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Thin-film-growth models: roughness and correlation functions

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Surfaces arising in amorphous thin-film-growth are often described by certain classes of stochastic PDEs. In this paper we address the question of existence of a solution and statistical quantities (e.g. mean interface width or correlation functions). Moreover, we discuss the approximations of such statistical quantities by the spectral Galerkin method. This is an important question, as the numerical computation of statistical quantities plays a key role in the verification of the models.

1 Introduction

This paper considers the existence of solutions for nonlinear Stochastic Partial Differential Equations (SPDEs) arising in thin-film-growth via a Galerkin method. Moreover, we discuss the question whether the Galerkin method can be used to approximate statistical quantities like mean interface width or mean correlation function of the described surface. This is an important question, as experiments yield such quantities using *in situ* scanning tunneling microscopy [33, 34, 35] or X-ray scattering [36]. The model, described by some SPDEs, is usually phenomenologically derived, and then fitted to the experiment using numerical calculations of such quantities (cf. [23, 24, 32], where finite differences are used).

The temporal evolution of the roughness of a surface is described by its height profile $h(t,x)$ at time $t \geqslant 0$ and at position $x \in [0,L]$. We consider recently proposed models for thin-film-growth, where the evolution of the height profile h is modeled by a stochastic partial differential equation of the form

$$\partial_t h = Ah + \mathscr{L}(\partial_x h)^2 + \xi \quad \text{for} \quad t > 0\,, \qquad h(0) = h_0. \tag{1.1}$$

Here A and $\mathscr{L}$ denote linear differential operators and h_0 is some random initial condition. The additive stochastic term is given by a noise process $\xi = \{\xi(t)\}_{t\geqslant 0}$ which is the generalized derivative of a Wiener process in Hilbert space.

Some examples of thin-film-growth models are given by

$$\text{[9] (1995)} \quad \partial_t h = -a_1\partial_x^4 h - a_2\partial_x^2 h + a_4(\partial_x h)^2 + \xi \tag{1.2}$$

$$\text{[26] (1997)} \quad \partial_t h = -a_1\partial_x^4 h - a_2\partial_x^2 h - a_3\partial_x^2(\partial_x h)^2 + \xi \tag{1.3}$$

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$$[32]\ (2000)\ \ \partial_t h = -a_1\partial_x^4 h - a_2\partial_x^2 h - (a_3\partial_x^2 - a_4)(\partial_x h)^2 + \xi \tag{1.4}$$

The coefficients a_i are usually positive constants in all of the models, but it is possible to relax this constraint for $i > 1$. However, it is essential that the differential operator $(a_3\partial_x^2 - a_4)$ is either positive or negative. The above equations are considered on $[0, L]$ subject to periodic boundary conditions and deterministic initial value 0 corresponding to an initially flat surface.

For literature on (1.3) or (1.4) we refer the reader to Siegert & Plischke [37], Moske [26], Raible, Linz & Hänggi [29, 30, 21], or Raible *et al.* [32]. General surveys on surface growth processes and molecular beam epitaxy are, for example, Krug & Spohn [20], Meakin [25], Barabasi & Stanley [3], Halpin-Healy & Zhang [16], Marsili *et al.* [22] or Krug [19], to name a few.

As the height profile of one realization depends heavily on randomness, only *statistical* quantities are useful for a quantitative comparison of the experiments with the model. In the following we focus on two important examples: The mean interface width and the height-height correlation function. Both are usually estimated using data derived by scanning tunneling microscopy from different surface specimens.

A usual characteristic for the roughness of a surface is the *mean interface width* $w(t)$, scaled with respect to the length scale L of the underlying domain. See, for example, Marsili *et al.* [22]. The mean interface width is defined by

$$w^2(t) := \frac{1}{L} \cdot E \left\| h(t) - \overline{h}(t) \right\|^2_{L^2([0,L])} , \tag{1.5}$$

where

$$\overline{h}(t) := \frac{1}{L} \cdot \int_0^L h(t,x)dx \tag{1.6}$$

and E denotes expectation with respect to the underlying probability space.

Surface roughness was first discussed rigorously in Blömker *et al.* [6, 7] for equations involving linearly bounded nonlinearities. These results are obviously not applicable in our case.

Another important quantity is the *height-height correlation function* given by

$$C(t,r) := \frac{1}{2}[C_+(t,r) + C_-(t,r)] \tag{1.7}$$

with

$$C_\pm(t,r) := \frac{1}{L} \cdot E \int_0^L [h(t,x) - \overline{h}(t)][h(t,x \pm r) - \overline{h}(t)]dx\,. \tag{1.8}$$

The smallest zero of $C(t,\cdot)$ or the smallest local maximum for $r \neq 0$ give characteristic length scales of the surface at time $t > 0$, which determine, for instance, the size of characteristic patterns.

There are no results available for any thin-film-growth equation ((1.2), (1.3), or (1.4)) in dimension $d \geqslant 2$. Even the existence of a solution is not settled completely, mainly, because it is not known how to derive any *a priori* estimates in the presence of the quadratic terms. Note that the existence of local solutions could be proved by standard fixed-point arguments (e.g. see Blömker [4] for a detailed discussion), but local solutions are not useful, as we want to obtain finite expectations of the solution.

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The existence of solutions in the case $d = 1$ for (1.3) has been addressed recently by Blömker & Gugg [5]. They prove the existence of a bounded L^2-valued solution, but it was not possible to show the continuity of the solution. The existence and uniqueness of solutions for (1.2) in case $d = 1$ is established by Duan & Ervin [13]. In both cases, it is straightforward to derive *a priori* bounds either for the L^2-norm in case of (1.3) or for the H^1-norm in case of (1.2). However, these methods rely on the fact that only the conservative (in (1.3)) or the nonconservative term (in (1.2)) is present.

One aim of our paper is to provide a proof of existence for (1.4). For this we will follow the results of Blömker & Gugg [5], which in turn are based on Capinski & Gatarek [8]. The main problem will be to derive an *a priori* estimate for our equation. The straightforward ideas involving L^2 or H^1-norm, which were used in Blömker & Gugg [5] or Duan & Ervin [13], do not work. We have to consider a norm on $L^2([0, L])$ given by the quadratic form of the operator $-\partial_x^2 \mathscr{L}^{-1}$. Moreover, we lose mass-conservation, which is another technical obstacle for our proofs.

Another goal is the question whether the spectral Galerkin-method used in the proof of existence is useful to provide approximations of mean interface width (1.5) and correlation functions (1.7) that are greatly needed in the numerical verification of thin-film-growth models. In the sequel we will focus on equation (1.4), as it is the relevant model in amorphous thin-film-growth of $Zr_{65.0}Al_{7.5}Cu_{27.5}$ alloy films (cf. Raible *et al.* [32]). We could treat (1.3) analogously; it should be even simpler in this case, as we additionally have conservation of mass, which is not present in (1.4). For (1.2) the results should be much better, as we have a unique global solution there.

A drawback of the convergence result is that we are not able to consider space-time white noise, which is often used as a simple model in physical applications. Although we obtain the existence of a solution in that case, we cannot verify that the statistical quantities of interest are finite.

The paper is organized as follows. § 2 provides basic notations and assumptions. In § 3 we formulate our main results. § 4 contains the a priori estimates of Lemma 3.1 that are the foundations of our proofs. § 5 contains the proofs of convergence for the statistical quantities mentioned above, and in § 6 we present numerical results.

2 Problem formulation

Let A be the linear differential operator $-a_1\partial_x^4 - a_2\partial_x^2$ subject to periodic boundary conditions in $L^2([0, L])$. We define $A_\alpha := A - \alpha \cdot I$ for some $\alpha \geqslant 0$ fixed later. It is well known that A_α generates an analytic semigroup $\{e^{tA_\alpha}\}_{t\geqslant 0}$. By $\{f_k\}_{k\in\mathbb{Z}}$ we denote some complete orthonormal set in $L^2([0, L])$ of eigenfunctions of A. Hereby, f_0 is the constant function, and f_{-k} and f_k belong to the same eigenvalue λ_k of A, $k \in \mathbb{N}$, i.e. $\lambda_0 = 0$ and λ_k grows like $-k^4$ for $k \to \infty$.

Now $D(A_\alpha) = H^4_{\text{per}}$, which is the subset of functions from the Sobolev space $H^4([0, L])$ fulfilling periodic boundary conditions. We will denote by X the L^2-orthogonal complement of f_0 equipped with the L^2-norm.

In the following we consider mainly (1.4). Therefore, define $\mathscr{L} = -a_3\partial_x^2 + a_4$, which is a positive self-adjoint operator in $L^2([0, L])$ since $a_3 > 0$ and $a_4 > 0$.

The perturbation ξ is the generalized time-derivative of some Wiener process W on some

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Hilbert space. To reformulate the problem as an integral equation we need the generalized stochastic process $\{\xi(t)\}_{t\geqslant 0}$ to be the generalized derivative of some Q-Wiener process $W = \{W(t)\}_{t\geqslant 0}$ on some probability space $(\Omega, \mathscr{A}, P)$. Here Q denotes the covariance operator of the Wiener process. For a detailed discussion of the class of noises covered by the following assumptions see Remark 2.3.

Moreover, we suppose that W is P-almost sure $C([0,T], H^{s_0}_{\mathrm{per}})$-valued for some $s_0 \in \mathbb{R}$, and that the stochastic convolution

$$W_{A_\alpha}(t) = \int_0^t e^{(t-\tau)A_\alpha} dW(\tau), \quad t \geqslant 0$$

is well-defined. To be more precise, we will assume that one of the following assumptions is true.

Assumption 2.1 *Suppose that Q and A have the same complete orthonormal system of eigenfunctions $\{f_k\}_{k\in\mathbb{Z}}$, and denote the corresponding eigenvalues of Q by α_k^2, $k \in \mathbb{Z}$. Hence, $Qf_k = \alpha_k^2 f_k$. This ensures that $W(t) = \sum_{k\in\mathbb{Z}} \alpha_k \beta_k(t) f_k$, where $\{\beta_k\}_{k\in\mathbb{Z}}$ is a family of stochastically independent standard Brownian motions. Moreover, we assume*

$$\sum_{k\neq 0} \alpha_k^2 |k|^{-2+\delta} < \infty$$

for some $\delta > 0$. This ensures, for example, that W_{A_α} possesses P-almost sure continuous trajectories in $C^1_{\mathrm{per}}([0,L])$.

Assumption 2.2 *Let Q be as in Assumption* 2.1. *Assume additionally that*

$$\sum_{k\neq 0} \alpha_k^2 |k|^{\delta} < \infty$$

for some $\delta > 0$. This ensures, for example, that W_{A_α} possesses P-almost sure continuous trajectories in $C^2_{\mathrm{per}}([0,L])$.

Before we proceed, let us first discuss which classes of noise are covered by the preceding assumptions.

Remark 2.3 *It is straightforward using, for example, a formal calculation from da Prato & Zabczyk* [11, Section 4.3.3] *to see which noises we can consider (cf. Blömker* [4, Section 2.1.3] *for a detailed discussion). Under our assumptions above, we cover Gaussian noise which is white in time and spatially homogeneous with sufficiently regular covariance functional. To be more precise, our noise ξ is Gaussian with mean-value-functional $E\xi(t,x) = 0$ and covariance-functional $E\xi(t,x)\xi(s,y) = \delta(t-s)q(|x-y|)$, where δ is the Delta-distribution. The covariance operator Q is then the convolution with the function (or functional) q. If q is the Delta-distribution, as in the case of space-time white noise, we obviously obtain $\alpha_k = Const.$ for any $k \in \mathbb{Z}$, and we are in the situation of Assumption* 2.1.

Remark 2.4 *The regularity of the stochastic convolution, which is asserted in Assumptions* 2.1 *and* 2.2, *is proved similarly to da Prato & Zabczyk* [11, Section 5.5.1], *where the celebrated Kolmogorov test is used. This reference shows only continuity in space and time, but*

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it is straightforward to apply this result, if we define $\mathscr{W}$ by $\partial_x W_{A_\alpha} = \mathscr{W}_{A_\alpha}$. Obviously, as ∂_x and Q commute, the process $\mathscr{W}$ is a Wiener process with covariance operator $-\partial_x^2 Q$. For a detailed discussion see Blömker [4, Section 2.2.2]. *Note that the assumption of A and Q having a joint base of eigenfunctions is essential for the regularity, as the proof explicitly uses the Fourier series expansion of the stochastic convolution with respect to the eigenfunctions of A. However, as discussed in the preceding remark, this assumption is satisfied in our applications.*

It is a straightforward calculation using Ito's Isometry that $E\|W_{A_\alpha}(t)\|_{H^m}^2 < \infty$ if and only if $\sum_{k\neq 0} \alpha_k^2 |k|^{-4+2m} < \infty$. It is a remarkable fact that the result sketched in the previous remark implies that under our slightly stronger assumption, we already obtain that the stochastic convolution is in $C([0,T], C^m([0,L]))$.

The problem is now to construct a solution $h : [0,T] \times \Omega \to L^2([0,L])$ of the equation

$$dh(t) = Ah(t)dt + \mathscr{L}([\partial_x h(t)]^2)dt + dW(t), \qquad h(0) = h_0 \tag{2.1}$$

for $t \in [0,T]$ with arbitrary $T > 0$. Here, h_0 is some random initial condition $h_0 \in L^2([0,L])$ fulfilling one of the following assumptions.

Assumption 2.5 *Let $E[\ln^+(\|h_0\|_{L^2})]^\kappa < \infty$ for some $\kappa \geqslant 1$, with $\ln^+(x) = \max\{0, \ln(x)\}$.*

Assumption 2.6 *For some $\kappa \geqslant 1$, suppose either $E\|h_0\|_{L^2}^{4\kappa+\delta} < \infty$ for some small $\delta > 0$ or that h_0 and W are stochastically independent and $E\|h_0\|_{L^2}^{4\kappa} < \infty$.*

Equation (2.1) is the Ito interpretation of (1.4) for $d = 1$. For technical reasons, we add $(-\alpha h + \alpha h)dt$ in (2.1). The constant α is usually large and will be fixed later on.

Usually, we do not have enough regularity of W to solve (2.1), so we only get solutions in some generalized sense. Therefore, we consider *mild solutions*, defined as Hilbert-space-valued solutions of the corresponding integral equation, called the variation of constants formula,

$$h(t) = e^{tA_\alpha} h_0 + \int_0^t \mathscr{L} e^{(t-\tau)A_\alpha}[(\partial_x h(\tau))^2]d\tau + \alpha \int_0^t e^{(t-\tau)A_\alpha} h(\tau)d\tau + W_{A_\alpha}(t). \tag{2.2}$$

To prove the existence of solutions for (2.2) we first define $v = h - W_{A_\alpha}$. Hence,

$$v(t) = e^{tA_\alpha} h_0 + \int_0^t \mathscr{L} e^{(t-\tau)A_\alpha}[(\partial_x v(\tau) + \partial_x W_{A_\alpha}(\tau))^2]d\tau + \alpha \int_0^t e^{(t-\tau)A_\alpha}[v(\tau) + W_{A_\alpha}(\tau)]d\tau. \tag{2.3}$$

We cannot prove the existence of a stochastic process v on $(\Omega, \mathscr{A}, P)$ that satisfies (2.3), but we will change our underlying probability space to construct a *martingale solution* (cf. da Prato & Zabczyk [11, Section 8]).

A stochastic process $\tilde{h}$ defined on a different probability space $\mathscr{M} = (\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{P})$ is called a martingale solution of (2.2), if the corresponding $\tilde{v}$ (for some fixed α large enough) satisfies (2.3) with $\tilde{h}_0$ as initial condition defined on $\mathscr{M}$ that has the same probability distribution as h_0 and with some Q-Wiener process on $\mathscr{M}$, that has the same distribution

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than the original Q-Wiener process W. Note that therefore the stochastic convolutions W_{A_α} and $\tilde{W}_{A_\alpha}$ have the same distribution, too.

We will use the Galerkin-method to prove the existence of $\tilde{v}$ or $\tilde{h}$. By $P^{(n)}$ we will denote the L^2-orthogonal projection onto the span $V^{(n)}$ of the first $2n+1$ eigenfunctions, i.e. $V^{(n)} := \text{span}\{f_{-n}, \dots, f_0, \dots, f_n\}$. This is the space of eigenfunctions corresponding to the first $n+1$ eigenvalues $\lambda_0, \dots, \lambda_n$. Note that with this definition ∂_x and $\mathscr{L}$ commute with $P^{(n)}$. Furthermore, any function in $V^{(n)}$ is smooth. Define $W^{(n)}_{A_\alpha}(t) := P^{(n)} W_{A_\alpha}(t)$ and $h^{(n)}_0 := P^{(n)} h_0$, and suppose that $v^{(n)}$ is the solution of

$$\begin{aligned} \partial_t v^{(n)} &= A v^{(n)} + \alpha W^{(n)}_{A_\alpha} + P^{(n)} \mathscr{L} (\partial_x v^{(n)} + \partial_x W^{(n)}_{A_\alpha})^2 \\ v^{(n)}(0) &= h^{(n)}_0 \end{aligned} \tag{2.4}$$

or in the mild formulation

$$\begin{aligned} v^{(n)}(t) = e^{tA_\alpha} h^{(n)}_0 &+ \int_0^t \mathscr{L} e^{(t-\tau)A_\alpha} P^{(n)} [(\partial_x v^{(n)}(\tau) + \partial_x W^{(n)}_{A_\alpha}(\tau))^2] d\tau \\ &+ \alpha \int_0^t e^{(t-\tau)A_\alpha} [v^{(n)}(\tau) + W^{(n)}_{A_\alpha}(\tau)] d\tau. \end{aligned} \tag{2.5}$$

The existence and uniqueness of a solution $v^{(n)}$ of the stochastic ordinary differential equation is standard (cf. Arnold, [2, Section 6.3], or Øksendahl [27]). The solution $v^{(n)}(\cdot)$ is continuous with values in $V^{(n)}$, which is a space of spatially smooth functions. Of course this solution is only a local solution, and the global existence follows from the *a priori* estimates we derive in the fourth section.

3 Results

The main technical result in our theory are the following *a priori* bounds that are proved in the next section.

Lemma 3.1 *For some $\kappa \geqslant 1$ let Assumptions 2.1 and 2.5 be true and fix $T > 0$. Then there is a positive constant C_T independent of $n \in \mathbb{N}$, such that*

$$E[\ln^+(\sup_{t\in[0,T]} \|v^{(n)}(t)\|_{L^2})]^\kappa \leqslant C_T \quad \text{and} \quad E[\ln^+(\int_0^T \|v^{(n)}(t)\|^2_{H^2} dt)]^\kappa \leqslant C_T. \tag{3.1}$$

If Assumptions 2.2 and 2.6 are true for some $\kappa \geqslant 1$, we obtain for sufficiently large α

$$E \sup_{t\in[0,T]} \|v^{(n)}(t)\|^{2\kappa}_{L^2} \leqslant C_T \quad \text{and} \quad E[\int_0^T \|v^{(n)}(t)\|^2_{H^2} dt]^\kappa \leqslant C_T \tag{3.2}$$

with some positive constant C_T independent of $n \in \mathbb{N}$.

If (1.4) is perturbed by space-time white noise (i.e. $\alpha_k \equiv Const$ for all $k \in \mathbb{Z}$) only Assumption 2.1 is true, and we cannot prove the convergence of statistical quantities like the mean interface width or the height-height correlation function (cf. (1.5) and (1.7)) for the approximations $v^{(n)}$. It is not even possible to get some uniform upper bound with respect to n for the quantities of interest. To obtain such a bound we need (3.2) for $\kappa = 1$ to be true. Hence, Assumption 2.2 has to be true.

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However, even in the case of space-time white noise, the *a priori* estimate (3.1) implies the following theorem for the existence of solutions. It is proved similar to Theorem 3.1 of Blömker & Gugg [5] up to some straightforward modifications and for simplicity of presentation we skip the proof. But we briefly recall the basic ideas of the proof.

Theorem 3.2 *Consider* (1.4) *for* $d = 1$ *with positive* a_i, $i = 1, \dots, 4$, *subject to random initial condition* $h_0 \in L^2([0, L])$. *Fix* $L > 0$ *and* $T > 0$ *and assume* 2.1 *and* 2.5. *Then there exists a martingale solution* $\tilde{h}$ *on a probability space* $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{P})$ *such that*

$$\tilde{v} = \tilde{h} - \tilde{W}_A \in L^2([0, T], H^1_{\text{per}}([0, L]))$$

$\tilde{P}$*-almost sure and* $\tilde{v}$ *satisfies (2.3) with* $\alpha = 0$ *and with* $\tilde{W}_A$ *and* $\tilde{h}_0$ *instead of* W_A *and* h_0 *in the space* $L^2([0, T], H^1_{\text{per}}([0, L]))$, $\tilde{P}$*-almost sure.*

Moreover, there exists a sequence $\tilde{v}^{(n_k)}$ *of random variables on* $(\tilde{\Omega}, \tilde{\mathscr{A}})$ *that have the same distribution as* $v^{(n_k)}$, *and* $\tilde{v}^{(n_k)}$ *converges to* $\tilde{v}$ *in* $L^2([0, T], H^1_{\text{per}}([0, L]))$ $\tilde{P}$*-almost sure. The same is true for* $\tilde{h}^{(n_k)} := \tilde{v}^{(n_k)} + \tilde{W}_A^{(n_k)}$, *which has the same distribution as* $h^{(n_k)} := v^{(n_k)} + P^{(n_k)} W_A$ *and the limit* $\tilde{h} = \tilde{v} + \tilde{W}_A$.

We will now recall the basic ideas for proving this theorem. For the first step we use the *a priori* estimate of Lemma 3.1 and some compactness of the semigroup e^{tA_α} to obtain using the variation of constants formula that the probability distributions of the approximations $v^{(n)}$ are tight. Now Prokhorov's theorem implies the existence of a weakly convergent subsequence, but only in the space $L^2([0, T], H^1_{\text{per}}([0, L]))$. Then we change the probability space to obtain the P-a.s. convergence via Skohorod's theorem. It remains to show that the limit is a solution of the variation of constants formula. This is at first only achieved in the space $L^2([0, T], H^{-3}_{\text{per}}([0, L]))$, therefore we have to add some regularity arguments.

As mentioned before, if we want to investigate the convergence of statistical quantities of interest, Assumption 2.2 has to be true. Otherwise we obtain no bound for the sequence. As in (1.5) we will define the mean interface width corresponding to $h^{(n)} = v^{(n)} + W_A^{(n)}$ by w_n. The mean interface width w is given by

$$w^2 = \frac{1}{L} \tilde{E} \|\tilde{h} - \overline{\tilde{h}}\|^2_{L^2},$$

where $\tilde{E}$ denotes the expectation with respect to $\tilde{P}$. Moreover, as in (1.8) we will define $C^{(n)}_{\pm}$ to be the height-height correlation function corresponding to $h^{(n)}$ and

$$C_{\pm}(t, r) = \frac{1}{L} \cdot \tilde{E} \int_0^L [\tilde{h}(t, x) - \overline{\tilde{h}}(t)][\tilde{h}(t, x \pm r) - \overline{\tilde{h}}((t)] dx.$$

Theorem 3.3 *Suppose Assumptions* 2.2 *and* 2.6 *are true for some* $\kappa > 1$. *Then we obtain for the subsequence of Theorem* 3.2 *that*

$$w^2_{n_k} \to w^2 \qquad \text{for} \quad k \to \infty \tag{3.3}$$

in any $L^p([0, T])$ *with* $p \in [1, \infty)$.

If additionally $\kappa \geqslant \frac{3}{2}$ *then* $\{w^2_{n_k}\}_{k \in \mathbb{N}}$ *is bounded in* $W^{1,1}([0, T])$, *which is the Sobolev space of* L^1*-functions with weak derivatives in* $L^1([0, T])$.

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Suppose Assumptions 2.2 *and* 2.6 *are true for some* $\kappa > 1$, *then for any* $p \in [1, \infty)$ *we obtain*

$$C_{\pm}^{(n_k)}(\cdot, r) \to C_{\pm}(\cdot, r) \quad \text{in } L^p([0, T])$$

and the convergence is uniform in $r \in [0, L]$.

Numerical results on the convergence are presented in § 6.

Remark 3.4 *If, for example, the initial condition* $h_0 \in L^2([0, L])$ *is deterministic (e.g.* $h_0 = 0$, *as in* (1.4)*), then Assumption* 2.6 *is true for any* κ, *and* (3.3) *holds in any* $L^p([0, T])$ *with* $p \geqslant 1$.

Using the compact embedding of $W^{1,1}([0, T])$ *in* $L^p([0, T])$ *for arbitrary* $p \geqslant 1$ *(cf. Adams* [1, Theorem 6.2]*), we obtain again the same result provided Assumption* 2.6 *is true for* $\kappa = \frac{3}{2}$. *But this argument relies on* $\mathrm{Tr}(Q) < \infty$ *(cf. Remark* 5.1*).*

It will be the topic of further research to check other numerical algorithms, for example finite differences as used in Raible *et al.* [32]. The results of Raible *et al.* [32, 30] even suggest that w_n converges to w in stronger topologies for space-time white noise and in dimension $d = 2$, i.e. we consider (1.4) on $[0, L]^2$ instead of $[0, L]$.

Remark 3.5 *If we additionally suppose* $\partial_t w_{n_k} \geqslant 0$ *on* $[0, T]$, *which is often true in numerical simulations (see* § 6 *or Raible* et al. [30]*), we get better results. For example under Assumptions* 2.2 *and* 2.6 *with* $\kappa = \frac{7}{3}$, *it is straightforward to verify that we can improve* (5.5) *and* (5.6) *to derive a uniform bound on* $\{w_{n_k}^2\}_{k \in \mathbb{N}}$ *in* $W^{1, \frac{8}{5}}([0, T])$. *Now Sobolev embedding (cf.* [1]*) establishes that the convergence in* (3.3) *is already uniform in* $t \in [0, T]$.

4 *A priori* estimates

This section basically provides the proof of Lemma 3.1. As the operator $\mathscr{L}$ is a positive self-adjoint operator, the operator $\mathscr{L}^{-\frac{1}{2}} : L^2([0, L]) \to L^2([0, L])$ is well-defined. As any power of $\mathscr{L}$ commutes with ∂_x, it is straightforward to verify that $\mathscr{L}^{-\frac{1}{2}}\partial_x : X \to X$ is a bounded linear operator. Define the space $X_{\mathscr{L}} = X$ equipped with the norm $\|\cdot\|_{\mathscr{L}} := \|\mathscr{L}^{-\frac{1}{2}}\partial_x \cdot\|_{L^2}$. The space $X_{\mathscr{L}}$ is a Hilbert space with inner product $< \cdot, \cdot >_{\mathscr{L}} := < \mathscr{L}^{-\frac{1}{2}}\partial_x \cdot, \mathscr{L}^{-\frac{1}{2}}\partial_x \cdot >_{L^2}$.

It is straightforward to verify that $\|\cdot\|_{\mathscr{L}}$ and $\|\cdot\|_{L^2}$ are equivalent norms on X. Moreover, $\|\cdot\|_{\mathscr{L}} + |\int_0^L \cdot\, dx|$ and $\|\cdot\|_{L^2}$ are equivalent norms on $L^2([0, L])$. For a shorthand notation we will extend the mappings $u \to \|u\|_{\mathscr{L}}$ and $(u_1, u_2) \to < u_1, u_2 >_{\mathscr{L}}$ to the entire space $L^2([0, L])$. On this space, $\|\cdot\|_{\mathscr{L}}$ is no longer a norm (due to $\|u\|_{\mathscr{L}} = \|u - \overline{u}\|_{\mathscr{L}}$ for any function $u \in L^2([0, L])$), but the Cauchy–Schwarz inequality still holds. Furthermore, note that $\overline{u} = 0$ for any $u \in X$ and $\overline{u - \overline{u}} = 0$ for any $u \in L^2([0, L])$. Note that in the remainder of this section, constants C may vary from line to line, but do not depend on n and α.

In the following we will use that for arbitrary functions $u \in V^{(n)}$ (i.e. $P^{(n)}u = u$)

$$\begin{aligned} < P^{(n)}\mathscr{L}(\partial_x u)^2, u >_{\mathscr{L}} &= < \partial_x(\partial_x u)^2, \partial_x u >_{L^2} \\ &= -\int_0^L (\partial_x u)^2 \partial_x^2 u\, dx = 0. \end{aligned} \tag{4.1}$$

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Because this result is the crucial point, everything is restricted to $d = 1$. Moreover,

$$\begin{aligned} < Au, u >_{\mathscr{L}} &= -a_1 \|\partial_x^2 u\|_{\mathscr{L}}^2 + a_2 \|\partial_x u\|_{\mathscr{L}}^2 \\ &\leqslant -a_1 \|\partial_x^2 u\|_{\mathscr{L}}^2 + C\|\partial_x(u - \overline{u})\|_{L^2}^2 \\ &\leqslant -\frac{a_1}{2}\|\partial_x^2 u\|_{\mathscr{L}}^2 + C\|u - \overline{u}\|_{L^2}^2 \\ &\leqslant -\frac{a_1}{2}\|\partial_x^2 u\|_{\mathscr{L}}^2 + C\|u\|_{\mathscr{L}}^2, \end{aligned} \tag{4.2}$$

where we used an interpolation inequality [1] and the equivalence of norms. We note that it is possible to establish (4.2) with constant $C = a_2^2/(2a_1)$.

We now take the $X_{\mathscr{L}}$-inner product of $v^{(n)}$ and (2.4). For simplicity, we suppress the arguments t and x in the notation. We obtain using (4.1)

$$\begin{aligned} \frac{1}{2}\partial_t \|v^{(n)}\|_{\mathscr{L}}^2 &= < v^{(n)}, \partial_t v^{(n)} >_{\mathscr{L}} \\ &= < Av^{(n)}, v^{(n)} >_{\mathscr{L}} + \alpha \cdot < W_{A_\alpha}^{(n)}, v^{(n)} >_{\mathscr{L}} \\ &\qquad + < P^{(n)} \mathscr{L}[(\partial_x v^{(n)} + \partial_x W_{A_\alpha}^{(n)})^2], v^{(n)} >_{\mathscr{L}} \\ &= < Av^{(n)}, v^{(n)} >_{\mathscr{L}} + \alpha \cdot < W_{A_\alpha}^{(n)}, v^{(n)} >_{\mathscr{L}} \\ &\quad -2\int_0^L \partial_x v^{(n)} \partial_x W_{A_\alpha}^{(n)} \partial_x^2 v^{(n)} dx - \int_0^L (\partial_x W_{A_\alpha}^{(n)})^2 \partial_x^2 v^{(n)} dx. \end{aligned} \tag{4.3}$$

Now Lemma 4.1 of Blömker & Gugg [5] implies for sufficiently small $\varepsilon > 0$ and some constant C_ε depending on ε

$$\left| \int_0^L \partial_x v^{(n)} \partial_x W_{A_\alpha}^{(n)} \partial_x^2 v^{(n)} dx \right| \leqslant \varepsilon \|\partial_x^2 v^{(n)}\|_{\mathscr{L}}^2 + C_\varepsilon (1 + M_{1,n}^4) \|v^{(n)}\|_{\mathscr{L}}^2, \tag{4.4}$$

where

$$M_{l,n} := \sup\{\|\partial_x^l (W_{A_\alpha}^{(n)}(t))\|_{L^\infty} : t \in [0, T]\} \tag{4.5}$$

for $l = 0, 1, 2$. This result is a straightforward but technical calculation using Cauchy–Schwarz and an interpolation inequalities together with Sobolev embeddings.

We use this together with (4.2), Cauchy–Schwarz, Young and an interpolation inequality, and the equivalence of norms to obtain from (4.3)

$$\frac{1}{2}\partial_t \|v^{(n)}\|_{\mathscr{L}}^2 \leqslant -c\|\partial_x^2 v^{(n)}\|_{\mathscr{L}}^2 + C\|v^{(n)}\|_{\mathscr{L}}^2 (1 + M_{1,n}^4) + C(\alpha^2 M_{0,n}^2 + M_{1,n}^4) \tag{4.6}$$

for arbitrary constant $c \in (0, \frac{a_1}{2})$.

Now Gronwall's lemma (Henry [17, Lemma 7.1.1]) implies

$$\|v^{(n)}\|_{\mathscr{L}}^2 \leqslant C(\|h_0^{(n)}\|_{\mathscr{L}}^2 + \alpha^2 M_{0,n}^2 + M_{1,n}^4) e^{C(1+M_{1,n}^4)} \tag{4.7}$$

and

$$\begin{aligned} &\int_0^T \|\partial_x^2 v^{(n)}\|_{L^2}^2 dt \leqslant C \int_0^T \|\partial_x^2 v^{(n)}\|_{\mathscr{L}}^2 dt \\ &\leqslant C(\|h_0^{(n)}\|_{\mathscr{L}}^2 + \alpha^2 M_{0,n}^2 + M_{1,n}^4)(1 + M_{1,n}^4) e^{C(1+M_{1,n}^4)}. \end{aligned} \tag{4.8}$$

To furnish a bound on $\int_0^L v^{(n)} dx$, we take the L^2-inner product of (2.4) with the constant

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function. We obtain

$$\partial_t \int_0^L v^{(n)} dx = \int_0^L \left[A v^{(n)} + \alpha W_{A_\alpha}^{(n)} + P^{(n)} \mathscr{L}(\partial_x v^{(n)} + \partial_x W_{A_\alpha}^{(n)})^2 \right] dx$$
$$= \int_0^L \alpha W_{A_\alpha}^{(n)} dx + \int_0^L a_4 (\partial_x v^{(n)} + \partial_x W_{A_\alpha}^{(n)})^2 dx. \tag{4.9}$$

Hence,

$$\left| \int_0^L v^{(n)} dx \right| \leqslant \int_0^L |v^{(n)}(0)| dx + \int_0^T \int_0^L |\alpha W_{A_\alpha}^{(n)}| + a_4 (\partial_x v^{(n)} + \partial_x W_{A_\alpha}^{(n)})^2 dx dt$$
$$\leqslant C \cdot \left[\|h_0^{(n)}\|_{L^2} + \int_0^T \|\partial_x v^{(n)}\|_{\mathscr{L}}^2 dt + M_{1,n}^2 + \alpha M_{0,n} \right]. \tag{4.10}$$

Using (4.8),

$$\left| \int_0^L v^{(n)} dx \right|^2 \leqslant C \cdot \Big[\|h_0^{(n)}\|_{L^2}^2 + M_{1,n}^4 + \alpha^2 M_{0,n}^2$$
$$+ (\|h_0^{(n)}\|_{L^2}^4 + M_{1,n}^8 + \alpha^4 M_{0,n}^4)(1 + M_{1,n}^8) e^{C(1+M_{1,n}^4)} \Big]$$
$$\leqslant C \cdot \left[1 + \|h_0^{(n)}\|_{L^2}^4 + \alpha^4 M_{0,n}^4 + M_{1,n}^8 \right] \cdot [1 + M_{1,n}^8] e^{C(1+M_{1,n}^4)}.$$

This, together with (4.7), implies

$$\|v^{(n)}\|_{L^2}^2 \leqslant C(1 + \|h_0^{(n)}\|_{L^2}^4 + \alpha^4 M_{0,n}^4 + M_{1,n}^8)(1 + M_{1,n}^8) e^{C(1+M_{1,n}^4)}. \tag{4.11}$$

Now we use that $\|h_0^{(n)}\|_{L^2}^2 \leqslant \|h_0\|_{L^2}^2$, and that $E M_{1,n}^4 < C$ and $E \ln^+(M_{0,n}) < C$ for some constant C independent of n (this is proved similarly to Gugg *et al.* [15, Corollary 2.3]). Taking the ln in (4.8) and (4.11) easily implies the first assertion of Lemma 3.1.

To prove the second assertion of Lemma 3.1 let Assumption 2.2 be true. Using the higher regularity of W_{A_α} we obtain by integration by parts

$$\left| \int_0^L \partial_x v^{(n)} \partial_x W_{A_\alpha} \partial_x^2 v^{(n)} dx \right| \leqslant \frac{1}{2} \int_0^L |\partial_x^2 W_{A_\alpha}| |\partial_x v^{(n)}|^2 dx$$
$$\leqslant C M_{2,n} \|\partial_x v^{(n)}\|_{\mathscr{L}}^2$$
$$\leqslant C_\varepsilon M_{2,n}^2 \|v^{(n)}\|_{\mathscr{L}}^2 + \varepsilon \|\partial_x^2 v^{(n)}\|_{\mathscr{L}}^2, \tag{4.12}$$

where we used an interpolation inequality in the last step. In the following, we proceed analogously to the first part. Hence,

$$\frac{1}{2} \partial_t \|v^{(n)}\|_{\mathscr{L}}^2 + c \|\partial_x^2 v^{(n)}\|_{\mathscr{L}}^2 \leqslant C \|v^{(n)}\|_{\mathscr{L}}^2 (1 + M_{2,n}^2) + C(\alpha^2 M_{0,n}^2 + M_{1,n}^4) \tag{4.13}$$

and Gronwall's lemma implies

$$\|v^{(n)}\|_{\mathscr{L}}^2 \leqslant C(\|h_0^{(n)}\|_{\mathscr{L}}^2 + \alpha^2 M_{0,n}^2 + M_{1,n}^4) e^{C(1+M_{2,n}^2)}.$$

Moreover,

$$\int_0^T \|\partial_x^2 v^{(n)}\|_{L^2}^2 dt \leqslant C(\|h_0^{(n)}\|_{\mathscr{L}}^2 + \alpha^2 M_{0,n}^2 + M_{1,n}^4)(1 + M_{2,n}^2) e^{C(1+M_{2,n}^2)}. \tag{4.14}$$

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This, together with (4.10), yields

$$|\int_0^L v^{(n)}dx|^2 \leqslant C(1+\|h_0^{(n)}\|_{L^2}^4+\alpha^4 M_{0,n}^4+M_{1,n}^8)(1+M_{2,n}^4)e^{C(1+M_{2,n}^2)} \tag{4.15}$$

and finally,

$$\|v^{(n)}\|_{L^2}^2 \leqslant C(1+\|h_0^{(n)}\|_{L^2}^4+\alpha^4 M_{0,n}^4+M_{1,n}^8)(1+M_{2,n}^4)e^{C(1+M_{2,n}^2)}. \tag{4.16}$$

Similarly to Gugg [14, Theorem 4.12] or the proof of Gugg *et al.* [15, Theorem 2.5], we ascertain that for fixed constant $C_0 > 0$ there exists a (sufficiently large) $\alpha > 0$ such that $Ee^{C_0 M_{2,n}^2} < C_1$ with constant C_1 independent of $n \in \mathbb{N}$. Now some straightforward calculations yield the second assertion of the theorem.

5 Convergence

This section basically provides the proof of Theorem 3.3.

Let $u_k := \tilde{h}^{(n_k)} - \overline{\tilde{h}^{(n_k)}}$, where $\tilde{h}^{(n_k)} := \tilde{v}^{(n_k)} + \tilde{W}_{A_\alpha}^{(n_k)}$ in the notation of Theorem 3.2. Furthermore, set $u = \tilde{h} - \frac{1}{L}\int_0^L \tilde{h}dx$. From Theorem 3.2 we obtain that $u_k \to u$ in $L^2([0,T],L^2([0,L]))$, $\tilde{P}$—a. s. Moreover, (3.2) of Lemma 3.1 implies $\tilde{E}[\int_0^T \|u_k\|_{L^2}^2 dt]^{1+\delta} \leqslant K$ uniformly in $k \in \mathbb{N}$ for some small $\delta > 0$. Fatou's lemma easily yields

$$\tilde{E}\|u\|_{L^2(0,T,L^2)}^{2(1+\delta)} = \tilde{E}[\int_0^T \|u\|_{L^2}^2 dt]^{1+\delta} \leqslant K.$$

By a well known argument of Vitali, we conclude

$$\tilde{E}\|u_k - u\|_{L^2(0,T,L^2)}^2 \overset{k\to\infty}{\longrightarrow} 0. \tag{5.1}$$

The Cauchy–Schwarz inequality implies

$$\begin{aligned}\int_0^T |\tilde{E}\|u_k(t)\|_{L^2}^2 - \tilde{E}\|u(t)\|_{L^2}^2|dt &\leqslant \int_0^T \tilde{E}\|u_k(t)-u(t)\|_{L^2}\|u_k(t)+u(t)\|_{L^2}dt \\ &\leqslant 2K^{\frac{1}{2}}\left(\int_0^T \tilde{E}\|u_k(t)-u(t)\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &\to 0 \qquad \text{for} \quad k\to\infty.\end{aligned}$$

Hence,

$$\tilde{E}\|u_k\|_{L^2}^2 \overset{k\to\infty}{\longrightarrow} \tilde{E}\|u\|_{L^2}^2 \qquad \text{in} \quad L^1([0,T]). \tag{5.2}$$

We obtain by Lemma 3.1 for any $t \in [0,T]$

$$\tilde{E}\|u_k(t)\|_{L^2}^2 \leqslant \tilde{E}\sup_{t\in[0,T]}\|u_k(t)\|_{L^2}^2 \leqslant C \in L^\infty([0,T]) \tag{5.3}$$

uniformly in $k \in \mathbb{N}$. Using (5.2) and (5.3), the same bound holds for the limit $\tilde{E}\|u(t)\|_{L^2}^2$ but only for almost all $t \in [0,T]$. As the sequence is uniformly bounded in $L^\infty([0,T])$ and convergent in $L^1([0,T])$, we readily derive the asserted convergence in $L^p([0,T])$ of Theorem 3.3.

To prove the second assertion of Theorem 3.3, we define $u_k := h^{(n_k)} - \overline{h^{(n_k)}}$ from

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Theorem 3.2. We immediately obtain that u_k is the weak solution of

$$du_k = \left[Au_k + P^{(n_k)} \mathscr{L}(\partial_x u_k)^2 - \frac{1}{L} a_4 \|\partial_x u_k\|_{L^2}^2 \right] dt + d[P^{(n_k)} W - \alpha_0 \beta_0 f_0]$$

Now Ito's formula (cf. Øksendahl [27, Theorem 4.6]) implies

$$\begin{aligned} & E\|u_k(t)\|_{L^2}^2 - E\|u_k(0)\|_{L^2}^2 \\ &= 2E \int_0^t < Au_k + P^{(n_k)} \mathscr{L}(\partial_x u_k)^2 - \frac{1}{L} a_4 \|\partial_x u_k\|_{L^2}^2, u_k >_{L^2} dt + t[\mathrm{Tr}(Q_k) - \alpha_0^2] \\ &= 2 \cdot \int_0^t \left[a_2 E\|\partial_x u_k\|_{L^2}^2 - a_1 E\|\partial_x^2 u_k\|_{L^2}^2 + a_4 E < (\partial_x u_k)^2, u_k >_{L^2} \right] dt \\ &\qquad + t[\mathrm{Tr}(Q_k) - \alpha_0^2], \end{aligned} \tag{5.4}$$

where $Q_k := P^{(n_k)} Q$, and where we used $\overline{u_k} = 0$. Hence,

$$\begin{aligned} \partial_t E\|u_k\|_{L^2}^2 &= 2a_2 E\|\partial_x u_k\|_{L^2}^2 - 2a_1 E\|\partial_x^2 u_k\|_{L^2}^2 \\ &\quad + 2a_4 E < (\partial_x u_k)^2, u_k >_{L^2} + \mathrm{Tr}(Q_k) - \alpha_0^2. \end{aligned} \tag{5.5}$$

It remains to prove that the right-hand side of (5.5) is an L^1-function. The proof that $E\|u_k\|_{L^2}^2$ is in $L^1([0,T])$ is similar. First, Assumption 2.2 implies that $\mathrm{Tr}(Q_k) \nearrow \mathrm{Tr}(Q) < \infty$. Together with Lemma 3.1, we readily deduce that $E\|\partial_x^2 u_k\|_{L^2}^2$ and $E\|\partial_x u_k\|_{L^2}^2$ are w.r.t. k uniformly bounded in $L^1([0,T])$. Finally, we obtain from an interpolation inequality [28, Theorem 2.6.10]

$$\|u_k\|_{H^{\frac{1}{4}}}^2 \leqslant C\|u_k\|_{H^2}^{\frac{1}{4}} \|u_k\|_{L^2}^{\frac{7}{4}}.$$

Hence, using the embedding of $H^{\frac{1}{4}}([0,L])$ in $L^4([0,L])$ [1, Theorem 7.57] and integration by parts,

$$\begin{aligned} |E < (\partial_x u_k)^2, u_k >_{L^2} | &\leqslant \frac{1}{2} E\|u_k\|_{H^2} \|u_k\|_{L^4}^2 \\ &\leqslant CE\|u_k\|_{H^2}^{\frac{5}{4}} \|u_k\|_{L^2}^{\frac{7}{4}}. \end{aligned} \tag{5.6}$$

Now the Young and Hölder inequalities imply

$$\begin{aligned} \int_0^T |E < (\partial_x u_k)^2, u_k >_{L^2} | dt &\leqslant CE \sup_{t\in[0,T]} \|u_k\|_{L^2}^3 + CE \left(\int_0^T \|u_k\|_{H^2}^{\frac{5}{4}} dt \right)^{\frac{12}{5}} \\ &\leqslant CE \sup_{t\in[0,T]} \|u_k\|_{L^2}^3 + CE \left(\int_0^T \|u_k\|_{H^2}^2 dt \right)^{\frac{3}{2}} \end{aligned}$$

which is w.r.t. k uniformly bounded, where we used Theorem 3.1 with $\kappa = \frac{3}{2}$. Hence, we established that $\{E\|u_k\|_{L^2}^2\}_{k\in\mathbb{N}}$ is uniformly bounded in $W^{1,1}([0,T])$. This implies the second assertion of Theorem 3.3.

Remark 5.1 *The result of the last part of the proof immediately implies convergence in any L^p, as discussed in Remark* 3.4. *But it relies on Ito's formula, which is only available in the case* $\mathrm{Tr}(Q) < \infty$. *The first part of the proof is based mainly on the a priori estimate. If this estimate could be established for* $\mathrm{Tr}(Q) = \infty$, *which covers space-time white noise, the whole proof could be generalized to this case.*

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For the correlation function, as in the first part we define $u_k := \tilde{h}^{(n_k)} - \overline{\tilde{h}^{(n_k)}}$. Using the periodicity of u_k and u we obtain

$$\begin{aligned}
&\left|\tilde{E}\int_0^L u_k(t,x)u_k(t,x+r)dx - \tilde{E}\int_0^L u(t,x)u(t,x+r)dx\right| \\
&\leqslant \tilde{E}\int_0^L |u_k(t,x)u_k(t,x+r) - u(t,x)u(t,x+r)|\,dx \\
&\leqslant \tilde{E}[\|u_k(t)\|_{L^2}\|\,\|u_k(t)-u(t)\|_{L^2}] + \tilde{E}[\|u(t)\|_{L^2}\|\,\|u_k(t)-u(t)\|_{L^2}]. \qquad (5.7)
\end{aligned}$$

From the beginning of the section we infer $\tilde{E}\|u\|^2_{L^2(0,T,L^2)} \leqslant C$ and $\tilde{E}\|u_k\|^2_{L^2(0,T,L^2)} \leqslant C$ with a constant independent of k. Now the Cauchy–Schwarz inequality implies

$$\begin{aligned}
&\int_0^T \left|\tilde{E}\int_0^L u_k(t,x)u_k(t,x+r)dx - \tilde{E}\int_0^L u(t,x)u(t,x+r)dx\right| dt \\
&\leqslant C\left(\tilde{E}\|u_k-u\|^2_{L^2(0,T,L^2)}\right)^{\frac{1}{2}} \to 0 \quad \text{for} \quad k\to\infty, \qquad (5.8)
\end{aligned}$$

where we used (5.1). From (5.3) we obtain using the periodicity of u_k

$$\left|\tilde{E}\int_0^L u_k(t,x)u_k(t,x+r)dx\right| \leqslant \tilde{E}\|u_k(t)\|^2_{L^2} \leqslant C, \qquad (5.9)$$

and the same bound holds for u, but now only for almost all $t \in [0,T]$. Note that the constant is independent of r. From (5.8) and (5.9) we readily obtain for fixed $r \in [0,L]$ that

$$\tilde{E}\int_0^L u_k(\cdot,x)u_k(\cdot,x+r)dx \to \tilde{E}\int_0^L u(\cdot,x)u(\cdot,x+r)dx$$

in $L^p([0,T])$ for any $p \in [1,\infty)$, and the convergence is uniform w.r.t. r. This immediately implies

$$C_+^{(n_k)}(\cdot,r) \to C_+(\cdot,r)$$

in $L^p([0,T])$ uniformly in r for any $p \in [1,\infty)$.

6 Numerical results

To determine whether it is possible to derive better results on the convergence of the statistical quantities w_n and $C^{(n)}$ than those of Theorem 3.3, we did some numerical simulations of (1.4) using the spectral Galerkin method, because our analytical result shows convergence only for a subsequence. There are already a few simulations (cf. Mayr *et al.* [23] or Raible *et al.* [32]) using finite difference methods to solve (1.3) and (1.4), but to our knowledge there is no rigorous result that justifies the use of this numerical method for our equation. However, as indicated by the numerical results the method seems to work quite well. For (1.2) there are a lot of simulations, but all results available for the stochastic version of this equation use finite difference methods (e.g. see Cuerno *et al.* [10] and Drotar *et al.* [12]).

For the finite difference method the interval is split by N equidistant points into subintervals, and a function is represented by N discrete values. Especially the noise is approximated by means over these subintervals, which gives well-defined real-valued random variables. The derivatives and the Laplacian are discretized using standard finite

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difference approximations. See, for example, Raible *et al.* [31] for a detailed presentation of numerical methods for (1.4) in dimension two. Here also finite element methods are briefly discussed.

Finite difference and spectral Galerkin methods both seem to work quite well. It is the experience of the authors that for (1.4) spectral Galerkin methods require less degrees of freedom than finite difference methods, with the exception of small times where many modes are involved due to the noise. A further advantage of the spectral Galerkin method is the possibility to implement a semi-implicit time discretization (implicit in the linear parts of the equation) which allows larger time steps in comparison to an explicit scheme in the time range that is dominated by the linear terms of (1.4). On the other hand, the main disadvantage of the spectral Galerkin method is the costly fast Fourier transform which we use to calculate the nonlinearity in (1.4). Therefore, for the same numerical accuracy, we need for both methods roughly the same order of magnitude in computation time on a PC.

To implement the spectral Galerkin method we solve the following system of the stochastic ordinary differential equations (e.g. see Kloeden *et al.* [18] for literature).

$$\partial_t h^{(n)} = -a_1 \partial_x^4 h^{(n)} - a_2 \partial_x^2 h^{(n)} + P^{(n)}(-a_3 \partial_x^2 + a_4)(\partial_x h^{(n)})^2 + P^{(n)} \xi, \tag{6.1}$$

$$h^{(n)}(0) = 0.$$

For the time discretization we applied a semi-implicit Euler scheme, which was implicit in the linear part $(-a_1 \partial_x^4 - a_2 \partial_x^2) h^{(n)}$ of the right-hand side of equation (6.1) and explicit in the nonlinear part $P^{(n)} \mathscr{L} (\partial_x h^{(n)})^2$. The error that was caused by the finite time discretization was estimated by a comparison of the results of two simultaneous computations using time steps of single and double length. The nonlinearity $P^{(n)} \mathscr{L} (\partial_x h^{(n)})^2$ was calculated using the Fast Fourier Transform on M equidistant grid points on the interval $[0, L]$. M was a power of 2 and satisfied the condition $M \geqslant 3n + 1$. We applied space-time white noise, i.e. $\alpha_k = \sqrt{2D} = Const.$ for any $k \in \mathbb{Z}$, yielding $E\xi(t, x)\xi(s, y) = 2D\delta(t - s)\delta(x - y)$. The chosen parameters were $L = 200$, $a_1 = 0.4044$, $a_2 = 0.1045$, $a_3 = 0.13$, $a_4 = 0.07$, and $D = 0.0035$. Note that the coefficients a_i of the equation are up to a rescaling of time given by the experiments [32].

Figure 1 shows the resulting interface width w_n for various values of n on a time interval $[0, 1000]$. The numbers of simulation runs were $R = 40$, 39 100, 136 400, 155 000 and 142 348 for $n = 21$, 26, 32, 42 and 85, respectively. Note that we actually compute $2n + 1$ modes.

The interface width w_n is obviously in very good agreement for n =42 and 85. On the other hand, $w_{21}(t)$ and $w_{26}(t)$ are too large at larger time t. The deviation between $w_{42}(t)$ and $w_{85}(t)$ was not larger than $2.1 \cdot 10^{-3} \cdot w_{85}(t)$ for $t \in [60, 1000]$ and not larger than $1.7 \cdot 10^{-2} \cdot w_{85}(t)$ for t =5, 15, and 30. The relative inaccuracies of w_{42} and w_{85} due to the finite time discretization were not larger than $9 \cdot 10^{-4}$ and due to the finite number of simulation runs were not larger than $4 \cdot 10^{-4}$.

We also estimated the deviation between w_{85} and w_{170} and between w_{170} and w_{341}. However, we were not able to determine w_{170} and w_{341} with a sufficient accuracy of the time discretization and a sufficiently large number of simulation runs at the same time. Therefore, we bypassed the latter problem by a simultaneous calculation of w_{85} and

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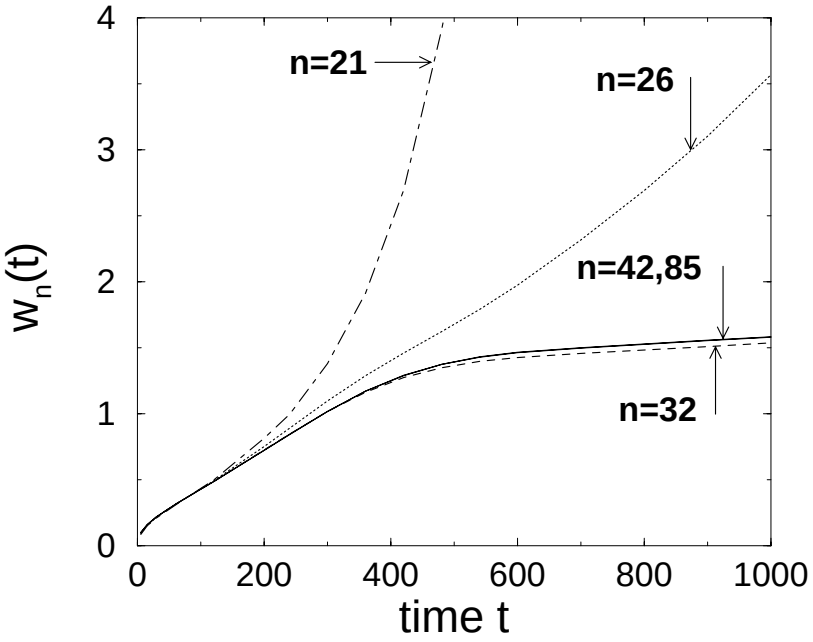


FIGURE 1. Interface width $w_n(t)$ resulting from equation (6.1) for various values of n: n =21 (dash-dotted line), n =26 (dotted line), n =32 (dashed line), n =42 and 85 (solid lines). The parameters are given in the text.

w_{170} using the *same* R =247 realizations of the noise ξ. The resulting deviation between $w_{85}(t)$ and $w_{170}(t)$ was not larger than $1.9 \cdot 10^{-4} \cdot w_{170}(t)$ for $t \in [60, 1000]$ and not larger than $2.0 \cdot 10^{-3} \cdot w_{170}(t)$ for t =5, 15 and 30, whereas the relative inaccuracies of w_{85} and w_{170} due to the finite length of the time steps were not larger than $5.7 \cdot 10^{-5}$ and due to the finite number of simulation runs were not larger than $1.0 \cdot 10^{-2}$. The latter does not affect our estimation of the deviation between w_{85} and w_{170}, since we used the same 247 simulation runs. The difference between the interface widths $w_{170}(t)$ and $w_{341}(t)$, that resulted from R =50 simultaneous simulation runs, was not larger than $2.6 \cdot 10^{-5} \cdot w_{341}(t)$ for $t \in [60, 1000]$ and not larger than $2.5 \cdot 10^{-4} \cdot w_{341}(t)$ for t =5, 15 and 30, whereas the relative inaccuracies of $w_{170}(t)$ and $w_{341}(t)$ due to the finite time discretization were not larger than $7.7 \cdot 10^{-6}$. So the difference between w_{85} and w_{170} is about 10 times smaller than the difference between w_{42} and w_{85}, and the difference between w_{170} and w_{341} is approximately by a factor of 7–8 smaller than the difference between w_{85} and w_{170}. Therefore, our numerical results suggest a convergence of $w_n(t)$ uniformly in $t \in [0, 1000]$. Additionally, our numerical simulations confirmed, that the condition $\partial_t w_n \geqslant 0$, that was mentioned in Remark 3.5, is satisfied on the time interval $[0, 1000]$.

The height-height correlation functions $C^{(n)}(t, r)$ at the times $t = 100$ and $t = 1000$ that resulted from the same simulations of equation (6.1) for various values of n are shown in the Figures 2 and 3. They are obviously in good agreement for n =42 and 85. So it seems that $C^{(n)}(t, r)$ converges for the chosen times $t = 100$ and $t = 1000$ uniformly in $r \in [0, L/2]$.

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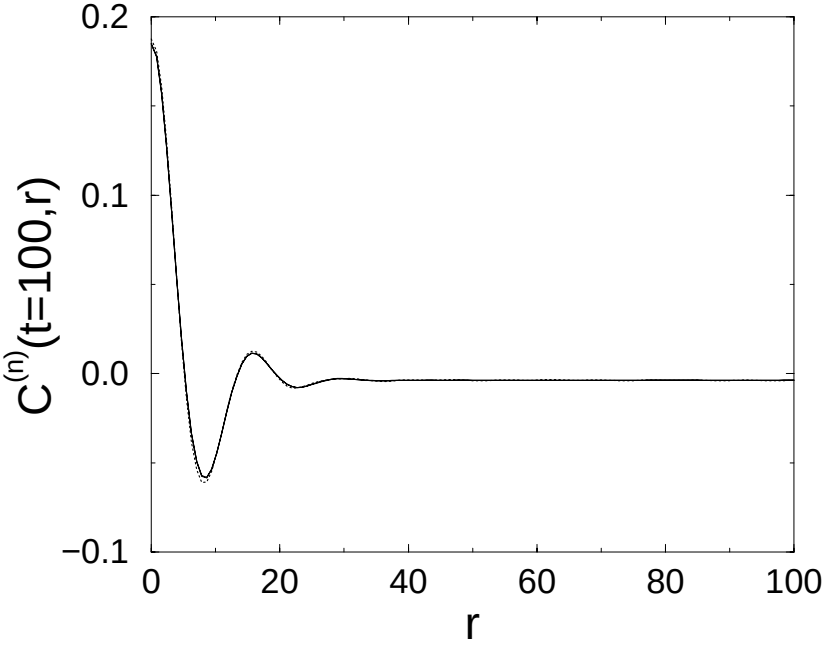


FIGURE 2. Height-height correlation function $C^{(n)}(t,r)$ at the time $t = 100$ resulting from equation (6.1) for various values of n: n =26 (dotted line), n =42 and 85 (solid lines). The parameters are given in the text.

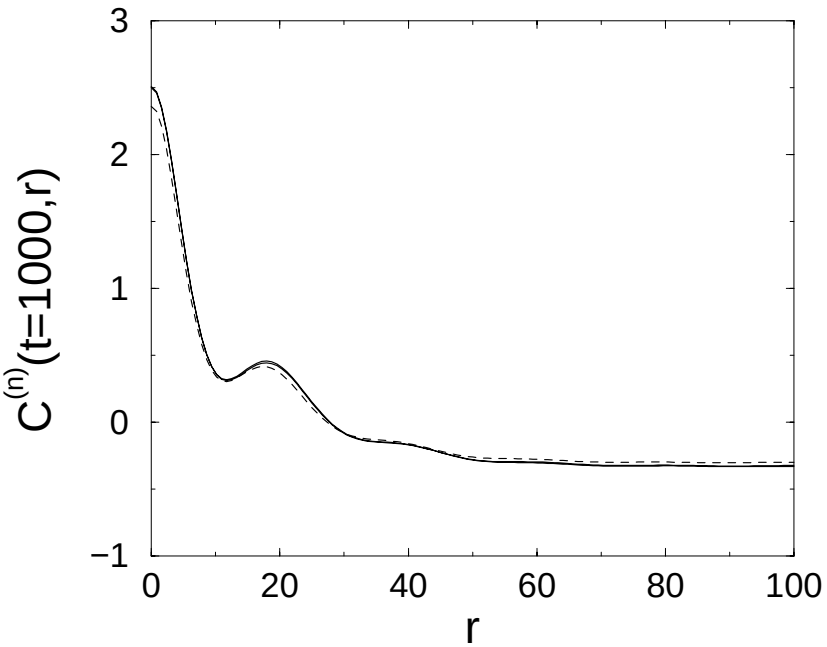


FIGURE 3. Height-height correlation function $C^{(n)}(t,r)$ at the time $t = 1000$ resulting from equation (6.1) for various values of n: n =32 (dashed line), n =42 and 85 (solid lines). The parameters are given in the text.

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7 Conclusions

In a one-dimensional model arising in surface growth of thin amorphous films, we use the spectral Galerkin method to verify the existence of a martingale solution of the corresponding mild formulation. The uniqueness of solutions is still an open problem, mainly, because it is difficult to derive useful *a priori* bounds due to the quadratic nonlinearity.

As an important result for the application we consider the convergence of the numerical approximations of the mean interface width and the height-height correlation function, which are used to compare theory with experiments. Our analytical result shows the convergence for the subsequence arising in the proof of existence of solutions.

Finally, we present numerical results indicating that the statistical quantities determined by the approximations actually converge. But this is out of reach for an analytical treatment due to the possible nonuniqueness of solutions.

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