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Perfect powers in values of certain polynomials at integer points

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1. For an integer $\nu > 1$, we define $P(\nu)$ to be the greatest prime factor of ν and we write $P(1) = 1$. Let $m \geqslant 0$ and $k \geqslant 2$ be integers. Let $d_1, \ldots, d_t$ with $t \geqslant 2$ be distinct integers in the interval $[1, k]$. For integers $l \geqslant 2$, $y > 0$ and $b > 0$ with $P(b) \leqslant k$, we consider the equation

$$(m+d_1)\ldots(m+d_t) = by^l. \tag{1}$$

Put

$$\nu_l = \frac{1}{2}\left(1+\frac{1}{l-2}\right), \quad l = 4, 5, \ldots$$

so that $\frac{1}{2} < \nu_l \leqslant \frac{3}{4}$. If $\alpha > 1$ and $k^\alpha < m \leqslant k^l$, then equation (1) implies that

$$P(m+d_i) \leqslant k$$

for $1 \leqslant i \leqslant t$ and hence

$$t < \alpha^{-1}k + \pi(k).$$

See Erdös and Turk [9], lemma 2·1. For $m > k^l$, we apply the theory of linear forms in logarithms and a theorem of Baker [1] on the approximations of certain algebraic numbers by rationals to prove

THEOREM 1. (*a*) *Let* $\epsilon > 0$. *Equation* (1) *with*

$$l \geqslant 4, m > k^l, t \geqslant (\nu_l + \epsilon)k$$

implies that k *is bounded by an effectively computable number depending only on* ϵ.

(*b*) *Let* $\epsilon > 0$. *Let* $l = 3$ *and* $m > k^3$ *with* $m \notin (k^4(\log k)^3, k^{4+\epsilon})$. *There exist effectively computable positive numbers* $C_1 < 1$ *and* C_2 *depending only on* ϵ *such that equation* (1) *with* $t \geqslant C_1 k$ *implies that* $k \leqslant C_2$.

Theorem 1 is a considerable improvement of a result of Erdös [7] that for $\epsilon > 0$ and integers m, k, l with $l \geqslant 3$, $m > k^l$ and k exceeding an effectively computable number depending only on ϵ, any product of at least

$$k - (1-\epsilon)k\frac{\log\log k}{\log k}$$

distinct integers from $m+1, \ldots, m+k$ is never an lth power. A conjecture of Erdös and Turk [9], p. 151, Remark 2·4 implies that the restriction on t in Theorem 1 cannot be relaxed to $t/k \to 0$ as $k \to \infty$. In Lemma 6, we prove Theorem 1 with the assumption on t replaced by $t/k \to 0$ as $\min(k, l) \to \infty$. Theorem 1 contributes to a problem of Erdös and Selfridge [8], p. 300.

If $l = 2$, we apply Brun's Sieve and an estimate of Sprindžuk [18] on the magnitude of integral solutions of a hyperelliptic equation to prove

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THEOREM 2. *Let* $\epsilon > 0$. *Equation* (1) *with*

$$l = 2, \quad m > k^2, \quad t \geqslant k - (1-\epsilon)k\frac{\log\log\log k}{\log k} \tag{2}$$

implies that k *is bounded by an effectively computable number depending only on* ϵ.

Theorem 2 is an improvement of a result of Erdös [7] and Erdös and Turk [9] that for certain effectively computable absolute constants $C_3 > 0$ and $C_4 > 0$, any product of at least $k - C_3 k/\log k$ with $k \geqslant C_4$ distinct integers from $m+1, \ldots, m+k$ is not a square.

If $t = k$, Erdös and Selfridge [8] confirmed an old conjecture by proving that the product of two or more consecutive positive integers is never a power. For earlier results in this direction, see Erdös and Selfridge [8]. We give a different proof of the following result of Erdös [7].

COROLLARY 1. *There exists an effectively computable absolute constant* $C_5 > 0$ *such that for* $k \geqslant C_5$ *and* $l \geqslant 2$, *the product of* k *consecutive positive integers is not an* lth *power.*

The proof of Corollary 1 is new in the sense that it does not depend on the fact that whenever $l \geqslant 3$, $a_i a_j$ are distinct for distinct pairs (i,j). See the proof of Lemma 6 for the definition of a_i's. This fact is crucial in the proof of Erdös [7] and Erdös and Selfridge [8]. The proof of Corollary 1 with $l = 2$ also differs is an essential way.

For a given k, it follows from theorem 2 of [17] and a theorem of Baker [3] that equation (1) with $l \geqslant 3$ and $y > 1$ implies that $\max(m, y, l)$ is bounded by an effectively computable number depending only on k. Also, by Baker [3], equation (1) with $l = 2$ and $t \geqslant 3$ implies that $\max(m, y)$ is bounded by an effectively computable number depending only on k. Further we remark that equation (1) includes.

$$\binom{m+k}{k} = y^l.$$

Erdös [6] proved that this equation has no solution in positive integers $m, k > 3, y > 1$ and $l > 1$ with $m \geqslant k$.

Next, we consider the equation

$$z^q = \frac{x^n - 1}{x - 1} \quad \text{in integers} \quad x > 1, z > 1, n > 2 \quad \text{and} \quad q > 1. \tag{3}$$

We have

THEOREM A. *Equation* (3) *has only finitely many solutions if at least one of the following conditions is satisfied.*

(i) $2|q$ (Nagell [13], Ljunggren [11]).

(ii) x *is fixed* (Shorey and Tijdeman [16]).

(iii) n *has a fixed prime divisor* (Shorey and Tijdeman [16]).

Further the result is effective.

In fact Ljunggren [11] determined all the solutions of (3) if $2|q$. Further Nagell [12] and Ljunggren [11] determined all the solutions of (3) if $4|n$ and $3|n$, respectively.

It suffices to consider equation (3) with q prime. We write

$$n = q^e n', \quad n' = p_1^{e_1} \ldots p_r^{e_r}, \quad (n', q) = 1 \tag{4}$$

where $e \geqslant 0, e_1 > 0, \ldots, e_r > 0$ are integers and $p_1 < p_2 < \ldots < p_r$ are prime numbers. Except for § 2, the letters x, z, n, q will denote integers $x > 1$, $z > 1$, $n > 2$, $q > 1$

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satisfying (3) and (4) with $n' \geqslant 2$ and q is a prime number. For an integer $\nu > 1$, denote by $\omega(\nu)$ the number of district prime divisors of ν and by $Q(\nu)$ the greatest square free factor of ν. Recall that $\phi(\nu)$ is the number of positive integers less than or equal to ν and coprime to ν. We shall write

$$Q_\nu = \phi(Q(\nu)).$$

For $1 \leqslant \lambda \leqslant \mu \leqslant r$, put

$$n_{\lambda,\mu} = p_\lambda^{e_\lambda} \dots p_\mu^{e_\mu},$$

so that $n_{1,r} = n'$ and $n_{\lambda,\lambda} = p_\lambda^{e_\lambda}$ for $1 \leqslant \lambda \leqslant r$. We again apply a theorem of Baker [1] on the approximations of certain algebraic numbers by rationals and an estimate of Baker [4] on linear forms in logarithms to prove

THEOREM 3. *Suppose that at least one of the following conditions is satisfied.*

(i) *There is a* $n_{\lambda,\mu}$ *with* $1 \leqslant \lambda \leqslant \mu \leqslant r$ *such that* $n_{\lambda,\mu} \equiv 1 \pmod q$.

(ii) $\omega(n') > q-2$.

(iii) $\omega(n') > (q-3)/2$ *and* n' *is a square.*

Then equation (3) *implies that* $\max(x,z,n,q)$ *is bounded by an effectively computable absolute constant.*

The proof of Theorem 3 also depends on Theorem A. For a fixed q, we see from Theorem A (iii) that it remains to consider equation (3) with $(n,q) = 1$. Then Theorem 3 implies the following result.

COROLLARY 2. *Suppose that at least one of the following conditions is satisfied.*

(i) $\omega(n) > q-2$.

(ii) $\omega(n) > (q-3)/2$ *and* n *is a square.*

(iii) $q = 3$ *and* $n \not\equiv 5 \pmod 6$ *if* $\omega(n) = 1$.

(iv) $q = 5$, $n \not\equiv 4 \pmod 5$ *if* $\omega(n) = 1$ *and* n *is a square. Then equation* (3) *implies that* $\max(x,z,n)$ *is bounded by an effectively computable number depending only on* q.

Ljunggren [11] determined all the solutions of (3) with $q = 3$ and $n \not\equiv 5 \pmod 6$. See also Inkeri [10]. Except for § 2, we shall follow the notation of this section without reference.

2. The height of a rational number v/w with $(v,w) = 1$ is defined as $\max(|v|,|w|)$. We apply several times the following estimate of Baker [4] on linear forms in logarithms.

LEMMA 1. *Let* $\alpha_1, \dots, \alpha_n$ *be positive rational numbers of heights not exceeding* $A_1, \dots, A_n$, *respectively, where* $A_j \geqslant 3$ *for* $1 \leqslant j \leqslant n$. *Put*

$$\Omega' = \prod_{j=1}^{n-1} \log A_j, \quad \Omega = \Omega' \log A_n.$$

There exist effectively computable absolute constants $C_6 > 0$ *and* $C_7 > 0$ *such that the inequalities*

$$0 < |b_1 \log \alpha_1 + \dots + b_n \log \alpha_n| < \exp(-(C_6 n)^{C_7 n} \Omega \log \Omega' \log B)$$

have no solution in rational integers $b_1, \dots, b_n$ *of absolute values not exceeding* $B(\geqslant 3)$.

It is shown in [14] that sharper estimates for linear forms in logarithms can be obtained if α_i's are very close to one. The proof of Theorem 1 depends on the following estimate for linear forms in logarithms with α_i's very close to one.

LEMMA 2. *Let* $u > 0$, $\tau_1 \geqslant 1$ *and* $\tau_2 \geqslant 1$. *Let* $\alpha_1, \dots, \alpha_{n-1}$ *and* α_n *be positive rational*

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numbers of heights not exceeding A_1 and A, respectively, where we assume that $A_1 \geqslant 3$, $A \geqslant 3$ and

$$(\log A)(\log A_1)^{-1} \geqslant \tau_1^{-1}.$$

Further assume that

$$|\log \alpha_i| \leqslant A_1^{-1/\tau_2} \quad (1 \leqslant i \leqslant n).$$

Let $b_1, \ldots, b_n$ with $b_n \neq 0$ be rational integers of absolute values not exceeding A_1^u. There exist effectively computable numbers $C_8 > 0$, $C_9 > 0$ depending only on u and there exists a prime $p \leqslant C_8 \tau_2 n^3$ such that the inequality

$$|b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \exp(-(C_9 \tau_2 n^3)^{3n+3} \tau_1 \log A) \tag{5}$$

implies that

$$\alpha_1^{j_1} \ldots \alpha_n^{j_n} = \eta^p \tag{6}$$

where η is a rational number and $j_1, \ldots, j_n$ are integers satisfying $0 \leqslant j_i < p$ and $j_n = 1$.

Lemma 2 includes theorem 3 of [15] where $A = A_1$. A result of Waldschmidt [19] implies that inequality (5) with $\log A$ replaced by $\log A(\log A_1 + \log\log A)(\log A_1)^{-1}$ is not possible. This result of Waldschmidt is not sufficient for the proof of Theorem 1.

Proof. For an integer $K \geqslant 1$, we write

$$\Delta(x; K) = (x+1)\ldots(x+K)/K!$$

and put $\Delta(x; 0) = 1$. We may assume that $n > 1$, otherwise the theorem follows immediately. Further, by observing $A^{\tau_1} \geqslant A_1$, there is no loss of generality in assuming that $\tau_1 = 1$. Put $\beta_j = -b_j/b_n$ for $1 \leqslant j \leqslant n$ and

$$|\Lambda| = |\beta_1 \log \alpha_1 + \ldots + \beta_{n-1} \log \alpha_{n-1} - \log \alpha_n|.$$

Denote by $u_1, \ldots, u_4$ effectively computable positive numbers depending only on u. Put

$$h = [u_1 \tau_2 n^3], \quad h_1 = h, h_{r+1} = h_r h, \quad 1 \leqslant r < 4n(n+3).$$

Further set $E' = (\log A)(\log A_1)^{-1}$ and

$$L = L_n = h^{2n-1}, \quad L_1 = L_2 = \ldots = L_{n-1} = [LE']$$

and

$$k = (\tfrac{1}{4}h^{2n+1}E'], \quad k_1 = k, \quad k_{r+1} = \left[\left(1 - \frac{1}{n^2}\right)k_r\right], \quad 1 \leqslant r < 4n(n+3).$$

Assume that

$$|\Lambda| < \exp(-(u_2 \tau_2 n^3)^{3n+3} \log A).$$

The numbers u_1 and u_2 are sufficiently large. We shall, first, choose u_1 and then u_2 depending on u_1. We shall make no reference to this choice of u_1 and u_2 in the argument. For any non-negative integers $m_1, \ldots, m_{n-1}$, we define

$$\phi(z_1, \ldots, z_{n-1}) = \sum_{\lambda_1=0}^{L_1} \ldots \sum_{\lambda_n=0}^{L_n} p(\lambda_1, \ldots, \lambda_n) \Delta \alpha_1^{\gamma_1 z_1} \ldots \alpha_{n-1}^{\gamma_{n-1} z_{n-1}}$$

where $p(\lambda_1, \ldots, \lambda_n)$ are integers to be determined, $\gamma_j = \lambda_j + \lambda_n \beta_j$ $(1 \leqslant j < n)$ and

$$\Delta = \prod_{j=1}^{n-1} \Delta(b_n \gamma_j; m_j).$$

Denote by $f(z) = \phi(z, \ldots, z)$ and $g(z)$ the function obtained from $f(z)$ by substituting α_n for $\alpha_1^{\beta_1} \ldots \alpha_{n-1}^{\beta_{n-1}}$. By Siegel's lemma, we determine integers $p(\lambda_1, \ldots, \lambda_n)$, not all zero,

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satisfying $g(l) = 0$ for all integers l with $1 \leqslant l \leqslant h$ and non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k$ and

$$\log |p(\lambda_1, \ldots, \lambda_n)| \leqslant u_3 h^{2n+1} \log A$$

for all $(\lambda_1, \ldots, \lambda_n)$. We understand $\log 0 = -\infty$.

Suppose that there exist integers l with $h_1 < l \leqslant h_2$ and nonnegative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_2$ such that $g(l) \neq 0$. Then

$$\log |f(l)| \geqslant -u_4 n L h_2 \log A. \tag{7}$$

On the other hand, we use inductive hypothesis and the formula

$$\frac{1}{2\pi i}\int_{\Gamma:|z|=5h_2 A_1^{\frac{1}{2}}/\tau_1} \frac{f(z)F(l)}{(z-l)F(z)}dz = f(l) + \sum_{r=1}^{h_1}\sum_{m=0}^{k_1-k_2}\frac{f^{(m)}(r)}{m!\,2\pi i}\int_{\Gamma_r:|z-r|=\frac{1}{2}} \frac{(z-r)^m F(l)}{(z-l)F(z)}dz$$

where $F(z) = ((z-1)\ldots(z-h_1))^{k_1-k_2+1}$, to obtain an upper bound for $\log |f(l)|$ which is inconsistent with (7). We proceed inductively to conclude that $g(l) = 0$ for all integers l with $1 \leqslant l \leqslant h_{n+1}$ and nonnegative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{n+1}$. Then we derive that $g(a/q) = 0$ for all rational numbers a/q with $0 \leqslant a/q \leqslant h$, $0 < q \leqslant h$ and all nonnegative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{n+2}$. For details, we refer to Baker [4] and the author [15]. Now argue as in [15] to conclude that the number of primes p, $h^{\frac{1}{2}} < p \leqslant h$, not satisfying (6) is at most $4n$. For this assertion, we need to use the non-vanishing of determinants $|\Delta(b_n \gamma_j; m_j)|_{0 \leqslant \lambda_j \leqslant L,\, 0 \leqslant m_j \leqslant L}$ in place of the determinants $|\gamma_j^{m_j}|_{0 \leqslant \lambda_j \leqslant L,\, 0 \leqslant m_j \leqslant L}$. Now Lemma 2 follows immediately.

Further, we state three Theorems that we shall apply in the proof of our results, Lemmas 3 and 4 depend on the theory of linear forms in logarithms.

LEMMA 3 (Baker [2]). *Let $F(X, Y)$ be a binary form with integer coefficients. Suppose that $F(X, 1)$ has at least three distinct roots. For integers $M \neq 0$, x and y, the equation*

$$F(x, y) = M$$

implies that $\max(|x|, |y|)$ *is bounded by an effectively computable number depending only on F and M.*

LEMMA 4 (Sprindžuk [18]). *Suppose that $f \in \mathbb{Z}[X]$ has at least three simple roots. Denote by $H (\geqslant 2)$ an upper bound for the height of f. For integers $a \neq 0$, x and y, the equation*

$$f(x) = ay^2$$

implies that

$$\log \max(|x|, |y|) \leqslant (|a|\, H)^{C_{10}}$$

where $C_{10} = C_{10}(n) > 0$ is an effectively computable number depending only on the degree n of f.

LEMMA 5 (Baker [1]). *Let m and n be integers such that $n \geqslant 3$ and $1 \leqslant m < n$. Put*

$$\mu_n = \prod_{\substack{p \mid n \\ p \text{ prime}}} p^{1/(p-1)}.$$

Let a, b be positive integers for which $\frac{7}{8}a \leqslant b < a$ and suppose that $a - b$ is divisible by n. Suppose also that $\lambda = 4b(a-b)^{-2}\mu_n^{-1} > 1$. Let $\alpha = (a/b)^{m/n}$ be an algebraic number of degree $\geqslant 2$. Then

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^\kappa}$$

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for all integers $p, q (q > 0)$, where κ and c are given by

$$\lambda^{\kappa-1} = 2\mu_n(a+b), \quad c^{-1} = 2^{\kappa+2}(a+b).$$

3. In this section, we consider equation (1) with large l.

LEMMA 6. *There exist effectively computable absolute constants $C_{11} > 0$ and $C_{12} > 0$ such that equation (1) with*

$$k > C_{11}, \quad m > k^l, \quad t \geqslant kl^{-1/11} + \pi(k) + 2 \tag{8}$$

implies that $l \leqslant C_{12}$.

Proof. Put $\delta = 1/11$. Denote by $c_1, c_2, \ldots, c_7$ effectively computable absolute positive constants. We may assume that $k \geqslant c_1$ with c_1 sufficiently large. By (1),

$$m + d_i = a_i x_i^l \quad (1 \leqslant i \leqslant t), \tag{9}$$

where a_i and x_i are positive integers with

$$P(a_i) \leqslant k, \quad (x_i, \prod_{p \leqslant k} p) = 1. \tag{10}$$

Let $a_i = a_j$ with $i \neq j$. Then, by (8) and (9),

$$m^{1/l} > k > |a_i(x_i^l - x_j^l)| > m^{(l-1)/l}$$

which implies that $l < 2$. Thus $a_1, \ldots, a_t$ are distinct. Put $S = \{a_1, \ldots, a_t\}$. For every prime $p \leqslant k$, we choose an $f(p) \in S$ such that p does not appear to a higher power in the factorisation of any other element of S. Denote by S_1 the subset of S obtained by deleting from S all $f(p)$ with $p \leqslant k$. For a prime $p \leqslant k$ and an integer $e' \geqslant 1$, observe that there are at most $[k/p^{e'}]$ multiples of $p^{e'}$ in S_1. Thus

$$\prod_{a_i \in S_1} a_i \leqslant \prod_{p \leqslant k} p^{[k/p]+[k/p^2]+\ldots} = k!$$

Hence

$$\prod_{a_i \in S_1} a_i \leqslant k^k, \quad |S_1| \geqslant kl^{-\delta} + 2. \tag{11}$$

This argument is due to Erdös [7], lemma 3.

Now we show that

$$l \leqslant k(\log k)^4. \tag{12}$$

Assume that $l > k$. Let a_i and a_j with $i \neq j$ be elements of S_1. Notice that $x_i > 1$ and $x_j > 1$, otherwise, by (8), (9) and (11), $k^k < k^l < m < k^k$. By (9),

$$0 \neq |a_i x_i^l - a_j x_j^l| < k. \tag{13}$$

We apply Lemma 1 with $n = 2$, $A_1 = k^k$, $A_2 = (2m)^{1/l}$ and $B = l$ to

$$|\log(a_i/a_j) + l \log(x_i/x_j)|$$

to conclude that the left hand side of (13) exceeds

$$m^{1-c_2 l^{-1} k (\log l)(\log k)^2} > k^{l - c_2 k (\log l)(\log k)^2}$$

which, together with (13), implies (12).

By (11), there exists a subset S_2 of S_1 such that $2|S_2| \geqslant kl^{-\delta}$ and

$$a_i \leqslant k^{2l^\delta} \quad (a_i \in S_2). \tag{14}$$

Then, in view of (9) and (8), we may assume that $x_i > 1$ whenever $a_i \in S_2$. By (12),

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$s_2 := |S_2| \geqslant 2$. Let a_i and a_j with $i \neq j$ be elements of S_2. We apply Lemma 1 to conclude that the left hand side of (13) exceeds

$$k^{l-c_3 l^{\delta}(\log l)^2 (\log k)^2}$$

which, together with (13), sharpens (12) to

$$l \leqslant (\log k)^3. \tag{15}$$

Consequently

$$s_2 > k(\log k)^{-1}. \tag{16}$$

By permuting the suffixes of $d_1, \ldots, d_t$, there is no loss of generality in assuming that $a_i \in S_2$ for $1 \leqslant i \leqslant s_2$ and $a_1 < a_2 \ldots < a_{s_2}$. By (14) and (15),

$$\sum_{j=1}^{s_2-1} \log\left(\frac{a_{j+1}}{a_j}\right) \leqslant (\log k)^2$$

which, together with (16), implies that there exists μ with $1 \leqslant \mu < s_2$ such that

$$0 < \log\left(\frac{a_{\mu+1}}{a_\mu}\right) \leqslant 2k^{-1}(\log k)^3 \leqslant k^{-\frac{1}{2}}. \tag{17}$$

By (9).

$$0 \neq \left|\log\left(\frac{a_{\mu+1}}{a_\mu}\right) + l \log\left(\frac{x_{\mu+1}}{x_\mu}\right)\right| < 2km^{-1} \tag{18}$$

which, together with (17) and (8), implies that

$$\left|\log\left(\frac{x_{\mu+1}}{x_\mu}\right)\right| \leqslant 3k^{-1}(\log k)^3 \leqslant k^{-\frac{1}{2}}.$$

We shall apply Lemma 2 with $n = 2$, $A_1 = k^{2l^{\delta}}$, $A = m^{1/l}$, $\tau_1 = 2l^{\delta}$, $\tau_2 = 4l^{\delta}$ and $u = 1$ to the linear form (18). Then the right hand side of inequality (5) exceeds

$$\exp(-c_4 l^{-\delta} \log m).$$

If $m^{-c_4 l^{-\delta}} \leqslant 2km^{-1}$, then $k^{l-c_4 l^{1-\delta}} \leqslant 2k$ which implies that $l \leqslant c_5$. Thus we may assume that the right hand side of inequality (5) exceeds $2km^{-1}$. Hence, by (18) and Lemma 2, we conclude that

$$\left(\frac{a_{\mu+1}}{a_\mu}\right)^{j_1}\left(\frac{x_{\mu+1}}{x_\mu}\right) = \eta^p$$

where $p \leqslant \log k$ is prime and $0 \leqslant j_1 < p$, $j_1 \in \mathbb{Z}$ and η is rational. Then by (10), we see that $x_\mu = y_\mu^p$ and $x_{\mu+1} = y_{\mu+1}^p$ for some positive integers y_μ and $y_{\mu+i}$. Consequently, by (10), $x_\mu > k^p$, $x_{\mu+1} > k^p$ and hence

$$m > k^{lp}.$$

We re-write (18) as

$$0 \neq \left|\log\left(\frac{a_{\mu+1}}{a_\mu}\right) + lp \log\left(\frac{y_{\mu+1}}{y_\mu}\right)\right| < 2km^{-1}.$$

We apply again Lemma 2 with $n = 2$, $A_1 = k^{2l^{\delta}}$, $A = m^{1/(lp)}$, $\tau_1 = 2l^{\delta}$, $\tau_2 = 4l^{\delta}$ and $u = 1$ to conclude that y_μ and $y_{\mu+1}$ are p_1th powers from some prime $p_1 \leqslant \log k$. We proceed inductively. We carry out $[(\log\log k)^2] + 1$ inductive steps. Then we may assume that

$$m + d_\mu = a_\mu z_\mu^{lL}, \quad m + d_{\mu+1} = a_{\mu+1} z_{\mu+1}^{lL}$$

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for some positive integers z_μ, $z_{\mu+1}$ and $L \geqslant 2^{(\log\log k)^2}$ and

$$m > k^{lL}.$$

For carrying out inductive steps, it is important to observe that $lL \leqslant A_1 = k^{2l^2}$. Now

$$0 \neq \left|\log\left(\frac{a_{\mu+1}}{a_\mu}\right) + lL\log\left(\frac{z_{\mu+1}}{z_\mu}\right)\right| < 2km^{-1} < m^{-\frac{1}{2}}. \tag{19}$$

By Lemma 1 with $n = 2$, $A_1 = k^{2l^2}$, $A_2 = m^{1/(lL)}$, $B = lL$ and (15), we conclude that the left hand side of inequality (19) exceeds

$$\exp(-c_6 L^{-1}(\log L)(\log k)^2(\log m))$$

which, together with (19), implies that $L \leqslant (\log k)^4$. Therefore $k \leqslant c_7$. This completes the proof of Lemma 6.

4. *Proof of Theorem* 1. Let $0 < \epsilon < \frac{1}{4}$. Denote by $c_8, \ldots, c_{24}$ effectively computable positive numbers depending only on ϵ. We assume that $k \geqslant c_8$ with c_8 sufficiently large. Then it follows from Lemma 6 that $l \leqslant c_9$. As in the proof of Lemma 6, we obtain (9), (10) and $a_1, \ldots, a_t$ are distinct. Put $S = \{a_1, \ldots, a_t\}$.

(*a*) We argue as in the proof of Lemma 6 to conclude that there exists a subset S_3 of S such that $|S_3| \geqslant c_{10}k$ and

$$a_i \leqslant k^{\nu_i^{-1}(1-\frac{1}{4}\epsilon)} \quad (a_i \in S_3).$$

Consequently $x_i > k$ whenever $a_i \in S_3$. We may assume that $a_i \in S_3$ for $1 \leqslant i \leqslant s_3$ and $a_1 < a_2 < \ldots < a_{s_3}$. Then there exists μ with $1 \leqslant \mu < s_3$ such that

$$0 < \log\left(\frac{a_{\mu+1}}{a_\mu}\right) \leqslant c_{11}k^{-1}\log k.$$

Now
$$0 \neq \left|\frac{a_{\mu+1}}{a_\mu} - \left(\frac{x_\mu}{x_{\mu+1}}\right)^l\right| < \frac{k}{a_\mu x_{\mu+1}^l} < \frac{2k}{a_{\mu+1}x_{\mu+1}^l}.$$

Therefore
$$0 \neq \left|\left(\frac{a_{\mu+1}}{a_\mu}\right)^{1/l} - \frac{x_\mu}{x_{\mu+1}}\right| < \frac{2k}{a_{\mu+1}x_{\mu+1}^l}. \tag{20}$$

By Lemma 5, the left hand side of inequality (20) exceeds

$$(c_{12}a_{\mu+1}x_{\mu+1}^\kappa)^{-1}, \quad \kappa = (1-\tfrac{1}{4}\epsilon)(l-2)+1.$$

Thus $k^{l-\kappa} < x_{\mu+1}^{l-\kappa} < 2c_{12}k$ which implies that $k \leqslant c_{13}$.

(*b*) Put $\theta = \frac{1}{24}\epsilon$ and $t \geqslant (1-\theta)k$. There exists a subset S_4 of S such that $s_4 := |S_4| \geqslant c_{14}k$ and

$$a_i < k^{1+2\theta} \quad (a_i \in S_4).$$

Consequently $x_i > k$ whenever $a_i \in S_4$.

First, assume that $m > k^{4+\epsilon}$. We may assume that $a_i \in S_4$ for $1 \leqslant i \leqslant s_4$,

$$a_1 < a_2 < \ldots < a_{s_4}$$

and there exists μ with $1 \leqslant \mu < s_4$ such that

$$0 < \log\left(\frac{a_{\mu+1}}{a_\mu}\right) \leqslant c_{15}k^{-1}\log k.$$

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Further (20) is valid. Now we apply Lemma 5 to conclude that the left hand side of inequality (20) exceeds $(c_{16}a_{\mu+1}x_{\mu+1}^{\kappa})^{-1}$ with $\kappa = 2+6\theta$. Therefore $x_{\mu+1}^{1-6\theta} \leqslant 2c_{16}k$. Then

$$k^{4+24\theta} < m < a_{\mu+1}x_{\mu+1}^3 \leqslant c_{17}k^{4+23\theta}$$

which implies that $k \leqslant c_{18}$.

Now assume that $k^3 < m \leqslant k^4(\log k)^3$. Then x_i is prime whenever $a_i \in S_4$. Therefore there exists a subset S_5 of S_4 such that $|S_5| \geqslant c_{19}k$ and $x_i \geqslant c_{20}k\log k$, whenever, $a_i \in S_5$. Then $a_i \leqslant c_{21}k$ for all $a_i \in S_5$. Now we apply Lemma 1 of Erdös [5] to conclude that there exist $a_i, a_j \in S_5$ with $i \neq j$ such that

$$0 < \left|\frac{a_i}{(a_i,a_j)}x_i^3 - \frac{a_j}{(a_i,a_j)}x_j^3\right| < \frac{k}{(a_i,a_j)}, \quad (a_i,a_j) \geqslant c_{22}k. \tag{21}$$

Now it follows from Lemma 3 that $\max(x_i,x_j) \leqslant c_{23}$. Hence $k \leqslant c_{24}$. This completes the proof of Theorem 1.

5. *Proof of Theorem* 2. Let $0 < \epsilon < \frac{1}{4}$. Put $C = (16C_{10}(4))^{-1}$ where $C_{10} = C_{10}(n)$ is the constant appearing in Lemma 4. Denote by c_{25}, c_{26} and c_{27} effectively computable positive numbers depending only on ϵ. We may assume that $k > c_{25}$ with c_{25} sufficiently large. By (1) with $l = 2$,

$$m + d_i = A_i X_i^2 \quad (1 \leqslant i \leqslant t), \tag{22}$$

where A_i and X_i are positive integers such that $P(A_i) \leqslant k$ and A_i is square free. Further, by $m > k^2$, we see, as in Lemma 6, that $A_1, \ldots, A_t$ are distinct. Put $S_6 = \{A_1, \ldots, A_t\}$. Observe that

$$A_1 \ldots A_t \leqslant \prod_{p \leqslant k} p^{[k/p]+1} \leqslant (4k)^k. \tag{23}$$

Consequently, there exists a subset S_7 of S_6 such that $|S_7| \geqslant \epsilon k/2$ and

$$A_i \leqslant k(\log\log k)^{1-\frac{1}{2}\epsilon} \quad (A_i \in S_7), \tag{24}$$

otherwise, by (2),

$$A_1 \ldots A_t \geqslant (k(\log\log k)^{1-\frac{1}{2}\epsilon})^{t-\frac{1}{2}\epsilon k}[\tfrac{1}{2}\epsilon k]! > k^k(\log\log k)^{\frac{1}{8}\epsilon^2 k}$$

which contradicts (23). Further we see from (22), $m > k^2$ and (24) that

$$X_i > k^{\frac{1}{4}}, \quad A_i \in S_7. \tag{25}$$

Denote by $b_1, \ldots, b_s$ all integers between $k(\log k)^{-2C}$ and $k(\log\log k)^{1-\frac{1}{2}\epsilon}$ such that every proper divisor of b_i is less than or equal to $k(\log k)^{-2C}$. If $b_i > k(\log k)^{-C}$, then every prime divisor of b_i exceeds $(\log k)^C$. Hence, by Brun's Sieve,

$$s \leqslant k(\log k)^{-C} + c_{26}k(\log\log k)^{-\frac{1}{2}\epsilon} \leqslant k(\log\log k)^{-\frac{1}{4}\epsilon}.$$

Denote by S_8 all $A_i \in S_7$ such that $A_i > k(\log k)^{-2C}$. Observe that every element of S_8 is divisible by at least one b_i. Further notice that

$$|S_8| \geqslant |S_7| - k(\log k)^{-2C} \geqslant \frac{\epsilon k}{4}. \tag{26}$$

If every b_i appears in at most two elements of S_8, then

$$|S_8| \leqslant 2s \leqslant 2k(\log\log k)^{-\frac{1}{4}\epsilon}$$

which contradicts (26) if c_{25} is sufficiently large.

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Thus we may assume that there exists b_ν with $1 \leqslant \nu \leqslant s$ such that three distinct elements of S_8 are divisible by b_ν. By permuting the suffixes of $d_1, \ldots, d_t$, there is no loss of generality in assuming that A_1, A_2 and A_3 are divisible by b_ν. Put

$$B_i = b_\nu^{-1}A_i, \quad (i = 1, 2, 3),$$

and

$$R = b_\nu^{-1}(d_2 - d_1), \quad R' = b_\nu^{-1}(d_3 - d_1).$$

Observe that B_1, B_2, B_3 and R, R' are integers of absolute values not exceeding $(\log k)^{3C}$. By (22), we have

$$B_2B_3(X_2X_3)^2 = (B_1X_1^2 + R)(B_1X_1^2 + R').$$

Notice that all the roots of the polynomial on the right hand side are simple. By (25) and Lemma 4,

$$(\log k)/4 \leqslant \log X_1 \leqslant (\log k)^{12CC_{10}} \leqslant (\log k)^{\frac{3}{4}}$$

which implies that $k \leqslant c_{27}$. This completes the proof of Theorem 2.

Proof of Corollary 1. As in [5], we may assume that $m > k^l$. Then, in view of Theorem 2, we may assume that $l > 2$. Further, by Theorem 1, it remains to consider only the case $l = 3$ and $m \in (k^4(\log k)^3, k^{4+\epsilon})$. In fact the argument that we give to deal with this case is also valid for the general case. Thus we present a proof which is independent of Theorem 1.

We may assume that C_5 is sufficiently large. Denote by c_{28} and c_{29} effectively computable absolute positive constants. Put $S = \{a_1, \ldots, a_k\}$. Notice that $|S| = k$. There exists a subset S_9 of S such that $|S_9| \geqslant k - \pi(k)$ and $\prod_{a_i \in S_9} a_i \leqslant k^k$. Thus there exists a subset S_{10} of S_9 such that $|S_{10}| \geqslant k/2$ and $a_i \leqslant c_{28}k, x_i > k$ whenever $a_i \in S_{10}$. For this assertion, we need to use that the product of ν distinct elements of S is greater than or equal to $\nu!$. Then we apply Lemma 1 of Erdös [5] to obtain (21) with 3 replaced by l. Now it follows from Lemma 1 and Lemma 3 that $k \leqslant c_{29}$.

6. In this section, we prove three Lemmas that we shall apply in the proof of Theorem 3. The first Lemma may be of independent interest.

LEMMA 7. *Let D be a positive divisor of n such that $(D, n/D) = (D, Q_{n/D}) = 1$. Then equation* (3) *implies that*

$$z_1^q = \frac{(x^D)^{n/D} - 1}{x^D - 1}, \quad z_2^q = \frac{x^D - 1}{x - 1}$$

for some positive integers z_1 and z_2.

Proof. Put $A = (x^n - 1)/(x^D - 1)$ and $B = (x^D - 1)/(x - 1)$. Observe, by (3), that $AB = z^q$. It suffices to show that $(A, B) = 1$. Let $(A, B) > 1$. Let p be a prime dividing (A, B) and denote by ν the least positive integer such that $x^\nu \equiv 1 \pmod p$. Then

$$p \Big| \frac{n}{D}, \quad \nu | D \quad \text{and} \quad \nu | (p - 1)$$

which implies that $\nu = 1$, since $(D, Q_{n/D}) = 1$. Consequently $p | D$ which is not possible, since $(D, n/D) = 1$. Hence $(A, B) = 1$. This completes the proof of Lemma 7.

LEMMA 8. *Equation* (3) *with $n \equiv 1 \pmod q$ implies that* $\max(x, z, n, q)$ *is bounded by an effectively computable absolute constant.*

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Proof. In view of Theorem A(i), we may assume that $q > 2$. Further, by Theorem A(ii), it suffices to show that $x \leqslant c_{30}$ where c_{30} and the subsequent letters c_{31}, c_{32} and c_{33} are effectively computable absolute positive constants. We may assume that $x > c_{31}$ with c_{31} sufficiently large.

We may write $n = 1 + Nq$ for some positive integer N. By (3),

$$|(x-1)z^q - x(x^N)^q| = 1. \tag{27}$$

Now we apply Lemma 1 to conclude that the left hand side of (27) exceeds

$$z^{q - c_{32} \log q (\log x)^2}.$$

Therefore
$$q \leqslant (\log x)^3. \tag{28}$$

By (27), we have

$$\left|\left(\frac{x}{x-1}\right)^{1/q} - \frac{z}{x^N}\right| < 2x^{-Nq-1}. \tag{29}$$

Now, by Lemma 5 together with (28), we obtain

$$\left|\left(\frac{x}{x-1}\right)^{1/q} - \frac{z}{x^N}\right| > x^{-(\frac{5}{2})N-1}. \tag{30}$$

Combining (30) and (29), we see that $x \leqslant c_{33}$, since $q > 2$. This completes the proof of Lemma 8.

LEMMA 9. *Suppose that n has a prime factor congruent to one mod q. Then equation* (3) *implies that* $\max(x, z, n, q)$ *is bounded by an effectively computable absolute constant.*

Proof. Let P be the greatest prime factor of n. Put $D = P^{\operatorname{ord}_P(n)}$. Then

$$(D, n/D) = (D, Q_{n/D}) = 1.$$

Therefore we see from Lemma 7 that $(x^D - 1)/(x-1)$ is a qth power. If $D \equiv 1 \pmod q$, then Lemma 9 follows from Lemma 8 and Theorem A(ii). Thus we may assume that $D \not\equiv 1 \pmod q$. Hence $P \not\equiv 1 \pmod q$. If n is not a prime power, we repeat this argument to

$$((x^D)^{n/D} - 1)/(x^D - 1)$$

which, by Lemma 7, is a q-th power. Thus we may assume that the greatest prime factor of n/D is not congruent to one mod q. We proceed inductively, as described above, to complete the proof of Lemma 9.

7. *Proof of Theorem* 3. By Theorem A(iii), we may assume that $n' > 2$. In view of Theorem A(ii), it suffices to show that $x \leqslant c_{34}$ where c_{34} and c_{35} are effectively computable absolute positive constants. Further, by Lemma 9, we may assume that $(q^e, Q_{n'}) = 1$. Then it follows from Lemma 7 with $D = q^e$ that

$$z_3^q = \frac{X^{n'} - 1}{X - 1}, \quad X = x^{q^e}$$

for some positive integer z_3. Thus there is no loss of generality in assuming that $(n, q) = 1$, i.e. $n' = n$.

(i) For $1 \leqslant \lambda \leqslant \mu \leqslant r$, let $n_{\lambda,\mu} \equiv 1 \pmod q$. Put

$$D_1 = p_1^{e_1} \dots p_{\lambda-1}^{e_{\lambda-1}}, \quad D_2 = n_{\lambda,\mu}, \quad D_3 = p_{\mu+1}^{e_{\mu+1}} \dots p_r^{e_r}.$$

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Observe that $D_1D_2D_3 = n$. Since $(D_3, D_1D_2) = (D_3, Q_{D_1D_2}) = 1$, we apply Lemma 7 with $D = D_3$ to obtain

$$z_4^q = \frac{X_1^{D_1D_2} - 1}{X_1 - 1}, \quad X_1 = x^{D_3}$$

for some positive integer z_4. Further, since $(D_2, D_1) = (D_2, Q_{D_1}) = 1$, we again apply Lemma 7 with $D = D_2$ to conclude that

$$z_5^q = \frac{X_1^{D_2} - 1}{X_1 - 1}$$

for some positive integer z_5. Now, since $D_2 \equiv 1 \pmod q$, it follows from Lemma 8 that $X_1 \leqslant c_{35}$. Hence $x \leqslant c_{35}$.

(ii) Let $\omega(n) > q-2$. Put $S' = \{n_{1,1}, \ldots, n_{1,r}\}$. Observe that no element of S' is divisible by q. Further, by case (i), we may assume that no element of S' is congruent to one mod q. If $n_{1,i} \equiv n_{1,j} \pmod q$ with $i < j$, then $n_{i+1,j} \equiv 1 \pmod q$. Therefore, by case (i), we may assume that the elements of S' are incongruent mod q. Hence $r = \omega(n) \leqslant q-2$. This is a contradiction.

(iii) Let $\omega(n) > (q-3)/2$ and n is a square. If $q = 2$, the assertion follows from Theorem A(i). Assume that $q > 2$. We argue as in case (ii) to conclude that we can assume that the elements of S' are incongruent quadratic residue mod q and none of them is congruent to one mod q. Since there are $(q-1)/2$ quadratic residues mod q, we see that $r = \omega(n) \leqslant (q-3)/2$. This is a contradiction.

Remarks. (i) One would like to show that equation (3) implies that $\max(x, z, n, q)$ is bounded by an effectively computable absolute constant. In view of Lemma 7 and Theorem A(ii), it suffices to prove the above assertion with n restricted to prime powers.

(ii) If $(n', Q_{n'}) = 1$, then the assertion of Theorem 3 is valid if n' has a divisor $d > 1$ with $(d, n'/d) = 1$ and $d \equiv 1 \pmod q$.

(iii) The exponent $-1/11$ of l in Lemma 6 is not the best possible obtainable by the arguments of this paper.

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