of the expansion factors above the edge e, divided by their LCM.

For the moduli space $\overline{Adm}^{\mathtt{trop}}_{1\to 0}((3),(2,1)^4)$ that we are studying, all local Hurwitz numbers appearing are nonzero, hence the map $\mathtt{T}$ is surjective. Every time there are multiple edges mapping to an edge of a target graph, their expansion factors are pairwise coprime. All local Hurwitz numbers appearing are equal to one, except for two:

- in the tropical cover parameterized by cone σ_4 the genus-one vertex has local Hurwitz number equal to $1/3$; this corresponds to the unique genus-one curve admitting a degree 3 automorphism;
- the central vertex in the tropical cover parameterized by cone σ_5 has local Hurwitz number equal to 2, corresponding to different labelings of the three unramified points: there are in fact six possible labelings, but a factor of three is divided out because of the automorphism of the cover.

The covers parameterized by cones of type σ_3 and σ_5 have a group of automorphisms of order two, while all other types have no nontrivial automorphisms. This information is systematically collected in Table 1.

In conclusion, if we denote by $\sigma_i^{\mathtt{tot}}$ the formal sum of all two-dimensional cones of type σ_i, we obtain:

$$\mathtt{T}_*([\Sigma]) = \sigma_1^{\mathtt{tot}} + \sigma_2^{\mathtt{tot}} + \frac{1}{2}\sigma_3^{\mathtt{tot}} + \frac{1}{3}\sigma_4^{\mathtt{tot}} + \sigma_5^{\mathtt{tot}}. \tag{6.15}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

The cycle in (6.15) plays the role of the fundamental class in the forthcoming computations, and we therefore dub it the *fundamentalish cycle* of the moduli space of tropical admissible covers.

6.4. The tropical ψ class

In this section we study the class ψ_1, supported on the mark with expansion factor 3, in the two-dimensional family of tropical admissible covers introduced in the previous sections. Our first result is that the tropical ψ class is in fact the tropicalization of the algebraic ψ class.

Proposition 6.3. *Consider the families of genus-one, 9-marked algebraic and tropical curves given by the morphisms* src *and* $\mathtt{src}$ *introduced in the previous sections. Then we have the following equality of operational classes with $\mathbb{Q}$-coefficients:*

$$\psi_1^{\mathtt{trop}} = \mathtt{Trop}(\psi_1), \tag{6.16}$$

where we have omitted explicitly writing the pull-back via the source morphism in order to not burden the notation unnecessarily.

Proof. By Corollary 5.11 it suffices to show that the affine structure on the family of tropical admissible covers induced by tropicalization contains enough information to define $\psi_1^{\mathtt{trop}}$. In fact, as the desired equality has coefficients in $\mathbb{Q}$, it suffices to show that there are enough affine functions to define $3\psi_1^{\mathtt{trop}}$.

Denote by $\hat{\psi}_1^{\mathtt{trop}}$ the ψ class at the first mark of $\overline{\mathcal{M}}_{0,5}^{\mathtt{trop}}$. By Theorem 3.17, the affine structure on $\overline{\mathcal{M}}_{0,5}^{\mathtt{trop}}$ induced by tropicalization agrees with the one induced by tropical cross-ratios. In particular, the ψ class $\hat{\psi}_1^{\mathtt{trop}}$ is defined with respect to the affine structure induced by tropicalization.

Consider the tropical branch morphisms $\mathtt{br}$. By Proposition 3.5, affine structures are functorial, so it follows that there are enough affine functions to define $\mathtt{br}^*(\hat{\psi}_1^{\mathtt{trop}})$. But by the same argument as in the proof of [6, Prop 7.8, Eq. (145)], we have

$$\mathtt{br}^*(\hat{\psi}_1^{\mathtt{trop}}) = 3\psi_1^{\mathtt{trop}}, \tag{6.17}$$

hence there are also enough affine functions to define the class $3\psi_1^{\mathtt{trop}}$, which is what we needed to show. □

Remark 6.4. As a sanity check, we observe that the steps in the proof of Proposition 6.3 are compatible with functoriality of logarithmic tropicalization ([26, Proposition 6.3]) and Ionel's lemma [14, Lemma 1.17] which provides the comparison statement on the algebraic side:

$$\mathrm{br}^*(\hat{\psi}_1) = 3\psi_1. \tag{6.18}$$

Having established the tropicalization statement, we now describe the cycle obtained by capping the $\psi_1^{\mathtt{trop}}$ with the fundamentalish cycle (6.15).

Proposition 6.5. *Consider the family of tropical curves corresponding to the moduli function* $\mathtt{src}$: $\overline{Adm}_{1\to 0}^{\mathtt{trop}}((3),(2,1)^4) \to \overline{\mathcal{M}}_{1,9}^{\mathtt{trop}}$. *For $x = a, b, c, d$, denote by $\rho_x^{\mathtt{tot}}$ the formal sum of all rays of type ρ_x, as described in Section 6.2. We have*

$$\psi_1^{\mathtt{trop}} \frown \mathtt{T}_*([\Sigma]) = \frac{2}{3}\rho_a^{\mathtt{tot}} + \rho_b^{\mathtt{tot}}. \tag{6.19}$$

Proof. The tropical cycle $\psi_1^{\mathtt{trop}} \frown \mathtt{T}_*([\Sigma]) = \sum c_\rho \rho$ is a linear combination of the rays of $\overline{Adm}_{1\to 0}^{\mathtt{trop}}((3),(2,1)^4)$. The coefficient c_ρ in front of a given ray ρ is obtained as follows: let $\psi_{1,\rho}$ be a

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

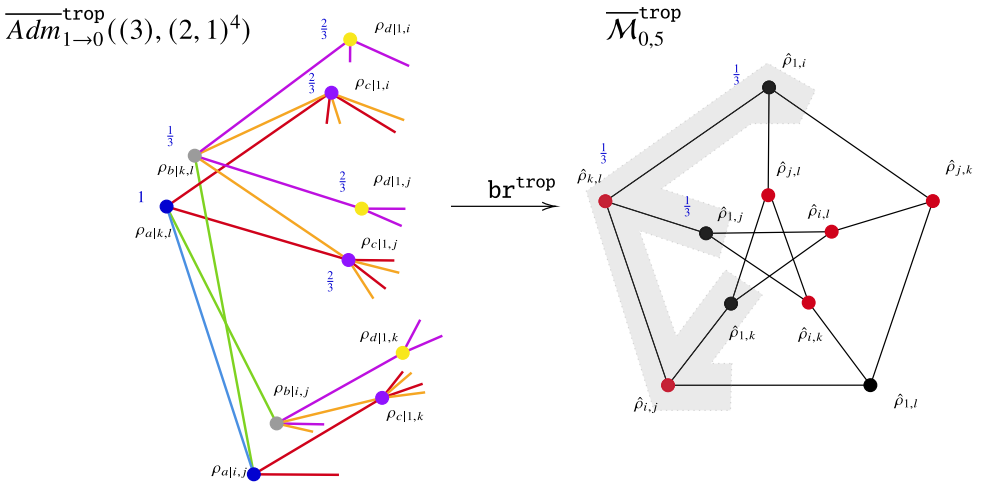


Figure 9. *The restriction of the map* $\mathtt{br}^{trop}$ *to the inverse image of the shaded region. In blue are the nonzero slopes of the piecewise linear function representing* $\frac{1}{3}\psi_1^{\mathrm{trop}}$ *and of its pullback to the space of tropical admissible covers. The rays and faces of the space of tropical admissible covers are color coded by the type of cones as in Figure 8.*

piecewise linear function representing the cycle ψ_1^{trop} with the property that it has slope zero along ρ; then

$$c_\rho = -\sum_{\sigma\succ\rho} \omega(\sigma)\psi_{1,\rho}(u_{\rho/\sigma}), \tag{6.20}$$

where $u_{\rho/\sigma}$, the lattice normal vector to ρ in σ, may in this case be taken to be the primitive vector for the other ray of σ. We perform this computation for the four types of rays described in Section 6.2. First we introduce some notation: for $i,j\in[5]$, denote by $\hat{\rho}_{i,j}$ the ray in $\overline{\mathcal{M}}^{\mathrm{trop}}_{0,5}$ dual to the divisor $D_{\{i,j\}}$[6], and $\varphi_{\hat{\rho}_{i,j}}$ the piecewise linear function with slope -1 along $\hat{\rho}_{i,j}$ and zero along all other rays. For $x\in\{a,b,c,d\}$, we denote by $\rho_{x|i,j}$ the ray of type ρ_x in $\mathtt{br}^{-1}(\hat{\rho}_{i,j})$, and by $\varphi_{\rho_{x|i,j}}$ the corresponding piecewise linear function with slope -1 along the ray, and zero along all other rays. Figure 9 illustrates the situation and it may be helpful in following the computation.

Type ρ_{A}: for $i,j\neq 1$, consider $\rho_{a|i,j}$. In $\overline{M}_{0,5}$ one may represent the class ψ_1 as the following linear combination of the boundary divisors:

$$\psi_1 = D_{1,i} + D_{1,j} + D_{k,l}, \tag{6.21}$$

where $\{i,j,k,l\}=\{2,3,4,5\}$. Correspondingly, the piecewise linear function on $\overline{\mathcal{M}}^{\mathrm{trop}}_{0,5}$ given by

$$\hat{\rho}_{1,i} + \hat{\rho}_{1,j} + \hat{\rho}_{k,l}, \tag{6.22}$$

may be chosen to represent $\hat{\psi}_1^{\mathrm{trop}}$. By (6.17), $\psi_1^{\mathrm{trop}} = \frac{1}{3}\mathtt{br}^*(\hat{\psi}_1^{\mathrm{trop}})$, and pulling back the representative from (6.22) we have a piecewise linear function representing ψ_1^{trop} which has slope zero along the ray $\rho_{a|i,j}$. Precisely:, ψ_1^{trop} is represented by

$$\frac{2}{3}\varphi_{\rho_{c|1,i}} + \frac{2}{3}\varphi_{\rho_{d|1,i}} + \frac{2}{3}\varphi_{\rho_{c|1,j}} + \frac{2}{3}\varphi_{\rho_{d|1,j}} + \varphi_{\rho_{a|k,l}} + \frac{1}{3}\varphi_{\rho_{b|k,l}}. \tag{6.23}$$

[6]the marks labeled i,j are on one component of a nodal rational curve, and the remaining marks on the other.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

There are only two two-dimensional cones containing $\rho_{a|i,j}$ such that the function representing $\psi_1^{\mathtt{trop}}$ is not identically zero: a cone of type σ_4 whose other ray is $\rho_{a|k,l}$ and a cone of type σ_5 whose other ray is $\rho_{b|k,l}$. We evaluate (6.20) to obtain:

$$c_{\rho_{a|i,j}} = \frac{1}{3} \cdot 1 + 1 \cdot \frac{1}{3} = \frac{2}{3}. \tag{6.24}$$

Type ρ_{B}: for $i, j \neq 1$, consider $\rho_{b|i,j}$. We choose the same representative for $\psi_1^{\mathtt{trop}}$ as in (6.23). There is only one two-dimensional cone containing $\rho_{b|i,j}$ such that the function representing $\psi_1^{\mathtt{trop}}$ is not identically zero: it is of type σ_5 and its other ray is $\rho_{a|k,l}$. From (6.20) we have:

$$c_{\rho_{b|i,j}} = 1 \cdot 1 = 1. \tag{6.25}$$

Types $\rho_{\mathrm{C}}, \rho_{\mathrm{D}}$: let $\{i, j, k, l\} = \{2, 3, 4, 5\}$; rays of these two types live in the inverse image via $\mathtt{br}$ of rays of the form $\hat{\rho}_{1,k}$. The piecewise linear function (6.22) representing $\hat{\psi}_1^{\mathtt{trop}}$ is identically zero on the star of $\hat{\rho}_{1,k}$, and therefore its pullback via $\mathtt{br}$ is identically zero on the stars of the rays $\rho_{c|1,k}, \rho_{d|1,k}$. It follows immediately that:

$$c_{\rho_{c|1,k}} = c_{\rho_{d|1,k}} = 0. \tag{6.26}$$

The statement in the Proposition is obtained by adding up (6.24), (6.25) and (6.26). □

6.5. *Families on* $\mathcal{M}_{1,2}^{\mathtt{trop}}$

We push-forward the ψ class and the fundamentalish cycle via two different forgetful morphisms to obtain cycles on $\mathcal{M}_{1,2}^{\mathtt{trop}}$. We verify that the resulting tropical cycles satisfy relations analogous to those in Lemma 6.2.

6.5.1. Remember a 1-end

We consider the tropical forgetful morphisms analogous to the map φ_1 from Section 6.1: we denote $\varphi_1^{\mathtt{trop}}$ the map that assigns to a tropical cover the stabilization of the source curve after forgetting all the ends with expansion factor two and all but one of the marked ends with expansion factor one. The end with expansion factor 3 is labeled to be the first end, and the remaining end with expansion factor one is labeled to be the second marked end.

Lemma 6.6. *The function* $\varphi_1^{\mathtt{trop}}$ *has degree* 24.

Proof. We recall that $\mathcal{M}_{1,2}^{\mathtt{trop}}$ is a cone stack (see [5]). Each of the two top-dimensional cones is the target of a degree two quotient map from $\mathbb{R}^2_{\geq 0}$. For the cone parameterizing curves where the two marked ends are on different vertices, the μ_2 action is reflection along the diagonal. For the cone where both marked ends are adjacent to the same vertex, the μ_2 action is trivial.

In order to show that the degree of $\varphi_1^{\mathtt{trop}}$ is equal to 24, we show that for each general interior point of $\mathcal{M}_{1,2}^{\mathtt{trop}}$, the sum of the local degrees of the map at its inverse images equals 24.

Recall that for a cover $\xi = [f : \Gamma \to T]$ belonging to a two-dimensional cone σ_ξ, and such that $\varphi_1^{\mathtt{trop}}(\xi) = \tilde{\Gamma}$ is in a two-dimensional cone $\tilde{\sigma}_{\varphi_1^{\mathtt{trop}}(\xi)}$, the local degree of $\varphi_1^{\mathtt{trop}}$ is defined to be:

$$\deg_\xi(\varphi_1^{\mathtt{trop}}) = \frac{\omega(\sigma_\xi)}{\omega(\tilde{\sigma}_{\varphi_1^{\mathtt{trop}}(\xi)})} |\det(M)|, \tag{6.27}$$

where $\omega(\sigma_\xi)$ is the weight of the cone σ_ξ in the fundamentalish cycle of $\overline{Adm}_{1\to 0}^{\mathtt{trop}}((3),(2,1)^4)$, $\omega(\tilde{\sigma}_{\varphi_1^{\mathtt{trop}}(\xi)}) = 1/|\operatorname{Aut}(\tilde{\Gamma})|$ and M is the matrix giving a local expression for $\varphi_1^{\mathtt{trop}}$ in integral lattice bases for the two cones. There are three distinct situations that need to be considered. Refer to Figure 10.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

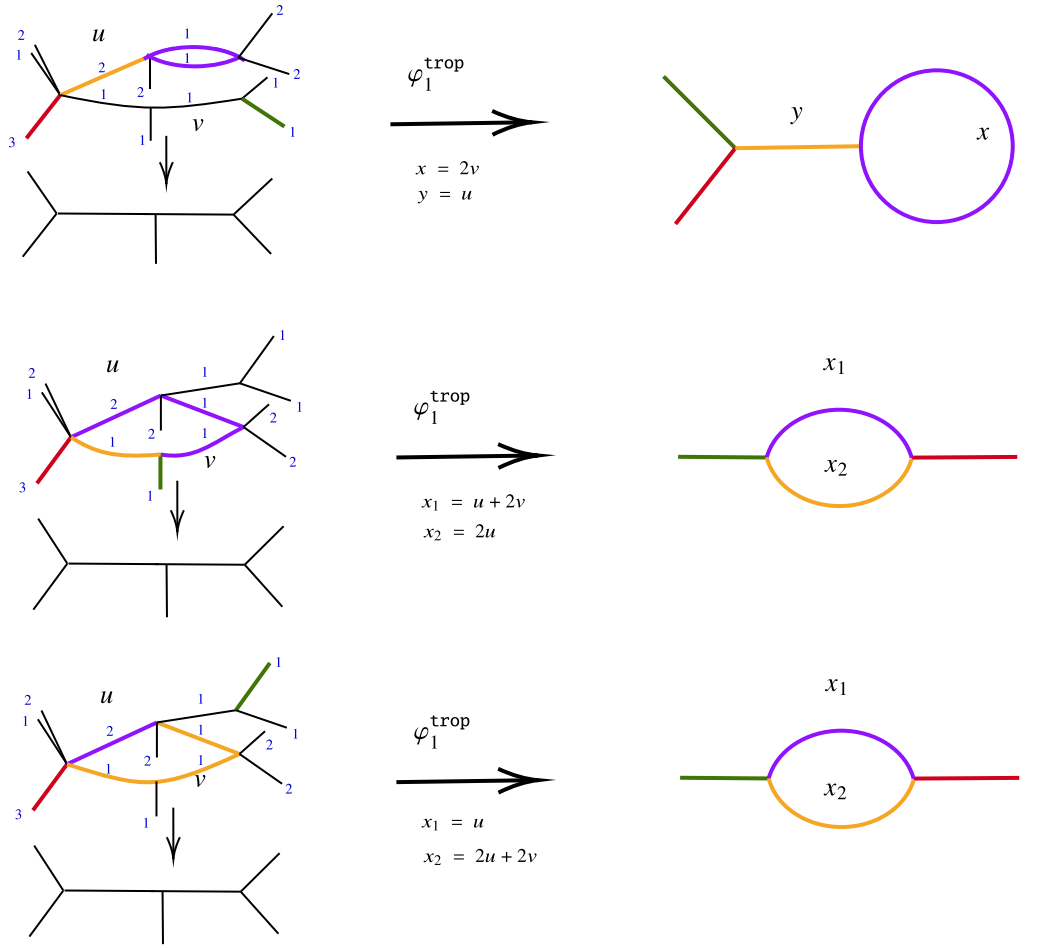


Figure 10. *The type of admissible covers contributing to the computation of the degree of the map $\varphi_1^{\mathrm{trop}}$. We have colored the ends that are remembered, as well as the edges whose lengths are relevant for the computation of the degree.*

Case I: $\tilde{\xi}$ belongs to the interior of the two-dimensional cone of $\mathcal{M}_{1,2}^{\mathrm{trop}}$ parameterizing tropical curves where the two marks emanate from the same vertex. Then there are twelve inverse images for $\tilde{\xi}$, one belonging to each cone of type σ_3 (see the first line of Figure 10 for one example). For any such inverse image ξ, the local degree of $\varphi_1^{\mathrm{trop}}$ is

$$\deg_{\xi}(\varphi_1^{\mathrm{trop}}) = \frac{2}{2}\left|\det\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right| = 2. \tag{6.28}$$

Since there are twelve inverse images, the sum of the local degrees equals 24, as desired.

Case II: $\tilde{\xi}$ belongs to the interior of the two-dimensional folded cone $\tilde{\sigma}$ of $\mathcal{M}_{1,2}^{\mathrm{trop}}$. Consider the orthant with coordinates (x_1, x_2) mapping with degree two to $\tilde{\sigma}$ and assume $\tilde{\xi}$ lies in the region whose inverse image is $\{x_2 < x_1/2\} \cup \{x_2 > 2x_1\}$.

Then $\tilde{\xi}$ has a total of 9 inverse images. Three inverse images belong to cones of type σ_2 where the 1-end that is remembered maps to the middle edge of the target graph (see second line of Figure 10). For each of these inverse images the local degree is:

$$\deg_{\xi}(\varphi_1^{\mathrm{trop}}) = \frac{1}{1}\left|\det\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\right| = 4. \tag{6.29}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

The remaining six inverse images belong to cones of type σ_2 with the 1-end mapped to an external edge of the target tree (third line of Figure 10). The local degree for these inverse images is:

$$\deg_{\tilde{\xi}}(\varphi_1^{\mathtt{trop}}) = \frac{1}{1}\left|\det\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\right| = 2. \tag{6.30}$$

It follows that the sum of the local degrees over all inverse images equals 24.

Case III: $\tilde{\xi}$ belongs to the interior of the two-dimensional folded cone $\tilde{\sigma}$ of $\mathcal{M}_{1,2}^{\mathtt{trop}}$. Now assume $\tilde{\xi}$ lies in the region whose inverse image is $\{x_1/2 < x_2 < 2x_1\}$. There are 12 inverse images, two for each of the six cones of type σ_2 where the 1-edge maps to an external end of the target tree (third line of Figure 10). The local degree computation is identical to (6.30), which again yields a total of 24.

We have thus verified that the degree of $\varphi_1^{\mathtt{trop}}$ is equal to 24. □

Lemma 6.7. *Denote by ρ_{irr} the ray of $\mathcal{M}_{1,2}^{\mathtt{trop}}$ parameterizing curves with one vertex of genus zero and a self loop. We have*

$$\varphi_{1,*}^{\mathtt{trop}}(\psi_1^{\mathtt{trop}} \frown \mathtt{T}_*([\Sigma])) = 12\,\rho_{irr}. \tag{6.31}$$

Proof. From the computation in Proposition 6.5, we only need to compute the pushforward of the rays of type ρ_a and ρ_b. It is immediate to see that all rays of type ρ_a are contracted to the cone point of $\overline{\mathcal{M}}_{1,2}^{\mathtt{trop}}$, and therefore their pushforward vanishes. All the rays of type ρ_b map to the ray we have denoted ρ_{irr}, and the function $\varphi_1^{\mathtt{trop}}$ restricts to each of these rays as a linear function of slope 2. Both the tropical covers $\Gamma \to T$ parameterized by the rays of type ρ_b and the tropical curves $\tilde{\Gamma}$ parameterized by ρ_{irr} have a group of automorphisms of order two. Recalling that we have 6 rays of type ρ_b we can conclude

$$\varphi_{1,*}(\rho_b^{\mathtt{tot}}) = \left(\frac{|\operatorname{Aut}(\tilde{\Gamma})|}{|\operatorname{Aut}(\Gamma \to T)|} \cdot \operatorname{slope}_{\rho_b}(\varphi_1) \cdot |\text{rays of type } \rho_b|\right)\rho_{irr} = \left(\frac{2}{2} \cdot 2 \cdot 6\right)\rho_{irr} = 12\rho_{irr}. \tag{6.32}$$

This computation concludes the proof of the lemma. □

We observe that this result is compatible with the algebraic computation from Lemma 6.2. Denote by ρ_{sec} the other ray of $\mathcal{M}_{1,2}^{\mathtt{trop}}$ (corresponding to graphs with a genus-one vertex and a genus zero vertex with the two legs attached), and for $x = sec, irr$, denote by D_x the divisor in $\overline{\mathcal{M}}_{1,2}$ parameterizing curves whose dual graph correspond to the ray ρ_x. It is a standard fact from the intersection theory of ψ classes that $\psi_1 \cdot D_{sec} = 0$ and $\psi_1 \cdot D_{irr} = \frac{1}{2}$. By (6.7), we have $\varphi_{1,*}(\psi_1) = 24\psi_1$, hence its operational tropicalization agrees with the computation from Lemma 6.7.

6.5.2. Remember a 2-end

In this section we compute the pushforward of $\psi_1^{\mathtt{trop}}$ to $\mathcal{M}_{1,2}^{\mathtt{trop}}$ via the forgetful morphism $\varphi_2^{\mathtt{trop}}$ (remembering the degree 3 end and one of the degree 2 ends of the tropical covers), and check that the result is consistent with the operational tropicalization of (6.8). We begin by computing the operational tropicalization of the Weierstrass divisor W. Since W is not dimensionally transverse to the boundary of $\overline{\mathcal{M}}_{1,2}$, before we apply tropicalization we resolve this issue by blowing up the stratum δ_{00} of self-intersection of the irreducible divisor in $\overline{\mathcal{M}}_{1,2}$.

Lemma 6.8. *Denote by ρ_{irr} and ρ_{sec} the rays of $\mathcal{M}_{1,2}^{\mathtt{trop}}$ dual to the irreducible divisor and the section of $\overline{\mathcal{M}}_{1,2}$, and by ρ_E the ray coinciding with the diagonal of the folded cone (corresponding to the exceptional divisor E in the blow up of the point δ_{00}). We have*

$$\mathtt{Trop}(W) = \frac{1}{2}(\rho_{irr} + \rho_E). \tag{6.33}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

Proof. To compute the operational tropicalization of W we must compute the intersection of W with all boundary curves of $Bl_{\delta_{00}}\overline{\mathcal{M}}_{1,2}$. We identify W with the (proper transform of the) locus of genus-one admissible covers of $\mathbb{P}^1$ of degree 2 with two labeled and two unlabeled branch points. This family has exactly two points parameterizing singular curves. One point p_1 corresponds to covers where one of the unlabeled branch points has collided with one of the labeled branch points. The source curve of such a cover is a banana curve. The cover has an automorphism group of order two. Altogether p_1 contributes $\frac{1}{2}$ to the intersection of W and E. The second point p_2 arises when the two unlabeled branch points have collided. In this case the source curve of the admissible cover is unstable. Its stabilization is a nodal rational curve with two marked points, that is, it is a general point in the interior of δ_{irr}. The automorphism group of the cover has order two, and therefore p_2 contributes $\frac{1}{2}$ to the intersection of W with δ_{irr}. Since W does not contain any other point parameterizing singular curves, we deduce that

$$W \cdot \delta_{sec} = 0, \quad W \cdot \delta_{irr} = \frac{1}{2}, \quad W \cdot \delta_E = \frac{1}{2}, \tag{6.34}$$

which proves the assertion of the lemma. □

Lemma 6.9. *The function $\varphi_2^{\mathrm{trop}}$ has degree* 6.

Proof. We proceed analogously to Lemma 6.6. Again, we need to analyze three distinct situations. Refer to Figure 11.

Case I: $\tilde{\xi}$ belongs to the interior of the two-dimensional cone of $\mathcal{M}_{1,2}^{\mathrm{trop}}$ parameterizing tropical curves where the two marks emanate from the same vertex. There are three inverse images for $\tilde{\xi}$, belonging to cones of type σ_3 where the 2-end we remember is on the same vertex as the 3-end (see the first line of Figure 11). For any such inverse image ξ, the local degree of $\varphi_1^{\mathrm{trop}}$ is

$$\deg_{\xi}(\varphi_2^{\mathrm{trop}}) = \frac{2}{2}\left|\det\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right| = 2. \tag{6.35}$$

Since there are three inverse images, the sum of the local degrees equals 6.

Case II: $\tilde{\xi}$ belongs to the interior of the two-dimensional folded cone $\tilde{\sigma}$ of $\mathcal{M}_{1,2}^{\mathrm{trop}}$. Assume $\tilde{\xi}$ lies in the region whose inverse image is $\{x_1/2 < x_2 < 2x_1\}$. Then $\tilde{\xi}$ has a total of 6 inverse images, belonging to cones of type σ_2 with the 2-end mapped to an external edge of the target tree (second line of Figure 11). The local degree for these inverse images is:

$$\deg_{\xi}(\varphi_2^{\mathrm{trop}}) = \frac{1}{1}\left|\det\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\right| = 1. \tag{6.36}$$

It follows that the sum of the local degrees over all inverse images equals 6.

Case III: $\tilde{\xi}$ belongs to the interior of the two-dimensional folded cone $\tilde{\sigma}$ of $\mathcal{M}_{1,2}^{\mathrm{trop}}$, in the region $\{x_2 < x_1/2\} \cup \{x_2 > 2x_1\}$. There are three inverse images, that belong to cones of type σ_2 where the 2-end that is remembered maps to the middle edge of the target graph (see third line of Figure 11). For each of these inverse images the local degree is:

$$\deg_{\xi}(\varphi_2^{\mathrm{trop}}) = \frac{1}{1}\left|\det\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\right| = 2. \tag{6.37}$$

We have thus verified that the degree of $\varphi_2^{\mathrm{trop}}$ is equal to 6. □

Lemma 6.10. *Let ρ_{irr}, ρ_{sec} and ρ_E be as in Lemma 6.8. We have*

$$\varphi_{2,*}^{\mathrm{trop}}(\psi_1^{\mathrm{trop}} \frown \mathrm{T}_*([\Sigma])) = 6\rho_{irr} + 3\rho_E. \tag{6.38}$$

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

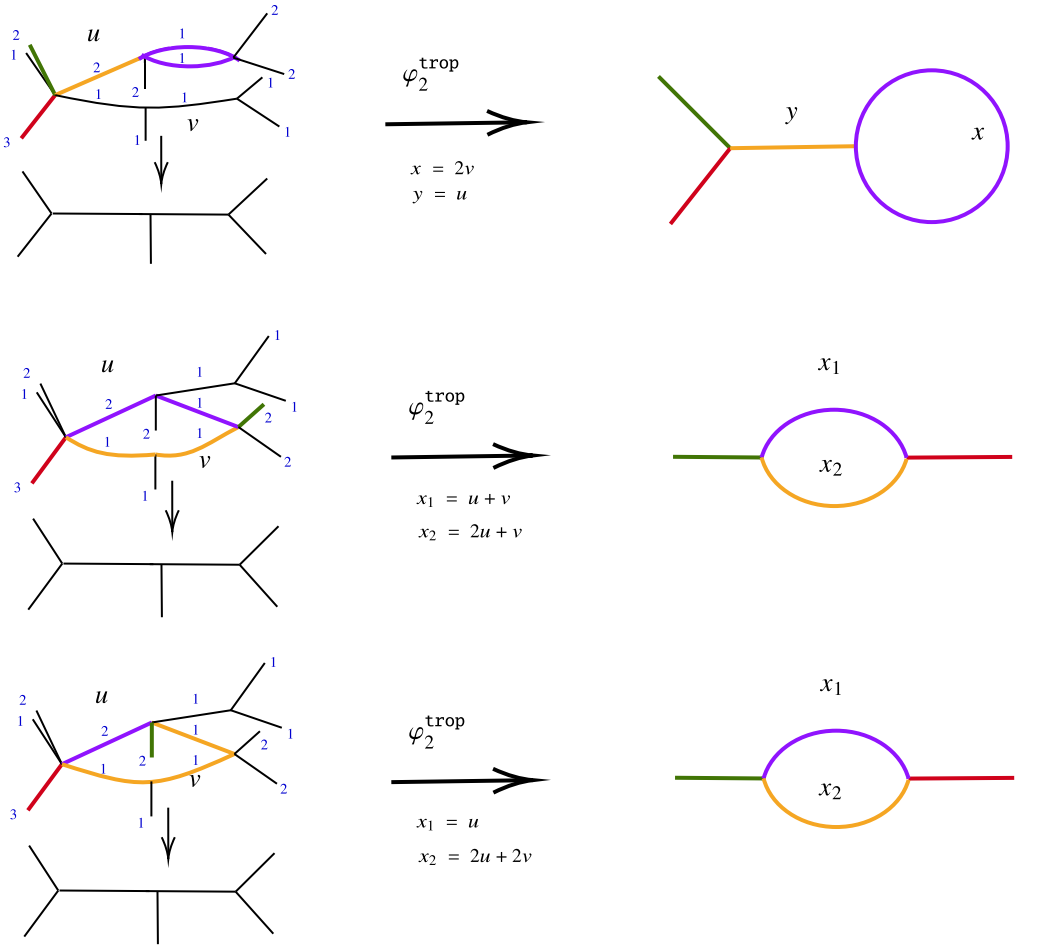


Figure 11. *The type of admissible covers contributing to the computation of the degree of the map $\varphi_2^{\mathrm{trop}}$. We have colored the ends that are remembered, as well as the edges whose lengths are relevant for the computation of the degree.*

Proof. As before, all rays of type ρ_a are contracted to the cone point of $\overline{\mathcal{M}}_{1,2}^{\mathrm{trop}}$, and therefore their pushforward vanishes.

Three of the rays of type ρ_b, where the 2-end is on the same vertex as the 3-end, map to the ray we have denoted ρ_{irr}, and the function $\varphi_2^{\mathrm{trop}}$ restricts to each of these rays as a linear function of slope 2. Three of the rays of type ρ_b, where the 2-end is on a different vertex than the 3-end, map to the ray we have denoted ρ_E, and the function $\varphi_2^{\mathrm{trop}}$ restricts to each of these rays as a linear function of slope 1.

Both the tropical covers $\Gamma \to T$ parameterized by the rays of type ρ_b and the tropical curves $\tilde{\Gamma}$ parameterized by ρ_{irr} and ρ_E have a group of automorphisms of order two. Hence:

$$\varphi_{2,*}^{\mathrm{trop}}(\rho_b^{\mathrm{tot}}) = \frac{|\operatorname{Aut}(\tilde{\Gamma})|}{|\operatorname{Aut}(\Gamma \to T)|} \cdot (2 \cdot 3\rho_{irr} + 1 \cdot 3\rho_E) = 6\rho_{irr} + 3\rho_E. \tag{6.39}$$

□

We now observe that this result is compatible with the algebraic computation (6.8) from Lemma 6.2. We have computed the operational tropicalization of ψ to be $\frac{1}{2}\rho_{irr}$ in Lemma 6.7, and of W to be $\frac{1}{2}(\rho_{irr} + \rho_E)$ in Lemma 6.8.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press

It follows immediately that

$$\varphi_{2,*}^{\mathtt{trop}}(\psi_1^{\mathtt{trop}}) = \mathtt{Trop}(6(\psi_1 + W)). \tag{6.40}$$

Acknowledgments. We wish to thank Hannah Markwig, Dhruv Ranganathan and Martin Ulirsch for interesting conversations related to this project.

Competing interests. The authors have no competing interest to declare.

Financial support. Renzo Cavalieri is grateful for support from NSF grant DMS - 2100962 and Simons Collaboration Grant MPS-TSM-00007937. Andreas Gross has received funding from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) TRR 326 *Geometry and Arithmetic of Uniformized Structures*, project number 444845124, from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Sachbeihilfe *From Riemann surfaces to tropical curves (and back again)*, project number 456557832, and from the Marie-Sk\l{}odowska-Curie-Stipendium Hessen (as part of the HESSEN HORIZON initiative).

References

[1] D. Abramovich, L. Caporaso, and S. Payne, 'The tropicalization of the moduli space of curves', *Ann. Sci. Ec. Norm. Super.* **48**(4) (2015), 765–809.
[2] L. J. Billera, S. P. Holmes and K. Vogtmann, 'Geometry of the space of phylogenetic trees', *Adv. in Appl. Math.* **27**(4) (2001), 733–767.
[3] L. Caporaso, 'Gonality of algebraic curves and graphs', in *Algebraic and Complex Geometry*, volume 71 of *Springer Proc. Math. Stat.* (Springer, Cham, 2014), 77–108.
[4] D. Cartwright. Tropical complexes. *Manuscr. Math.*, **163**(1-2): 227–261, 2020.
[5] R. Cavalieri, M. Chan, M. Ulirsch and J. Wise, A moduli stack of tropical curves, 2017.
[6] R. Cavalieri, A. Gross and H. Markwig, 'Tropical ψ classes', *Geom. Topol.* **26**(8) (2022), 3421–3524.
[7] R. Cavalieri, H. Markwig and D. Ranganathan, 'Tropicalizing the space of admissible covers', *Math. Ann.* **364**(3–4) (2016), 1275–1313.
[8] M. Chan, S. Galatius and S. Payne, 'Topology of moduli spaces of tropical curves with marked points', in *Facets of Algebraic Geometry. A Collection in Honor of William Fulton's 80th Birthday. Volume 1* (Cambridge: Cambridge University Press, 2022), 77–131.
[9] E. Maria Feichtner and B. Sturmfels, 'Matroid polytopes, nested sets and Bergman fans', *Port. Math. (N.S.)* **62**(4) (2005), 437–468.
[10] G. Francois, 'Cocycles on tropical varieties via piecewise polynomials', *Proc. Am. Math. Soc.* **141**(2) (2013), 481–497.
[11] A. Gross 'Intersection theory on tropicalizations of toroidal embeddings', *Proc. Lond. Math. Soc. (3)* **116**(6) (2018), 1365–1405.
[12] W. Gubler, P. Jell and J. Rabinoff, 'Forms on berkovich spaces based on harmonic tropicalizations', Preprint: arXiv:2111.05741, 2021.
[13] D. Holmes and R. Schwarz, 'Logarithmic intersections of double ramification cycles', *Algebr. Geom.* **9**(5) (2022), 574–605.
[14] E.-N. Ionel, 'Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$', *Invent. Math.* **148**(3) (2002), 627–658.
[15] P. Jell, J. Rau and K. Shaw, 'Lefschetz (1, 1)-theorem in tropical geometry', *Épijournal Geom. Algébrique*, **2** (2018), Art. 11, 27.
[16] T. Kajiwara, 'Tropical toric geometry', in *Toric Topology*, volume 460 of *Contemp. Math*. (Amer. Math. Soc., Providence, RI, 2008), 197–207.
[17] E. Katz, 'Tropical intersection theory from toric varieties', *Collect. Math.* **63**(1) (2012), 29–44.
[18] G. Kempf, F. Faye Knudsen, D. Mumford and B. Saint-Donat, *Toroidal Embeddings. I*, Vol. 339 of *Lecture Notes in Mathematics* (Springer-Verlag, Berlin-New York, 1973).
[19] J. Kock. Notes on psi classes. Notes. http://mat.uab.es/~kock/GW/notes/psi-notes.pdf, 2001.
[20] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, volume 161 of *Graduate Studies in Mathematics* (American Mathematical Society, Providence, RI, 2015).
[21] G. Mikhalkin, 'Moduli spaces of rational tropical curves', in *Proceedings of Gökova Geometry-Topology Conference 2006*, pages 39–51. Gökova Geometry/Topology Conference (GGT), Gökova, 2007.
[22] G. Mikhalkin and I. Zharkov, 'Tropical curves, their Jacobians and theta functions', in *Curves and Abelian Varieties*, volume 465 of *Contemp. Math.* (Amer. Math. Soc., Providence, RI, 2008), 203–230.
[23] S. Molcho, R. Pandharipande and J. Schmitt, 'The Hodge bundle, the universal 0-section, and the log Chow ring of the moduli space of curves', *Compos. Math.* **159**(2) (2023), 306–354.
[24] S. Molcho and D. Ranganathan, 'A case study of intersections on blowups of the moduli of curves', *Algebra Number Theory*, **18**(10) (2024), 1767–1816.
[25] M. Ulirsch, 'Functorial tropicalization of logarithmic schemes: the case of constant coefficients', *Proc. Lond. Math. Soc. (3)* **114**(6) (2017), 1081–1113.
[26] M. Ulirsch, 'Non-archimedean geometry of artin fans', *Adv. Math.* **345** (2019), 346–381.

https://doi.org/10.1017/fms.2026.10191 Published online by Cambridge University Press