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Correction to 'A class of unique g-measures'

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Theorem 1.1 in [**4**] is false: the hypothesis $\sum v_n(g) < \infty$ needs to be replaced by a stronger condition. A corrected version of Theorem 1.1 is given below; a counterexample to the original result is given in [**5**].

The problem is in the proof of Proposition 4.2: the claim that $E(\bar{f}_m|T^{-i}\mathcal{B})(z) \le \varepsilon^m$ if $z_i \ne 0$ is false. However (in the notation of [**4**]), if the sequence $\{a_n\}$ used to define γ in §4 is such that $\sum_{n=0}^{\infty} a_n < 1$, and $\gamma(1z) = \sum_{i=0}^{\infty} a_i z_i$, then

$$\mathcal{L}_\gamma f_1(z) = \sum_{i=0}^{\infty} a_i z_i.$$

Hence,

$$\nu(f_1) = \sum_{i=0}^{\infty} a_i \nu(f_1),$$

and so $\nu(f_1) = 0$. Thus, Proposition 4.2 holds in this case.

To deduce Theorem 1.1 from Proposition 4.2, we need the probability measures $P(\cdot \mid x,y)$ $(x,y \in X)$ to satisfy

$$P(\{(\xi,\eta) : \xi \ne \eta\} \mid x,y) \le \gamma(1\delta(x,y)) \qquad (*)$$

for all $x,y \in X$. Note that

$$|g(sx) - g(sy)| \le \sum_{i=0}^{\infty} v_i\big(g(s\,\cdot)\big)\mathbf{1}\{x_i \ne y_i\} \quad (s \in S). \qquad (\dagger)$$

If the state space $S = \{0, 1\}$ then $P(\xi \ne \eta \mid x,y) = |g(1x) - g(1y)|$. Therefore, Theorem 1.1 holds if $\sum_{n=0}^{\infty} v_n(g(1\,\cdot)) < 1$, since we can take $a_n = v_n(g(1\,\cdot))$, and then $(*)$ is satisfied. The examples mentioned in the introduction to [**4**] satisfy this criterion when the sequence $\{a_n\}$ used to define them (not the same as the above a_n) satisfies $\sum_{n=1}^{\infty} a_n < 1$.

In general, if $|S| > 2$, the hypothesis that follows from $(*)$ is more complicated. However, note that the proof of Theorem 1.1 works with any set of probability measures $P(\cdot \mid x,y)$ such that

$$P(\{(\xi,\eta) : \xi = i\} \mid x,y) = g(ix), \quad P(\{(\xi,\eta) : \eta = j\} \mid x,y) = g(jy) \qquad (\ddagger)$$

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for all $i, j \in S$. Redefining the $P(\cdot \mid x,y)$ so that $P(\xi \neq \eta \mid x,y)$ is minimised gives a slightly stronger result, and one more easily stated. Let

$$P((s,s) \mid x,y) = \min\{g(sx),\, g(sy)\} \quad (s \in S),$$

and

$$P(A_{i,j} \mid x,y) = \min\left\{\sum_{s\geq i}(g(sx) - g(sy))^+,\ \sum_{s\geq j}(g(sy) - g(sx))^+\right\},$$

where

$$A_{i,j} = \{(\xi,\, \eta) : \xi \geq i,\ \eta \geq j,\ \xi \neq \eta\} \quad (i,\, j \in S)$$

and $(t)^+ = \max\{t,\, 0\}$. It is easily checked that (‡) holds, and

$$\begin{aligned} P(\{(\xi,\, \eta) : \xi \neq \eta\} \mid x,y) &= \frac{1}{2}\sum_{s\in S} P(\{\xi = s\}\triangle\{\eta = s\} \mid x,y) \\ &= \frac{1}{2}\sum_{s\in S} |g(sx) - g(sy)|. \end{aligned}$$

Combining this with $(*)$ and (†), and taking $a_n = \frac{1}{2}\sum_{s\in S} v_n(g(s\,\cdot))$ gives the following.

THEOREM 1.1. *Let $g \in \mathcal{G}_X$ be positive and continuous. If $\sum_{n=0}^{\infty}\sum_{s\in S} v_n(g(s\,\cdot)) < 2$, there is a unique g-measure μ. The natural extension of $(T,\, \mu)$ is Bernoulli, and $\mathcal{L}_g^n f \to \mu(f)$ uniformly as $n \to \infty$, for all $f \in C(X)$.*

No changes need to be made to §3 as a result of redefining the $P(\cdot \mid x,y)$, although a more natural definition of the σ_j would be $\sigma_j(x,y) = |g(jx) - g(jy)|$. However, note that although Proposition 3.1 is true as stated, it is assumed in the proof that the convergence is uniform. This is the case in the proof of Theorem 1.1 anyway.

Note that $(*)$ can be used to deduce other results on uniqueness. For example, if

$$r_n = \sup\left\{\frac{1}{2}\sum_{s\in S} |g(sx) - g(sy)| : x_i = y_i,\ 0 \leq i < n\right\}$$

and

$$\gamma(1z) = \sum_{n=0}^{\infty} r_n \mathbf{1}[0^n 1](z),$$

then $(*)$ holds; furthermore,

$$\mathcal{L}_\gamma^n \mathbf{1}[0^n 1](z) = \prod_{i=0}^{n-1}(1 - r_i)\mathbf{1}[1](z) \quad (n \geq 1,\ z \in Z),$$

and so,

$$v(f_m) = v(f_1)\left(1 + \sum_{n=1}^{m-1}\prod_{i=0}^{n-1}(1 - r_i)\right).$$

The left-hand side is finite for all m; thus if $\sum_{n=1}^{\infty}\prod_{i=0}^{n-1}(1 - r_i) = \infty$ then $v(f_1) = 0$, in which case the conclusions of Theorem 1.1 hold (cf. [**1**], see also [**2, 6**]).

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The referee has pointed out a recent paper by Fernández and Maillard [**3**] in which they study non-stationary chains with complete connections. They prove the uniqueness of such chains subject to a hypothesis which, in the present context, is equivalent to that of Theorem 1.1. Mixing properties are also obtained, although not the Bernoulli property.

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