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Calculating the Lagrange point L_2 for the JWST

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1. *Introduction*

On December 25th, 2021, NASA, in collaboration with the European and Canadian space agencies, achieved a remarkable feat by successfully launching the James Webb Space Telescope (JWST), marking it as the largest and most powerful telescope ever constructed.* The JWST, a monumental collaborative project spanning over two decades, involved scientists and engineers from 300 universities and 14 countries, with a total investment of ten billion dollars.

The momentous occasion occurred at 7:20 ET on Christmas Day 2021, as the Ariane 5 rocket carrying the JWST thundered into space, poised to rewrite cosmic history. The launch was met with elation as research teams and mission control members erupted in cheers, cameras flashed, and news reports flooded social media. It was a pivotal moment that etched its place in history. Since its momentous launch, the JWST has significantly reshaped our fundamental understanding of the universe, unveiling a cascade of fascinating discoveries that continue to fascinate the scientific community and beyond.

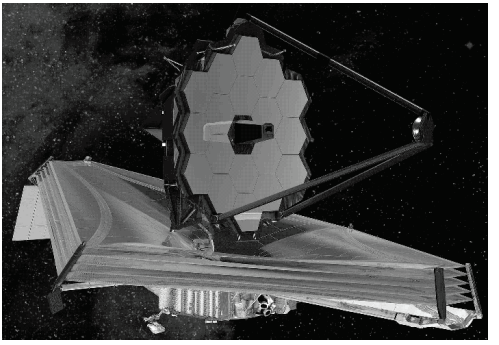


FIGURE 1: An artist's rendering of the JWST. (Image credit: NASA)

* Notably, this date holds significance as it coincides with the birthday of Sir Isaac Newton, December 25th, 1642,

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JWST has been parked at its planned destination L_2, a Lagrange point about a million miles away from the Earth, where it is orbiting the Sun. What is a Lagrange point? How can we calculate the distance to this particular Lagrange point L_2? Where is L_2 in relation to the Earth's umbra and penumbra? Last but not least, why was this spot chosen to park JWST? We will answer these questions from a mathematical standpoint. Our exposition will not assume any background in physics or astronomy and can profitably be read and enjoyed by mathematics undergraduates.

This paper has generated stimulating discussions among my friends and colleagues, and I am sincerely grateful to them for their valuable feedback. Additionally, I would like to express my appreciation to an anonymous referee for their insightful comments and new approaches, which have significantly contributed to enhancing the clarity, content, and presentation of this paper.

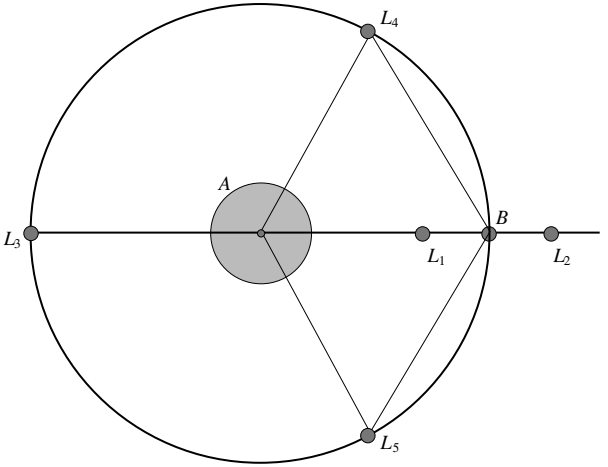


FIGURE 2: The five Lagrange points of the Sun-Earth system

2. *Lagrange points*

Suppose we have two objects, A and B (e.g. the Sun and the Earth), of masses M and m, respectively. Since $M >> m$, we may assume that A is fixed and B revolves around A in a circular orbit. A Lagrange point is a point in the orbital plane of A and B where an object x (of small mass) remains stationary relative to A and B. More precisely, at any Lagrange point, the net gravitational force exerted by A and B on x equals the centripetal force necessary to lock the position of x relative to A and B. In other words, an object placed at a Lagrange point gets a ‘free ride’. Therefore these are ideal places to park communication satellites and space telescopes because they minimise fuel consumption.

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In their work on the three-body problem*, Euler and Lagrange have shown that precisely five equilibrium points exist in the gravitational field created by any two masses, as shown in Figure 2 (Sun (A) and Earth (B) in this case). Euler discovered L_1, L_2 and L_3 (located on the line joining A and B), a few years before Lagrange discovered L_4 and L_5 (points that form the apex of the two equilateral triangles with base AB) in his landmark paper (Essai sur le problème des trois corps, 1772).

Of the five Lagrange points, L_4 and L_5 are stable equilibrium points provided $M/m > 24.96$. This condition is satisfied by the Sun-Earth system, for instance. Stability here means that although any displacement of an object at one of the Lagrange points will move it away from the point, eventually it will follow a closed orbit around the original point. On the other hand, L_1, L_2 and L_3 are unstable, meaning that objects orbiting at these points will gradually move out of orbit unless station-keeping is used to maintain their orbits.

L_2 is the preferred Lagrange point for space telescopes for several reasons. First, it is relatively close to the Earth, thus allowing quick and easy communication. Second, it provides an unimpeded view of the deep sky because the Sun, the Earth, and the Moon remain, at all times, behind the telescope. Finally, L_2 being an unstable Lagrange point does not collect space objects like L_3 and L_4. This ensures the safety of the space telescope. Moreover, the JWST has to maintain a very low temperature (-233 degrees Celsius) to observe in the infrared zone. A five-layer sunshield helps maintain this low temperature while giving it the necessary solar power as it orbits around L_2.

3. *Calculating L_2*

We will concentrate on L_2 because it is the home for the JWST. The reader can refer to [1] for a more in-depth analysis of all Lagrange points.

3.1. *Physical model and laws*.

To determine the position of L_2 relative to the Sun (S) and the Earth (E), we employ a simplifying assumption that the Sun remains stationary while the Earth orbits around it in a circular path. It is worth noting that the actual orbit of the Earth is elliptical but with a very low eccentricity of approximately 0.0167, resulting in an orbit that closely resembles a circle. Hence our assumption is reasonable in this context.

Now let us revisit some pertinent laws of physics established by Newton. The first is a fundamental law of motion that states that when a force $\vec{F}$ acts on a mass m, it produces an acceleration $\vec{a}$ such that

$$\vec{F} = m\vec{a}.$$

The second is one of the most famous equations in physics and astronomy. It is Newton's universal inverse square law of gravity, which states that the

* The problem of determining the paths of three bodies that interact mutually under gravity given their initial positions and velocities.

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magnitude F of the force of attraction between objects of masses M and m that are separated by a distance r is given by

$$F = \frac{GMm}{r^2},$$

where G is the universal constant of gravity whose value is approximately equal to $6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$.

These two laws alone perform wonders; to some extent they explain the motion of all objects in the universe, from falling apples to orbiting planets and satellites.

In addition, we need one more fact. An object moving in a circular orbit of radius r at constant speed v has an acceleration whose magnitude is given by $a = v^2/r$. (This is established in standard calculus texts.) If the angular velocity is ω, then using the relation $v = r\omega$, we have $a = r\omega^2$. We can now begin the calculation of L_2.

3.2. *Deriving the equation for L_2.* Let the masses of the Sun and Earth be M and m respectively and let r be the distance between their centres. The force exerted by the Sun on the Earth is given by

$$F = \frac{GMm}{r^2}.$$

The natural angular velocity ω_E of the Earth resulting from this force is such that

$$\frac{GMm}{r^2} = mr\omega_E^2.$$

Simplifying this equation gives

$$\omega_E^2 = \frac{GM}{r^3}. \tag{1}$$

Let m' denote the mass of the JWST, and x be the distance between the Earth and the JWST. The magnitude of the net force exerted at a point C by the Sun and the Earth is equal to

$$\frac{GMm'}{(r+x)^2} + \frac{Gmm'}{x^2}.$$

If ω_J is the angular velocity of the JWST resulting from these two forces at this point, then we have

$$\frac{GMm'}{(r+x)^2} + \frac{Gmm'}{x^2} = m'(r+x)\omega_j^2.$$

Simplifying this as before gives

$$\omega_j^2 = \frac{GM}{(r+x)^3} + \frac{Gm}{x^2(r+x)}. \tag{2}$$

The key point is that x will be a Lagrange point (i.e. the JWST at x will

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orbit the Sun in line with the Earth) if, and only if, $\omega_E = \omega_j$. So to find x we equate the expressions for the squares of these two quantities from (1) and (2). This gives

$$\frac{GM}{r^3} = \frac{GM}{(r + x)^3} + \frac{GM}{x^2 (r + x)}. \tag{3}$$

Simplifying the above expression results in the following 5th degree polynomial

$$J(X) : X^5 + 3X^4 + 3X^3 - \mu X^2 - 2\mu X - \mu = 0,$$

where $X = x/r$ and $\mu = m/M$. The units of X will be AU (Astronomical Unit), the distance between the Sun and the Earth.

3.3. *Roots of* $J(X) = 0$. A position of L_2 will correspond to a real root of this equation. Since every odd-degree polynomial has a real root, we know that at least one real root exists. In fact, $J(0) = -\mu < 0$ and $J(1) = 7 - 4\mu > 0$. Therefore, by the intermediate value theorem, we know that at least one positive root of this equation exists in the interval (0, 1). Moreover, there is only one sign-change in coefficients of the polynomial (X) when the terms are arranged in the descending order of the exponents. Descartes' rule of signs then tells us that there is at most one positive root to this equation. Combining these two facts, we conclude that this equation has only one positive real root, and that root belongs to the interval (0, 1). The physical interpretation of this is that there is only one Lagrange point outside the orbit of the Earth in the direction of the radial vector from the Sun to the Earth.

The 5th-degree equation $J(X) = 0$ has five complex roots, counting multiplicities by the fundamental theorem of algebra. However, there is no physical interpretation of the remaining four roots of this equation that are not on the positive real line. Although we know there is a positive root of this 5th-degree polynomial, there is no formula for this root in terms of radicals. A theorem in abstract algebra, due to Abel, states that it is impossible to solve a general polynomial degree 5 or above in radicals.

However, we can get an approximate closed-form expression for our root. To this end, note that our polynomial

$$J(X) : X^5 + 3X^4 + 3X^3 - \mu X^2 - 2\mu X - \mu = 0$$

can be approximated by $3X^3 - \mu = 0$ after dropping all but the dominant terms. The latter can be readily solved to obtain: $X \approx \sqrt[3]{\frac{\mu}{3}}$. One can also arrive at this via Newton's binomial approximation.

Substituting the values for m and M gives $X \approx 0.01$. The upshot is that the distance from L_2 and the Earth's centre is approximately 0.01 AU, or 1.496×10^6 km. For a more accurate approximation, one can start with this initial approximate root $\left(X \approx \sqrt[3]{\frac{\mu}{3}} \approx 0.01\right)$ and converge to the actual root

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(≈ 0.01003657) using the Newton-Raphson method, with three iterations.

One can also arrive at an approximation of the root quickly without the Newton-Raphson method, as pointed out to us by the referee. For instance, consider the recursive formula given by:

$$X_0 = \sqrt[3]{\frac{\mu}{3}}, \qquad X_{n+1} = X_0 \sqrt[3]{1 + \frac{X_n\left(1 + \frac{2}{3}X_n\right)}{1 + X_n + \frac{1}{3}X_n^2}} \text{ for } n = 0, 1, 2, 3, \ldots$$

Using mathematical induction, it can be shown that $\{X_n\}$ is a monotonic sequence in the interval (0, 1). Therefore, by the monotone convergence theorem, this sequence converges. Taking limits on both sides of the recursive formula, we see that the limit is a root of our equation $J(X) = 0$. Moreover, this approach also gives a very good approximation to the root, namely

$$X = \sqrt[3]{\frac{\mu}{3}\left(1 + \sqrt[3]{\frac{\mu}{3}}\right)}.$$

Remark: In our model, we made the simplifying assumption that the Sun remains stationary. This choice significantly streamlined our calculations by allowing us to work within an inertial frame of reference, which does not involve any pseudo forces. However, the analysis presented in [1] considers a more general scenario where the two objects under examination, with masses M and m respectively, exert rotational influence on each other under the inverse square law of gravity. A non-inertial frame was utilised to accommodate this general scenario, incorporating two pseudo forces: the coriolis and centrifugal forces. Consequently, the polynomial employed to solve for the position of L_2 in [1] differs slightly from the one we used in our model. Nonetheless, this disparity becomes inconsequential when the mass m is significantly smaller than M, as in the Earth-Sun system. In fact, as the ratio m/M approaches zero, the two polynomials converge and become indistinguishable.

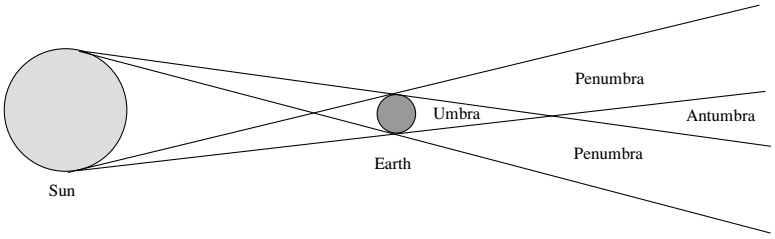


FIGURE 3: The three regions of the Earth's shadow

In our effort to ensure accessibility to mathematics students without a specialised background in physics, we simplified the physical model by not introducting pseudo forces. By doing so, we aimed to streamline our calculations and present the material in a less intricate and more comprehensible manner.

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3.4. *The Moon*. Note that in our calculations, we have ignored the Moon. This raises the question of how the Moon influences the position of L_2. To answer this, we can adjust our equation by substituting the Earth's mass with the Earth-Moon system's combined mass. By solving this modified equation, we find that L_2 is drawn closer to the Earth at a distance of approximately 1501390 km. This results in a difference of around 6000 km compared to the original calculation. To provide some context, this disparity is roughly equivalent to the distance between the west and east coasts of the United States. However, it is important to note that this discrepancy is considered negligible on an astronomical scale, and both values would be rounded to 0.01 AU.

The next question is, where exactly is the Moon positioned in relation to the JWST? Let us consider the average distance to the Moon, approximately 384,400 km. Moreover, $\frac{1.496 \times 10^6}{384{,}400} \approx 4$. This means that the distance to the JWST at L_2 is roughly four times greater than the average distance to the Moon. Consequently, when viewed from the JWST at L_2, the maximum angular separation between the Earth and the Moon is approximately $\tan^{-1}\frac{1}{4} \approx 14°$. To put this in perspective, it is equivalent to about three-fourths of the width of Ursa Major.

4. *JWST in relation to the Earth's umbra and penumbra*

The Earth's shadow caused by the Sun has three distinct parts (umbra, penumbra and antumbra), as shown in Figure 3. We have shown that L_2 is outside the orbit of the Earth on the line joining the centres of the Sun and the Earth. So, the next question is whether L_2 is inside the Earth's umbra or antumbra. To answer this, consider the geometry of the Earth's umbra as

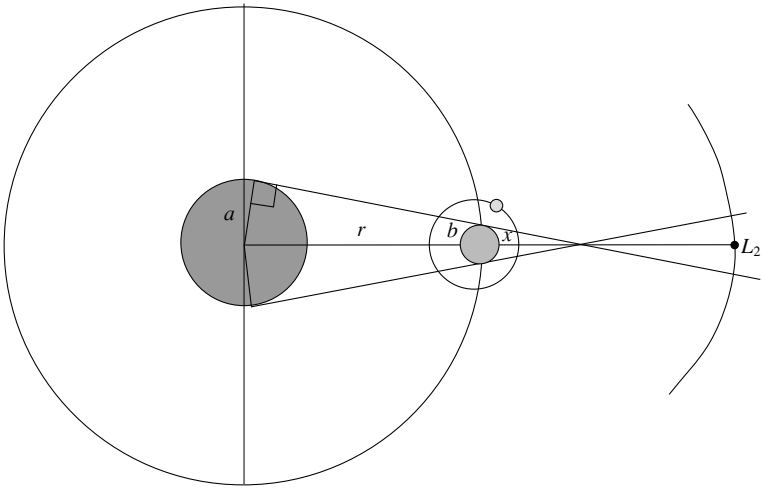


FIGURE 4: The Sun, Earth, Moon, and the JWST

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shown in Figure 4. Let a and b denote the radius of the Sun and the Earth, respectively. Let r be the distance between the Sun and the Earth, and let x be the length of the Earth's umbra measured from the planet's centre.

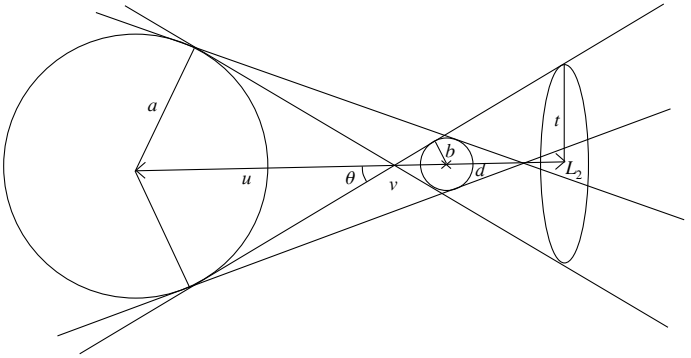


FIGURE 5: Transverse common tangents to Sun and Earth

The similarity of the top two right triangles gives

$$\frac{a}{r + x} = \frac{b}{x}.$$

From this we get $x = \dfrac{br}{a - b}$. In kilometres, we have $b = 6378$, $a = 695700$ and $r = 1.37 \times 10^8$. Substituting these in the formula for x, we get $x = 1.37 \times 10^6$. This means L_2 will miss Earth's umbra by approximately 0.12 million km. As shown in Figure 4, L_2 is located inside Earth's antumbra.

It is worth pointing out that JWST will orbit at L_2 while going around the Sun. This orbit has a radius large enough to keep the telescope away from the Earth's penumbra. By staying in this orbit, the JWST's six-metre-long solar array will receive continuous, direct sunlight, providing the necessary solar power for its operations. Additionally, the telescope is protected by a sunshield the size of a tennis court, which helps maintain an extremely low temperature required for collecting infrared light.

A simple geometric argument gives this minimum radius of the orbit for this job. Consider the Sun and the Earth along with the tangents, as shown in Figure 5. Let a and b be the radii of the Sun and Earth, respectively. The similarity of right triangles gives

$$\frac{a}{u} = \frac{b}{v}.$$

Moreover, $u + v$ equals 1 AU (The mean distance between the Sun and the Earth). This gives $v = \dfrac{b}{a + b}$ AU or approximately 1.35×10^6 km. We also have $\sin\theta = \frac{b}{v} \approx 0.00471$. For small θ, $\sin\theta \approx \theta \approx \tan\theta$. And

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finally, we have $\tan\theta = \dfrac{t}{v+d}$. This tells us that $t = (v+d)\theta \approx 13398$ km. Any circular orbit for the JWST, as shown in Figure 5, with a radius of at least 13585 km and centre at L_2, will keep it outside the penumbra of the Earth. The actual circumference of the JWST's orbit is about 800,000 km and the orbital period is about six months.

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1. Neil Cornish, The Lagrange points, *WMAP Education and Outreach* (1998), available at https://map.gsfc.nasa.gov/ContentMedia/lagrange.pdf

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