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Hyperbolic evolution equations for moving boundary problems

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Short-time existence and uniqueness results in Sobolev spaces are proved for Hele-Shaw flow with kinetic undercooling and for Stokes flow without surface tension. In both cases, the flow is driven by arbitrarily distributed sources and sinks in the interior of the liquid domain. The proofs are based on a general approach consisting of the reformulation of the problem as a Cauchy problem for a nonlinear, nonlocal evolution equation on the unit sphere, quasilinearization by equivariance, investigation of the linearization, and Galerkin approximations. In the situation discussed here, the linearized evolution operator is a first-order differential operator, and thus the evolution equation is of hyperbolic type. Finally, a brief survey of the properties of the evolution equations that arise from Hele-Shaw flow and Stokes flow with and without regularization is given.

1 Introduction

In the treatment of moving boundary problems, a widely used technique is to represent the moving domain as the image of a time-dependent, *a priori* unknown diffeomorphism which is defined on some fixed reference domain. If the motion of the domain is completely determined by the domain itself, as it is the case for quasisteady motion, for example, the problems can be reformulated as nonlinear evolution equations for the diffeomorphisms (more precisely, for their restriction to the boundary of the reference domain). The analysis of the moving boundary problems (at least as far as short-time solutions and their smoothness properties are concerned) can be carried out then by investigating these evolution equations.

Due to the nonlocal character of moving boundary problems, the resulting evolution equations are also nonlocal; in particular, they are not differential equations. Moreover, the strong nonlinearity which enters the problems via the transformations that have to be carried out necessitates the use of techniques for the treatment of so-called fully nonlinear evolution equations, i.e. equations in which neither a semilinear nor a quasilinear structure is immediately available.

In Prokert [12], a general framework for the analysis of a class of moving boundary problems by the method described above is introduced. Essentially, the analysis given there consists of the following steps:

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(1) description of the mapping properties of the evolution operator;

(2) application of a chain rule which is derived from the invariance of the problem with respect to a class of reparametrizations;

(3) study of the linearized problems;

(4) derivation of *a priori* estimates and proof of existence and uniqueness results based on Galerkin approximations in Sobolev spaces H^s of sufficiently high order.

It should be pointed out that Step 2 is completely independent of the particular moving boundary problem under investigation. In the other steps, the arguments also follow similar lines for different problems. The results, however, can be essentially different.

In Step 1, the *order* of the evolution problem is determined while in Step 3 one obtains information about its *type*, i.e. whether the considered evolution equation is forward or backward parabolic or hyperbolic. (These concepts are to be understood in the sense that the linearizations, although nonlocal, allow energy estimates analogous to the standard cases of parabolic and first-order symmetric hyperbolic equations or systems.) Both for first-order hyperbolic equations and for parabolic equations of arbitrary order, the techniques of Step 4 are applicable. Moreover, for forward parabolic problems it is possible to show the smoothing effect of the evolution, i.e. it can be proved that even for initial data having finite smoothness only, the boundary of the moving domain will be smooth for all positive times within the existence interval.

The procedure sketched above is applied in Prokert [12] to three moving boundary problems, namely

(1) Stokes flow driven by surface tension.

(2) Hele-Shaw flow driven by surface tension.

(3) classical Hele-Shaw flow driven by internal sources and sinks.

(The precise statement of these problems as well as references to related work can be found in Prokert [12].) If the surface tension coefficient is positive, the first two problems lead to forward parabolic equations of first and third order, respectively. In the case of classical Hele-Shaw flow, the evolution equation is of first order and forward or backward parabolic, depending on whether the liquid front is advancing or receding (cf. [1, 5]). This gives a general functional analytic explanation of the well-known dichotomy in the character of Hele-Shaw flow in the injection and suction cases.

It is the aim of the present paper to apply the same methods with the necessary changes to two further moving boundary problems, namely, Hele-Shaw flow with kinetic undercooling regularization and Stokes flow without surface tension. In both cases, the flow is driven by internal sinks or sources having positive distance to the boundary.

The (one-phase) Hele-Shaw flow problem with kinetic undercooling has the following formulation: We look for a family of (sufficiently smooth) domains $\Omega(t) \subset \mathbb{R}^N$ whose evolution is described by the kinematic boundary condition

$$V_n = \frac{\partial v(t)}{\partial n} \quad \text{on } \partial\Omega(t).$$

Here, V_n denotes the velocity of the boundary in normal direction, $v = v(t)$ is a function

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on $\Omega(t)$ defined as solution of a so-called Robin problem for the Laplacian,

$$\Delta v = g \quad \text{in } \Omega(t), \tag{1.1}$$

$$\frac{\partial v}{\partial n} + \gamma v = 0 \quad \text{on } \partial\Omega(t), \tag{1.2}$$

and γ is a positive constant. The given fixed source term g represents the driving mechanism; we assume that g vanishes near the moving boundary.

As the name 'kinetic undercooling' suggests, the corresponding boundary condition (1.2) was originally applied to the temperature in the closely related Stefan problem to account for nonequilibrium thermodynamic effects. Apparently, for Hele-Shaw flow it has first been used in Romero [14]. Qualitatively, it introduces a penalization for large normal velocities of the boundary, and therefore one expects a regularizing effect, compared to the homogeneous Dirichlet condition which is known to lead to an ill-posed problem in the case of a receding liquid boundary. Our analysis can provide an understanding of the nature of this regularization from an analytic point of view (see § 5). Analytic results concerning short-time solvability of this moving boundary problem have been proved in Hohlov & Reissig [8] and Reissig & Rogosin [13] for the plane case and analytic data by means of abstract Cauchy–Kovalevskaya type theorems. Existence and uniqueness of solutions which are global in time have been shown in Friedman & Hu [6] for a strip-like two-dimensional geometry under the assumption that the initial data are near an essentially one-dimensional situation. A corresponding two-phase problem in arbitrary space dimension has been discussed in Xu [18]; short-time existence and uniqueness of the solution are proved in Hölder spaces of functions. The existence is shown by means of the Schauder fixed point theorem. Similar results are obtained in Yu [20] for a two-phase problem where both kinetic undercooling and surface tension regularization are applied on the moving boundary. For results on the one-phase problem with both regularizations, we refer to Deckelnick & Elliott [4] and Yi [19].

Using the same notation as above, the Stokes flow problem can be described by the kinematic boundary condition

$$V_n = v(t) \cdot n(t) \quad \text{on } \partial\Omega(t)$$

where $n(t)$ denotes the outer normal vector field on $\partial\Omega(t)$, and $v = v(t)$ is an $\mathbb{R}^N$-valued function on $\Omega(t)$ such that (v, q) solves the system of the inhomogeneous Stokes equations

$$\left.\begin{aligned} -\Delta v + \nabla q &= 0 \\ \operatorname{div} v &= g \end{aligned}\right\} \text{ in } \Omega(t), \tag{1.3}$$

$$\mathscr{T}(v, q)n = 0 \quad \text{on } \partial\Omega(t). \tag{1.4}$$

These equations describe the quasisteady motion of a liquid occupying the domain Ω in the absence of exterior forces. The sources and sinks that drive the flow are represented by g. As in the case of Hele-Shaw flow, we assume that g is fixed and vanishes near the moving boundary. Furthermore,

$$\mathscr{T}(v, q) = \nabla v + (\nabla v)^T - q\mathscr{I}$$

denotes the stress tensor field of the liquid. The boundary condition (1.4) is the natural dynamic boundary condition for a free surface without surface forces acting on it; it

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expresses the equilibrium of the forces normal to the boundary which result from the pressure jump across the surface and from the local deformation of the liquid. (In particular, no surface tension effects are included.)

Free boundary problems of this type occur, for example, in the modeling of glass and ceramics production processes at relatively low temperatures [9, 17]. The main interest in the analysis of such problems has been concentrated on problems with surface tension [2, 7, 9, 12]. For the zero surface tension case, the only results known to the author concern families of explicit solutions [10, 15] and conserved quantities comparable to the Richardson moments in Hele-Shaw flow [3], both in two dimensions.

For both problems which are addressed here, the crucial observation from an analytic point of view is the following: The evolution equation is of first order and the leading order term in the linearization is a differential operator. Although its coefficients depend nonlocally on the unknown function, this fact together with the chain rule can be employed for the derivation of the necessary a priori estimates, i.e. the problem can be treated as a quasilinear (symmetric) hyperbolic system.

2 The evolution equation and quasilinearization

This section gives an outline of the derivation of the evolution equations that describe our moving boundary problems and of the chain rule on which the quasilinearization is based. We closely follow Prokert [12] and refer to that article for further details and the proofs.

As mentioned earlier, one has to determine a family of domains whose evolution is described by

$$V_n = \mathscr{U}(\Omega(t)) \cdot n(t) \qquad \text{on } \partial\Omega(t),$$

where the same notation as in the introduction is used and $\mathscr{U}$ denotes an operator which assigns to any (sufficiently smooth, bounded) domain a vector-valued function defined on its boundary. In quasistationary free boundary flow problems, $\mathscr{U}(\Omega)$ can be interpreted as the velocity of the liquid particles at the boundary of the liquid domain Ω.

We restrict our attention to domains which are diffeomorphic images of balls. Hence, let $B_\delta = \{x \in \mathbb{R}^N \,|\, |x| < 1 + \delta\}$ and let

$$\Phi \in C^\omega\left(B_\delta, \mathbb{R}^N\right) \cap \mathrm{Diff}^\infty(B_\delta, \Omega_\delta)$$

be given where $\Omega_\delta := \Phi[B_\delta]$, $\delta > 0$. Here and in the sequel, $C^\omega(E, F)$ denotes (real) analytic mappings between open sets in Banach spaces, i.e. we demand that Φ is represented by an absolutely convergent Taylor series in a neighborhood of any point of B_δ. We assume analyticity here mainly for technical reasons, the basic results on well-posedness of the evolution equation in a fixed Sobolev space remain true if Φ is C^∞ or has finite (but sufficient) smoothness only.

The diffeomorphism Φ will be used to 'translate' small perturbations of the domain $\Omega_0 := \Phi[B_0]$ to small perturbations of the unit ball B_0. The latter can be described in the following way: there is a mapping Ψ such that $\Psi \in \mathscr{L}\left(H^s(S^{N-1}), (H^s(B_0))^N\right)$ for all $s > \frac{N+1}{2}$, $\Psi(\mathbf{1})$ is the identity on B_0, and if $r \in H^s(S^{N-1})$, $s > \frac{N+1}{2}$, is sufficiently small then the mapping $\psi_r \in (H^s(B_0))^N$ defined by $\psi_r = \Psi(r + \mathbf{1})$ is a global diffeomorphism of

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B_0 onto its image $\psi_r[B_0] = B_r$, and

$$\psi_r(x) = (1 + r(x))x, \qquad x \in S^{N-1}.$$

To describe perturbed domains near Ω_0 we define $\Omega_r := \Phi[B_r]$ (this makes sense if $\|r\|_s$ is small). Thus, we can look now for a family of functions $r(t)$ such that

$$\Omega(t) = \Omega_{r(t)}$$

and derive an evolution equation for r. (With a slight abuse of notation, we will denote by r both elements of $H^s(S^{N-1})$ and functions of t valued in this space. In the respective contexts, however, confusion is not likely to occur from this.) One finds

$$\dot{r} = \rho(r) = F(\Phi, r) = \frac{(\Phi^\star \mathscr{U}[\Omega_r]) \circ \psi_r \cdot \nu(r)}{n \cdot \nu(r)} \qquad \text{on } S^{N-1}, \tag{2.1}$$

where $\Phi^\star$ denotes the pull-back of a vector field from Ω_r to B_r, $\nu(r) = \tilde{\nu}(r) \circ \psi(r)$, $\tilde{\nu}(r)$ denotes the outer unit normal vector field on B_r, and $n = \nu(0) = \mathrm{id}_{S^{N-1}}$ is the outer normal vector field on S^{N-1}. This evolution equation has to be complemented with the initial condition

$$r(0) = r_0 \tag{2.2}$$

describing the initial domain $\Omega(0)$.

Our task now is to analyse the Cauchy problem (2.1), (2.2), i.e. the operators ρ and F. At first, we formulate an equivariance property with respect to rotations. We choose a basis $\{Q_1, \ldots, Q_{\binom{N}{2}}\}$ of the linear space of all skew-symmetric (N, N)-matrices and define for any $\theta \in \mathbb{R}^{\binom{N}{2}}$ the rotation around the origin

$$S_\theta = e^{\theta_1 Q_1} \ldots e^{\theta_{\binom{N}{2}} Q_{\binom{N}{2}}}. \tag{2.3}$$

It can be shown then that

$$F(\Phi, r) \circ S_\theta = F(\Phi \circ S_\theta, r \circ S_\theta)$$

or, writing $F(\Phi \circ S_\theta, r) =: f(\theta, r)$

$$f(\varphi, r) \circ S_\theta = f(T(\varphi, \theta), r \circ S_\theta) \tag{2.4}$$

for θ, φ near 0, where T is smooth. Differentiating this equation with respect to θ_j at $\theta = 0$ yields, under suitable differentiability assumptions,

$$D_j f(\varphi, r) = f'(\varphi, r) \left[\frac{\partial T}{\partial \theta_j}(\varphi, 0), D_j r \right], \tag{2.5}$$

where D_j is a (globally defined) differential operator on S^{N-1}, corresponding to a smooth vector field. Note that (2.5) can be seen as a generalized chain rule.

We will employ the usual multi-index notation for compositions of the operators D_j, $j = 1, \ldots, \binom{N}{2}$. On the Sobolev spaces $H^\sigma(S^{N-1})$, $\sigma \in \mathbb{N}$, we work with the scalar products

$$(u, v)_\sigma = \sum_{|\alpha| \leqslant \sigma} (D^\alpha u, D^\alpha v)_0,$$

where $(\cdot, \cdot)_0$ denotes the L^2-scalar product. These scalar product generate the usual

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topologies. Moreover, it will be convenient to work with the additional scalar products

$$(u,v)_{\sigma,n} = \sum_{|\alpha|\leqslant n} (D^\alpha u, D^\alpha v)_\sigma, \qquad n \in \mathbb{N}.$$

The corresponding norms $\|\cdot\|_{\sigma,n}$ are equivalent norms on the spaces $H^{\sigma+n}(S^{N-1})$.

Generalizing (2.5) to higher derivatives and setting $\varphi = 0$ yields the following results for the operator ρ. We will not give the most general ones but those which are oriented at the present applications.

Lemma 1 (Properties of ρ) *For fixed $s > \frac{N+3}{2}$ and a neighbourhood V of 0 in $\mathbb{R}^{\binom{N}{2}} \times H^s(S^{N-1})$, assume that*

$$f \in C^\omega\left(V, H^{s-1}(S^{N-1})\right). \tag{2.6}$$

Then there is a neighbourhood W in $H^s(S^{N-1})$ such that for all positive integer n:

(1) $\rho \in C^\omega\left(W \cap H^{s+n}(S^{N-1}), H^{s+n-1}(S^{N-1})\right)$,

(2) *ρ is weakly sequentially continuous from $W \cap H^{s+n}(S^{N-1})$ to $H^{s+n-1}(S^{N-1})$,*

(3) *For all nonzero multiindices α, $|\alpha| \leqslant n$, there is an operator*

$$G_\alpha \in C^\omega\left(W \cap H^{s+n-1}(S^{N-1}), H^{s-1}(S^{N-1})\right)$$

such that

$$D^\alpha \rho(r) = \rho'(r)[D^\alpha r] + G_\alpha(r) \tag{2.7}$$

and

$$\|G_\alpha(r)\|_{s-1} \leqslant C_{s,n}\left(\|r\|^2_{s+n-1} + 1\right). \tag{2.8}$$

Note that the third statement provides a quasilinear decomposition for the operators $D^\alpha \circ \rho$ because G_α is of order $|\alpha|$ only while $D^\alpha \circ \rho$ and the linear operator $\rho'(r) \circ D^\alpha$ are of order $|\alpha| + 1$. This crucial observation will be the basis for the estimates on the nonlinear operators. For the particular moving boundary problems we want to study now, the main tasks are to prove (2.6) and to investigate the operator $\rho'(r)$.

3 Hele-Shaw flow with kinetic undercooling

The moving boundary problem of Hele-Shaw flow with kinetic undercooling regularization is obtained in our general framework by setting

$$\mathcal{U}(\Omega) = (\nabla v)|_{\partial\Omega}, \tag{3.1}$$

where v solves (1.1), (1.2) (with $\Omega(t)$ replaced by Ω).

We will use the following standard existence, uniqueness, and regularity results on this Robin problem:

Lemma 2 (Regularity of the Robin BVP) *Let Ω be smooth and bounded, $\Gamma = \partial\Omega$, $s \geqslant 2$. Then*

$$\left(\Delta, \frac{\partial}{\partial n} + \gamma \mathrm{id}_\Gamma\right) \in \mathscr{L}_{is}\left(H^s(\Omega), H^{s-2}(\Omega) \times H^{s-\frac{3}{2}}(\Gamma)\right).$$

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Proof It follows from the general theory of elliptic BVP that $\left(\Delta, \dfrac{\partial}{\partial n} + \gamma \mathrm{id}_\Gamma\right)$ is a Fredholm operator of index 0 between the spaces indicated above (e.g. see Taylor [16], Proposition 5.12.6). Moreover, using $\gamma > 0$ one easily obtains uniqueness of the (weak) solution to the homogeneous problem by discussing its variational formulation

$$\int_\Omega \nabla v \cdot \nabla w \, dx + \gamma \int_\Gamma v w \, d\Gamma = 0 \qquad \forall w \in H^1(\Omega).$$

This implies the assertion. □

To include distributed and point sources as well as multipoles of arbitrary order we consider $g \in \mathscr{E}'(\Omega_0)$, i.e. g is a distribution with compact support in Ω_0. We set $u_S = G * g$ where G is a fundamental solution for the Laplacian. Decomposing $v = \overline{u} + u_S$ we find

$$\begin{aligned} \Delta \overline{u} &= 0 && \text{in } \Omega, \\ \frac{\partial \overline{u}}{\partial n} + \gamma \overline{u} &= -\frac{\partial u_S}{\partial n} - \gamma u_S && \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

and $\mathscr{U}(\Omega) = (\nabla(\overline{u} + u_S))|_{\partial\Omega}$. Note that u_S is a real-analytic function outside supp g, hence the boundary condition makes sense if Γ is near Γ_0.

Our first task is to establish (2.6) for this choice of $\mathscr{U}$. Transformation of (3.2) from $\Omega = \Omega_{\theta,r} := \Phi \circ S_\theta[B_r]$ to B_0 by means of the diffeomorphism $\Phi_{\theta,r} := \Phi \circ S_\theta \psi_r$ yields

$$L(\theta, r) u(\theta, r) = \left(0, -((\nabla u_S) \circ \Phi_{\theta,r})|_{S^{N-1}} \cdot \overline{\nu}(\theta, r) - \gamma (u_S \circ \Phi_{\theta,r})|_{S^{N-1}}\right),$$

where

$$\begin{aligned} u(\theta, r) &= \overline{u} \circ \Phi_{\theta,r} = \Phi^*_{\theta,r} \overline{u}, \\ L(\theta, r) &= \left(\Phi^*_{\theta,r} \Delta \left(\Phi^*_{\theta,r}\right)^{-1}, \mathrm{Tr}_{S^{N-1}} \circ \left(\Phi^*_{\theta,r} \nabla \left(\Phi^*_{\theta,r}\right)^{-1}\right) \cdot \overline{\nu}(\theta, r) + \gamma \mathrm{Tr}_{S^{N-1}}\right), \\ \overline{\nu}(\theta, r) &= n(\Omega_{\theta,r}) \circ \Phi_{\theta,r}, \end{aligned}$$

and $n(\Omega_{\theta,r})$ denotes the outer unit normal field on $\Omega_{\theta,r}$.

By explicitly carrying out the coordinate transformations, using the analyticity of Φ and the Banach algebra property of the spaces $H^\tau(B_0)$ for $\tau > \frac{N}{2}$ and $H^\sigma(S^{N-1})$ for $\sigma > \frac{N-1}{2}$ one shows that for $s > \frac{N+3}{2}$

$$L \in C^\omega\left(V, \mathscr{L}\left(H^{s+\frac{1}{2}}(B_0), H^{s-\frac{3}{2}}(B_0) \times H^{s-1}(S^{N-1})\right)\right),$$

where V is a sufficiently small neighbourhood of 0 in $\mathbb{R}^{\binom{N}{2}} \times H^s(S^{N-1})$ (cf. the proof of Lemma 2 in Prokert [12]). By similar arguments, we get

$$\begin{aligned} \overline{\nu} &\in \left(C^\omega\left(V, H^{s-1}(S^{N-1})\right)\right)^N, \\ (\theta, r) \mapsto (u_S \circ \Phi_{\theta,r})|_{S^{N-1}} &\in C^\omega\left(V, H^s(S^{N-1})\right), \\ (\theta, r) \mapsto ((\nabla u_S) \circ \Phi_{\theta,r})|_{S^{N-1}} &\in \left(C^\omega\left(V, H^s(S^{N-1})\right)\right)^N, \end{aligned}$$

and consequently

$$(\theta, r) \mapsto ((\nabla u_S) \circ \Phi_{\theta,r})|_{S^{N-1}} \cdot \overline{\nu}(\theta, r) + \gamma (u_S \circ \Phi_{\theta,r})|_{S^{N-1}} \in C^\omega\left(V, H^{s-1}(S^{N-1})\right).$$

Moreover, as $L(0, 0)$ represents the BVP (3.2) on Ω_0 after transformation by the smooth

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diffeomorphism $\Phi_{0,0}$,

$$L(0,0) \in \mathscr{L}_{is}\left(H^{s+\frac{1}{2}}(B_0), H^{s-\frac{3}{2}}(B_0) \times H^{s-1}(S^{N-1})\right)$$

by Lemma 2. Consequently, by (the analytic version of) the Implicit Function theorem,

$$u \in C^{\omega}\left(V, H^{s+\frac{1}{2}}(B_0)\right). \tag{3.3}$$

Note that

$$\overline{v}(\theta, r)(\xi) = \frac{T(\Phi \circ S_\theta)^{-1}(\Phi \circ S_\theta(\psi_r(\xi)))^T v(r)(\xi)}{|T(\Phi \circ S_\theta)^{-1}(\Phi \circ S_\theta(\psi_r(\xi)))^T v(r)(\xi)|}.$$

Hence we find, using the boundary condition in (3.2),

$$\begin{aligned} f(\theta, r)(\xi) &= \frac{T(\Phi \circ S_\theta)^{-1}(\Phi \circ S_\theta(\psi_r(\xi)))(\nabla v)(\Phi \circ S_\theta(\psi_r(\xi))) \cdot v(r)(\xi)}{n(\xi) \cdot v(r)(\xi)} \\ &= \frac{(\nabla v)(\Phi \circ S_\theta(\psi_r(\xi))) \cdot \overline{v}(\theta, r)(\xi)|T(\Phi \circ S_\theta)^{-1}(\Phi \circ S_\theta(\psi_r(\xi)))^T v(r)(\xi)|}{n(\xi) \cdot v(r)(\xi)} \\ &= \frac{-\gamma(u(r)(\xi) + u_S(\Phi \circ S_\theta(\psi_r(\xi))))|T(\Phi \circ S_\theta)^{-1}(\Phi \circ S_\theta(\psi_r(\xi)))^T v(r)(\xi)|}{n(\xi) \cdot v(r)(\xi)}. \end{aligned}$$

From this we obtain by the analyticity of Φ and the Banach algebra property of $H^{s-1}(S^{N-1})$

$$f \in C^{\omega}\left(V, H^{s-1}(S^{N-1})\right).$$

Hence, Lemma 1 will be applicable. Note, moreover, that in local coordinates we have

$$\rho(r)(\xi) = a(\xi, r(\xi), \partial r(\xi))u(r)(\xi) + b(\xi, r(\xi), \partial r(\xi)), \tag{3.4}$$

where $\partial r := (\partial_1 r, \ldots, \partial_{N-1} r)$ and a and b are smooth functions of their arguments. Calculating the Frechet derivative of this yields

$$\begin{aligned} \rho'(r)[h] = a_j(\cdot, r(\cdot), \partial r(\cdot), u(r)(\cdot))\partial_j h + a_0(\cdot, r(\cdot), \partial r(\cdot), u(r)(\cdot))h \\ + a(\cdot, r(\cdot), \partial r(\cdot))u'(r)[h] \end{aligned} \tag{3.5}$$

with functions a_j, a_0 that are smooth with respect to all arguments.

To give a (local) energy estimate for $\rho'(r)$ in the scalar products defined above, we start with a generalization of the well-known L^2-energy estimate for first-order differential operators to $H^\sigma(S^{N-1})$, $\sigma \in \mathbb{N}$, $\sigma > \frac{N+1}{2}$.

Lemma 3 (Energy estimate for first-order differential operators) *Let $L = a^j D_j$ be a first-order differential operator on S^{N-1}, $a^j \in C^\sigma(S^{N-1})$. Then*

$$(Lh, h)_\sigma \leqslant C\|h\|_\sigma^2 \qquad \forall h \in H^{\sigma+1}(S^{N-1}),$$

where C depends only upon $\|a^j\|_{C^\sigma}$.

Proof We have

$$\begin{aligned} (Lh, h)_\sigma &= \sum_{|\alpha| \leqslant \sigma} (D^\alpha Lh, D^\alpha h)_0 = \sum_{|\alpha| \leqslant \sigma} \left((LD^\alpha h, D^\alpha h)_0 + ([D^\alpha, L]h, D^\alpha h)_0\right) \\ &\leqslant C \sum_{|\alpha| \leqslant \sigma} \|D^\alpha h\|_0^2 \leqslant C\|h\|_\sigma^2, \end{aligned}$$

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where we have used that

$$(Lw, w)_0 \leqslant C\|w\|_0^2 \qquad \forall w \in H^1(S^{N-1})$$

and that $[D^\alpha, L] = D^\alpha L - LD^\alpha$ is a differential operator of order σ with continuous (hence bounded) coefficients. □

This lemma enables us to give the necessary estimates for $\rho'(r)$ and, together with the chain rule, also for $\rho(r)$.

Lemma 4 (Local energy estimate for $\rho'(r)$) *Let $s > \sigma + 1 + \frac{N}{2}$. There are positive constants ε and C depending only upon Φ, s, and σ such that*

$$(\rho'(r)[h], h)_\sigma \leqslant C\|h\|_\sigma^2 \qquad \forall h \in H^{\sigma+1}(S^{N-1}),\ r \in \mathscr{B}\left(\varepsilon, H^s(S^{N-1})\right).$$

Here and in the following, for a normed space X and a positive number d we denote by $\mathscr{B}(d, X)$ the open ball around 0 of radius d in X.

Proof Due to (3.3) and the continuous embedding $H^s(S^{N-1}) \hookrightarrow C^{\sigma+1}(S^{N-1})$, the functions $a(\cdot, r(\cdot), \partial r(\cdot))$, $a_j(\cdot, r(\cdot), \partial r(\cdot), u(r)(\cdot))$, and $a_0(\cdot, r(\cdot), \partial r(\cdot), u(r)(\cdot))$ are in $C^\sigma(S^{N-1})$, uniformly bounded with respect to $r \in \mathscr{B}\left(\varepsilon, H^s(S^{N-1})\right)$. Thus, clearly

$$\|a_0(\cdot, r(\cdot), \partial r(\cdot), u(r)(\cdot))h\|_\sigma \leqslant C\|h\|_\sigma. \tag{3.6}$$

Moreover, it follows from (3.3) that the operators $u'(r)$ are uniformly bounded in $\mathscr{L}\left(H^\sigma(S^{N-1})\right)$, hence

$$\|a(\cdot, r(\cdot), \partial r(\cdot))u'(r)[h]\|_\sigma \leqslant C\|h\|_\sigma. \tag{3.7}$$

Finally, note that the first term on the right in (3.5) constitutes a first-order differential operator (applied to h), with coefficients in $C^\sigma(S^{N-1})$, uniformly bounded with respect to $r \in \mathscr{B}\left(\varepsilon, H^s(S^{N-1})\right)$, to which Lemma 3 applies. Together with (3.6) and (3.7), this implies the assertion. □

Lemma 5 (Local *a priori* estimate for ρ) *Let n, k be integer, $n \geqslant k > \frac{N}{2} + 1$. There are positive constants ε and C depending only on σ, k, and Φ such that*

$$(\rho(r), r)_{\sigma,n} \leqslant C\left(\|r\|_{\sigma,n}^3 + 1\right) \qquad \forall r \in \mathscr{B}\left(\varepsilon, H^{\sigma+k}(S^{N-1})\right) \cap H^{\sigma+n+1}(S^{N-1}).$$

Proof The proof is very similar to the proof of Lemma 7(ii) in Prokert [12]. We decompose, using (2.7),

$$\begin{aligned}(\rho(r), r)_{\sigma,n} &= \sum_{|\alpha|\leqslant n} (D^\alpha \rho(r), D^\alpha r)_\sigma \\ &= (\rho(r), r)_\sigma + \sum_{1\leqslant|\alpha|\leqslant n} \left((\rho'(r)[D^\alpha r], D^\alpha r)_\sigma + (G_\alpha(r), D^\alpha r)_\sigma\right)\end{aligned}$$

and obtain by (2.8) and Lemma 4

$$\begin{aligned}(\rho(r), r)_{\sigma,n} &\leqslant C\|r\|_{\sigma+1} + C\left(\|r\|_{\sigma+n}^2 + C(\|r\|_{\sigma+n}^2 + 1)\|r\|_{\sigma+n}\right) \\ &\leqslant C\left(\|r\|_{\sigma+n}^3 + 1\right).\end{aligned}$$

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Together with the equivalence of the norms $\|\cdot\|_{\sigma,n}$ and $\|\cdot\|_{\sigma+n}$, this yields the assertion. □

On the basis of this estimate, the following result can be proved in analogy to the proofs of Propositions 1 and 2 in Günther & Prokert [7] or Theorem 1 in Prokert [11]. The proof is based on Galerkin approximations. For a corresponding abstract result, see Theorem 30.B in Zeidler [21]. (In our case, however, we need a 'local version' of this result.) For $T > 0$ and X a Banach space, we will denote by $C_w([0,T],X)$ and $C_w^1([0,T],X)$ the spaces of weakly continuous and weakly differentiable functions $u : [0,T] \longrightarrow X$, i.e. the functions for which $\langle \varphi, u\rangle \in C([0,T])$ and $\langle \varphi, u\rangle \in C^1([0,T])$ for all $\varphi \in X'$, respectively.

Theorem 1 (Existence and uniqueness for Hele-Shaw flow with kinetic undercooling) *Let $\mathscr{U}$ be given by* (1.1), (1.2), (3.1). *Let σ, k be integers with $\sigma > \frac{N+1}{2}$, $k > \frac{N}{2} + 1$. There are positive constants ε and T such that for all integer $n \geqslant k$ and all*

$$r_0 \in \mathscr{B}\left(\varepsilon, H^{\sigma+k}(S^{N-1})\right) \cap H^{\sigma+n}(S^{N-1})$$

there is a solution

$$r \in C_w\left([0,T], H^{\sigma+n}(S^{N-1})\right) \cap C_w^1\left([0,T], H^{\sigma+n-1}(S^{N-1})\right)$$

of (2.1), (2.2) *satisfying* $r(t) \overset{H^{\sigma+n}(S^{N-1})}{\to} r_0$ *as* $t \to 0$.

There is at most one solution of (2.1), (2.2) *in*

$$C^1\left([0,T], H^{\sigma+k}(S^{N-1})\right) \cap L^\infty\left((0,T), H^{\sigma+k+1}(S^{N-1})\right).$$

Remarks (1) It has to be pointed out that the smoothness of an initial datum $r_0 \in H^{\sigma+n}(S^{N-1})$ with $n > k$ is preserved during the evolution without smallness assumption on $\|r_0\|_{s+n}$. In particular, due to this fact and the smoothness of the mapping ρ, for $r_0 \in \mathscr{B}\left(\varepsilon, H^{\sigma+k}(S^{N-1})\right) \cap C^\infty(S^{N-1})$ we have $r \in C^\infty\left([0,T], C^\infty(S^{N-1})\right)$.

(2) It is clear that the same estimates as for ρ can also be given for $-\rho$, therefore the Hele-Shaw problem with kinetic undercooling has the same solvability properties 'backwards in time'.

4 Stokes flow driven by interior sources

To describe Stokes flow driven by interior sources and sinks we have to set

$$\mathscr{U}(\Omega) = v|_{\partial\Omega}, \tag{4.1}$$

where (v,q) solves (1.3), (1.4) (with $\Omega(t)$ replaced by Ω).

At first, some technical preparations are needed. We introduce the bilinear operator

$$\otimes : \mathbb{R}^{\binom{N}{2}} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$$

by

$$(a \otimes b)_i = \sum_{j=1}^{i-1} a_{K(j,i)} b_j - \sum_{j=i+1}^{N} a_{K(i,j)} b_j \qquad (i = 1,\dots,N),$$

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where

$$K : \{(i,j) \mid 1 \leqslant i < j \leqslant N\} \longrightarrow \{1, \ldots, \tbinom{N}{2}\}$$

is an arbitrary but fixed bijection. Moreover, we define the $\binom{N}{2}$- vector-valued differential operator rot, acting on differentiable N-vector fields, by

$$(\operatorname{rot} v)_{K(i,j)} = \partial_i v_j - \partial_j v_i \qquad (1 \leqslant i < j \leqslant N).$$

Clearly, for $N = 3$ we essentially recover the usual concepts of the vector product and the curl of a vector field. Moreover, we introduce the $\left(N + \binom{N}{2}\right)$-dimensional space

$$V_0 = \{v \in C^\infty(\Omega)^N \mid v_i(x) = s_{ij}x_j + c_i,\ i,j = 1, \ldots, N,\ s_{ij}, c_i \in \mathbb{R},\ s_{ij} = -s_{ji}\}$$

which corresponds to the rigid body motions in $\mathbb{R}^N$. It is easy to check that any solution of (1.3), (1.4) is determined only up to elements of V_0. To remove the corresponding degrees of freedom, it is natural to demand

$$\int_\Omega v \, dx = 0, \quad \int_\Omega \operatorname{rot} v \, dx = 0. \tag{4.2}$$

(It will be shown below that this makes sense even if g is a distribution.)

Concerning solvability of (1.3), we have the following preliminary result:

Lemma 6 (Existence of solutions to (1.3)) *Let Ω be a bounded C^1-domain, $\Gamma = \partial\Omega$. For arbitrary $g \in \mathscr{E}'(\Omega)$, a solution of* (1.3) *is given by*

$$v = u_S = G * \nabla g,$$
$$q = p_S = g.$$

This solution satisfies

$$\int_\Gamma \mathscr{T}(u_S, p_S) n \cdot v \, d\Gamma = 0 \qquad \forall v \in V_0. \tag{4.3}$$

Remarks (1) The convolution is to be applied to ∇g 'componentwise'.

(2) Outside supp g, p_S vanishes and u_S is smooth. Hence, the left side of (4.3) is well-defined.

Proof of Lemma 6 We clearly have $-\Delta u_S + \nabla p_S = 0$, and $\operatorname{div} u_S = g$ follows from $G * \nabla g = \nabla(G * g)$. Let $\Omega' \subset\subset \Omega$ be a domain containing supp g. To show (4.3) we assume first $g \in C_0^\infty(\Omega')$. This implies that u_S and p_S are smooth, and by the Green formula for the Stokes equations we have for arbitrary $v \in V_0$

$$\begin{aligned}\int_\Gamma \mathscr{T}(u_S, p_S) n \cdot v \, d\Gamma &= -\int_\Omega (-\Delta u_S + \nabla p_S - \nabla \operatorname{div} u_s) \cdot v \, dx \\ &\quad + \frac{1}{2}\int_\Omega \left(\frac{\partial u_{Si}}{\partial x_j} + \frac{\partial u_{Sj}}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx - \int_\Omega p_S \operatorname{div} v \, dx \\ &= \int_\Omega \nabla g \cdot v \, dx = -\int_\Omega g \operatorname{div} v \, dx + \int_\Gamma g \, v \cdot n \, d\Gamma = 0. \end{aligned} \tag{4.4}$$

Let Ω'' be an open set containing Γ and having positive distance to Ω'. Note that p_S vanishes on Ω'' and the mapping $g \mapsto u_s$ from $C_0^\infty(\Omega')$ to $\left(C^\infty(\Omega'')\right)^N$ is given by

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an integral operator with a smooth kernel $K \in \left(C^\infty(\Omega' \times \Omega'')\right)^N$. This implies that the mapping $g \mapsto u_s$ has an extension to a linear continuous map from $\mathscr{E}'(\Omega')$ to $\left(C^\infty(\Omega'')\right)^N$. Thus, (4.3) for general g follows from (4.4) by a density argument. □

As in the treatment of Hele-Shaw flow with kinetic undercooling, we decompose $(v, q) = (\overline{u}, \overline{p}) + (u_S, p_S)$ and obtain

$$\mathscr{U}(\Omega) = (\overline{u} + u_S)|_{\partial\Omega},$$

$$\left.\begin{aligned} -\Delta\overline{u} + \nabla\overline{p} &= 0 && \text{in } \Omega, \\ \operatorname{div}\overline{u} &= 0 && \text{in } \Omega, \\ \mathscr{T}(\overline{u}, \overline{p})n &= -\mathscr{T}(u_S, p_S)n && \text{on } \Gamma, \\ \int_\Omega \overline{u}\,dx &= -\int_\Omega u_S\,dx = -\int_\Gamma (G * g)n\,dx, \\ \int_\Omega \operatorname{rot}\overline{u}\,dx &= -\int_\Omega \operatorname{rot} u_S\,dx = 0. \end{aligned}\right\} \tag{4.5}$$

The following lemma provides a regularity result for an augmented version of the BVP (4.5). For its formulation we introduce the notations

$$X_s(\Omega) = (H^s(\Omega))^N \times H^s(\Omega) \times \left(\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}\right),$$
$$Y_s(\Omega) = \left(H^{s-2}(\Omega)\right)^N \times H^{s-1}(\Omega) \times \left(H^{s-\frac{3}{2}}(\partial\Omega)\right)^N \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}},$$

and P_3 for the projection of X_s onto its third component.

Lemma 7 (Regularity of the Stokes–Neumann BVP) *Let Ω be smooth and bounded, $s \geqslant 2$. The operator $\overline{L} \in \mathscr{L}(X_s, Y_s)$, given by*

$$\overline{L}(u, p, (\lambda_1, \lambda_2)) = \left(-\Delta u + \nabla p + \lambda_1, \operatorname{div} u, \mathscr{T}(u, p)n + \lambda_2 \otimes n, \int_\Omega u\,dx, \int_\Omega u\,dx\right)$$

is an isomorphism between the spaces X_s and Y_s. Moreover, if

$$\int_\Gamma f v\,d\Gamma = 0 \qquad \forall v \in V_0$$

then

$$P_3\overline{L}^{-1}(0, 0, f, \mu_1, \mu_2) = 0.$$

For a proof we refer to Günther & Prokert [7] (Section 2). It rests essentially on the fact that the Stokes equations with boundary conditions prescribing the normal component of the stress tensor can be treated in strict analogy to the Neumann problem for the Laplacian. From the second part of the lemma in connection with (4.3) it follows that (4.5) is uniquely solvable.

If (4.5) is augmented in the way described in Lemma 7 and transformed from $\Omega = \Omega_{\theta,r}$

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to B_0 by the diffeomorphism $\Phi_{\theta,r}$ one gets (cf. Prokert [12])

$$\begin{aligned} &L(\theta,r)(u(\theta,r),p(\theta,r),\lambda(\theta,r)) \\ &= \left(0,0,(\mathscr{T}(u_S,0)\circ\Phi_{\theta,r})\overline{v}(\theta,r),\int_{S^{N-1}}((G*g)\circ\Phi_{\theta,r})\chi_{\theta,r}\,d\sigma,0\right), \end{aligned}$$

where

$$L:\ V\longrightarrow \mathscr{L}\left(X_{s+\frac{1}{2}}(B_0),Y_{s+\frac{1}{2}}(B_0)\right),$$

V is a neighbourhood of 0 in $\mathbb{R}^{\binom{N}{2}}\times H^s(S^{N-1})$,

$$\begin{aligned} L(\theta,r)(u,p,(\lambda_1,\lambda_2)) = \Big(&-\Phi^*_{\theta,r}\Delta(\Phi^*_{\theta,r})^{-1}u+\Phi^*_{\theta,r}\nabla(\Phi^*_{\theta,r})^{-1}p+\lambda_1, \\ &\Phi^*_{\theta,r}\operatorname{div}(\Phi^*_{\theta,r})^{-1}u,\ \mathscr{T}_{\theta,r}(u,p)+\lambda_2\otimes\overline{v}(\theta,r), \\ &\int_{B_0}u\det D\Phi_{\theta,r}\,dx,\int_{B_0}\Phi^*_{\theta,r}\operatorname{rot}(\Phi^*_{\theta,r})^{-1}u\det D\Phi_{\theta,r}\,dx\Big), \\ \mathscr{T}_{\theta,r}(u,p) = {}&\Phi^*_{\theta,r}\nabla(\Phi^*_{\theta,r})^{-1}u+\left(\Phi^*_{\theta,r}\nabla(\Phi^*_{\theta,r})^{-1}u\right)^T-p\mathscr{I}, \\ u(\theta,r) = {}&\Phi^*_{\theta,r}\overline{u}, \\ p(\theta,r) = {}&\Phi^*_{\theta,r}\overline{p}. \end{aligned}$$

Moreover, $d\sigma$ denotes the surface measure on S^{N-1} and $\chi_{\theta,r}$ is a scalar function on S^{N-1} accounting for the 'change of the surface element' under the transformation from S^{N-1} to $\Phi_{\theta,r}[S^{N-1}]$.

In analogy to the treatment of Hele-Shaw flow, one straightforwardly shows

$$\begin{aligned} L&\in C^\omega\left(V,\mathscr{L}\left(X_{s+\frac{1}{2}}(B_0),Y_{s+\frac{1}{2}}(B_0)\right)\right), \\ (\theta,r)&\mapsto(\mathscr{T}(u_S,0)\circ\Phi_{\theta,r})\overline{v}(\theta,r)\in C^\omega\left(V,H^{s-1}(S^{N-1})\right), \\ (\theta,r)&\mapsto\int_{S^{N-1}}((G*g)\circ\Phi_{\theta,r})\chi_{\theta,r}\,d\sigma\in C^\omega\left(V,\mathbb{R}^N\right). \end{aligned}$$

It follows from Lemma 7 that

$$L(0,0)\in\mathscr{L}_{is}\left(X_{s+\frac{1}{2}}(B_0),Y_{s+\frac{1}{2}}(B_0)\right).$$

Hence, by the Implicit Function theorem,

$$u(\theta,r)\in C^\omega\left(V,\left(H^{s+\frac{1}{2}}(B_0)\right)^N\right). \tag{4.6}$$

Writing, as above, $u(r):=u(0,r)$ and $\overline{v}(r):=\overline{v}(0,r)$ and using local coordinates we have

$$\begin{aligned} \rho(r)(\xi) &= \frac{T\Phi^{-1}(\Phi(\psi_r(\xi)))(u(r)(\xi)+u_S(\Phi(\psi_r(\xi)))\cdot\overline{v}(r)(\xi)}{n\cdot v(r)} \\ &= \tilde{a}(\xi,r(\xi),\partial r(\xi))\cdot u(r)(\xi)+\tilde{b}(\xi,r(\xi),\partial r(\xi)), \end{aligned} \tag{4.7}$$

where $\tilde{a}$ and $\tilde{b}$ are smooth functions of their arguments. Note that (4.6) and (4.7) correspond to (3.3) and (3.4). Hence, in the same way as above, we arrive at the following result:

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Table 1

	Unregularized	Surface tension	Kin. undercooling
Hele-Shaw	$\rho'(0) \sim -\frac{\partial v_0}{\partial n} A_{DN}^{\Delta} + L$ $d = 1$ Forward/backward parabolic $\left(\text{dep. on sign of} \frac{\partial v_0}{\partial n}\right)$	$\rho'(0) \sim \sigma A_{DN}^{\Delta} \Delta_{\Gamma_0}$ $d = 3$ Forward parabolic $(\sigma > 0)$	$\rho'(r) \sim L(r)$ $d = 1$ Hyperbolic
Stokes	$\rho'(r) \sim L(r)$ $d = 1$ Hyperbolic	$\rho'(0) \sim \sigma n^T A_{ND}^{\text{Stokes}} n \Delta_{\Gamma_0} + L$ $d = 1$ Forward parabolic ($\sigma > 0$)	—

Theorem 2 (Existence and uniqueness for Stokes flow driven by interior sources) *Let* $\mathscr{U}$ *be given by* (1.3), (1.4), (4.1), (4.2). *Then the same results as in Theorem 1 hold.*

Clearly, the remarks after Theorem 1 are also valid in the present situation.

5 Conclusion

We want to point out once more that the technique of reformulating a (quasistationary) moving boundary problem as an evolution equation for a real-valued function on S^{N-1} (or another reference manifold) is not restricted to the two problems discussed above but can be applied to a broad class of problems including, for example, the problems of Stokes flow and Hele-Shaw flow driven by surface tension as well as 'classical' Hele-Shaw flow [12]. The study of the linearizations is the crucial tool in the analysis of the occurring evolution equations also in these cases. The corresponding results concerning the structure of this linearization and order and type of the evolution equations are summarized in Table 1 (see also Abergel [1] and Escher & Simonett [5]):

For the sake of simplicity, we consider here evolution equations on $\Gamma_0 = \partial\Omega_0$ rather than on S^{N-1} and their linearizations; order, type, and structure of the evolution equations are not influenced by this. The order of the problems is denoted by d. The symbol $\sim$ means equality of the highest order terms. L and $L(r)$ denote first-order differential operators, while A_{DN}^{Δ} and A_{ND}^{Stokes} denote the Dirichlet–Neumann operator for the Laplacian and the Neumann–Dirichlet operator for the Stokes system. These are elliptic pseudodifferential operators of orders 1 and -1, respectively (the latter is a (N, N)-matrix of scalar pseudo-differential operators). In the problems with surface tension, σ denotes the surface tension coefficient and Δ_{Γ_0} denotes the Laplace–Beltrami operator on Γ_0 which arises from the linearization of the curvature with respect to the perturbation.

In the case of Hele-Shaw flow without regularization, v_0 denotes the solution of the elliptic fixed-time problem in Ω_0. The sign is chosen in such a way that

$$\frac{\partial v_0}{\partial n} > 0$$

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corresponds to an advancing liquid domain while for

$$\frac{\partial v_0}{\partial n} < 0$$

the liquid domain is receding. Thus, the well-known dichotomy of the well-posed problem of Hele-Shaw flow with injection and the ill-posed problem of Hele-Shaw flow with suction can be explained by the corresponding behavior of forward and backward parabolic equations, respectively. Moreover, it should be noted that the application of kinetic undercooling to a Hele-Shaw flow problem with injection will replace a forward parabolic problem by a hyperbolic one, and the smoothing property of the evolution will be lost. Therefore, one can speak about kinetic undercooling regularization only for Hele-Shaw flow with an (at least partially) receding boundary.

The relatively strong smoothness assumptions on r_0 in Theorems 1 and 2 are made here for technical purposes. The conjecture seems reasonable that the problems discussed above are (locally) well-posed also in function spaces with lower smoothness. Finally, it has to be pointed out that due to the local character of our considerations no answers to the interesting questions about blow-up, changes of topology, or global existence in time can be expected.

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