*On Quantum Algebra.* By Dr P. A. M. DIRAC, St John's College. (Communicated by Mr R. H. FOWLER.)

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For the purposes of atomic physics it has been found convenient to introduce the idea of quantities that do not in general satisfy the commutative law of multiplication, but satisfy all the other laws of ordinary algebra*. These quantities are called $q$-numbers, and the numbers of ordinary mathematics $c$-numbers, while the word number alone is used to denote either a $q$-number or a $c$-number. Both $q$-numbers and $c$-numbers can occur together in the same piece of analysis, and even in the same equation, as numbers of the two kinds can be added together or multiplied. A $c$-number may, in fact, be regarded as a special case of the more general $q$-number. In the particular case when two numbers $x$ and $y$ satisfy $xy = yx$, we shall say that $x$ commutes with $y$. A $c$-number is assumed to commute with every number.

The development of the algebra of $q$-numbers has up to the present been greatly hampered by the fact that there has been no general definition of a function of a $q$-number variable. One has been obliged to restrict the word function to apply only to analytic functions specified by algebraic equations involving the operations of addition, multiplication and reciprocation (which last applied to $q$-numbers can easily be given a definite meaning), and one has usually contented oneself with taking a theorem involving arbitrary functions to be true generally when one has proved it true for power series.

In the present paper a general definition of a function is proposed which appears to enable one to establish theorems involving arbitrary functions with rigour. The differential coefficient of a function of a single variable can also be conveniently defined, its most important properties being contained in, or easily deducible from, its definition.

*Functions of a q-number variable.*

Consider a simple algebraic function of a single $q$-number variable $x$, such as $x^2$. Evidently every number that commutes with $x$ must also commute with $x^2$. This property provides us with the general definition of a function of a $q$-number variable.

Let $x$ and $X$ be two $q$-numbers such that any number that commutes with $x$ also commutes with $X$. We then say that $X$ is a function of the single variable $x$, and consider the two numbers

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$X$ and $x$ as specifying a function. A function of a $q$-number variable is thus specified by a pair of $q$-numbers that satisfy a certain condition and is quite different from a function of a real $c$-number variable, for the specification of which an infinite number of pairs of numbers are required, which need not satisfy any conditions.

In order that we may consider the two $q$-numbers $x$ and $X$ as really specifying a function, we must be able to say what is the value of this function for any particular value of the variable, $y$ say. We take the rule for this to be that if $b$ satisfies the equation

$$y = bxb^{-1} \quad \text{.....(1),}$$

then $Y$ defined by

$$Y = bXb^{-1} \quad \text{.....(2)}$$

is the same function of $y$ that $X$ is of $x$. This rule completes the definition of a function of a single $q$-number variable. (It is taken as an axiom that there always exists a $q$-number $b$ satisfying (1) when $x$ and $y$ are $q$-numbers.)

In order that our definition of a function may be tenable, one must be able to deduce the following theorems:

(i) All values of $b$ that satisfy (1) must give the same value for $Y$ defined by (2).

(ii) $Y$ defined by (2) must be a function of the single variable $y$, *i.e.* any number that commutes with $y$ must also commute with $Y$.

(iii) Functions of a single variable can be formed by the addition or multiplication of functions of that variable.

(iv) A function of a function of a certain variable must be a function of that variable.

To prove (i) we observe that if $b'$ is another root of (1), we have

$$bxb^{-1} = y = b'xb'^{-1},$$

whence, dividing by $b$ in front and multiplying by $b'$ behind, we get

$$xb^{-1}b' = b^{-1}b'x.$$

Thus $b^{-1}b'$ commutes with $x$, and therefore also with $X$, since by hypothesis every number that commutes with $x$ also commutes with $X$. Hence

$$b^{-1}b'X = Xb^{-1}b',$$

so that, multiplying by $b$ in front and dividing by $b'$ behind, we find

$$b'Xb'^{-1} = bXb^{-1} = Y$$

from (2), which proves the theorem.

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To prove (ii), let $a$ be any number that commutes with $y$, so that

$$ay = ya,$$

or

$$abxb^{-1} = bxb^{-1}a,$$

from (1). This gives

$$b^{-1}abx = xb^{-1}ab,$$

so that $b^{-1}ab$ commutes with $x$, and hence also with $X$. Thus

$$b^{-1}abX = Xb^{-1}ab,$$

which gives

$$abXb^{-1} = bXb^{-1}a,$$

or

$$aY = Ya,$$

which proves the theorem.

To prove (iii), we observe that if $X_1$ and $X_2$ are two functions of the single $q$-number variable $x$, then $X_1 + X_2$ and $X_1X_2$ are also functions of the single variable $x$, since any number that commutes with $x$ must commute with $X_1$ and $X_2$, and therefore commutes with $X_1 + X_2$ and $X_1X_2$. To complete the proof that functions of a single variable may be formed by the addition or multiplication of functions of that variable, we must show that if $Y_1$ and $Y_2$ are the same functions of $y$ that $X_1$ and $X_2$ are of $x$, then $Y_1 + Y_2$ and $Y_1Y_2$ are the same functions of $y$ that $X_1 + X_2$ and $X_1X_2$ are of $x$. By hypothesis we have

$$y = bxb^{-1}, \qquad Y_1 = bX_1b^{-1}, \qquad Y_2 = bX_2b^{-1},$$

and hence

$$Y_1 + Y_2 = b(X_1 + X_2)b^{-1},$$

$$Y_1Y_2 = bX_1b^{-1}.bX_2b^{-1} = bX_1X_2b^{-1},$$

which proves the theorem.

Finally, to prove (iv), we see that if $X$ is a function of the single variable $x$ and $\xi$ is a function of the single variable $X$, then any number that commutes with $x$ must commute with $X$, and hence must also commute with $\xi$, so that $\xi$ is a function of the single variable $x$. Also, if $Y$ is the same function of $y$ that $X$ is of $x$, we have

$$y = bxb^{-1}, \qquad Y = bXb^{-1},$$

and if in addition $\eta$ is the same function of $Y$ that $\xi$ is of $X$, we have

$$\eta = b\xi b^{-1}.$$

It follows that $\eta$ is the same function of $y$ that $\xi$ is of $x$.

Theorem (iii) shows that algebraic functions defined by the operations of addition and multiplication are still functions in the new sense of the word. Also it is easily verified that $x^{-1}$ (defined by $x^{-1}x = 1$ or $xx^{-1} = 1$) is a function of $x$, and hence from (iv) the reciprocal of any function of $x$ is a function of $x$. Any function of

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a single variable must commute with any other function of that variable. A $c$-number may be regarded as a particular case of a function of any $q$-number variable.

The extension of the theory to functions of more than one variable is rather restricted. We define $X$ to be a function of the variables $x_1, x_2, \ldots x_n$ only, if $X$ commutes with every number that commutes with $x_1, x_2, \ldots x_n$. If we now have

$$y_1 = bx_1b^{-1},\ y_2 = bx_2b^{-1},\ \ldots\ y_n = bx_nb^{-1} \quad \ldots\ldots\ldots(3),$$

and put

$$Y = bXb^{-1},$$

we can define $Y$ to be the same function of the variables $y_1, y_2, \ldots y_n$ that $X$ is of the variables $x_1, x_2, \ldots x_n$, and can easily deduce the theorems corresponding to (i)...(iv) for functions of several variables. It is, however, only when the $y$'s satisfy the conditions (3) that one can give a meaning to the expression "the same function of the $y$'s that $X$ is of the $x$'s," owing to the fact that one cannot consider the $x$'s to vary independently, as one could if they were $c$-numbers. Suppose, for instance, that $x_1$ and $x_2$ satisfy

$$x_1x_2 - x_2x_1 = 1.$$

Then one would have to count $x_1x_2$ and $(1 + x_2x_1)$ as two ways of writing the same function of $x_1$ and $x_2$; but one would not in general have $y_1y_2$ equal to $(1 + y_2y_1)$, except when the $y$'s satisfy (3), so that it is only in this case that one can hope to give a meaning to the same function of the $y$'s.

*Differential coefficients.*

Let $Q$ be a function of the single variable $q$, and let $p$ be a number that satisfies

$$qp - pq = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Put

$$Qp - pQ = Q' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Then we define $Q'$ to be the differential coefficient of $Q$ with respect to $q$, and write it $dQ/dq$.

In order that our definition may be tenable, one must be able to deduce the following theorems:

(i) All values of $p$ that satisfy (4) must give the same value for $Q'$ defined by (5).

(ii) $Q'$ defined by (5) must be a function of the single variable $q$.

(iii) The ordinary rules for the differential coefficient of a sum and a product must still hold.

(iv) The ordinary rule for the differential coefficient of a function of a function must still hold.

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To prove (i), we observe that if $p'$ is another root of (4), we have

$$qp - pq = 1 = qp' - p'q,$$

or

$$q(p - p') = (p - p')\, q.$$

Thus $(p - p')$ commutes with $q$, and therefore also with $Q$. Hence

$$Q(p - p') = (p - p')\, Q,$$

or

$$Qp' - p'Q = Qp - pQ = Q',$$

which proves the theorem.

To prove (ii), let $a$ be any number that commutes with $q$. It must therefore commute with $Q$. We have to prove that it commutes with $Q'$. From (4), since $(qp - pq)$ is a $c$-number, we have

$$\begin{aligned} 0 &= a(qp - pq) - (qp - pq)\, a \\ &= aqp - apq - qpa + pqa \\ &= qap - apq - qpa + paq \\ &= q(ap - pa) - (ap - pa)\, q. \end{aligned}$$

Hence $(ap - pa)$ commutes with $q$, and therefore also with $Q$, so that

$$\begin{aligned} 0 &= Q(ap - pa) - (ap - pa)\, Q \\ &= Qap - Qpa - apQ + paQ \\ &= aQp - Qpa - apQ + pQa \\ &= a(Qp - pQ) - (Qp - pQ)\, a \\ &= aQ' - Q'a. \end{aligned}$$

Thus $a$ commutes with $Q'$. To complete the proof of (ii), we must show that if $Q_1$ is the same function of $q_1$ that $Q_2$ is of $q_2$, then $Q_1'$ is the same function of $q_1$ that $Q_2'$ is of $q_2$, *i.e.* if

$$q_2 = bq_1b^{-1}, \qquad Q_2 = bQ_1b^{-1},$$

$$q_1p_1 - p_1q_1 = 1, \qquad q_2p_2 - p_2q_2 = 1,$$

and

$$Q_1p_1 - p_1Q_1 = Q_1', \qquad Q_2p_2 - p_2Q_2 = Q_2',$$

then

$$Q_2' = bQ_1'b^{-1}.$$

We have

$$1 = q_2p_2 - p_2q_2 = bq_1b^{-1}p_2 - p_2bq_1b^{-1},$$

from which, by multiplying by $b$ behind and $b^{-1}$ in front, we obtain

$$1 = q_1b^{-1}p_2b - b^{-1}p_2bq_1.$$

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This shows that $b^{-1}p_2b$ may be taken to be $p_1$. Hence

$$Q_1' = Q_1 b^{-1}p_2 b - b^{-1}p_2 b Q_1,$$

or

$$bQ_1'b^{-1} = bQ_1b^{-1}p_2 - p_2bQ_1b^{-1}$$
$$= Q_2p_2 - p_2Q_2 = Q_2',$$

which proves the theorem.

To prove (iii) let $Q_1$ and $Q_2$ be two functions of the single variable $q$. It is obvious that

$$(Q_1 + Q_2)' = Q_1' + Q_2'.$$

Further

$$(Q_1Q_2)' = Q_1Q_2p - pQ_1Q_2$$
$$= (Q_1p - pQ_1)Q_2 + Q_1(Q_2p - pQ_2)$$
$$= Q_1'Q_2 + Q_1Q_2'.$$

The ordinary laws for the differentiation of a sum and a product thus still hold.

To prove (iv), let $Q$ be a function of the single variable $q$ and $\xi$ a function of the single variable $Q$, and let

$$qp - pq = 1,$$

and

$$QP - PQ = 1 \quad \ldots\ldots(6).$$

We then have by definition

$$Qp - pQ = dQ/dq \quad \ldots\ldots(7),$$

$$\xi p - p\xi = d\xi/dq \quad \ldots\ldots(8),$$

and

$$\xi P - P\xi = d\xi/dQ \quad \ldots\ldots(9).$$

Now $dQ/dq$ must commute with $Q$, since they are both functions of the single variable $q$, and hence, dividing (7) by $dQ/dq$ in front, we obtain

$$1 = \left(\frac{dQ}{dq}\right)^{-1}(Qp - pQ) = Q\left(\frac{dQ}{dq}\right)^{-1}p - \left(\frac{dQ}{dq}\right)^{-1}pQ.$$

This shows that $(dQ/dq)^{-1}p$ may be taken to be the $P$ of equation (6). Hence, using this value for $P$ in equation (9), we obtain

$$\frac{d\xi}{dQ} = \xi\left(\frac{dQ}{dq}\right)^{-1}p - \left(\frac{dQ}{dq}\right)^{-1}p\xi = \left(\frac{dQ}{dq}\right)^{-1}(\xi p - p\xi) = \left(\frac{dQ}{dq}\right)^{-1}\frac{d\xi}{dq}$$

using (8) and the fact that $(dQ/dq)^{-1}$ commutes with $\xi$. Hence

$$\frac{d\xi}{dq} = \frac{dQ}{dq}\,\frac{d\xi}{dQ},$$

which is the ordinary rule for the differentiation of a function of a function.

From (iii) and (iv) it follows that the differential coefficients of simple algebraic functions have their ordinary values. In order



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that the converse process of integration applied to simple algebraic functions may also lead to the usual results, it is necessary and sufficient that the only functions whose differential coefficients are zero shall be constants (*i.e.* $c$-numbers). This requires that the only numbers that commute with $p$ and also with every number that commutes with $q$, when $p$ and $q$ satisfy (4), shall be $c$-numbers. We take this to be an axiom in the new algebra. It is a particular case of a more general axiom which may be enunciated thus: If a number commutes with every number that commutes with the $q$-numbers $q$, $q_1$, $q_2$, ... $q_n$ and with one number that does not commute with $q$ but commutes with $q_1, q_2, \ldots q_n$, then it commutes with every number that commutes with $q_1$, $q_2$, ... $q_n$*. The axiom may also be stated geometrically thus: Each $q$-number may be represented by a point in space of an infinite number of dimensions in such a way that the condition that a $q$-number shall commute with certain given $q$-numbers is that it shall lie in a certain flat space.

The definition of a differential coefficient may be extended to the partial differential coefficients of a function of several variables that commute with one another. If $Q$ is a function of the variables $q$, $q_1$, $q_2$, ... $q_n$ that commute with one another, to define $\partial Q/\partial q$ one requires a $p$ that satisfies (4) and that commutes with $q_1$, $q_2$, ... $q_n$. One can then define $\partial Q/\partial q$ by the left-hand side of (5), and can show that it satisfies theorems corresponding to (i)...(iv). It is impossible, however, to extend the definition to the partial differential coefficients of functions of several arbitrary variables. Suppose, for example, that $x_1$ and $x_2$ are $q$-numbers such that

$$x_1 x_2 = x_2 (x_1 + c),$$

where $c$ is a $c$-number; then $x_1 x_2$ and $x_2 (x_1 + c)$ must be two ways of writing the same function of $x_1$ and $x_2$, and it would obviously be impossible to give a rational definition of the partial differential coefficient of this function with respect to $x_2$.

* Note that a number $x$ that commutes with every number must be a $c$-number, on account of the axiom that if $x$ is a $q$-number, there must be a $q$-number $b$ that makes $bxb^{-1}$ equal an arbitrary $q$-number.

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