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# Summability fields which span the bounded sequences

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1. If $A = (a_{m,n})$ is a regular matrix, then for any sequence $x = \{x_n\}$, $A_m(x)$ will denote the transform $\sum_{n=1}^{\infty} a_{mn}x_n$ and $A\text{-lim}\, x$ denotes $\lim_{m\to\infty} A_m(x)$, if that limit exists. We shall denote by $\mathscr{A}$ the set of bounded sequences which are summed by $A$. If $B$ is another regular matrix with $\mathscr{B} \supset \mathscr{A}$ then we say that $B$ is $b$-stronger than $A$. In that case $B$ must be *b-consistent* with $A$ (see (4) and (6)), i.e. if $x \in \mathscr{A} \cap \mathscr{B}$, then

$$A\text{-lim}\, x = B\text{-lim}\, x.$$

If $\{\mu_n\}$ is a sequence of positive real numbers with $\mu_n \nearrow \infty$, we say that $A$ and $B$ are $(\mu_n)$-*consistent* if every sequence $x = \{x_n\}$ satisfying $x_n = O(\mu_n)$ which is summed by both $A$ and $B$ is summed to the same value by both matrices. A finite set of matrices $A^1, A^2, \ldots, A^N$ is said to be *simultaneously* $(\mu_n)$-*consistent* (*b-consistent*) if whenever $\{x_n^i\}$ is summed by $A^i$ with $x_n^i = O(\mu_n)$ ($\{x_n^i\}$ bounded) $(i = 1, 2, \ldots, N)$ then $\sum_{i=1}^{N} x^i = 0$ implies that $\sum_{i=1}^{N} A^i\text{-lim}\, x^i = 0$. The set of sequences, $x = \sum_{i=1}^{N} x^i$, $x^i \in \mathscr{A}^i$ $(i = 1, 2, \ldots, N)$, is denoted by $\mathscr{A}^1 + \mathscr{A}^2 + \ldots + \mathscr{A}^N$.

If $x = \{x_n\}$ is a bounded sequence then $\|x\|$ will denote $\sup_n |x_n|$. For this norm, the bounded sequences form a Banach Space, and $\mathscr{A}$ is a closed vector subspace. We shall denote the unit sequence $\{1\}$ by $u$.

Let $A^1, A^2, \ldots, A^n, \ldots$ be a countable set of regular matrices. We say that these matrices *span the bounded sequences* if every bounded sequence $x$ can be expressed as a finite sum $x = \sum_{i=1}^{N} x^i$, where $x^i \in \mathscr{A}^i$ $(i = 1, 2, \ldots, N)$. We say that these matrices *span the bounded sequences densely* if for each bounded sequence $x$ there is a constant $M$ (independent of $\epsilon$) such that for each $\epsilon > 0$, there exist sequences $x^i \in A^i$ $(i = 1, 2, \ldots N)$ with

$$\left\| x - \sum_{i=1}^{N} x^i \right\| < \epsilon \tag{1}$$

and

$$\sum_{i=1}^{N} \|x^i\| < M. \tag{2}$$

If the set of matrices is finite, then a similar terminology may be used. In that case $N$ can be taken to be the number of matrices in the set.



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The following lemma is proved (but not stated explicitly ) in (1), p. 706.

LEMMA 1. *Let* $\{A^i\}$ $(i = 1, 2, \ldots)$ *be a countable (possibly finite) family of matrices. There exists increasing sequences of natural numbers,* $\{\lambda_\nu\}$, $\{p_\mu\}$ *and* $\{q_\mu\}$ *with* $p_\mu < q_\mu$ $(\mu = 1, 2, \ldots)$, *such that if the matrices* $\{B^i\}$ $(i = 1, 2, \ldots)$ *are defined by*

$$b^i_{\mu,\nu} = a^i_{\mu,\nu} \quad \text{if} \quad p_\mu \leqslant \nu < q_\mu,$$
$$b^i_{\mu,\nu} = 0 \quad \text{otherwise},$$

*for* $i, \mu, \nu = 1, 2, \ldots$, *then for every sequence* $x = \{x_\nu\}$ *with* $x_\nu = O(\lambda_\nu)$, *we have*

$$\lim_{\mu\to\infty} (A^i_\mu(x) - B^i_\mu(x)) = 0.$$

A finite set of matrices $\{A^i\}$ $(i = 1, 2, \ldots, N)$ is said to have a singularity $S_2$ if, for every $\epsilon > 0$, there exist $M(\epsilon)$, and $m(\epsilon)$ such that for every $n > m$, there exist bounded sequences $x^1(\epsilon, n)$, $x^3(\epsilon, n)$, $\ldots$, $x^N(\epsilon, n)$ which satisfy the following relations:

$$\sum_{i=1}^{N} \|x^i(\epsilon, n)\| \leqslant M, \tag{3}$$

$$\left|1 - \sum_{i=1}^{N} x^i_\nu(\epsilon, n)\right| < \epsilon \quad \text{whenever} \quad m \leqslant \nu < n, \tag{4}$$

$$\sum_{i=1}^{N} |A^i_\mu(x^i)| < \epsilon \quad \text{for all} \quad \mu \quad \text{with} \quad m \leqslant p_\mu \quad \text{and} \quad n > q_\mu. \tag{5}$$

If $\{A^i\}$ $(i = 1, 2, \ldots)$ is a countable set of matrices such that for each $\epsilon > 0$, there is an $N(\epsilon)$, $m(\epsilon)$ and $M(\epsilon)$ for which (3), (4) and (5) are satisfied for some set $x^1, x^2, \ldots, x^N$ of bounded sequences, the set of matrices are said to have a singularity $S_3$. If a countable set of matrices has a singularity $S_3$ and the $M$ of (3) can be taken idependently of $\epsilon$, the matrices are said to have a $\beta$-singularity.

2. In (3) the authors have shown that if $A^1, A^2, \ldots, A^N$ span the bounded sequences then they are not simultaneously $b$-consistent. A study of the proof of this result will reveal that it is only necessary for $A^1, A^2, \ldots, A^N$ to span the bounded sequences densely. The following theorem will show that this distinction is unnecessary. However it would seem that the two concepts differ in the case of infinite sets of matrices.

THEOREM 1. *Let* $A^1, A^2, \ldots, A^N$ *be a finite set of regular matrices. If these matrices span the bounded sequences densely, then they span the bounded sequences.*

*Proof.* Let $x$ be a bounded sequence. By hypothesis, for each natural number $t$ there exist bounded sequences $x^1(t) \in \mathscr{A}^1$, $x^2(t) \in \mathscr{A}^2, \ldots, x^N(t) \in \mathscr{A}^N$ for which

$$\left\| x - \sum_{i=1}^{N} x^i(t) \right\| < \frac{1}{t}$$

and

$$\sum_{i=1}^{N} \|x^i(t)\| \leqslant M,$$

where $M$ is a natural number independent of $t$. We shall construct bounded sequences $y^i \in \mathscr{A}^i$ $(i = 1, 2, \ldots, N)$ such that

$$x = \sum_{i=1}^{N} y^i. \tag{6}$$

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Choose a natural number $P$ such that $\sup_{i,m} \sum_{n=1}^{\infty} |a^i_{m,n}| \leqslant P$. Then

$$|A^i\text{-lim}\, x^i(t)| \leqslant MP \quad (i = 1, 2, \ldots, N; \quad t = 1, 2, \ldots).$$

Hence, there exists a sequence $\{t_p\}$, $t_p \nearrow \infty$, for which $A^i\text{-lim}\, x^i(t_p) \to L^i$ as $p \to \infty$, for some number $L^i$ $(i = 1, 2, \ldots, N)$. Clearly, we lose no generality in assuming that $t_p = p$ $(p = 1, 2, \ldots)$. Since,

$$x - \sum_{i=1}^{N} x^i(t) = x - Lu - \sum_{i=1}^{N} (x^i(t) - L^i u),$$

where $L = \sum_{i=1}^{N} L^i$, (1) and (2) are true for the sequence $x' = x - Lu$ but now

$$A^i\text{-lim}\, [x^i(t)]' \to 0 \quad \text{as} \quad t \to \infty,$$

$(i = 1, 2, \ldots, N)$ where $[x^i(t)]' = x^i(t) - L^i u$. As the sequence $Lu$ is convergent, if $x - Lu$ may be represented as in (6), the sequence $x$ may be so represented. Hence, we lose no generality in assuming that $L^i = 0$ $(i = 1, 2, \ldots, N)$. After Lemma 1, there is no loss of generality in assuming that there exist sequences $\{r(m)\}$, $r(m) \nearrow \infty$, and $\{q(m)\}$, $q(m) \nearrow \infty$ such that $a^i_{m,n} = 0$ whenever $n \geqslant r(m)$ or $n < q(m)$.

Since $A^i\text{-lim}\, x^i(t) \to 0$ as $t \to \infty$, there exists a sequence $\{\epsilon_\nu\}$ of real numbers, $\epsilon_\nu \searrow 0$ such that

$$|A^i_m(x^i(\nu))| < \epsilon_\nu \quad \text{if} \quad m \geqslant m_0(\nu) \quad (i = 1, 2, \ldots, N).$$

Select a sequence $\{m(\nu)\}$ of natural numbers, $m(\nu) \nearrow \infty$, such that

$$q(m(\nu+1)) \geqslant r(m(\nu)) \quad \text{and} \quad m(\nu) \geqslant m_0(\nu+2) \quad (\nu = 1, 2, \ldots).$$

The sequences $z^i = \{z^i_n\}$ are constructed by interweaving the sequences $x^i(t)$ as follows:

$$\text{if} \quad n < q(m(1)) \quad \text{we define} \quad z^i_n = x^i_n(1);$$
$$\text{if} \quad q(m(\nu)) \leqslant n < q(m(\nu+1))$$

we express $\nu$ in the form $(\alpha - 1)\, MP + \beta$, where $\alpha$ and $\beta$ are positive integers with $0 \leqslant \beta < MP$. We then define

$$z^i_n = \left(1 - \frac{\beta}{\alpha MP}\right) x^i_n(\alpha) + \frac{\beta}{\alpha MP} x^i_n(\alpha+1).$$

Clearly, we have,

$$\left|x_n - \sum_{i=1}^{N} z^i_n\right| \leqslant \left(1 - \frac{\beta}{\alpha MP}\right)\left|x_n - \sum_{i=1}^{N} x^i_n(\alpha)\right| + \frac{\beta}{\alpha MP}\left|x_n - \sum_{i=1}^{N} x^i_n(\alpha+1)\right| < 1/\alpha.$$

Hence, the sequence $x - \sum_{i=1}^{N} z^i$ is convergent to zero and is summed by each of the matrices $\{A^i\}$ $(i = 1, 2, \ldots, N)$.

Now suppose that $m(\nu) \leqslant m < m(\nu+1)$. Express $\nu$ in the form $(\alpha - 1)\, MP + \beta$. Then

$$A^i_m(z^i) = \sum_{n=q(m)}^{r(m)-1} a^i_{m,n} z^i_n.$$

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Also, $$q(m(\nu)) \leqslant q(m) < r(m) \leqslant r(m(\nu+1)) \leqslant q(m(\nu+2)):$$

if $n$ is in $[q(m(\nu)), q(m(\nu+1)))$, then

$$z_n^i = \left(1-\frac{\beta}{\alpha MP}\right)x_n^i(\alpha)+\frac{\beta}{\alpha MP}x_n^i(\alpha+i);$$

if $n$ is in $[q(m(\nu+1)), q(m(\nu+2)))$ then either,

$$z_n^i = \left(1-\frac{\beta+1}{\alpha MP}\right)x_n^i(\alpha)+\frac{\beta+1}{\alpha MP}x_n^i(\alpha+1)$$

or $$z_n^i = x_n(\alpha+1).$$

Hence, if $n$ is in $[q(m), r(m)-1]$,

$$\left|z_n^i-\left[\left(1-\frac{\beta}{\alpha MP}\right)x_n^i(\alpha)+\left(\frac{\beta}{\alpha MP}\right)x_n^i(\alpha+1)\right]\right| < \frac{1}{\alpha MP}(|x_n^i(\alpha)|+|x_n^i(\alpha+1)|),$$

and $$|A_m^i(z^i)| < |A_m^i(x_n^i(\alpha))|+|A_m^i(x_n^i(\alpha+1))|+\frac{1}{\alpha MP}.2MP.$$

Since, $$m \geqslant m(\nu) = m((\alpha-1)MP+\beta) \geqslant m((\alpha-1)) \geqslant m_0(\alpha+1) \geqslant m_0(\alpha),$$

we have $$|A_m^i(z^i)| \leqslant 2\epsilon_\alpha+(2/\alpha),$$

and as $m \to \infty$, $\alpha \to \infty$, so that

$$2\epsilon_\alpha+(2/\alpha) \to 0.$$

Therefore, $z^i \in \mathscr{A}^i$ $(i = 1, 2, \ldots, N)$ and the sequences $\{y^i\}$ have the required properties where

$$y^1 = x-\sum_{i=2}^{N} z^i = \left(x-\sum_{i=1}^{N} z^i\right)+z^1$$

and $$y^i = z^i \ (i = 2, 3, \ldots, N).$$

This completes the proof of our theorem.

3. Let $E$ be a norm-closed vector subspace of the bounded sequences, then $E$ will be a Banach space. For example, if $A$ is a regular matrix we may take the space $E$ to be $\mathscr{A}$.

THEOREM 2. *Let $\{A^i\}$ $(i = 1, 2, \ldots, n, \ldots)$ be a countable family of regular matrices which span $E$, i.e. every $x$ in $E$ can be expressed in the form $x = \sum_{i=1}^{N} x^i$, $x^i \in \mathscr{A}^i$ $(i = 1, 2, \ldots, N)$. Then, for some natural number $n_0$,*

$$\mathscr{A}^1+\mathscr{A}^2+\ldots+\mathscr{A}^{n_0}$$

*contains a dense susbspace of $E$.*

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*Proof.* Since $E$ is a Banach space, $E$ is of the second category in itself. Let

$$\mathscr{Z}^n = E \cap (\mathscr{A}^1 + \ldots + \mathscr{A}^n) \quad (n = 1, 2, \ldots).$$

By hypothesis,
$$E = \bigcup_{n=1}^{\infty} \mathscr{Z}^n$$

so that for some $n$, $\mathscr{Z}^n$ is somewhere dense in $E$. Since $\mathscr{Z}^n$ is a vector space, it is everywhere dense in $E$.

We now obtain some corollaries to this result. It has been shown in (1) that if $A^1, A^2, \ldots, A^N$ are regular matrices, simultaneously $(\mu_n)$-consistent for some sequence $\mu_n \nearrow \infty$, then there exists a regular matrix $B$ which is $b$-stronger than each member of the set. Since $\mathscr{B}$ is closed in the bounded sequences, it is clear that $\mathscr{A}^1 + \mathscr{A}^2 + \ldots + \mathscr{A}^n$ cannot be a dense subspace of the bounded sequences. Therefore, if we apply the above theorem, with $E$ the space of bounded sequences we obtain the following:

COROLLARY 1. *If $A^1, A^2, \ldots, A^n, \ldots$ span the bounded sequences then for some $n_0$, $A^1, A^2, \ldots, A^{n_0}$ are not simultaneously $(\mu_n)$-consistent for any sequence $\mu_n \nearrow \infty$.*

The corollary may be rephrased in terms of singularities. A set of matrices, $A^1, A^2, \ldots, A^N$ has a singularity $S_2$ if and only if they are not simultaneously $(\mu_n)$-consistent for any $\mu_n \nearrow \infty$. (See (1), (2) and (5).)

COROLLARY 2. *If $A^1, A^2, \ldots, A^n, \ldots$ span the bounded sequences then, for some $n_0$, $A^1, A^2, \ldots, A^{n_0}$ have a singularity $S_2$.*

4. In this section, we shall improve on Corollary 2 and show that if $A^1, A^2, \ldots, A^n, \ldots$ span the bounded sequences then, for some $v_0$, $A^1, A^2, \ldots, A^{v_0}$ are not $b$-consistent. If a finite set of matrices have a singularity $(S_2)$ and the $M$ of (3) can be chosen independent of $\epsilon$, the matrices are said to have a *$b$-singularity*. A $b$-singularity implies that the matrices are not $b$-consistent, see (2). If the matrices have a singularity $(S_2)$ but do not have a $b$-singularity, they are said to have a singularity $S_2'$.

THEOREM 3. *If $A^1, A^2, \ldots, A^n, \ldots$ span the bounded sequences, then for some $v_0$, $A^1, A^2, \ldots, A^{v_0}$ are not $b$-consistent.*

*Proof.* From Corollary 2, we have that the matrices $A^1, A^2, \ldots, A^{n_0}$ have a singularity $S_2$. It is clear that the matrices $A^1, A^2, \ldots, A^n$, $n > n_0$ have a singularity $S_2$ as well. We shall suppose that the matrices $A^1, A^2, \ldots, A^v$ have a singularity $S_2'$ for every $v \geqslant n_0$ and show that this leads to a contradiction.

In (7) it has been shown (although not explicitly stated) that if $A^1, A^2, \ldots, A^v$ have a singularity $S_2'$, there exist constants $M_v'(\epsilon)$, $M_v'(\epsilon) \nearrow \infty$ as $\epsilon \searrow 0$, such that for each $\epsilon$, there is a pair of integers $r$ and $n(r, \epsilon)$ where $r$ may be chosen larger than any given number, for which

$$\left| \sum_{i=1}^{v} x_k^i - 1 \right| < \epsilon \quad \text{for} \quad r \leqslant k < n(r, \epsilon), \tag{7}$$

$$\sum_{i=1}^{v} |A_\mu^i(x^i)| < \epsilon \quad \text{for all } \mu \quad \text{with} \quad r \leqslant p_\mu;\ q_\mu < n \tag{8}$$

implies
$$\sum_{i=1}^{v} \|x^i\| \geqslant \tfrac{1}{2} M_v'(\epsilon). \tag{9}$$

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Let $N_1$ be such that for some $r$ and $n$ less than $N_1$

$$\left|\sum_{\nu=1}^{v} x_k^\nu - 1\right| < \tfrac{1}{2} \text{ for } r \leqslant k < n,$$

$$\sum_{\nu=1}^{v} |A_\mu^\nu(x^\nu)| < \tfrac{1}{2} \quad \text{for all } \mu \text{ with } \quad r \leqslant p_\mu, \quad q_\mu < n$$

implies

$$\sum_{\nu=1}^{v} \|x^\nu\| \geqslant \tfrac{1}{2} M_2'(\tfrac{1}{2}).$$

If the sequence of integers $(N_i)\,(i = 1, 2, \ldots, j-1)$ has been chosen, $N_j$ is chosen so that there exist pairs $(r_1, n_1), \ldots, (r_j, n_j)$

$$N_{j-1} < r_t < n_t \leqslant N_j \quad (t = 1, 2, \ldots, j)$$

and the following is true for each $t\,(t = 1, 2, \ldots, j)$

$$\left|\sum_{\nu=1}^{v} x_k^\nu - 1\right| < 2^{-t} \quad \text{for} \quad r_t \leqslant k < n_t,$$

$$\sum_{\nu=1}^{v} |A_\mu^\nu(x^\nu)| < 2^{-t} \quad \text{for all } \mu \text{ with} \quad r_t \leqslant p_\mu, \quad q_\mu < n_t$$

implies

$$\sum_{\nu=1}^{v} \|x^\nu\| \geqslant \tfrac{1}{2} M_v'(2^{-t})$$

for all $v$ $(v = 1, 2, \ldots, j)$.

We note that if $r_t(v)$, $n_t(v)$ are the indices in (7), (8), and (9) corresponding to $\epsilon = 2^{-t}$ and the matrices $A^1, A^2, \ldots, A^v$ the above construction is effected by a choice

$$r_t \leqslant \max_{1 \leqslant r \leqslant j} (r_t(v))$$

and a similar choice for $n_t$.

Let $s_h' = 0$ if $N_{2k-1} < h \leqslant N_{2k}\,(k = 1, 2, \ldots)$ and $s_h' = 1$ if $h \leqslant N_1$ or $N_{2k} < h \leqslant N_{2k+1}$ $(k = 1, 2, \ldots)$ Then suppose that for some $v$

$$\sum_{\nu=1}^{v} x_h^\nu = s_h' \quad (h = 1, 2, \ldots), \tag{10}$$

$x^\nu \in \mathscr{A}^\nu\,(\nu = 1, 2, \ldots, v)$. It is always possible to assume that

$$A^\nu\text{-lim}\, x^\nu = 0 \quad (\nu = 1, 2, \ldots, v-1), \tag{11}$$

since, if $A^\nu\text{-lim}\, x^\nu = L_\nu\,(\nu = 1, 2, \ldots, v-1)$, then the sequences

$$(x_h^\nu - L_\nu)\,(\nu = 1, 2, \ldots, v-1) \text{ and the sequence } (x_h^v + L), L = \sum_{\nu=1}^{v-1} L_\nu,$$

satisfy (10) and (11).

Some cases must now be distinguished.

(i) It may be that $\quad A^\nu\text{-lim}\,(x^\nu) = 0 \quad (\nu = 1, 2, \ldots, v).$ (12)

Then $$\Sigma |A_\mu^\nu(x^\nu)| < 2^{-t} \tag{13}$$

for all $\mu > \mu(t)$. But if $\quad N_{2k_0} > \max(p_{\mu(t)}, N_t, N_v),$

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then for all $\mu$ with $p_\mu > N_{2k_0}$, (13) is satisfied (since $p_\mu \nearrow$) and (10) implies

$$\left| \sum_{\nu=1}^{v} x_h^\nu - 1 \right| = 0 \quad (r_t \leqslant h < n_t)$$

so that

$$\sum_{\nu=1}^{v} \|x^\nu\| \geqslant \tfrac{1}{2} M'_v(2^{-t}).$$

Since this is true for all $t > v$, one of the sequences $(x_h^\nu)$ $(\nu = 1, 2, \ldots, v)$, is unbounded.

(ii) It may be that

$$A^\nu\text{-lim}\,(x^\nu) = 0 \quad (\nu = 1, 2, \ldots, v-1) \tag{14}$$

and

$$A^v\text{-lim}\,(x^v) = -1. \tag{15}$$

Then

$$\sum_{\nu=1}^{v-1} |A_\mu^\nu(x^\nu)| + |A_\mu^v(x^v + u)| < 2^{-t} \tag{16}$$

for all $\mu > \mu(t)$. Also,

$$\left| \sum_{\nu=1}^{v-1} x_h^\nu + (x_h^v + 1) - 1 \right| = 0, \tag{17}$$

$N_{2k-1} < h \leqslant N_{2k}$ $(k = 1, 2, \ldots)$. If $N_{2k_0-1} > \max[p_{\mu(t)}, N_t, N_v]$, (16) and (17) imply

$$\sum_{\nu=1}^{v-1} \|x^\nu\| + \|x^v + u\| \geqslant \tfrac{1}{2} M'_v(2^{-t}).$$

Since this is true for all $t$, either one of the sequences $(x_h^\nu)$ $(v = 1, 2, \ldots, v-1)$, or the sequence $(x_h^v + 1)$ is unbounded. The case

$$A^\nu\text{-lim}\, x^\nu = 0 \quad (\nu = 1, 2, \ldots, v-1),$$
$$A^v\text{-lim}\, x^v = L \quad (L \neq 0)$$

can be reduced to the above by considering the sequences $(y_h^\nu)$ $(\nu = 1, 2, \ldots, v)$;

$$y_h^\nu = -\frac{1}{L} x_h^\nu \quad (h = 1, 2, \ldots; \nu = 1, 2, \ldots, v).$$

Then we have

$$\left| \sum_{\nu=1}^{v-1} y_h^\nu + (y_h^v + 1) - 1 \right| = 0$$

for $N_{2k-1} < h \leqslant N_{2k}$ $(k = 1, 2, \ldots,)$. It now follows that one of the sequences

$$y^\nu \ (\nu = 1, 2, \ldots, v)$$

is unbounded, hence, one of the sequences $x^\nu$ $(\nu = 1, 2, \ldots, v)$ is unbounded. We have a contradiction in all cases and the proof of the theorem is complete.

The proof of Theorem 3 is much the same as the proof of the Theorem in (7), but because of some slight differences, it was better to write it out in full.

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