Combinatorics, Probability and Computing (2002) 11, 1–10. © 2002 Cambridge University Press
DOI: 10.1017/S0963548301004965 Printed in the United Kingdom

The Chromatic Number of Graph Powers

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Received 4 September 2000; revised 28 June 2001

It is shown that the maximum possible chromatic number of the square of a graph with maximum degree d and girth g is $(1+o(1))d^2$ if $g = 3, 4, 5$ or 6, and is $\Theta(d^2/\log d)$ if $g \geqslant 7$. Extensions to higher powers are considered as well.

1. Introduction

The *square* G^2 of a graph $G = (V, E)$ is the graph whose vertex set is V and in which two distinct vertices are adjacent if and only if their distance in G is at most 2. What is the maximum possible chromatic number of G^2, as G ranges over all graphs with maximum degree d and girth g?

Our (somewhat surprising) answer is that for $g = 3, 4, 5$ or 6 this maximum is $(1+o(1))d^2$ (where the $o(1)$ term tends to 0 as d tends to infinity), whereas for all $g \geqslant 7$, this maximum is of order $d^2/\log d$.

To state this result more precisely, for every two integers $d \geqslant 2$ and $g \geqslant 3$, define $f_2(d, g)$ to be the maximum possible value of $\chi(G^2)$ over all graphs with maximum degree d and girth g. Since the maximum degree of G^2 is at most $d + d(d-1) = d^2$, it follows that, for every g,

$$f_2(d, g) \leqslant d^2 + 1. \tag{1.1}$$

Equality holds in (1.1) for $d = 2$ and $g \leqslant 5$, as shown by the 5-cycle, for $d = 3$ and

† Research supported in part by a USA–Israel BSF grant and by a grant from the Israel Science Foundation.
‡ Supported in part by the Ministry of Science and Technology of Slovenia, Research Project J1–0502–0101–98.

https://doi.org/10.1017/S0963548301004965 Published online by Cambridge University Press

$g \leqslant 5$, as shown by the Petersen graph, and for $d = 7$ and $g \leqslant 5$, as shown by the Hoffman–Singleton graph; see [10]. Moreover, by Brooks' theorem (*cf.*, *e.g.*, [5]) it follows that equality can hold in (1.1) only for $g \leqslant 5$ and only if there exists a d-regular graph of diameter 2 on $d^2 + 1$ vertices, since G^2 must then contain a complete graph on $d^2 + 1$ vertices. As proved in [10], if such a graph exists then $d \in \{2, 3, 7, 57\}$, and it is known to exist for $d \in \{2, 3, 7\}$, whereas the case $d = 57$ is still open.

It is also not difficult to see that $f_2(2, g) = 4$ for all $g \geqslant 6$, as shown, for example, by the disjoint union of a g-cycle and a cycle of length $l \geqslant g$, where l is not divisible by 3.

In this short paper we prove the following.

Theorem 1.1.

(i) *There exists a function $\varepsilon(d)$ that tends to 0 as d tends to infinity such that, for all $g \leqslant 6$,*

$$(1 - \varepsilon(d))d^2 \leqslant f_2(d, g) \leqslant d^2 + 1.$$

(ii) *There are absolute positive constants c_1, c_2 such that, for every $d \geqslant 2$ and every $g \geqslant 7$,*

$$c_1 \frac{d^2}{\log d} \leqslant f_2(d, g) \leqslant c_2 \frac{d^2}{\log d}.$$

Theorem 1.1 exhibits an interesting 'phase transition': as g grows from 3 to 6, $f_2(d, g)$ stays roughly the same, while it drops significantly when g increases from 6 to 7, and then it stays essentially the same as g keeps increasing.

The problem of colouring the square of a graph is related to the channel assignment problem [7]. McCormick [15] proved that it is NP-hard to determine $\chi(G^2)$. Heggernes and Telle [9] proved the same for 4-colouring of the square of cubic graphs, and Agnarsson and Halldórsson [1] considered powers of planar graphs and their chromatic numbers.

The rest of the paper contains a proof of Theorem 1.1, and an extension of it for higher powers of graphs. Throughout the paper, all logarithms are in base e.

2. The upper bounds

The proof of the upper bounds in parts (i) and (ii) of Theorem 1.1 are rather simple (given the main result in [3]). As we have observed above, the upper bound in part (i) of Theorem 1.1 follows from (1.1). To prove the upper bound in (ii), we need the following result of [3].

Theorem 2.1 ([3]). *There is an absolute constant c such that, for every integer Δ and every real number t, $2 \leqslant t \leqslant \Delta^2$, the chromatic number of any graph H with maximum degree at most Δ in which, for every vertex v, the induced subgraph on the set of all neighbours of v spans at most Δ^2/t edges, satisfies*

$$\chi(H) \leqslant c \frac{\Delta}{\log t}.$$

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We can now prove that there is a constant c_2 such that, for all $g \geqslant 7$,

$$f_2(d,g) \leqslant c_2 \frac{d^2}{\log d}. \tag{2.1}$$

Let $G = (V,E)$ be a graph with maximum degree d and girth $g \geqslant 7$. Let $H = G^2$. Then the maximum degree of H is at most $d + d(d-1) = d^2$. It is not difficult to check that, if $u, v \in V$ are adjacent in H, then they have fewer than $2d$ common neighbours in H. It follows that each neighbourhood of a vertex of H spans less than $\frac{d^2 \cdot 2d}{2} = d^3$ edges. (In fact, the induced subgraph on each neighbourhood consists of at most $d+1$ edge-disjoint cliques of size at most d each, implying it spans at most $(d+1)d(d-1)/2 < d^3/2$ edges.) Applying Theorem 2.1 with $\Delta = d^2$ and $t = d$, we conclude that (2.1) holds.

3. The lower bounds

In this section we prove the lower bounds in Theorem 1.1. The bound in part (i) is simple. If $d = q + 1$ for some prime power q, then the bipartite incidence graph of the lines and points of the finite projective plane of order q form a d-regular bipartite graph G on $2(q^2 + q + 1)$ vertices. The girth of G is 6 and its square contains two cliques of size $q^2 + q + 1 = d^2 - d + 1$ each, and hence $\chi(G^2) \geqslant d^2 - d + 1$. (It is, in fact, easy to check that $\chi(G^2) = d^2 - d + 1$.)

Remark. It is not difficult to check that, if G has girth 6 and maximum degree d, then the maximum clique in G^2 is of size at most $d^2 - d + 1$. To see this, fix a vertex v in a maximum clique of G^2. Then, in G^2, v is connected to the set $N(v)$ of its neighbours in G, where $|N(v)| \leqslant d$, and to the set $N^2(v)$ of the vertices at distance 2 from v, where $|N^2(v)| \leqslant d^2 - d$. Since the girth is 6, any member of $N(v)$ is connected in G^2 to at most $d-1$ members of $N^2(v)$. It follows that if the clique contains more than $d-1$ members of $N^2(v)$ then it contains no member of $N(v)$. Thus, the size of the clique is at most $1 + d^2 - d$. The main result of Reed in [16] asserts that there is some absolute positive constant ϵ such that, for every graph H with maximum degree Δ, maximum clique size ω and chromatic number χ, the inequality $\chi \leqslant \epsilon\omega + (1-\epsilon)(\Delta + 1)$ holds. This implies that if G has girth 6 and maximum degree $d > 2$ then $\chi(G^2) \leqslant d^2 - \Omega(d)$, showing that the example above is nearly tight.

Returning to the proof of the lower bound in part (i) of Theorem 1.1 for general d, we simply choose the largest prime power q satisfying $q + 1 \leqslant d$, and take the graph constructed above for q together with some extra pendant edges to make sure that the maximum degree is precisely d. By the known results on the distribution of primes (see, *e.g.*, [11]), $q^2 + q + 1 \geqslant (1 - \varepsilon(d))d^2$, where $\varepsilon(d)$ tends to 0 as d tends to infinity. This proves the lower bound in Theorem 1.1(i) for $g = 6$. The bounds for $g = 3, 4, 5$ follow by simply adding to the example for $g = 6$ a vertex-disjoint copy of a cycle of length g.

It remains to prove the lower bound in Theorem 1.1(ii). This is done by a probabilistic construction, which resembles the original proof of Erdős that there are graphs with arbitrarily large girth and chromatic number [8]. Note that we may assume, without loss

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of generality, that d is sufficiently large (by choosing $c_1 > 0$ sufficiently small), and we thus assume, from now on, that d is large (for example, $d \geqslant 10^{10}$ will be enough). We also omit all floor and ceiling signs whenever they are not crucial, to simplify the presentation.

Let n be a large integer ($n \gg d^g$), let $V' = \{1, \ldots, n\}$, define $p = \frac{d}{2n}$, and let $G' = (V', E')$ be a random graph on V' obtained by choosing each pair of distinct elements of V', randomly and independently, to be an edge with probability p. The expected value of each vertex degree in G' is $\frac{d}{2n}(n-1) < \frac{d}{2}$. However, G' will have, with high probability, about $n \cdot 2^{-\Theta(d)}$ vertices of degree higher than d. It will also have, with high probability, some cycles of length less than g (fewer than $O(d^g)$ of them). By omitting all vertices of degree $> d$ and by removing an arbitrarily chosen vertex from each cycle of length $< g$ we get a graph of girth $\geqslant g$ and maximum degree at most d. (If necessary, we can add a cycle of length g and some pendant edges to make sure that the maximum degree is equal to d and that the girth is precisely g.)

As we show next, with high probability, for the graph G obtained in this manner, G^2 does not contain an independent set of size bigger than $c\frac{n}{d^2}\log d$ for an appropriately chosen constant $c > 0$, and hence $\chi(G^2) \geqslant \Omega(d^2/\log d)$, as needed.

The detailed proof is given in the following claims (where we make no attempt to optimize the absolute constants).

Claim 3.1. *The expected number of vertices of degree $> d$ in G' is at most $n \cdot 2^{-d/10}$. Thus, with probability $\geqslant 0.9$, there are at most $10n \cdot 2^{-d/10}$ such vertices.*

Proof. The degree of any fixed vertex is a binomial random variable with parameters $n-1$ and $p = \frac{d}{2n}$, and hence its expected degree is less than $d/2$.

By the standard estimates for binomial distributions (see, *e.g.*, [2, Appendix A]), the probability that the degree of such a vertex exceeds d is smaller than $2^{-d/10}$. By linearity of expectation, the expected number of vertices of degree $> d$ is thus at most $n \cdot 2^{-d/10}$, and hence, by Markov's inequality, the probability that there are more than $10n \cdot 2^{-d/10}$ such vertices is at most 0.1. (In fact, this probability is exponentially small, but this is not needed here.) □

Claim 3.2. *The expected number of cycles of length $< g$ in G' is*

$$\sum_{i=3}^{g-1} \frac{n(n-1)\cdots(n-i+1)}{2i}\left(\frac{d}{2n}\right)^i < \frac{1}{2}\sum_{i=3}^{g-1}\left(\frac{d}{2}\right)^i < d^g,$$

and hence, with probability $\geqslant 0.9$, there are at most $10d^g$ such cycles.

Proof. Straightforward. □

We say that an event holds almost surely if its probability tends to 1 as n tends to infinity.

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Claim 3.3. *There exists a constant c, $0 < c < 1000$, such that the following holds almost surely. For every set of vertices $U \subset V'$ of cardinality $|U| = c\frac{n}{d^2}\log d$, there are at least $\frac{1}{30}c^2 n\frac{\log^2 d}{d^2}$ vertices outside U that have exactly 2 neighbours in U.*

Proof. Put $x = c\frac{n}{d^2}\log d$. Fix a set U, $|U| = x$, and fix a vertex $w \in V' \setminus U$. Note that, as d is large, $|V' \setminus U| \geqslant \frac{3n}{4}$. The probability of the event E_w that w has precisely two neighbours in U is

$$\begin{aligned} q &= \binom{x}{2} p^2 (1-p)^{x-2} \geqslant \frac{x(x-1)}{2}\left(\frac{d}{2n}\right)^2 \left(1-\frac{d}{2n}\right)^x \\ &\geqslant \frac{x(x-1)}{2}\left(\frac{d}{2n}\right)^2 \left(1-\frac{xd}{2n}\right) \geqslant \frac{1}{10}c^2\frac{\log^2 d}{d^2}. \end{aligned}$$

The number of vertices $w \in V' \setminus U$ is at least $\frac{3n}{4}$, and as the events E_w are mutually independent, the number of vertices $w \in V' \setminus U$ that have 2 neighbours in U is a binomial random variable with parameters $|V' \setminus U| \geqslant \frac{3n}{4}$ and $q \geqslant \frac{1}{10}c^2\frac{\log^2 d}{d^2}$. The expected value of this number is at least $\frac{3}{40}c^2 n\frac{\log^2 d}{d^2}$ and hence, by the standard estimates for binomial distributions (see, *e.g.*, [2, Appendix A]), the probability that it is smaller than $\frac{1}{30}c^2 n\frac{\log^2 d}{d^2}$ is at most, say, $\exp\left(-\frac{1}{300}c^2 n\frac{\log^2 d}{d^2}\right)$.

Therefore, for each fixed set U, the probability that there are fewer than $\frac{1}{30}c^2 n\frac{\log^2 d}{d^2}$ vertices $w \in V' \setminus U$ with precisely 2 neighbours in U is at most $\exp\left(-\frac{1}{300}c^2 n\frac{\log^2 d}{d^2}\right)$. As the total number of sets U is

$$\binom{n}{x} = \binom{n}{c\frac{n}{d^2}\log d} \leqslant \left(\frac{ed^2}{c\log d}\right)^{c\frac{n}{d^2}\log d} \leqslant \exp\left(3c\frac{n}{d^2}\log^2 d\right),$$

it follows that, if $\frac{c^2}{300} > 3c$, then with probability $1 - o(1)$ every set U has at least that many vertices w with 2 neighbours in U. This completes the proof of the claim. □

We can now complete the proof of Theorem 1.1. For any given g and (large) d, choose $n \gg d^g$. By Claims 3.1–3.3, there exists a graph G' on n vertices satisfying the conclusions of all three claims. Let G be the graph obtained from G' by omitting all vertices of degree $> d$ and an arbitrarily chosen vertex from each cycle of length $< g$. Then G has girth $\geqslant g$, maximum degree $\leqslant d$, and more than $n/2$ vertices. As G' satisfies the conclusion of Claim 3.3, and as the total number of vertices omitted (which is at most $10n \cdot 2^{-d/10} + 10d^g$) is smaller than $\frac{c^2}{30}n\frac{\log^2 d}{d^2}$, it follows that G^2 contains no independent set of size $x = c\frac{n}{d^2}\log d$. Thus $\chi(G^2) \geqslant \frac{d^2}{2c\log d}$.

By adding pendant edges to G (if needed), and a disjoint copy of a cycle of length g, we conclude that

$$f_2(d,g) \geqslant \frac{d^2}{2c\log d}$$

for all d, g, completing the proof of Theorem 1.1. □

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4. Higher powers

The kth *power* G^k of a graph $G = (V, E)$ is the graph whose vertex set is V in which two distinct vertices are adjacent if and only if their distance in G is at most k. Note that $G^1 = G$. Let $f_k(d, g)$ denote the maximum possible value of $\chi(G^k)$, as G ranges over all graphs of maximum degree d and girth g. Trivially, $f_1(d, 3) = d + 1$ for all $d \geqslant 2$. By the theorem of Johansson [12] and the known results about chromatic numbers of random graphs ([6, 13]), there are absolute constants c_1, c_2 such that, for every $d \geqslant 2$ and every $g \geqslant 4$,

$$c_1 \frac{d}{\log d} \leqslant f_1(d, g) \leqslant c_2 \frac{d}{\log d}.$$

The behaviour of $f_k(d, g)$ for $k = 2$ is determined by Theorem 1.1. For higher values of k, the situation is less clear, though the main parts of the proof of Theorem 1.1 can be extended. If G has maximum degree d, then the maximum degree of G^k is at most

$$d + d(d-1) + \cdots + d(d-1)^{k-1} = \frac{d}{d-2}((d-1)^k - 1).$$

Therefore, for every fixed k,

$$f_k(d, g) \leqslant \frac{d}{d-2}((d-1)^k - 1) + 1 = O(d^k).$$

The known constructions of large graphs with a given maximum degree and diameter at most k show that for every k there exists a constant $c_k > 0$ such that, for some small values of g,

$$f_k(d, g) \geqslant c_k d^k.$$

For example, the DeBruijn graphs show that

$$f_k(d, 3) \geqslant \lfloor d/2 \rfloor^k,$$

and the existence of generalized m-gons imply similar estimates for somewhat larger values of g. See also the constructions in [4], [14] and their references.

Finally, the proof in Section 2 and the random construction described in Section 3 can be extended to prove the following.

Theorem 4.1. *There exists an absolute constant $c > 0$ such that, for all integers $k \geqslant 1$, $d \geqslant 2$ and $g \geqslant 3k + 1$,*

$$f_k(d, g) \leqslant \frac{c}{k} \frac{d^k}{\log d}.$$

For every integer $k \geqslant 1$ there exists a positive number b_k such that, for every $d \geqslant 2$ and $g \geqslant 3$,

$$f_k(d, g) \geqslant b_k \frac{d^k}{\log d}.$$

The proof in Section 2 easily extends to a proof of the upper bound in Theorem 4.1, and the details are left to the reader.

The proof of the lower bound follows the basic approach in Section 3, but requires

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somewhat more sophisticated tools in order to show that the random graph G' satisfies the following property almost surely. For an appropriately chosen constant c_k, for every set $U \subseteq V'$ of size $|U| = c_k \frac{n}{d^k} \log d$, there are at least $\Omega(c_k^2 \frac{n}{d^k} \log^2 d)$ internally vertex-disjoint paths of length k, both of whose endpoints lie in U, and whose other vertices lie in $V' \setminus U$.

This can be proved using Talagrand's inequality [17] (*cf.* also [2, Chapter 7]) and implies the assertion of Theorem 4.1 following the reasoning of the proof in Section 3. We proceed with the details.

Throughout the proof we assume, whenever this is needed, that d is sufficiently large as a function of k, and omit, as before, all floor and ceiling signs whenever these are not crucial. Let $V' = \{1, 2, \ldots, n\}$, suppose $n \gg d^g$, and let $G' = (V', E')$ be a random graph on V' obtained by choosing each pair of distinct elements of V', randomly and independently, to be an edge with probability $p = \frac{d}{2n}$. Define $x = c_k \frac{n}{d^k} \log d$, where $c_k > 0$ will be chosen later.

Lemma 4.2. *For an appropriate choice of c_k, the following holds almost surely. For every set of vertices $U \subset V'$ of cardinality $|U| = x$, there are at least $\frac{c_k^2 n \log^2 d}{2^{k+5} d^k}$ internally vertex-disjoint paths of length k, both of whose endpoints lie in U, and whose other vertices lie in $V' \setminus U$.*

Proof. We shall apply Talagrand's inequality. To do so, fix a set $U \subset V'$, $|U| = x$. Call any path of length k, whose endpoints lie in U and whose internal vertices lie outside U, a *U-path*. Let $X = X(G')$ be the random variable counting the maximum number of internally vertex-disjoint U-paths. We first show that the expected value of X satisfies

$$E(X) \geqslant \frac{c_k^2 n \log^2 d}{2^{k+2} d^k}. \tag{4.1}$$

Indeed, the expected number of U-paths is

$$\begin{aligned} \mu &= \binom{x}{2}(n-x)(n-x-1)\cdots(n-x-k+2)p^k \\ &> 0.49 \frac{c_k^2 n^{k+1}}{d^{2k}} \log^2 d \; p^k = 0.49 \frac{c_k^2 n}{2^k d^k} \log^2 d, \end{aligned}$$

where we have used the fact that d is sufficiently large.

Let Δ denote the expected number of pairs of U-paths that share at least one common internal vertex. We claim that, as $n \gg d \gg k$, $\Delta < \mu/3$ (with room to spare).

Indeed, the expected number of such pairs that share only one endpoint and its unique neighbour is at most

$$\mu n^{k-2} x p^{k-1} = \frac{\mu c_k \log d}{2^{k-1} d}.$$

It is not difficult to check that, for $n \gg d \gg k$, the expected number of pairs of internally intersecting U-paths of any other type is much smaller, and the number of types is bounded by a function of k.

By omitting an arbitrarily chosen path from each pair of internally intersecting U-paths, we get a collection of internally pairwise vertex-disjoint U-paths. It follows, by linearity of expectation, that $E(X) \geqslant \mu - \Delta$, implying (4.1).

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To apply Talagrand's inequality (in the form presented, for example, in [2, Chapter 7]), note that X is a Lipschitz function, that is, $|X(G') - X(G'')| \leqslant 1$ if G', G'' differ in at most one edge. This is because X counts internally vertex-disjoint paths, which are edge-disjoint, and hence no single edge can change the value of X by more than 1. Note also that X is f-certifiable for $f(s) = ks$, that is, when $X(G') \geqslant s$ there is a set of at most ks edges of G' so that, for every graph G'' that agrees with G'' on these edges, $X(G'') \geqslant s$.

By Talagrand's inequality we conclude that, for every b and t,

$$\Pr[X \leqslant b - t\sqrt{kb}] \cdot \Pr[X \geqslant b] \leqslant \exp(-t^2/4). \tag{4.2}$$

Let B denote the median of X. Without trying to optimize the constants, we prove that

$$B \geqslant \frac{c_k^2 n \log^2 d}{2^{k+4} d^k}. \tag{4.3}$$

Indeed, assume this is false and apply (4.2) with $b = \frac{c_k^2 n \log^2 d}{2^{k+3} d^k}$ and $t = \frac{1}{2}\sqrt{\frac{b}{k}}$ to conclude that

$$\Pr\left[X \leqslant \frac{b}{2}\right] \cdot \Pr[X \geqslant b] \leqslant \exp\left(-\frac{b}{16k}\right).$$

Since, by assumption, $B \leqslant \frac{b}{2}$, it follows that $\Pr[X \leqslant b/2] \geqslant 1/2$, and hence $\Pr[X \geqslant b] \leqslant 2\exp(-\frac{b}{16k})$. However, as $X(G') < n$ for every G', this implies that

$$E(X) \leqslant b + n2\exp\left(-\frac{b}{16k}\right) = b + 2n\exp\left(-\frac{c_k^2 n \log^2 d}{2^{k+7} d^k k}\right) = b + o(1) = \frac{c_k^2 n \log^2 d}{2^{k+3} d^k} + o(1),$$

contradicting (4.1) and thus proving (4.3).

We can now apply (4.2) again with $b = \frac{c_k^2 n \log^2 d}{2^{k+4} d^k}$ and $t = \frac{1}{2}\sqrt{\frac{b}{k}}$ to conclude that

$$\Pr\left[X \leqslant \frac{b}{2}\right] \Pr[X \geqslant b] \leqslant \exp\left(-\frac{b}{16k}\right).$$

By (4.3), $\Pr[X \geqslant b] \geqslant 1/2$, and hence

$$\Pr\left[X \leqslant \frac{b}{2}\right] \leqslant 2\exp\left(-\frac{b}{16k}\right) = 2\exp\left(-\frac{c_k^2 n \log^2 d}{2^{k+8} d^k k}\right).$$

We have thus proved that, for every fixed U, the probability that there are fewer than $\frac{b}{2} = \frac{c_k^2 n \log^2 d}{2^{k+5} d^k}$ internally vertex-disjoint U-paths is at most $2\exp(-\frac{c_k^2 n \log^2 d}{2^{k+8} d^k k})$. Since the total number of sets U is at most

$$\binom{n}{x} \leqslant \left(\frac{en}{x}\right)^x \leqslant \left(\frac{ed^k}{c_k \log d}\right)^{c_k \frac{n}{d^k} \log d} \leqslant \exp\left(c_k k \frac{n}{d^k} \log^2 d\right),$$

it follows that if, say,

$$\frac{c_k^2}{2^{k+8} k} > 2kc_k$$

then, with probability $1 - o(1)$, for every set U there are at least $\frac{c_k^2 n \log^2 d}{2^{k+5} d^k}$ pairwise internally vertex-disjoint U-paths. This completes the proof of the lemma. □

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We can now complete the proof of Theorem 4.1. For any given g and k, and any (large) d and sufficiently large n there exists a graph G' satisfying the conclusions of Claim 3.1, Claim 3.2 and Lemma 4.2. Let G be the graph obtained from G' by omitting all vertices of degree greater than d and an arbitrarily chosen vertex from each cycle of length $< g$. Then G has girth $\geqslant g$, maximum degree $\leqslant d$, and more than $n/2$ vertices. As G' satisfies the conclusion of Lemma 4.2, and as the total number of vertices omitted from it to create G is at most

$$10n \cdot 2^{-d/10} + 10d^g < \frac{c_k^2 n \log^2 d}{2^{k+5} d^k},$$

it follows that G^k contains no independent set of size $x = c_k \frac{n \log d}{d^k}$. Therefore

$$\chi(G^k) \geqslant \frac{d^k}{2c_k \log d}.$$

By adding to G, if needed, pendant edges and a disjoint copy of a cycle of length g, this implies that, for an appropriately defined $b_k > 0$,

$$f_k(d, g) \geqslant b_k \frac{d^k}{\log d}$$

for all d, g, completing the proof of Theorem 4.1. □

Note that, for $k = 1$ and 2, the function $f_k(d, g)$ exhibits a drastic change when g changes from $3k$ to $3k + 1$. It would be interesting to decide if this is the case for higher values of k as well. The results mentioned in this section do suggest that for every k there is some integer g_k such that

$$f_k(d, g) = \Theta_k(d^k), \qquad \text{for } g \leqslant g_k,$$

and

$$f_k(d, g) = \Theta_k(d^k / \log d), \qquad \text{for } g > g_k.$$

At the moment, however, we are unable to prove the existence of such a g_k.

Acknowledgement

Part of this work was done during a visit to the workshop on Graph Coloring organized by Pavol Hell in the PIMS Institute at the Simon Fraser University in July 2000. We would like to thank Pavol and the other hosts at PIMS for their hospitality.

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https://doi.org/10.1017/S0963548301004965 Published online by Cambridge University Press