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Non-repetitive sequences

By P. A. B. PLEASANTS
University College, Cardiff

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1. This note is concerned with infinite sequences whose terms are chosen from a finite set of symbols. A *segment* of such a sequence is a set of one or more consecutive terms, and a *repetition* is a pair of finite segments that are adjacent† and identical. A *non-repetitive* sequence is one that contains no repetitions.

A sequence on two symbols with four or more terms always contains a repetition, but N. G. de Bruijn (unpublished) constructed an infinite non-repetitive sequence on three symbols. Such sequences have also been constructed by R. Peltesohn and Sutherland (unpublished), Morse and Hedlund (1), Hawkins and Mientka (2), B. J. Birch (unpublished), and Leech (3). Erdös (see (4), p. 240) has raised the question of the maximum length of a sequence on k symbols such that no two adjacent finite segments are permutations of each other. Such a sequence might be called *strongly non-repetitive*. For $k \leqslant 3$ this maximum length is $2^k - 1$; but for $k = 4$ there are strongly non-repetitive sequences that are longer than this (the longest known so far was found by M. R. Bird using the Chilton Atlas computer and has over 1600 terms) and it is conjectured that there exist infinite strongly non-repetitive sequences on four symbols.

The main purpose of this note is to construct an infinite sequence on five symbols that is strongly non-repetitive. We also give a construction,‡ which is slightly simpler than previous ones, for an infinite non-repetitive sequence on three symbols. Both constructions are the same in outline as those given in (2) and (3). The non-repetitive sequence is built up from three blocks of terms, these blocks being chosen in such a way that when they are substituted for the three symbols in any finite non-repetitive sequence the resulting sequence is still non-repetitive. Starting with a sequence consisting of only one term and applying this process repeatedly, we obtain an infinite chain of non-repetitive sequences each of which strictly contains its predecessor, and in the limit this gives an infinite non-repetitive sequence. In the same way, five blocks of terms are used to build up an infinite, strongly non-repetitive sequence on five symbols. It is possible that the same method could be used to construct an infinite, strongly non-repetitive sequence on four symbols, but a great deal of calculation would be necessary.

2. In this section we prove

THEOREM 1. *There exists a non-repetitive sequence on three symbols that extends to infinity in both directions.*

† That is, abutting but not overlapping.

‡ *Added in proof.* M. R. Bird tells me that he too has found an almost identical construction.

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Taking the three symbols to be a, b and c, we shall use the following three blocks of terms to build up the sequence:

$$A = abcab, \quad B = acabcb, \quad C = acbcacb.$$

Each of these blocks consists of two half-blocks, each headed by an a, and the six half-blocks are all different. It follows that in any sequence built up from these blocks the terms between two consecutive a's completely determine to what block the first of the a's belongs and whether it is the first or the second a in that block.

For any finite sequence, S, on the symbols a, b, c, $\mathscr{E}(S)$ will denote the sequence got by putting the blocks A, B, C in place of the symbols a, b, c, respectively, everywhere in S. Thus $\mathscr{E}(S)$ is made up entirely of these blocks. We shall show that $\mathscr{E}(S)$ is non-repetitive whenever S is. In what follows x, y and z will be used as unspecified symbols and X, Y and Z will denote the corresponding blocks.

Let S be a sequence such that $\mathscr{E}(S)$ contains a repetition, and let T_1 and T_2 be adjacent, identical segments of $\mathscr{E}(S)$. We need to show that S contains a repetition. Suppose, first of all, that two or more a's occur in T_1, and let X be the block the first a in T_1 belongs to. Then X and its position relative to T_1 are determined completely by the terms between the first two a's of T_1, and, since T_2 is the same as T_1, the first a of T_2 also belongs to a block X which is similarly situated relative to T_2. If these two blocks X are adjacent then S contains the repetition xx. If they are not adjacent let the blocks between them be $Y_1, Y_2, \ldots, Y_n$. The block Y_n may overlap T_2, but all the a's of the blocks $Y_1, \ldots, Y_n$ are in T_1, and so these blocks are determined completely by the part of T_1 to the right of the block X. Hence the corresponding part of T_2 determines the same sequence of blocks, and S contains the repetition $xy_1 \ldots y_n xy_1 \ldots y_n$.

Now suppose that T_1 contains just one a. Then T_2 also contains just one a, and the two a's are consecutive in the sequence $\mathscr{E}(S)$. Since the middle a in each of the blocks A, B and C is preceded by a c and each block ends with a b, consecutive a's in the sequence $\mathscr{E}(S)$ are preceded by different symbols, and so the a's in T_1 and T_2 must be the first terms of these segments. Thus T_1 is a half-block and T_2 is a segment of the following half-block. Inspection of the blocks A, B and C now shows that either T_1 is the second half-block of A and T_2 is part of the first half-block of A or T_1 is the second half-block of C and T_2 is part of the first half-block of C. Hence S contains either the repetition aa or the repetition cc.

The only remaining possibility is that T_1 and T_2 contain no a's. In this case T_1 and T_2 must each be just one term, since there are at most three terms of the sequence $\mathscr{E}(S)$ between any two consecutive a's; but this is impossible, as no two consecutive terms of $\mathscr{E}(S)$ can be equal.

This completes the proof that $\mathscr{E}(S)$ is non-repetitive whenever S is. There is a similar proof in which b plays the part that a does here, since interchanging a and b merely reverses the order of each block.

To generate an infinite non-repetitive sequence we start with a single symbol—say a. Now $\mathscr{E}(a) = A$, and so $\mathscr{E}(a)$ contains a as an interior segment (that is, a segment which does not contain the first or the last term of $\mathscr{E}(a)$). On applying the operation $\mathscr{E}$ again we see that $\mathscr{E}^2(a)$ contains $\mathscr{E}(a)$ as an interior segment, and repeated application

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of $\mathscr{E}$ shows that, for all positive integers n, $\mathscr{E}^n(a)$ contains $\mathscr{E}^{n-1}(a)$ as an interior segment. Also, considered as a sequence with only one term, a is trivially non-repetitive. It follows that $\{a, \mathscr{E}(a), \mathscr{E}^2(a), \ldots\}$ is a chain of non-repetitive sequences each containing its predecessor as an interior segment. This chain of sequences generates a sequence which is infinite in both directions, and the infinite sequence generated in this way cannot contain a repetition, as any such repetition would be contained in $\mathscr{E}^n(a)$ for some n.

3. We now prove

THEOREM 2. *There exists a strongly non-repetitive sequence on five symbols that extends to infinity in both directions.*

We shall take the symbols to be a, b, c, d and e, and denote by σ the cyclic permutation of these symbols, so that $\sigma(a) = b$, $\sigma(b) = c$, etc. For a sequence S made up of these symbols, $\sigma(S)$ will denote the sequence obtained from S by replacing every symbol x of S by $\sigma(x)$. The five blocks used in the construction of the strongly non-repetitive sequence are:†

$$A = bacaeacadaeadab, \quad B = \sigma(A), \quad C = \sigma^2(A), \quad D = \sigma^3(A), \quad E = \sigma^4(A).$$

Each block contains 15 terms, one symbol occurring 7 times and the others twice each.

Obtaining the other blocks from the first by permuting the symbols in this way is not an essential part of the proof, but it is helpful for reducing the amount of calculation necessary. (Leech uses this device in (3).)

Analogously to the corresponding definition in section 2, we define $\mathscr{E}(S)$, for any finite sequence S on the symbols $a, \ldots, e$, as the result of replacing each term of S by its corresponding block (that is, of replacing a by A, b by B, etc.). The main step in the proof of Theorem 2 is showing that $\mathscr{E}(S)$ is a strongly non-repetitive sequence whenever S is, and this depends on the block A having been chosen to have far more occurrences of the symbol a than of any other symbol. It will be convenient to extend the meaning of '*repetition*' from now on to include pairs of adjacent segments of a sequence which are merely permutations of each other (that is, which are such that each symbol occurs as many times in one segment as in the other).

If S is a finite sequence on the symbols $a, \ldots, e$ and T is a segment of $\mathscr{E}(S)$, we denote by $n_a(T)$ the number of occurrences of a in T, by $N_A(T)$ the number of A-blocks that T overlaps, and by $M_A(T)$ the number of A-blocks that are completely contained in T. Then the following two inequalities hold, where $|T|$ is the number of terms in T:

$$n_a(T) \leqslant \tfrac{2}{15}|T| + 5N_A(T) + \tfrac{32}{15}, \tag{1}$$

$$n_a(T) \geqslant \tfrac{2}{15}|T| + 5M_A(T) - \tfrac{32}{15}. \tag{2}$$

The first two terms on the right of each inequality represent the value of $n_a(T)$ when T consists of a number of complete blocks, and the third term on the right is the greatest correction that is needed to allow for partial blocks at each end of T.

† A, B, C and $\mathscr{E}$ are not the same in this section as in the preceding one, but their use is similar.

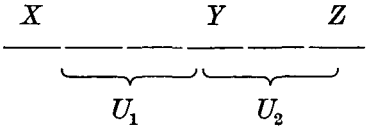
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Now suppose that two adjacent segments of $\mathscr{E}(S)$, say T_1 and T_2, are permutations of each other. We have to show that in this event S itself contains a repetition. Since T_1 and T_2 contain the same number of terms and the same number of occurrences of a, applying (1) to T_1 and (2) to T_2 and combining the inequalities obtained gives $N_A(T_1) \geqslant M_A(T_2) - 64/75$, and so, since $N_A(T_1)$ and $M_A(T_2)$ are integers, $N_A(T_1) \geqslant M_A(T_2)$. The same argument, with the roles of T_1 and T_2 reversed, shows that $N_A(T_2) \geqslant M_A(T_1)$, and it follows that if the A-blocks contained in T_1 are cancelled against A-blocks contained in T_2 as far as possible, then to each remaining A-block contained in either of these segments there corresponds at least one A-block that overlaps (but is not contained in) the other segment. The same reasoning applies to the blocks B, C, D and E, and it follows that when the blocks contained in T_1 have been cancelled against equal blocks contained in T_2 as far as possible, to each remaining block contained in either segment there corresponds a similar block overlapping, but not contained in, the other segment. The cancellation has the effect of removing certain pairs of equal terms from S to leave a subsequence, S' say, and if we denote by U_1 and U_2 the parts of T_1 and T_2 that remain after the cancellation, then U_1U_2 is a repetition in $\mathscr{E}(S')$. Also neither of the segments U_1 and U_2 contains more than two complete blocks, since no more than two blocks can be partially contained in the other segment.

The proof now reduces to a finite (but large) amount of checking, for U_1U_2 has bounded length and it is enough to show that no repetition whose length is below this bound can be derived from a strongly non-repetitive sequence S in the way described. The rest of this section consists of *ad hoc* arguments which avoid most of this checking. We divide the proof into cases according to the number of complete blocks contained in U_1 and U_2, denoting by (i,j) the case where U_1 contains i complete blocks and U_2 contains j complete blocks. The cases (i,j) with $0 \leqslant i,j \leqslant 2$ have to be considered. We may suppose that U_1 and U_2 are not empty, as the blocks cancelled from T_1 and T_2 would then correspond to a repetition in S.

Case (2,2) Corresponding to the two blocks contained in U_1 are two blocks partially contained in U_2. Let the block that overlaps the beginning of U_2 (and hence also the end of U_1) be X. Then one of the blocks contained in U_1 is equal to X, and so is one of the blocks contained in U_2. But this contradicts the assumption that equal blocks have been cancelled from U_1 and U_2 as far as possible, and so this case cannot occur.

Case (2,1) As in the previous case there are two blocks partially contained in U_2 (corresponding to the two blocks completely contained in U_1), and so the situation is as in the following diagram:†



The two blocks contained in U_1 are equal to Y and Z. The block contained in U_2 cannot be equal to Y (as in that case it would have been cancelled with the block Y

† The lines represent blocks, and X, Y and Z are not necessarily all different. This diagram and later ones are intended to include the possibility that the intersection of U_1 with the first block illustrated or of U_2 with the last block illustrated is empty.

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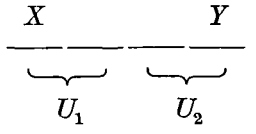
contained in U_1) and so must be equal to X. Hence the six blocks illustrated correspond to a repetition in S', which remains a repetition in S when the cancelled terms are replaced.

Case (1,2) This case can be treated in the same way as the previous case.

Cases (2,0) *and* (0,2) These cases cannot arise, since if a segment of $\mathscr{E}(S')$ contains two complete blocks any segment of equal length certainly contains at least one complete block.

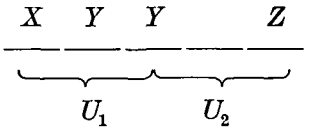
Case (1,1) Here there are four possibilities to be considered.

(i) *No block overlaps both* U_1 *and* U_2. This is illustrated by the following diagram.



The block contained in U_1 must be equal to Y and the block contained in U_2 must be equal to X. Hence S' contains the repetition $xyxy$, and when the cancelled pairs of terms are replaced S also contains a repetition.

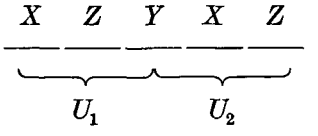
(ii) *There is a block overlapping* U_1 *and* U_2 *and it is equal to the block contained in* U_1. We now have the following diagram.



The block contained in U_2 cannot be equal to Y, or it would have been cancelled with the Y in U_1, so it must be equal to X, and this again gives rise to a repetition in S.

(iii) *There is a block overlapping* U_1 *and* U_2 *and it is equal to the block contained in* U_2. This is treated in the same way as (ii).

(iv) *There is a block overlapping* U_1 *and* U_2 *and it is equal to neither of the blocks contained in these segments*. Since each of the blocks contained in the segments U_1 and U_2 corresponds to a block partially contained in the other segment, the diagram for this situation is as follows.



If now the end of block X and the beginning of block Z are cancelled from each of U_1 and U_2, the remaining parts of U_1 and U_2 will form a repetition both halves of which overlap the block Y and which is contained in the sequence ZYX. This is impossible, however, when neither X nor Z is equal to Y, as will be shown in the treatment of Case (0,0).

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Case (1,0). In this case there is certainly a block that overlaps U_1 and U_2, and we may suppose that it is not the same as the block contained in U_1, as that would imply the existence of a repetition in S. Hence we have the following diagram.

$$\underbrace{X \quad \underbrace{Y \quad A}_{U_1} \quad \underbrace{Y}_{U_2}}$$

(By applying a suitable power of the permutation σ to the sequence we may assume, without loss of generality, that the block overlapping U_1 and U_2 is A.) We may also suppose that $X \neq A$, as that would again lead to a repetition in S.

We now cancel the part of Y that overlaps U_2 from each of U_1 and U_2, and denote by V_1 and V_2 the remaining parts of U_1 and U_2 respectively. Thus V_1 consists of a final segment, say X_F, of X, a final segment, Y_F, of Y, and an initial segment, A_I, of A, and V_2 consists of the remainder of A.

The next step is to determine how many a's are contained in V_2. V_2 cannot contain six or seven a's, as A_I would then contain at most one a and X and Y contain only two a's each. If V_2 contained five a's it would have at most 11 terms and A_I would contain two a's and have at least 4 terms. Then X_F and Y_F combined would have at most 7 terms and so could not contain three a's. (Neither X_F nor Y_F can contain three a's, and if one of them contains two a's then it has at least 7 terms, so that the other has no terms.) Hence V_2 must contain exactly four a's, since it is at least as long as A_I. Also V_2 must contain one of the c's in A, as otherwise V_1 would contain at least two c's and V_2 none.

It follows that $V_2 = cadaeadab$, and so $A_I = bacaea$ and X_F and Y_F together contain just the three terms a, d, d. This is impossible, however, since the last two terms of X are $x\sigma(x)$ and the last two terms of Y are $y\sigma(y)$, whereas $a \neq \sigma(d)$ and $d \neq \sigma(a)$.

Case (0,1). This can be dealt with in the same way as the previous case. In fact treating Case (0,1) is equivalent to treating Case (1,0) with each of the blocks $A, \ldots, E$ reversed, and this reversal affects the argument only to the extent of interchanging c and d throughout. Since $a \neq \sigma(c)$ and $c \neq \sigma(a)$ the argument remains valid.

Case (0,0). First, suppose that no block overlaps both U_1 and U_2. Then U_1 is a final segment of one block (which may be assumed, without loss of generality, to be A) and U_2 is an initial segment of another block (say X). If $X \neq A$ then U_2 contains at most two x's (since A contains only two x's) and so at most five terms altogether. Hence U_2 contains at most one a, and so U_1 contains at most one a and therefore at most three terms altogether. This is impossible, however, as it is easy to check that none of the three final segments of A consisting of the last one, two, or three terms is a permutation of an initial segment of the same length of any of the blocks B, C, D and E. Hence $X = A$, and it follows that S contains a repetition.

Now suppose that there is a block (which we may take to be A) overlapping both U_1 and U_2. U_1 and U_2 cannot both be contained in A since it is strongly non-repetitive (in fact it is strongly non-repetitive even when the a's are omitted). If U_1 is not contained in A then it contains the first term of A. If, on the other hand, U_1 is contained

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in A, then U_2 is not, and so contains the last term, b, of A. Now U_1 must also contain a b, and so it again contains the first term of A. Similarly U_2, whether or not it is contained in A, necessarily contains the last term of A. Hence the situation is represented by the following diagram, in which the intersection of U_1 with X or of U_2 with Y may be empty.

$$\begin{array}{ccc} X & A & Y \\ \hline \underbrace{\qquad\qquad}_{U_1} & & \underbrace{\qquad\qquad}_{U_2} \end{array}$$

If either X or Y is equal to A then the original sequence S must have contained a repetition.

It remains to show that when neither X nor Y is equal to A the sequence XAY cannot contain a repetition U_1U_2 in the position indicated by the diagram. The following argument remains valid even if U_1 contains the whole of X or U_2 contains the whole of Y, and so it disposes of the part of Case (1,1) that is still outstanding. The first step is to show that we need only look for repetitions U_1U_2 in which U_1 contains no more than 9 terms of X and U_2 contains no more than 9 terms of Y. If $Y \neq X$ then A and Y each contain only two x's and so U_2 (and hence also U_1) can contain at most four x's and U_1 can contain at most 9 terms of X. Similarly U_2 can contain at most four y's and so at most 9 terms of Y. Suppose now that $Y = X$, and denote by X_F the part of X contained in U_1 and by Y_I the part of Y contained in U_2. If, when considered as segments of the same block X, X_F and Y_I overlap, cancelling the terms in the overlap from each of U_1 and U_2 leaves a shorter but similarly placed repetition. Hence we may suppose that X_F and Y_I do not overlap in this way; that is, that the combined length of X_F and Y_I is not more than 15 terms. If, now, X_F contains 10 or more terms (and so five or more x's), Y_I contains not more than 5 terms (and so not more than two x's), and it follows that U_1 contains at least five x's but U_2 contains at most four x's, contradicting the fact that U_2 is a permutation of U_1. Hence X_F contains at most 9 terms, and similarly Y_I also contains at most 9 terms.

Next, let U_1U_2 be a repetition contained in XAY (where $X \neq A$ and $Y \neq A$) such that X_F and Y_I each contain at most 9 terms, and suppose that U_1 contains four or more of the a's in A. Then U_1 contains both the c's in A, and so Y_I must contain two c's. The only two blocks to have two c's among their first 9 terms are A and C, and since $Y \neq A$ we must have $Y = C$. Also, since U_1 contains more of the a's in A than U_2 does, Y_I must contain at least one a. Hence Y_I consists of exactly the first 9 terms of C and we have $Y_I = dcecbceca$. Since Y_I contains only one a, U_2 must contain three of the a's in A, and so we have $U_2 = (d)aeadabdcecbceca$, where the d in brackets may or may not be included. It follows that X_F contains no a's, one b, two c's, one or three d's and two e's. This is impossible, however; for if two symbols each occur more than once in X_F then it contains at least three x's, and if three symbols each occur more than once it contains at least five x's.

Finally, if U_1 contains less than four of the a's in A then U_2 contains at least four, and a similar argument can be used. U_2 now contains both the d's in A, and we must have $X = D$ in order that U_1 can contain two d's. The fact that X_F must contain an a

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shows it to be exactly the last 9 terms of D, and the rest of the argument can now be carried out just as before because of the symmetry of the blocks.

This completes the proof that $\mathscr{E}(S)$ is strongly non-repetitive whenever S is. A strongly non-repetitive sequence that extends to infinity in both directions can now be obtained by the process described at the end of section 2; namely, starting with a suitable strongly non-repetitive sequence (such as a) and repeatedly applying the operation $\mathscr{E}$.

In any infinite sequence constructed in this way the occurrences of each symbol have asymptotic density $\frac{1}{3}$. Also the occurrences of each symbol in an infinite sequence constructed in the way described in section 2 have asymptotic density $\frac{1}{3}$. More generally, if an infinite sequence on a finite number of symbols is formed by associating with each symbol a certain block of symbols and then repeatedly replacing each symbol in a finite sequence by its associated block, then (provided that each block contains all the symbols) the occurrences of each symbol will have a positive asymptotic density which is independent of the finite sequence used to start the process. Furthermore, the asymptotic densities of the symbols will all be equal if and only if each symbol is contained the same number of times in the aggregate of all the blocks.

4. There remains the question whether there exist infinite (or, at any rate, arbitrarily long) strongly non-repetitive sequences on four symbols. In view of the very long strongly non-repetitive sequence on four symbols that has been found and the ease with which these sequences can be generated by a computer, it seems certain that there is such an infinite sequence, and it may be possible to use the method of the previous section to construct one. It would be necessary to find a suitable strongly non-repetitive block in which b, c and d occurred equally often but which contained a greater number of a's. By permuting the symbols three other strongly non-repetitive blocks could be obtained, each with a preponderence of a different symbol, and a cancellation argument of the kind used in section 3 would then reduce the problem to a finite amount of checking. The search for the initial block and the checking would probably both be best done by computer. To minimize the amount of checking the number of excess a's in the initial block would have to be about six (the exact number depending on how evenly distributed the other symbols were), and this would probably require a block of thirty or more symbols.

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