WEAKLY ALMOST PERIODIC FUNCTIONS ON $\mathbb{N}$ WITH A NEGATIVE BASE

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0. *Introduction*

Weakly almost periodic compactifications have been seriously studied for over 30 years. In the pioneering papers of de Leeuw and Glicksberg [**4**] and [**5**], the approach adopted was operator-theoretic. The current definition is more likely to be created from the perspective of universal algebra (see [**1**, Chapter 3]). For a discrete group or semigroup S, the weakly almost periodic compactification wS is the largest compact semigroup which (i) contains S as a dense subsemigroup, and (ii) has multiplication continuous in each variable separately (where *largest* means that any other compact semigroup with the properties (i) and (ii) is a quotient of wS). A third viewpoint is to envisage wS as the Gelfand space of the C*-algebra of bounded weakly almost periodic functions on S (for the definition of such functions, see below).

In this paper, we are concerned only with the simplest semigroup $(\mathbb{N}, +)$. The three approaches described above give three methods of obtaining information about $w\mathbb{N}$. An early striking result about $w\mathbb{N}$, that it contains more than one idempotent, was obtained by T. T. West using operator theory [**13**]. He considered the weak operator closure of the semigroup $\{T, T^2, T^3, \ldots\}$ of iterates of a single operator T on the Hilbert space $L^2(\mu)$ for a particular measure μ on $[0, 1]$. Brown and Moran, in a series of papers culminating in [**2**], used sophisticated techniques from harmonic analysis to produce measures μ that permitted the detection of further structure in $w\mathbb{N}$; in particular, they found 2^c distinct idempotents. However, for many years, no other way of showing the existence of more than one idempotent in $w\mathbb{N}$ was found.

The breakthrough came in 1991, and it was made by Ruppert [**11**]. In his paper, he created a direct construction of a family of weakly almost periodic functions which could detect 2^c different idempotents in $w\mathbb{N}$. His method was very ingenious (he used a unique variant of the p-adic expansion of integers) and rather complicated. Our main aim in this paper is to construct weakly almost periodic functions which are easy to describe and so appear more 'natural' than Ruppert's. We also show that there are enough functions of our type to distinguish 2^c idempotents in $w\mathbb{N}$.

There are three main ingredients in our development. The first is an observation made by Papazyan [**9**], that each integer $\mathbf{x} \in \mathbb{N}$ can be expressed uniquely in the form

$$\mathbf{x} = \sum_{i=0}^{\infty} x_i(-2)^i \quad x_i \in \{0, 1\}$$

where $x_i = 0$ for all but finitely many i. Our main theorem can be stated easily as follows.

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THEOREM 0.1. *Let $\{c_i\}$ be any sequence with $c_i \downarrow 0$, $\sum_i c_i = \infty$. Then the function $f: \mathbb{N} \to \mathbb{R}$ defined by*

$$f(x) = \sum_{i=0}^{\infty} x_i c_i$$

is weakly almost periodic.

This statement is simple, but it is also misleading. We indicated above that weakly almost periodic functions were bounded, but this function f is not. The second feature of our work is that we ensure that our functions have relatively compact ranges by taking their codomain to be $[0, \infty]$. The idea of allowing functions determining compactifications to take values in spaces other than $\mathbb{R}$ or $\mathbb{C}$ is in [**8**], but it has not been used for weakly almost periodic functions before.

The third factor which enables our proof to work is that the function f described above, although not a homomorphism, extends to a continuous function on $w\mathbb{N}$ which is a homomorphism from a compact subsemigroup $\mathbf{H}$ of $w\mathbb{N}$ to the semigroup $([0, \infty], +)$. Since $([0, \infty], +)$ has two idempotents, we conclude that $\mathbf{H}$ does also. This technique was used by Hindman and Pym in discussing the structure of $\beta\mathbb{N}$ [**7**], and it lies behind recent work of Strauss [**12**]. It was also used in a slightly different way by Ruppert [**11**].

1. *Preliminaries*

In what follows, $\mathbb{N}$ denotes the discrete semigroup of positive integers with addition. For any two nonempty subsets A and B of $\mathbb{N}$, we write $A \prec B$ when each member of A is less than each member of B.

Define a bounded complex-valued function $f: \mathbb{N} \to \mathbb{C}$ to be a *weakly almost periodic function* if

$$\lim_{n \to \infty} \lim_{m \to \infty} f(\mathbf{x}_n + \mathbf{y}_m) = \lim_{m \to \infty} \lim_{n \to \infty} f(\mathbf{x}_n + \mathbf{y}_m)$$

whenever $(\mathbf{x}_n)$ and $(\mathbf{y}_m)$ are sequences in $\mathbb{N}$ such that all the limits involved exist. The set of all bounded weakly almost periodic functions is a commutative C*-algebra. It is therefore isomorphic with the space $\mathbf{C}(w\mathbb{N})$ of all continuous functions on a compact space $w\mathbb{N}$, which is known as the weakly almost periodic compactification of $\mathbb{N}$. It is a compact, Hausdorff space which contains $\mathbb{N}$ densely, and every bounded weakly almost periodic function on $\mathbb{N}$ is extendable to a continuous function on $w\mathbb{N}$. The semigroup operation $+$ in $w\mathbb{N}$ is determined by

$$\tau + \mu = \lim_{\alpha} \lim_{\beta} (\mathbf{x}_\alpha + \mathbf{y}_\beta) = \lim_{\beta} \lim_{\alpha} (\mathbf{x}_\alpha + \mathbf{y}_\beta)$$

in which $(\mathbf{x}_\alpha)$ and $(\mathbf{y}_\beta)$ are two nets in $\mathbb{N}$ converging to τ and μ, respectively. $w\mathbb{N}$ with the above operation is a commutative semigroup. Also, for fixed μ, the map which sends τ to $\tau + \mu$ is a continuous function on $w\mathbb{N}$, that is, the operation in $w\mathbb{N}$ is separately continuous. In fact, $w\mathbb{N}$ is the universal semitopological compactification of $\mathbb{N}$.

It is possible to put an order on the set of idempotents of $w\mathbb{N}$. (Of course, by an idempotent, we mean $e \in w\mathbb{N}$ such that $e + e = e$.) For any two idempotents e and e' in $w\mathbb{N}$, if $e + e' = e$, we write $e \leqslant e'$. The above order has been extensively studied in [**10**]. With the above order, each commutative compact semitopological semigroup

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has a minimal idempotent, as is easily seen, and so this applies in particular to closed subsemigroups of $w\mathbb{N}$. We refer the reader to [**1**] for general information about weakly almost periodic functions and compactifications.

Unfortunately, none of the functions that we are going to introduce are bounded, and naturally they are not in the C*-algebra of weakly almost periodic functions. This brings us to our first definition.

DEFINITION 1.1. Given a compact, Hausdorff space X, $f: \mathbb{N} \to X$ is called an *extended weakly almost periodic function* if, for each continuous complex-valued function $\phi : X \to \mathbb{C}$, $\phi \circ f$ is a weakly almost periodic function.

If X is a sequentially compact Hausdorff space, we can see that, if $f: \mathbb{N} \to X$ satisfies the following conditions, then it is an extended weakly almost periodic function

$$\lim_{n\to\infty} \lim_{m\to\infty} f(\mathbf{x}_n + \mathbf{y}_m) = \lim_{m\to\infty} \lim_{n\to\infty} f(\mathbf{x}_n + \mathbf{y}_m)$$

whenever $(\mathbf{x}_n)$ and $(\mathbf{y}_m)$ are sequences in $\mathbb{N}$ such that all limits involved exist. In fact, in this paper, we always use the above criterion to prove that f is an extended weakly almost periodic function.

Although the extended weakly almost periodic functions are not really weakly almost periodic functions, they are extendable to $w\mathbb{N}$ continuously.

PROPOSITION 1.2. *If $f: \mathbb{N} \to X$ is an extended weakly almost periodic function into a compact and Hausdorff space X, there exists a continuous function $\hat{f}: w\mathbb{N} \to X$ which is an extension of f.*

Proof. For $\phi \in \mathbf{C}(X)$, $\phi \circ f$ is a bounded weakly almost periodic function on $\mathbb{N}$, and so it can be identified with a function $\widehat{\phi \circ f} \in \mathbf{C}(w\mathbb{N})$. The map $\phi \mapsto \widehat{\phi \circ f}$ is a C*-algebra homomorphism and so it induces a map $\hat{f}$ from $w\mathbb{N}$ to X. This map extends f.

Throughout this paper, $[0, \infty]$ denotes the one-point compactification of $[0, \infty)$. This becomes a semigroup with an operation $+$, which is the following extension of the ordinary addition of nonnegative real numbers:

$$r + \infty = \infty + r = \infty \quad r \in [0, \infty].$$

To finish this section, we have to mention that, in what follows, $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

2. *Weakly almost periodic functions on $\mathbb{N}$ with negative base -2*

The negative base -2 was introduced for the first time in [**9**]. More general negative bases were used later in [**3**] to study $\beta\mathbb{Q}$. The following Lemma 2.1 is a consequence of [**3**, Theorem 4.1], and it is not hard to see that it is true in any case.

LEMMA 2.1. *Each integer $\mathbf{x} \in \mathbb{Z}$ has a unique expression*

$$\mathbf{x} = \sum_{i=0}^{\infty} x_i (-2)^i \quad 0 \leqslant x_i < 2$$

with $x_i = 0$ for all except finitely many i. If $\mathbf{x} \in \mathbb{N}$, the greatest index i such that $x_i \neq 0$ is an even number.

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NOTATION 2.2. Owing to Lemma 2.1, we sometimes write

$$\mathbf{x} = (x_0, x_1, \ldots) \quad \text{or} \quad \mathbf{x} = (x_0, x_1, \ldots, x_m)$$

where x_m is the last nonzero coefficient in the expansion of $\mathbf{x}$. We refer to

$$(x_0, x_1, \ldots) \quad \text{or} \quad (x_0, x_1, \ldots, x_m)$$

as the expansion of $\mathbf{x}$. The ith coordinate of $\mathbf{x}$ is denoted by $(\mathbf{x})_i$, and so $(\mathbf{x})_i = x_i$ in the notation of Lemma 2.1. By $\operatorname{Supp}\mathbf{x}$, we mean the set of subscripts of all nonzero coordinates of $\mathbf{x}$, that is, if $\mathbf{x} = (x_0, x_1, \ldots)$, then

$$\operatorname{Supp}\mathbf{x} = \{k \mid x_k \neq 0\}.$$

The arithmetic of negative bases is completely explained in [**3**]. In this paper, we only describe formally the rule for addition. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z} = \mathbf{x}+\mathbf{y}$ be fixed natural numbers with the expansions

$$\mathbf{x} = (x_0, x_1, \ldots)$$
$$\mathbf{y} = (y_0, y_1, \ldots)$$
$$\mathbf{z} = (z_0, z_1, \ldots).$$

Write $\delta_0 = 0$, and define δ_i for $i = 1, 2, \ldots$ inductively as follows:

$$\delta_{i+1} = \begin{cases} 0 & 0 = x_i + y_i + \delta_i < 2 \\ -1 & 2 \leqslant x_i + y_i + \delta_i \\ 1 & x_i + y_i + \delta_i < 0. \end{cases}$$

(Note that $x_i + y_i + \delta_i < 0$ is possible only if $x_i = y_i = 0$ and $\delta_i = -1$.) In comparison with positive bases, for example the dyadic expansion, δ_i is the 'carrying over' from the $(i-1)$th place into the ith place. The difference is that, in the dyadic expansion, when two numbers are added, only $+1$ and 0 can be carried over, but in the negative base -2 we can also carry -1. Then

$$z_i = (\mathbf{x}+\mathbf{y})_i = \begin{cases} x_i + y_i + \delta_i & 0 \leqslant x_i + y_i + \delta_i < 2 \\ x_i + y_i + \delta_i - 2 & 2 \leqslant x_i + y_i + \delta_i \\ 1 & x_i = y_i = 0, \delta_i = -1. \end{cases}$$

It is straightforward to check by induction that these rules are correct (although there are many cases to consider). We refer the reader who is interested in the details to [**3**, Section 4]. There is also a parallel with the formulae in Ruppert's paper [**11**, Section 1.3].

NOTATION 2.3. Let Ω be the set of all decreasing positive sequences $\{c_i\}_{i=0}^{\infty}$ such that $\lim_i c_i = 0$ and $\sum_{i=0}^{\infty} c_i = \infty$. Each member of Ω is denoted by a single letter, for example c for the sequence $\{c_i\}_{i=0}^{\infty}$. Given that $c \in \Omega$, define $f_c : \mathbb{N} \to [0, \infty]$ by

$$f_c(\mathbf{x}) = \sum_{i=0}^{\infty} c_i x_i$$

in which $c = \{c_i\}_{i=0}^{\infty}$ and $\mathbf{x} = (x_0, x_1, \ldots)$.

In the next lemma, Lemma 2.4, an inequality which is the key to proving that each f_c is an extended weakly almost periodic function is established.

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LEMMA 2.4. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{N}$. *Suppose that* k *and* l, *where* $k < l$, *are positive integers such that* $x_i = 0$ *for* $k < i < l$.

(i) *If, in the sequence* $\delta_k, \delta_{k+1}, \ldots, \delta_i, \delta_{i+1}, \ldots, \delta_{l-1}$, *in which* $k < i$, *we have* $\delta_i = \delta_{i+1}$, *then* $\delta_j = 0$ *for* $i \leqslant j \leqslant l-1$. (*Thus two consecutive equal carryings over stop any further nonzero carrying over.*)

(ii) *If* $f_c \in \Omega$,

$$\left| \sum_{i=k+1}^{l-1} c_i(\mathbf{x}+\mathbf{y})_i - \sum_{i=k+1}^{l-1} c_i(\mathbf{y})_i \right| \leqslant 4c_k.$$

Proof. (i) can be easily checked by considering different cases, and so we only prove (ii). In the situation that we are considering, $\mathbf{x}$ and $\mathbf{y}$ have the following forms:

$$\mathbf{x} = (x_0, \ldots, x_k, 0, 0, \ldots, 0, x_l, x_{l+1}, \ldots)$$
$$\mathbf{y} = (y_0, \ldots, y_k, y_{k+1}, y_{k+2}, \ldots, y_{l-1}, y_l, y_{l+1}, \ldots).$$

If there is an h with $k \leqslant h < l$ such that $\delta_{h+1} = 0$, then we find successively that $(\mathbf{x}+\mathbf{y})_{h+1} = y_{h+1}$, $\delta_{h+2} = 0, \ldots, \delta_{l-1} = 0$, $(\mathbf{x}+\mathbf{y})_{l-1} = y_{l-1}, \delta_l = 0$. Then, for any $c \in \Omega$,

$$\left| \sum_{i=h+1}^{l-1} c_i(\mathbf{x}+\mathbf{y})_i - \sum_{i=h+1}^{l-1} c_i y_i \right| = 0.$$

Take h to be the smallest integer that is larger than or equal to k with $\delta_{h+1} = 0$. If $h = k$, then there is nothing left to prove. Suppose that $h > k$; then $\delta_{h+1} = 0$. Two cases may occur, as follows.

Case (i) ($\delta_{k+1} = -1$): If $h > k+1$ (so that $\delta_{k+2} \neq 0$), we cannot have $y_{k+1} = 1$; therefore, $y_{k+1} = 0$, $(\mathbf{x}+\mathbf{y})_{k+1} = 1$ and $\delta_{k+2} = 1$. Next, if $h > k+2$ (so that $\delta_{k+3} \neq 0$), then we must have $y_{k+2} = 1$, and consequently $(\mathbf{x}+\mathbf{y})_{k+2} = 0$ and $\delta_{k+3} = -1$. Then, if $h > k+3$, we find that $y_{k+3} = 0$, $(\mathbf{x}+\mathbf{y})_{k+3} = 1$ and $\delta_{k+4} = 1$, and the pattern repeats itself. Thus, for $c \in \Omega$,

$$\left| \sum_{i=k+1}^{h-1} c_i(\mathbf{x}+\mathbf{y})_i - \sum_{i=k+1}^{h-1} c_i y_i \right| < c_{k+1} - c_{k+2} + \ldots + (-1)^{h-k} c_{h-1} < 2c_{k+1}$$

since c_i is decreasing. The term with suffix h depends on whether $h-k$ is even or odd. By the choice of h, $\delta_{h+1} = 0$ and $\delta_h \neq 0$. If $h = k+2r$, then $\delta_h = 1$. Therefore, to ensure that $\delta_{h+1} = 0$, we need $y_k = 0$, and then $(\mathbf{x}+\mathbf{y})_h = 1$. If $h-k$ is odd, then $\delta_h = -1$. We must have $y_h = 1$ and $(\mathbf{x}+\mathbf{y})_h = 0$. Thus, in both cases,

$$|c_h(\mathbf{x}+\mathbf{y})_h - c_h y_h| = c_h.$$

Putting our inequalities together (and remembering that c_i is decreasing), we get

$$\left| \sum_{i=k+1}^{l-1} c_i(\mathbf{x}+\mathbf{y})_i - \sum_{i=k+1}^{l-1} c_i y_i \right| \leqslant 3c_{k+1}. \tag{1}$$

Case (ii) ($\delta_{k+1} = 1$): If $h > k+1$ (so that $\delta_{k+2} \neq 0$), then we cannot have $y_{k+1} = 0$; therefore, $y_{k+1} = 1$ and $(\mathbf{x}+\mathbf{y})_{k+1} = 0$ and $\delta_{k+2} = -1$. Using case (i) for $k+2$ substituted for $k+1$ in (1), we have

$$\left| \sum_{i=k+2}^{l-1} c_i(\mathbf{x}+\mathbf{y})_i - \sum_{i=k+2}^{l-1} c_i y_i \right| \leqslant 3c_{k+2}.$$

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On the other hand, since $y_{k+1} = 1$ and $(\mathbf{x}+\mathbf{y})_{k+1} = 0$ and $\delta_{k+2} = -1$,

$$|c_{k+1}(\mathbf{x}+\mathbf{y})_{k+1} - c_{k+1}y_{k+1}| = c_{k+1}.$$

Consequently,

$$\left|\sum_{i=k+1}^{l-1} c_i(\mathbf{x}+\mathbf{y})_i - \sum_{i=k+1}^{l-1} c_i y_i\right| \leqslant 3c_{k+2} + c_{k+1}.$$

In both cases, since c_i is decreasing,

$$\left|\sum_{i=k+1}^{l-1} c_i(\mathbf{x}+\mathbf{y})_i - \sum_{i=k+1}^{l-1} c_i y_i\right| \leqslant 4c_k.$$

In the rest of this section, we prove that the functions $f_c, c \in \Omega$, introduced in Notation 2.3 are extendable to $w\mathbb{N}$ continuously.

THEOREM 2.5. *For each $c \in \Omega$, $f_c : \mathbb{N} \to [0, \infty]$ is an extended weakly almost periodic function, and hence it is extendable to $w\mathbb{N}$ continuously.*

Proof. Take two sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_m\}$ such that the following limits exist in $[0, \infty]$:

$$\lim_{m\to\infty}\lim_{n\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m) \quad \text{and} \quad \lim_{n\to\infty}\lim_{m\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m).$$

We have to show that they are equal. The expansion of each $\mathbf{x}_n$ in base -2 consists of a sequence of 0s and 1s. Therefore, $\{\mathbf{x}_n\}$ has a subsequence $\{\mathbf{x}_{n_k}\}$ for which $\{(\mathbf{x}_{n_k})_0\}$ is eventually a constant, say u_0. By replacing $\{\mathbf{x}_n\}$ by a subsequence, we can assume that $(\mathbf{x}_n)_0 = u_0$ for all $n \geqslant 1$. Then, by passing to a further subsequence, we can assume that $(\mathbf{x}_n)_1 = u_1$ for all $n \geqslant 2$. Continuing in this way, we can assume that $(\mathbf{x}_n)_k = u_k$ for $0 \leqslant k \leqslant n$, and consequently

$$\mathbf{x}_n = (u_0, u_1, \ldots, u_n, x^n_{n+1}, \ldots, x^n_{N(n)}, 0, 0, 0, \ldots).$$

Similarly, we can suppose that

$$\mathbf{y}_m = (v_0, v_1, \ldots, v_m, y^m_{m+1}, \ldots, y^m_{M(m)}, 0, 0, 0, \ldots).$$

For convenience in the proof, let w denote

$$(w_0, w_1, \ldots) = (u_0, u_1, \ldots) + (v_0, v_1, \ldots).$$

Neither $\{u_i\}$ nor $\{v_i\}$ need correspond to integers, but addition (carried out formally according to the rules in base -2) makes sense and is well defined. We divide the proof into four separate cases.

Case (i) ($\sum_{i=0}^{\infty} c_i w_i = \infty$): Under the assumption that $\sum_{i=0}^{\infty} c_i w_i = \infty$, it is easy to check that

$$\lim_{m\to\infty}\lim_{n\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m) = \infty = \lim_{n\to\infty}\lim_{m\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m).$$

Case (ii) ($\sum_{i=0}^{\infty} c_i u_i = \infty$ and $\sum_{i=0}^{\infty} c_i v_i = \infty$): Take a fixed n. When m is larger than $N(n)$, we have

$$\mathbf{x}_n = (u_0, u_1, \ldots, u_n, x^n_{n+1}, \ldots, x^n_{N(n)}, 0, 0, 0, \ldots)$$
$$\mathbf{y}_m = (v_0, v_1, \ldots, v_n, v_{n+1}, \ldots, v_{N(n)}, \ldots, v_m, y^m_{m+1}, \ldots, y^m_{M(m)}, 0, 0, 0, \ldots).$$

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Using Lemma 2.4 with $l = N(n)$ and $l = m+1$, we get

$$\left| \sum_{N(n)+1}^{m} c_i(\mathbf{x}_n + \mathbf{y}_m)_i - \sum_{N(n)+1}^{m} c_i v_i \right| < 4c_{N(n)}.$$

The right-hand side of the expression is a constant finite number, and so, when $m \to \infty$, since $\sum_{N(n)+1}^{\infty} c_i v_i = \infty$, we also have $\sum_{N(n)+1}^{\infty} c_i(\mathbf{x}_n + \mathbf{y}_m)_i = \infty$, so that $\lim_{m\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m) = \infty$ and $\lim_{n\to\infty} \lim_{m\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m) = \infty$. Since $\sum_{i=0}^{\infty} c_i u_i = \infty$, by a similar discussion for u_i instead of v_i, we have

$$\lim_{m\to\infty} \lim_{n\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m) = \infty = \lim_{n\to\infty} \lim_{m\to\infty} f_c(\mathbf{x}_n + \mathbf{y}_m).$$

Case (iii) ($\sum_{i=0}^{\infty} c_i u_i < \infty$ and $\sum_{i=0}^{\infty} c_i v_i = \infty$): If $\sum_{i=0}^{\infty} c_i u_i < \infty$ and $\sum_{i=0}^{\infty} c_i v_i = \infty$, then $\sum_{u_i \neq 0} c_i < \infty$. We observe that $\sum_{i=0}^{\infty} c_i u_i < \infty$ implies that the sequence $\{u_i\}$ has a large proportion of 0s. If $u_k \neq 0$ but $u_{k+1} = \ldots = u_{l-1} = 0$ for some k and l, by Lemma 2.4,

$$\left| \sum_{k+1}^{l-1} c_i w_i - \sum_{k+1}^{l-1} c_i v_i \right| \leqslant 4c_k$$

and so

$$\left| \sum_{\substack{i=0 \\ u_i=0}}^{\infty} c_i w_i - \sum_{\substack{i=0 \\ u_i=0}}^{\infty} c_i v_i \right| \leqslant \sum_{\substack{k=0 \\ u_k \neq 0}}^{\infty} 4c_k.$$

On the other hand, since u_i, v_i and w_i are always 0 or 1,

$$\left| \sum_{\substack{i=0 \\ u_i \neq 0}}^{\infty} c_i w_i - \sum_{\substack{i=0 \\ u_i \neq 0}}^{\infty} c_i v_i \right| \leqslant \sum_{\substack{k=0 \\ u_k \neq 0}}^{\infty} c_k.$$

Consequently,

$$\left| \sum_{i=0}^{\infty} c_i w_i - \sum_{i=0}^{\infty} c_i v_i \right| \leqslant 5 \sum_{\substack{k=0 \\ u_k \neq 0}}^{\infty} c_k < \infty.$$

Therefore, $\sum_{i=0}^{\infty} c_i v_i = \infty$ implies that $\sum_{i=0}^{\infty} c_i w_i = \infty$, and we can apply case (i).

Case (iv) ($\sum_{i=0}^{\infty} c_i u_i < \infty$ and $\sum_{i=0}^{\infty} c_i v_i < \infty$): Passing to subsequences, we can suppose that the following limits exist:

$$\lim_{n\to\infty} \sum_{i=n+1}^{N(n)} c_i x_i^n = L$$

and

$$\lim_{m\to\infty} \sum_{i=m+1}^{M(m)} c_i y_i^m = K.$$

Since (c_i) is decreasing and, for any r, $\sum_{i=r+1}^{\infty} c_i = \infty$ while $\sum_{i=r+1}^{\infty} c_i v_i < \infty$, there must exist $i_r > r$ such that $v_{i_r} = v_{i_r+1} = 0$. Now $x_i = 0$ for $i > N(n)$. If we take $r > N(n)$ and $m > i_r$, we find that $\delta_{i_r+1} = 0$, and then $\delta_i = 0$ for $i > i_r + 1$. Therefore, $(\mathbf{x}_n + \mathbf{y}_m)_i = y_i^m$ for $m < i \leqslant M(m)$, and

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=m+1}^{M(m)} c_i(\mathbf{x}_n + \mathbf{y}_m)_i = K.$$

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Now, still for $m > i_r (> N(n))$, we have

$$\left| \sum_{\substack{i=n+1 \\ v_i \neq 0}}^{N(n)} c_i (\mathbf{x}_n + \mathbf{y}_m)_i \right| \leqslant \sum_{\substack{i=n+1 \\ v_i \neq 0}}^{N(n)} c_i$$

and

$$\left| \sum_{\substack{i=n+1 \\ v_i \neq 0}}^{N(n)} c_i (\mathbf{x}_n)_i \right| \leqslant \sum_{\substack{i=n+1 \\ v_i \neq 0}}^{N(n)} c_i.$$

For the part corresponding to $v_i = 0$, we use Lemma 2.4 in the same way as we did in the previous case, and so

$$\left| \sum_{\substack{i=n+1 \\ v_i = 0}}^{N(n)} c_i (\mathbf{x}_n + \mathbf{y}_m)_i - \sum_{\substack{i=n+1 \\ v_i = 0}}^{N(n)} c_i (\mathbf{x}_n)_i \right| \leqslant 4 \sum_{\substack{i=n+1 \\ v_i \neq 0}}^{N(n)} c_i.$$

Hence

$$\left| \sum_{i=n+1}^{N(n)} c_i (\mathbf{x}_n + \mathbf{y}_m)_i - \sum_{i=n+1}^{N(n)} c_i (\mathbf{x}_n)_i \right| \leqslant 6 \sum_{\substack{i=n+1 \\ v_i \neq 0}}^{N(n)} c_i$$

and since the right-hand side of the expression tends to 0 when $n \to \infty$,

$$\lim_{n \to \infty} \lim_{m \to \infty} \sum_{i=n+1}^{N(n)} c_i (\mathbf{x}_n + \mathbf{y}_m)_i = L.$$

Again with $m > N(n)$, using Lemma 2.4 for $l = N(n)$ and $k = m+1$, we have

$$\left| \sum_{i=N(n)+1}^{m} c_i (\mathbf{x}_n + \mathbf{y}_m)_i - \sum_{i=N(n)+1}^{m} c_i v_i \right| \leqslant 4 c_{N(n)}$$

and so $\lim_{n \to \infty} \lim_{m \to \infty} \sum_{i=N(n)+1}^{m} c_i (\mathbf{x}_n + \mathbf{y}_m)_i = 0$. Combining all the above limits, we find that

$$\lim_{n \to \infty} \lim_{m \to \infty} f_c (\mathbf{x}_n + \mathbf{y}_m) = \sum_{i=0}^{\infty} c_i w_i + L + K.$$

The same discussion shows that $\lim_{m \to \infty} \lim_{n \to \infty} f_c (\mathrm{x}_n + \mathbf{y}_m) = \sum_{i=0}^{\infty} c_i w_i + K + L$.

We have covered all the possible cases. Therefore f_c is an extended weakly almost periodic function, and owing to Proposition 1.2, it is extendable to $w\mathbb{N}$.

REMARK 2.6. We point out that, if we replace the negative base -2 with an ordinary positive base, for example the dyadic expansion, then the function f_c in Theorem 2.5 is not a weakly almost periodic function. For example, assume that

$$\mathbf{x}_n = \sum_{m=0}^{n} 2^{2^m}$$

$$\mathbf{y}_n = \sum_{m=0}^{n} \{2^{2^m} + \ldots + (2^{2^{m+1}} - 2)\}$$

$$c_r = \frac{1}{2^{2^m}}$$

if $2^{2^m} \leqslant r < 2^{2^{m+1}}$, $m = 0, 1, \ldots$, and $c_r = 1$, $r < 4$. Then it is easy to see that one of the iterated limits is finite and the other one is infinity, and so f_c is not a weakly almost periodic function.

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3. *A homomorphism from* $\mathbf{H}$ *onto* $[0, \infty]$

We have proved, for each $c \in \Omega$, that, although $f_c : \mathbb{N} \to \mathbb{R}$ is not bounded, it is extendable to $w\mathbb{N}$. We denote the extension again by f_c, and so $f_c : w\mathbb{N} \to [0, \infty]$ is a continuous function. f_c is not a homomorphism, but, as we shall see below, the restriction of f_c to a suitable subset of $w\mathbb{N}$ is an onto homomorphism.

NOTATION 3.1. For $\mathbf{x} \in \mathbb{N}$ with the expansion $\mathbf{x} = (x_0, x_1, \ldots)$, define $\mathbf{m}(\mathbf{x})$ as follows:

$$\mathbf{m}(\mathbf{x}) = \min \{i \mid x_i \neq 0\}.$$

It is easy to use the iterated limit property to show that $\mathbf{m}$ is an extended weakly almost periodic function.

LEMMA 3.2. *The function* $\mathbf{m} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ *is an onto extended weakly almost periodic function.*

Therefore $\mathbf{m}$ is extendable to a continuous function from $w\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$, which is again denoted by $\mathbf{m}$.

NOTATION 3.3. With $\mathbf{H} = (\mathbf{m})^{-1}(\{\infty\})$ and for each $c \in \Omega$, let $\langle c \rangle : \mathbf{H} \to [0, \infty]$ denote the restriction of f_c to $\mathbf{H}$.

$\mathbf{H}$ is an important part of $w\mathbb{N}$ because of Proposition 3.4 below.

PROPOSITION 3.4. *There is no idempotent of* $w\mathbb{N}$ *outside* $\mathbf{H}$.

Proof. Take $e \in w\mathbb{N}$ such that $e + e = e$, but $\mathbf{m}(e) = k < \infty$. There is a net $(\mathbf{x}_\alpha)$ in $\mathbb{N}$ such that $\mathbf{m}(\mathbf{x}_\alpha) = k$ for each α and $\mathbf{x}_\alpha \to e$ in the $w\mathbb{N}$-topology. Thus, for any two indices α and β, we have $(\mathbf{x}_\alpha)_i = (\mathbf{x}_\beta)_i = 0$ for $i < k$ and $(\mathbf{x}_\alpha)_k = (\mathbf{x}_\beta)_k = 1$, so that $(\mathbf{x}_\alpha + \mathbf{x}_\beta)_k = 0$ and $\mathbf{m}(\mathbf{x}_\alpha + \mathbf{x}_\beta) > k$. Taking limits over α and then β gives $\mathbf{m}(e + e) > k$, which contradicts $e + e = e$.

THEOREM 3.5. *Given that* $c \in \Omega$, $\langle c \rangle : \mathbf{H} \to [0, \infty]$ *is an onto continuous homomorphism.*

Proof. Take $\tau, \mu \in \mathbf{H}$. There are nets $(\mathbf{x}_\alpha)$ and $(\mathbf{y}_\beta)$ that are convergent to τ and μ, respectively, in the $w\mathbb{N}$-topology. By definition,

$$\tau + \mu = \lim_\alpha \lim_\beta (\mathbf{x}_\alpha + \mathbf{y}_\beta).$$

Fix $\mathbf{x}_\alpha$. Since $\mathbf{m}(\mathbf{y}_\beta) \to \infty$ from some stage onward,

$$\operatorname{Supp}(\mathbf{x}_\alpha) \prec \operatorname{Supp}(\mathbf{y}_\beta)$$

and hence

$$f_c(\mathbf{x}_\alpha + \mathbf{y}_\beta) = f_c(\mathbf{x}_\alpha) + f_c(\mathbf{y}_\beta).$$

Taking the iterated limit first in β and then in α, we have

$$\langle c \rangle (\tau + \mu) = \langle c \rangle (\tau) + \langle c \rangle (\mu).$$

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Therefore $\langle c\rangle$ is a homomorphism. $\langle c\rangle$ is continuous, since it is the restriction of the continuous function f_c. We check that $\langle c\rangle$ is surjective. Given that $\lambda\in[0,\infty]$ and $\lambda\neq\infty$, there exists a sequence $\{\mathbf{x}_n\}$ of natural numbers such that $\mathbf{m}(\mathbf{x}_n)\geqslant n$ and $|f_c(\mathbf{x}_n)-\lambda|<1/n$. ($c_i\downarrow 0$ and $\sum c_i=\infty$ imply the existence of $c_{n_1},c_{n_2},\ldots,c_{n_k}$ such that $n<n_1<\ldots<n_k$ and n_k is even, satisfying $\lambda<c_{n_1}+\ldots+c_{n_k}<\lambda+1/n$. Then $\mathbf{x}_n=(-2)^{n_1}+\ldots+(-2)^{n_k}$ is a natural number, and it satisfies the conditions.) Then any limit point τ of $\{\mathbf{x}_n\}$ in $w\mathbb{N}$ is in $\mathbf{m}^{-1}(\infty)$ and $\langle c\rangle(\tau)=\lambda$. If $\lambda=\infty$, since $\sum c_i=\infty$, for each positive integer n there exists a natural number $\mathbf{x}_n$ such that $\mathbf{m}(\mathbf{x}_n)\geqslant n$ and $f_c(\mathbf{x}_n)>n$. Any limit point τ of the sequence $\{\mathbf{x}_n\}$ in $w\mathbb{N}$ is in $\mathbf{m}^{-1}(\infty)$ and $\langle c\rangle(\tau)=\infty$.

Given that $c\in\Omega$, $\langle c\rangle^{-1}(0)$ and $\langle c\rangle^{-1}(\infty)$ are compact semitopological semigroups, and each must contain an idempotent. Thus we have an easy elementary proof of West's theorem [**13**].

COROLLARY 3.6. *$w\mathbb{N}$ contains at least two idempotents.*

It is easy to check the following Lemma 3.7.

LEMMA 3.7. *Given that $c=\{c_i\}$ and $c'=\{c_i'\}$ in Ω such that, for each $i\in\mathbb{N}\cup\{0\}$, $c_i\leqslant c_i'$, we have*

$$\langle c'\rangle^{-1}(0)\subseteq\langle c\rangle^{-1}(0).$$

In the rest of this section, we introduce some suitable conditions which can guarantee that $\langle c\rangle^{-1}(0)\cap\langle c'\rangle^{-1}(\lambda)\equiv\varnothing$ for $c,c'\in\Omega$ and $0<\lambda\leqslant\infty$.

PROPOSITION 3.8. *Let $c,c'\in\Omega$. Then the following are equivalent:*

(i) *$\langle c\rangle^{-1}(0)\cap\langle c'\rangle^{-1}(\lambda)\neq\varnothing$ for one $\lambda\in(0,\infty]$.*

(ii) *$\langle c\rangle^{-1}(0)\cap\langle c'\rangle^{-1}(\lambda)\neq\varnothing$ for every $\lambda\in(0,\infty]$.*

(iii) *There exists a subsequence $\{n_i\}$ of $\mathbb{N}\cup\{0\}$ such that*

$$\sum_{i=0}^{\infty}c_{n_i}<\infty\quad\textit{but}\quad\sum_{i=0}^{\infty}c'_{n_i}=\infty.$$

Proof. That (i) implies (ii) is trivial. We prove that (i) implies (iii). Let $\langle c\rangle^{-1}(0)\cap\langle c'\rangle^{-1}(\lambda)\neq\varnothing$, for a given $\lambda\in(0,\infty]$, and take τ in the intersection. If $(\mathbf{x}_\alpha)$ is a net of $\mathbb{N}$ that converges to τ, we can find $\mathbf{x}_{\alpha_1},\mathbf{x}_{\alpha_2},\ldots$, a sequence of terms of the net, such that, if $\lambda=\infty$,

$$f_c(\mathbf{x}_{\alpha_n})<\frac{1}{n^2}\qquad f_{c'}(\mathbf{x}_{\alpha_n})>n$$

for each n, and

$$\operatorname{Supp}(\mathbf{x}_{\alpha_1})\prec\operatorname{Supp}(\mathbf{x}_{\alpha_2})\prec\ldots$$

while, if $0<\lambda<\infty$,

$$f_c(\mathbf{x}_{\alpha_n})<\frac{1}{n^2}\qquad f_{c'}(\mathbf{x}_{\alpha_n})>\frac{\lambda}{2}$$

for each n, and

$$\operatorname{Supp}(\mathbf{x}_{\alpha_1})\prec\operatorname{Supp}(\mathbf{x}_{\alpha_2})\prec\ldots.$$

If $\{n_i\,|\,i\in\mathbb{N}\cup\{0\}\}$ denotes $\bigcup_{i=1}^{\infty}\operatorname{Supp}(\mathbf{x}_{\alpha_i})$, then $\sum_{i=0}^{\infty}c_{n_i}<\infty$, but $\sum_{i=0}^{\infty}c'_{n_i}=\infty$. We next show that (iii) implies (ii). Take any $\lambda\in(0,\infty]$. Let $\{n_i\}$ be any sequence that satisfies the conditions in (iii), and put $A=\{n_i\,|\,i\in\mathbb{N}\cup\{0\}\}$. Using the conditions and

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the fact that c_i is decreasingly convergent to 0, let T_n denote the set of all natural numbers of the form $(-2)^{m_1}+\ldots+(-2)^{m_l}+(-2)^m$, in which $\{m_1,\ldots,m_l\}$ is a subset of A that satisfies $n<m_1<\ldots<m_l$, $m>m_l$ is an even natural number, and

$$c_{m_1}+c_{m_2}+\ldots+c_{m_l}+c_m<\frac{1}{n^2}$$

and, if $\lambda=\infty$,

$$c'_{m_1}+c'_{m_2}+\ldots+c'_{m_l}+c'_m>n$$

while, if $0<\lambda<\infty$,

$$\lambda<c'_{m_1}+c'_{m_2}+\ldots+c'_{m_l}+c'_m<\lambda+\frac{1}{n}.$$

(Notice that m is taken to be even only to ensure that $(-2)^{m_1}+\ldots+(-2)^{m_l}+(-2)^m\in\mathbb{N}$). Then

$$T=\bigcap_{n=1}^{\infty}\mathrm{cl}_{w\mathbb{N}}\,T_n\neq\varnothing$$

by compactness, and any $\tau\in T$ satisfies $\langle c\rangle(\tau)=0$, and $\langle c'\rangle(\tau)=\lambda$.

4. *The lattice of idempotents in* $w\mathbb{N}$

In this section, we prove that there are $2^{\mathfrak{c}}$ idempotents in $w\mathbb{N}$. This is a well known, rather old conclusion, which was proved for the first time by Brown and Moran [**2**]. Their proof was based on some techniques in harmonic analysis. Here we use only elementary calculations. Our proofs here are parallel to those used by Ruppert in [**11**] to reach the same conclusions. Whether Ruppert has determined a significantly larger set of functions than ours (in the sense that his functions separate more points of $w\mathbb{N}$ than ours), we do not know. We start with an elementary lemma, Lemma 4.1.

LEMMA 4.1 ([**6**, Exercise 6Q]). *There is a collection* $\mathfrak{F}$ *of infinite subsets of* $\mathbb{N}$ *such that the following is true:*

(i) *The cardinality of* $\mathfrak{F}$ *is* $\mathfrak{c}$.

(ii) *For each two different* M, $N\in\mathfrak{F}$, $M\cap N$ *is a finite set.*

Proof. Let us write $\mathbb{Q}=\{q_1,q_2,\ldots\}$ for the countable set of rational numbers. For each $r\in\mathbb{R}\setminus\mathbb{Q}$, there exists a sequence q_{n_i} such that $q_{n_i}\to r$ in the real line topology. Put

$$M_r=\{n_1,n_2,\ldots\}$$

and

$$\mathfrak{F}=\{M_r\,|\,r\in\mathbb{R}\setminus\mathbb{Q}\}.$$

$\mathfrak{F}$ satisfies (i) and (ii).

The next step is to construct a collection $\{\{\xi_M(i)\}_{i=0}^{\infty}\,|\,M\in\mathfrak{F}\}$ of sequences in Ω which can separate $2^{\mathfrak{c}}$ idempotents.

We start with a sequence $\{\alpha_m\}_{m=0}^{\infty}$, where $\alpha_m=2^{2^m}$. Note that $\alpha_m^2=\alpha_{m+1}$. Define $\xi=\{\xi(n)\}_{n=0}^{\infty}$ as follows:

$$\xi(n)=\begin{cases}\dfrac{1}{\alpha_0}=\dfrac{1}{2} & n\leqslant\alpha_0=2\\[2ex] \dfrac{1}{\alpha_m} & \alpha_{m-1}<n\leqslant\alpha_m,\ m\geqslant 1.\end{cases}\tag{2}$$

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Then $\xi(n)\downarrow 0$ (and, informally, ξ consists of constant blocks which get longer and longer).

$$\begin{aligned}\sum_{n=\alpha_{m-1}+1}^{\alpha_m} \xi(n) &= (\alpha_m - \alpha_{m-1})\frac{1}{\alpha_m} \\ &= 1 - \frac{\alpha_{m-1}}{\alpha_{m-1}^2} \\ &= 1 - \frac{1}{\alpha_{m-1}}\end{aligned} \tag{3}$$

and so the sum of these constant blocks increases to 1. On the other hand, if we replace the constant value $\xi(n) = 1/\alpha_m$ in the mth block by the 'wrong' value $1/\alpha_r$, where $r > m$, we find that

$$\begin{aligned}\sum_{n=\alpha_{m-1}+1}^{\alpha_m} \frac{1}{\alpha_r} &= (\alpha_m - \alpha_{m-1})\frac{1}{\alpha_r} \\ &= \frac{\alpha_m - \alpha_{m-1}}{\alpha_m}\frac{\alpha_m}{\alpha_r} \\ &< 1\left(\frac{\alpha_m}{\alpha_m^2}\right) \\ &= \frac{1}{\alpha_m}.\end{aligned} \tag{4}$$

Now let $M = (m_0, m_1, \ldots)$ be any strictly increasing sequence in $\mathbb{N} \cup \{0\}$. Define $\xi_M = \{\xi_M(n)\}_{n=0}^{\infty}$ as follows:

$$\xi_M(n) = \begin{cases} \dfrac{1}{\alpha_{m_0}} & n \leqslant \alpha_{m_0} \\ \dfrac{1}{\alpha_{m_k}} & \alpha_{m_{k-1}} < n \leqslant \alpha_{m_k},\ k \geqslant 1. \end{cases} \tag{5}$$

Then $\xi_M(n)\downarrow 0$. (Informally, ξ_M uses some of the values in ξ, but the constant blocks are longer. $\xi_M(n)$ and $\xi(n)$ are equal precisely in the blocks $\alpha_{m_k-1} < n \leqslant \alpha_{m_k}$.) Using (3) and (5), we see that

$$\sum_{k=1}^{\infty} \sum_{n=\alpha_{m_{k-1}}+1}^{\alpha_{m_k}} \xi_M(n) = \sum_{k=1}^{\infty} \left(1 - \frac{1}{\alpha_{m_k} - 1}\right) = \infty. \tag{6}$$

Hence we have the following Proposition 4.2.

PROPOSITION 4.2. *If $M = (m_0, m_1, \ldots)$ is any strictly increasing sequence in $\mathbb{N} \cup \{0\}$, then ξ_M defined as in* (5) *is in Ω, that is, $\xi_M \downarrow 0$ and $\sum_{i=0}^{\infty} \xi_M(i) = \infty$.*

Proof. $\xi_M\downarrow 0$ is trivial, and $\sum_{i=0}^{\infty} \xi_M(i) = \infty$ is a consequence of (6).

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Next, let $\mathfrak{F}$ be as stated in Lemma 4.1. We arrange $M \in \mathfrak{F}$ as a strictly increasing sequence, and define ξ_M as above.

LEMMA 4.3. *Given any $M \in \mathfrak{F}$, there exists a sequence $\{n_i\}$ in $\mathbb{N} \cup \{0\}$ such that $\sum_{i=0}^{\infty} \xi_m(n_i) = \infty$, but, for any $M' \in \mathfrak{F}$, if $M' \neq M$, $\sum_{i=0}^{\infty} \xi_{M'}(n_i) < \infty$. (Notice that $\{n_i\}$ is dependent only on M.)*

Proof. Given that $M' \neq M$ in $\mathfrak{F}$, we arrange them as strictly increasing sequences $M = (m_0, m_1, \ldots)$ and $M' = (m'_0, m'_1, \ldots)$. By Lemma 4.1, $M \cap M'$ is a finite set, and so there exists a natural number k such that

$$\{m_i \mid m_i \geqslant k\} \cap \{m'_i \mid m'_i \geqslant k\} = \varnothing. \tag{7}$$

Let m_i be larger than k. For any $n \in \mathbb{N} \cup \{0\}$ with $\alpha_{m_i - 1} < n \leqslant \alpha_{m_i}$, there exists a j such that $\alpha_{m'_{j-1}} < n \leqslant \alpha_{m'_j}$. We cannot have $m'_j < m_i$, since in this case

$$\alpha_{m_i - 1} < n \leqslant \alpha_{m'_j} < \alpha_{m_i}$$

and so $m_i - 1 < m'_j < m_i$, which is impossible. Hence $m'_j > m_i$, since by (7) equality cannot happen. Now, by (4),

$$\sum_{\substack{i=0 \\ m_i > k}}^{\infty} \sum_{n=\alpha_{m_i - 1}+1}^{\alpha_{m_i}} \xi_N(n) \leqslant \sum_{\substack{i=0 \\ m_i > k}}^{\infty} \frac{1}{\alpha_{m_i}} < \infty.$$

The corresponding sum for $m_i \leqslant k$ is finite, since the indexing set is finite. Therefore

$$\sum_{i=0}^{\infty} \sum_{n=\alpha_{m_i - 1}+1}^{\alpha_{m_i}} \xi_N(n) < \infty.$$

On the other hand, we know from (6) that

$$\sum_{i=0}^{\infty} \sum_{n=\alpha_{m_i - 1}+1}^{\alpha_{m_i}} \xi_M(n) = \infty.$$

If we write $\bigcup_{i=0}^{\infty} \{\alpha_{m_i - 1} + 1, \ldots, \alpha_{m_i}\}$ as a sequence $\{n_i\}_{i=0}^{\infty}$, it will satisfy the conditions.

The proof of the following Theorem 4.4 is the aim of this section.

THEOREM 4.4. *Define $\Phi : \mathbf{H} \to \prod_{\mathfrak{F}} [0, \infty]$ as follows:*

$$\Phi(\tau) = (\langle \xi_M \rangle(\tau))_{M \in \mathfrak{F}}.$$

Φ is an onto continuous homomorphism.

Proof. Since each coordinate is a continuous homomorphism, Φ is also continuous and a homomorphism. In order to prove surjectivity, take $\lambda = (\lambda_M)_{M \in \mathfrak{F}}$ in $\prod_{\mathfrak{F}} [0, \infty]$. We shall prove that $\bigcap_{\tau \in \mathfrak{F}} \langle \xi_M \rangle^{-1}(\lambda_M) \neq \varnothing$. Since each $\langle \xi_M \rangle^{-1}(\lambda_M)$ is a closed set in the compact space $\mathbf{H}$, it is sufficient to show that the set of all $\langle \xi_M \rangle^{-1}(\lambda_M)$, $M \in \mathfrak{F}$, has the finite intersection property. Let $M_1, M_2, \ldots, M_k$ be k arbitrary distinct sets in $\mathfrak{F}$. We have to show that

$$\bigcap_{i=1}^{k} \langle \xi_{M_i} \rangle^{-1}(\lambda_{M_i}) \neq \varnothing.$$

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For each fixed $1 \leqslant t \leqslant k$, let

$$c(n) = \xi_{M_1}(n) + \ldots + \xi_{M_{t-1}}(n) + \xi_{M_{t+1}}(n) + \ldots + \xi_{M_k}(n)$$

for each $n \in \mathbb{N} \cup \{0\}$. By Proposition 4.2, there exists a sequence $\{n_i\}$ such that

$$\sum_{i=0}^{\infty} \xi_{M_t}(n_i) = \infty \tag{8}$$

while, for any $N \in \mathfrak{F}$ not equal to M_t,

$$\sum_{i=0}^{\infty} \xi_N(n_i) < \infty.$$

Hence

$$\sum_{i=0}^{\infty} c(n_i) < \infty. \tag{9}$$

By Proposition 3.8,

$$\langle c \rangle^{-1}(0) \cap \langle \xi_{M_t} \rangle^{-1}(\lambda_{M_t}) \neq \varnothing.$$

If τ_t is the above intersection, then

$$\xi_{M_t}(\tau_t) = \lambda_{M_t}$$

and, by Lemma 3.7, since $\xi_{M_i} \leqslant c$ for $i \neq t$, we get

$$\xi_{M_i}(\tau_t) = 0.$$

Hence

$$\tau_1 + \tau_2 + \ldots + \tau_k \in \bigcap_{i=1}^{k} \langle \xi_{M_i} \rangle^{-1}(\lambda_{M_i}).$$

Therefore $\bigcap_{i=1}^{k} \langle \xi_{M_i} \rangle^{-1}(\lambda_{M_i}) \neq \varnothing$.

Now we are able to prove that there are 2^c idempotents in $w\mathbb{N}$.

COROLLARY 4.5. *$w\mathbb{N}$ contains a copy of the lattice of idempotents of $\prod_{\mathfrak{F}}[0, \infty]$, and consequently there are 2^c idempotents in $w\mathbb{N}$.*

Proof. By Theorem 4.4, for each idempotent $e \in \prod_{\mathfrak{F}}[0, \infty]$, there exists a unique minimal idempotent $e' \in \Phi^{-1}(e)$. The set of all such idempotents e' is a lattice, which has 2^c members.

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