Math. Proc. Camb. Phil. Soc. (1982), **91**, 79
Printed in Great Britain

A boundary estimate for harmonic functions

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(*Received* 20 *November* 1980, *revised* 6 *January* 1981)

1. *Introduction*

In this paper we extend to certain domains in m-dimensional Euclidean space R^m, $m \geqslant 3$, some results about the boundary behaviour of harmonic functions which, in R^2, are known to follow from distortion theorems for conformal mappings.

We suppose throughout that $P = (X, y)$ denotes a point of R^m, $m \geqslant 2$, with X in R^{m-1} and y real. We denote by $B(P, r)$ the open ball in R^m with centre P and radius r, and by O the origin in R^{m-1}.

Our principal result concerns a certain auxiliary domain D_ϕ.

THEOREM 1. *Let ϕ denote a positive, increasing, continuous function in* $(0, 1]$ *such that*

$$I = \int_0^1 \frac{\phi(x)}{x^2}\,dx < \infty, \tag{1·1}$$

and let

$$r_\phi = \begin{cases} \inf\{x: \phi(x) = 1\} & (\phi(1) \geqslant 1), \\ 1 & (\phi(1) < 1). \end{cases}$$

If u is harmonic and positive in

$$D_\phi = \{(X, y): |X| < r_\phi, \phi(|X|) < y < 1 + (r_\phi^2 - |X|^2)^{\frac{1}{2}}\}, \tag{1·2}$$

then

$$\frac{u(O, y)}{u(O, 1)} \leqslant \frac{A^{m(3I+1)(I+1)}}{y^{m-1}} \quad (0 < y \leqslant 1), \tag{1·3}$$

where A is an absolute constant.

Using this we can obtain a result on harmonic functions in the unit ball $B(1) = B((O, 0), 1)$ which, for the case $m = 2$, is implied by results of Hayman and Korenblum ((1), theorems 2, 3).

THEOREM 2. *Let k denote a positive, increasing, continuous function in* $[0, 1)$ *such that*

$$J = \int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/m} dr < \infty. \tag{1·4}$$

If u is harmonic in $B(1)$, vanishes at $(O, 0)$ and satisfies

$$u(P) \leqslant k(|P|) \quad (|P| < 1),$$

then

$$|u(P)| \leqslant \frac{(BJ)^m}{(1-|P|)^{m-1}} \quad (|P| < 1),$$

where B is an absolute constant.

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For $m = 2$ it is known ((1), theorem 4) that the restriction (1·4) on k is best possible. This is also true for $m > 2$.

THEOREM 3. *Given any positive, increasing, continuous function k in $[0, 1)$ such that $k(0) = 1$ and*

$$\int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/m} dr = \infty, \tag{1·5}$$

there exists a function u harmonic in $B(1)$ such that $u(O, 0) = 0$ and

$$u(P) \leqslant k(|P|) \quad (|P| < 1),$$

for which

$$(1-y)^{m-1} u(O, y) \to -\infty \quad (y \to 1).$$

2. *Proof of Theorem* 1

We shall need a version of Harnack's inequality.

LEMMA 1. *If u is positive and harmonic in $B(P_0, r)$ then*

$$\left|\frac{\nabla u(P)}{u(P)}\right| \leqslant \frac{m-1}{r-|P-P_0|} + \frac{1}{r+|P-P_0|} \quad (P \in B(P_0, r)), \tag{2·1}$$

where $\nabla u(P)$ denotes the gradient of u at P.

Proof. We only need to show that (2·1) holds for the Poisson kernel

$$K(P, Q) = \frac{r^2 - |P-P_0|^2}{|P-Q|^m} \quad (P \in B(P_0, r)),$$

of the ball. Here Q is a fixed point satisfying $|Q - P_0| = r$. The general case follows by use of the Poisson integral formula.

We obtain, after a routine calculation, that

$$\left|\frac{\nabla K(P, Q)}{K(P, Q)}\right| = \left[\frac{m^2-2m}{|P-Q|^2} + \frac{2m}{r^2-|P-P_0|^2} + \frac{4|P-P_0|^2}{(r^2-|P-P_0|^2)^2}\right]^{\frac{1}{2}},$$

and so, on using the fact that $|P-Q| \geqslant r - |P-P_0|$, that

$$\left|\frac{\nabla K(P, Q)}{K(P, Q)}\right| \leqslant \frac{[(m^2-2m)(r+|P-P_0|)^2 + 2mr^2 + (4-2m)|P-P_0|^2]^{\frac{1}{2}}}{r^2-|P-P_0|^2}$$

$$= \frac{mr + (m-2)|P-P_0|}{r^2-|P-P_0|^2} = \frac{m-1}{r-|P-P_0|} + \frac{1}{r+|P-P_0|},$$

as required.

We shall apply Lemma 1 to open balls lying in D_ϕ near the origin, with centres on the y-axis. As a first step in this direction note that

$$\phi(x) \leqslant 2Ix \quad (0 < x \leqslant \tfrac{1}{2}), \tag{2·2}$$

since, by (1·1),

$$I \geqslant \int_x^{2x} \frac{\phi(t)}{t^2}\, dt \geqslant \phi(x) \int_x^{2x} \frac{dt}{t^2} = \frac{\phi(x)}{2x}.$$

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Thus, for $0 < y \leqslant 1$, D_ϕ certainly contains the open ball with centre (O, y) and radius $y/(2I+2)$. Applying Lemma 1 to this ball with $P = (O, y)$ and letting u_y denote $\partial u/\partial y$ we see that

$$\left|\frac{u_y(O,y)}{u(O,y)}\right| \leqslant \frac{2m(I+1)}{y} \quad (0 < y \leqslant 1). \tag{2·3}$$

Integrating this estimate leads to a rather weak upper bound for $u(O, y)$. To improve on this we need to find open balls in D_ϕ whose boundaries approximate more closely to ∂D_ϕ near the origin. This is achieved with the aid of the following lemma, which may be of independent interest.

LEMMA 2. *Suppose that f is positive and continuous in $(0, 1]$ and that $x^p f(x)$ is increasing in $(0, 1]$ for some $p \geqslant 2$. If*

$$g(x) = \sup\{f(t) : x \leqslant t \leqslant \tfrac{1}{2}\} \quad (0 < x \leqslant \tfrac{1}{2}),$$

then

$$\int_0^{\frac{1}{2}} g(x)\,dx \leqslant (2p-1)\int_0^1 f(x)\,dx.$$

Proof. Consider

$$E = \{x \in (0, \tfrac{1}{2}] : g(x) > f(x)\},$$

which is the union of an at most countable collection $\mathcal{O}$ of disjoint open intervals. If $(a, b) \in \mathcal{O}$ then

$$g(x) = f(b) \quad (a < x \leqslant b).$$

Thus if $(a_1, b_1), \ldots, (a_N, b_N) \in \mathcal{O}$ and $E' = \cup_{n=1}^N (a_n, b_n)$ we have

$$\int_{E'} g(x)\,dx = \sum_{n=1}^N b_n f(b_n)\left(1 - \frac{a_n}{b_n}\right). \tag{2·4}$$

We shall suppose that $b_n \leqslant a_{n+1}$ for $n = 1, 2, \ldots, N-1$.

On the other hand, for any a, b satisfying $0 < a < b \leqslant 1$,

$$\int_a^b f(x)\,dx \geqslant a^p f(a)\int_a^b \frac{dx}{x^p} = \frac{af(a)}{p-1}\left(1 - \left(\frac{a}{b}\right)^{p-1}\right) \tag{2·5}$$

and so

$$\sum_{n=1}^{N-1} b_n f(b_n)\left(1 - \left(\frac{b_n}{b_{n+1}}\right)^{p-1}\right) \leqslant (p-1)\int_{b_1}^{b_N} f(x)\,dx.$$

Since $b_n \leqslant a_{n+1} < b_{n+1}$ and $p \geqslant 2$,

$$\sum_{n=1}^{N-1} b_n f(b_n)\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) \leqslant (p-1)\int_{b_1}^{b_N} f(x)\,dx,$$

so that, by (2·4),

$$\begin{aligned}\int_{E'} g(x)\,dx &= b_1 f(b_1)\left(1 - \frac{a_1}{b_1}\right) + \sum_{n=1}^{N-1} b_{n+1} f(b_{n+1})\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) \\ &\leqslant b_1 f(b_1) + \sum_{n=1}^{N-1} (b_{n+1} f(b_{n+1}) - b_n f(b_n))\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) + (p-1)\int_{b_1}^{b_N} f(x)\,dx\end{aligned}$$

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$$\leqslant b_1 f(b_1) + \sum_{n=1}^{N-1} [b_{n+1} f(b_{n+1}) - b_n f(b_n)]^+ + (p-1)\int_{b_1}^{b_N} f(x)\,dx$$

$$= b_N f(b_N) + \sum_{n=1}^{N-1} [b_n f(b_n) - b_{n+1} f(b_{n+1})]^+ + (p-1)\int_{b_1}^{b_N} f(x)\,dx,$$

where $[x]^+ = \max[x, 0]$.

Then, using the estimate

$$af(a) - bf(b) \leqslant (p-1)\int_a^b f(x)\,dx \quad (0 < a < b \leqslant 1),$$

which arises from (2·5), we have

$$\int_{E'} g(x)\,dx \leqslant \tfrac{1}{2} f(\tfrac{1}{2}) + (p-1)\int_{b_N}^{\frac{1}{2}} f(x)\,dx + 2(p-1)\int_{b_1}^{b_N} f(x)\,dx$$

$$\leqslant 2(p-1)\int_{\frac{1}{2}}^{1} f(x)\,dx + 2(p-1)\int_{b_1}^{\frac{1}{2}} f(x)\,dx.$$

Thus

$$\int_E g(x)\,dx \leqslant 2(p-1)\int_0^1 f(x)\,dx$$

and since $g = f$ in $(0, \frac{1}{2}]\backslash E$ the result follows.

Applying Lemma 2 with $f(x) = \phi(x)/x^2$, $0 < x \leqslant 1$, and $p = 2$, we see that ϕ has an increasing, continuous majorant ψ in $(0, \frac{1}{2}]$ such that $\psi(x)/x^2$ is decreasing and

$$\int_0^{\frac{1}{2}} \frac{\psi(x)}{x^2}\,dx \leqslant 3I, \tag{2·6}$$

where I is given by (1·1).

Now define

$$F = \{y \in (0, \tfrac{1}{2}] : \psi(y) \leqslant y/2\}$$

and, for $y \in F$, put

$$r(y) = \min[\tfrac{1}{2}, y^2/2\psi(y)] \quad \text{and} \quad P_y = (O, \psi(y) + r(y)).$$

LEMMA 3. *For $y \in F$ we have*

$$\psi(y) < y \leqslant r(y) \tag{2·7}$$

and

$$B(P_y, r(y)) \subset D_\phi. \tag{2·8}$$

Proof. For $y \in F$ the estimates $\psi(y) \leqslant y/2$, $y \leqslant \frac{1}{2}$ and $y \leqslant y^2/2\psi(y)$ hold and these give (2·7).

We also have $\psi(y) + r(y) \leqslant 1$ so that the proof of (2·8) will be complete if we show that, for $y \in F$,

$$\psi(x) \leqslant \psi(y) + r(y) - (r(y)^2 - x^2)^{\frac{1}{2}} \quad (0 < x \leqslant r(y)). \tag{2·9}$$

This estimate holds for $0 < x \leqslant y$ since ψ is increasing. For $y < x \leqslant r(y)$, on the other hand,

$$\psi(x) + (r(y)^2 - x^2)^{\frac{1}{2}} \leqslant \frac{x^2\psi(y)}{y^2} + r(y)\left(1 - \frac{x^2}{2r(y)^2}\right)$$

$$\leqslant r(y),$$

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since $\psi(x)/x^2$ is decreasing and $r(y) \leqslant y^2/2\psi(y)$. This proves (2·9). Note that (2·9) implies that $r(y) \leqslant r_\phi, y \in F$.

We can now prove Theorem 1. If $y \in F$ then, by (2·8), u is harmonic and positive in $B(P_y, r(y))$ and, by (2·7), the point (O, y) lies in this ball. Applying Lemma 1 gives

$$\begin{aligned}\left|\frac{u_y(O,y)}{u(O,y)}\right| &\leqslant \frac{m-1}{y-\psi(y)} + \frac{1}{\psi(y)+2r(y)-y} \\ &\leqslant \frac{m-1}{y} + 2(m-1)\frac{\psi(y)}{y^2} + \frac{1}{r(y)} \\ &\leqslant \frac{m-1}{y} + 2m\frac{\psi(y)}{y^2} + 2,\end{aligned}$$

since $\psi(y) \leqslant \frac{1}{2}y$ and $1/r(y) = \max[2, 2\psi(y)/y^2]$.

If $0 < y \leqslant \frac{1}{2}$ but $y \notin F$, then, by (2·3),

$$\left|\frac{u_y(O,y)}{u(O,y)}\right| \leqslant \frac{2m(I+1)}{y} \leqslant 4m(I+1)\frac{\psi(y)}{y^2}.$$

Also using (2·3) for the case $\frac{1}{2} < y \leqslant 1$, we see that

$$\left|\frac{u_y(O,y)}{u(O,y)}\right| \leqslant \begin{cases} \dfrac{m-1}{y} + 4m(I+1)\left(\dfrac{\psi(y)}{y^2}+1\right) & (0 < y \leqslant \tfrac{1}{2}), \\ 4m(I+1) & (\tfrac{1}{2} < y \leqslant 1). \end{cases} \tag{2·10}$$

Thus, by (2·6),

$$\begin{aligned}\log\frac{u(O,y)}{u(O,1)} &= -\int_y^1 \frac{u_t(O,t)}{u(O,t)}\,dt \\ &\leqslant (m-1)\log\frac{1}{y} + 4m(I+1)(3I+1),\end{aligned}$$

so that

$$\frac{u(O,y)}{u(O,1)} \leqslant \frac{\exp[4m(I+1)(3I+1)]}{y^{m-1}} \quad (0 < y \leqslant 1),$$

which gives (1·3) with $A = e^4$. This completes the proof of Theorem 1.

We remark that the above technique also gives a lower bound for u. In fact, with the same assumptions as in Theorem 1,

$$\frac{u(O,y)}{u(O,1)} \geqslant \frac{y}{A^{m(I+1)(3I+1)}} \quad (0 < y \leqslant 1), \tag{2·11}$$

where $A = e^4$ again. A similar result (though with a less precise constant) was obtained in (2) under somewhat weaker assumptions on the function ϕ.

To carry out the proof of (2·11) here we need another type of Harnack inequality. Namely, that if u is positive and harmonic in the ball $B(P_0, r)$ in R^m, then

$$-\frac{u_\rho(P)}{u(P)} \leqslant \frac{1}{r-|P-P_0|} + \frac{m-1}{r+|P-P_0|} \quad (0 < |P-P_0| < r). \tag{2·12}$$

Here $u_\rho(P)$ denotes the directional derivative of u outwards along the ray from P_0 through P. The proof of (2·12) is, like that of Lemma 1, a routine procedure involving

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the Poisson integral. Applying (2·12) instead of Lemma 1, in the balls $B(P_y, r(y))$, leads to (2·11).

The estimate (2·12) has another application. It says that if u is positive and harmonic in $B(P_0, r)$ then, for $Q \in \partial B(P_0, r)$, the function

$$\frac{(r+\rho)^{m-1}}{(r-\rho)}\, u\left(P_0 + \frac{\rho}{r}(Q-P_0)\right) \quad (0 < \rho < r),$$

is increasing with ρ. This provides an affirmative answer to a question left open in ((2), remark 2, p. 44).

3. *Proof of Theorem 2*

Let $P_1 = (O, 1)$ and let k be the real function described in Theorem 2. We shall prove that if u is harmonic in $B(P_1, 1)$, vanishes at P_1 and satisfies

$$u(P) \leqslant k(|P-P_1|) \quad (0 \leqslant |P-P_1| < 1), \tag{3·1}$$

then

$$|u(O, y)| \leqslant \frac{(BJ)^m}{y^{m-1}} \quad (0 < y \leqslant 1), \tag{3·2}$$

where B is an absolute constant. This gives Theorem 2.

There is no loss of generality in supposing that J (see (1·4)) takes a particular fixed value. We deduce the general case by considering a suitable multiple of u. It is convenient to take $J = \frac{1}{4}$. Also it is harmless to assume that $k(r) > 0$ for $0 \leqslant r < 1$.

Consider $\mu(y) = k(1-y/4)$, $0 < y \leqslant 1$, which is positive, decreasing and continuous in $(0, 1]$ and satisfies

$$\int_0^1 \left(\frac{\mu(y)}{y}\right)^{1/m} dy = 4^{1-1/m} \int_{\frac{3}{4}}^1 \left(\frac{k(r)}{1-r}\right)^{1/m} dr \leqslant 1. \tag{3·3}$$

Then define

$$f(y) = \left(\frac{y}{\mu(y)}\right)^{1/m} \quad (0 < y \leqslant 1).$$

Since, for $0 < y \leqslant 1$,

$$y\left(\frac{\mu(y)}{y}\right)^{1/m} \leqslant \int_0^y \left(\frac{\mu(t)}{t}\right)^{1/m} dt \leqslant 1, \tag{3·4}$$

we have $f(1) \geqslant 1$ so that $y = \phi_1(x) = f^{-1}(x)$ is positive, increasing and continuous in $(0, 1]$ with $\phi_1(1) \leqslant 1$. For $0 < \epsilon \leqslant 1$,

$$\begin{aligned}
\int_\epsilon^1 \frac{\phi_1(x)}{x^2}\, dx &= -\int_\epsilon^1 \phi_1(x)\, d(1/x) \\
&= -\int_{\phi_1(\epsilon)}^{\phi_1(1)} y\, d\left[\left(\frac{\mu(y)}{y}\right)^{1/m}\right] \\
&= \int_{\phi_1(\epsilon)}^{\phi_1(1)} \left(\frac{\mu(y)}{y}\right)^{1/m} dy - \left[y\left(\frac{\mu(y)}{y}\right)^{1/m}\right]_{\phi_1(\epsilon)}^{\phi_1(1)} \\
&\leqslant 1 + o(1) \quad (\epsilon \to 0),
\end{aligned}$$

in view of (3·4).

Finally put

$$\phi(x) = \phi_1(x) + 4x^2 \quad (0 < x \leqslant 1), \tag{3·5}$$

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so that ϕ satisfies the hypotheses of Theorem 1 with $I = 5$. Notice in particular that $r_\phi = \phi^{-1}(1) \leqslant \frac{1}{2}$.

Now let D_ϕ be given by (1·2) and suppose that $P = (X, y) \in D_\phi$. If $0 < y \leqslant 1$ then, by (3·5),

$$|P - P_1|^2 = |X|^2 + (1-y)^2 \leqslant y/4 + (1-y)^2 \leqslant (1 - y/4)^2$$

and so, using (3·1) and (3·4), we have

$$\begin{aligned} u(P) &\leqslant k(|P - P_1|) \\ &\leqslant \left[\frac{2}{1 + (y/f(y))^2}\right]^{m/2} \mu(y) \\ &= \frac{2^{m/2} y}{(f(y)^2 + y^2)^{m/2}} \\ &\leqslant \frac{2^m y}{(|X|^2 + y^2)^{m/2}}. \end{aligned}$$

If $P \in D_\phi$ but $y > 1$ then

$$u(P) \leqslant k(r_\phi) \leqslant k(\tfrac{3}{4}) = \mu(1) \leqslant 1,$$

by (3·4). Since $|P| \leqslant \frac{3}{2}$ in this case, we deduce that

$$u(P) \leqslant \frac{2^m y}{|P|^m} \quad (P \in D_\phi).$$

It follows that we can write

$$u(P) = u_1(P) - u_2(P) \quad (P \in D_\phi),$$

where u_1, u_2 are positive and harmonic in D_ϕ. Moreover, $u_1(P_1) = u_2(P_1) \leqslant 2^m$ since $u(P_1) = 0$.

According to Theorem 1 therefore, for $0 < y \leqslant 1$,

$$\begin{aligned} |u(O, y)| &\leqslant \max\left[u_1(O, y), u_2(O, y)\right] \\ &\leqslant \frac{(2A^{96})^m}{y^{m-1}}, \end{aligned}$$

which gives (3·2) as required.

4. *Proof of Theorem* 3

Let $D = \{(X, y) : X \in R^{m-1}, y > 0\}$ and let k be the real function described in Theorem 3. Putting

$$\lambda(y) = \begin{cases} k(1-y) & (0 < y \leqslant 1), \\ 1 & (y > 1), \end{cases}$$

so that, by (1·5),

$$\int_0^1 \left(\frac{\lambda(y)}{y}\right)^{1/m} dy = \infty,$$

we shall prove that there exists a function u harmonic in D such that $u(P_1) = 0$ (recall that $P_1 = (O, 1)$) and

$$u(P) \leqslant \lambda(y) \quad (P = (X, y) \in D), \tag{4·1}$$

for which

$$y^{m-1} u(O, y) \to -\infty \quad (y \to 0). \tag{4·2}$$

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The desired example may then be obtained by considering the restriction of u to $B(P_1, 1)$ and making a reflection in the hyperplane $\{y = \frac{1}{2}\}$.

As in the previous section we proceed to define an auxiliary function ϕ. First let

$$f(y) = \left(\frac{y}{\lambda(y)}\right)^{1/m} \quad (0 < y \leqslant 1),$$

and then put $y = \phi(x) = f^{-1}(x)$ which is positive, increasing and continuous in $(0, 1]$ with $\phi(1) = 1$. Note that

$$\frac{\phi(x)}{x^m} = \lambda(\phi(x)) \quad (0 < x \leqslant 1), \tag{4·3}$$

and

$$\int_0^1 \frac{\phi(x)}{x^2} = \infty,$$

which follows from the partial integration used earlier.

The following lemma occurs in the literature in various forms and we supply a proof only for the sake of completeness.

LEMMA 4. *If* $\phi^*(x) = \min[\phi(x), x]$, $0 < x \leqslant 1$, *then*

$$\int_0^1 \frac{\phi^*(x)}{x^2} dx = \infty.$$

Proof. Consider

$$E = \{x \in (0, 1] : \phi(x) > \phi^*(x)\}$$

which is the union of an at most countable collection $\mathcal{O}$ of disjoint open intervals. We may assume that $\mathcal{O}$ is infinite and also that

$$\int_{(0,1]\setminus E} \frac{\phi(x)}{x^2} dx < \infty,$$

for otherwise there is nothing to prove.

If $\mathcal{O} = \{(a_n, b_n) : n = 1, 2, \ldots\}$ then

$$\phi^*(x) = x \quad (a_n \leqslant x \leqslant b_n),$$

so that

$$\int_{a_n}^{b_n} \frac{\phi^*(x)}{x^2} dx = \log \frac{b_n}{a_n} \quad (n = 1, 2, \ldots).$$

Suppose now, by contradiction, that

$$\int_E \frac{\phi^*(x)}{x^2} dx < \infty.$$

Then we must have $b_n/a_n \to 1$ as $n \to \infty$ and so

$$\begin{aligned} \int_{a_n}^{b_n} \frac{\phi(x)}{x^2} dx &\leqslant \phi(b_n) \int_{a_n}^{b_n} \frac{dx}{x^2} \\ &= \frac{b_n}{a_n} - 1 \\ &\leqslant 2 \log \frac{b_n}{a_n}, \end{aligned}$$

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when n is sufficiently large. Thus

$$\int_E \frac{\phi(x)}{x^2}\,dx < \infty,$$

which is the required contradiction.

We then put

$$\psi(x) = \phi^*(\tfrac{1}{2}x) \quad (0 < x \leqslant 1),$$

so that

$$\int_0^1 \frac{\psi(x)}{x^2}\,dx = \infty. \tag{4·4}$$

Throughout the rest of this section $P = (X, y)$ denotes a point of D. Consider

$$U(P) = \int_{|X'|\leqslant 1} \frac{\psi(|X'|)}{|X'|^m} K(P, X')\,dX', \tag{4·5}$$

where $X' \in R^{m-1} = \partial D$, dX' denotes $(m-1)$-dimensional Lebesgue measure and

$$K(P, X') = y\left[\frac{1}{|P-X'|^m} - \frac{1}{|P|^m(1+|X'|^2)^{m/2}}\right]. \tag{4·6}$$

The kernel $K(P, X')$ is evidently harmonic in D for each fixed X' and so, therefore, is U if the integral in (4.5) is locally uniformly convergent. That this is the case follows from $\psi(|X'|) \leqslant |X'|$ and the estimate

$$\frac{1-(1+|X'|/|P|)^m}{|P-X'|^m(1+|X'|^2)^{m/2}} \leqslant \frac{K(P, X')}{y} \leqslant \frac{(1+|X'|)^m - |1-|X'|/|P||^m}{|P-X'|^m(1+|X'|^2)^{m/2}}, \tag{4·7}$$

which shows that if P remains in any compact subset of D then

$$K(P, X') = O(|X'|) \quad (|X'| \to 0),$$

uniformly in P. It is also clear that U is bounded outside any neighbourhood of the origin and that $U(P_1) = 0$.

To obtain upper bounds for U near the origin we note, in the right-hand side of (4·7) that, for $|X'| \leqslant 1$,

$$(1+|X'|)^m \leqslant 1 + 2^m|X'|,$$

and

$$|1-|X'|/|P||^m \geqslant 1 - m|X'|/|P|.$$

Thus

$$\frac{K(P, X')}{y} \leqslant \frac{2m|X'|}{|P-X'|^m|P|} \quad (|P| \leqslant m2^{-m}). \tag{4·8}$$

From now on we shall denote positive constants which depend only on the dimension m by $c_i(m)$, $C_i(m)$, $i = 1, 2, \ldots$.

LEMMA 5. *If* $|P| \leqslant c_1(m) = \min[1/5, m2^{-m}]$ *then*

$$U(P) \leqslant C_1(m)\frac{\phi(|X|)}{|X|^m} + \frac{y}{|P|^m}\left(C_2(m) - c_2(m)\int_{5|P|}^1 \frac{\psi(t)}{t^2}\,dt\right). \tag{4·9}$$

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If, in addition, $y \geqslant |X|$ *then*

$$U(P) \leqslant \frac{y}{|P|^m}\left(C_2(m) - c_2(m)\int_{5|P|}^1 \frac{\psi(t)}{t^2}\,dt\right). \tag{4·10}$$

Proof. If $|P| \leqslant c_1(m)$ we put

$$E_1 = \{X' \in R^{m-1}: |X' - X| \leqslant |X|/2\},$$
$$E_2 = \{X' \in R^{m-1}: |X' - X| \geqslant |X|/2, |X'| \leqslant 5|P|\},$$
$$E_3 = \{X' \in R^{m-1}: 5|P| \leqslant |X'| \leqslant 1\}$$

and then

$$U_i(P) = \int_{E_i} \frac{\psi(|X'|)}{|X'|^m} K(P, X')\,dX' \quad (i = 1, 2, 3),$$

so that

$$U(P) = U_1(P) + U_2(P) + U_3(P).$$

Notice first that, because

$$\psi(x) = \phi^*(\tfrac{1}{2}x) \leqslant \phi(\tfrac{1}{2}x) \quad (0 < x \leqslant 1),$$

we have, for $X' \in E_1$,

$$\frac{\psi(|X'|)}{|X'|^{m-1}} \leqslant 2^{m-1}\frac{\phi(|X|)}{|X|^{m-1}}.$$

Thus, if $|P| \leqslant c_1(m)$ then, by (4·8),

$$\begin{aligned} U_1(P) &\leqslant \frac{2my}{|P|}\int_{E_1} \frac{\psi(|X'|)}{|X'|^{m-1}|P - X'|^m}\,dX' \\ &\leqslant \frac{m2^m\phi(|X|)}{|P|\,|X|^{m-1}}\left[y\int_{E_1}\frac{dX'}{|P - X'|^m}\right] \\ &\leqslant (m2^{m-1}\sigma_m)\frac{\phi(|X|)}{|X|^m}, \end{aligned}$$

since the expression in square brackets is a Poisson integral. Here σ_m denotes the surface area of the unit ball in R^m.

Next suppose that $X' \in E_2$. Then $|X' - X| \geqslant |X|/2$ and so $|P - X'| \geqslant |P|/2$. Thus if $|P| \leqslant c_1(m)$ then, by (4·8),

$$\begin{aligned} U_2(P) &\leqslant \frac{m2^{m+1}y}{|P|^{m+1}}\int_{|X'| \leqslant 5|P|} \frac{\psi(|X'|)}{|X'|^{m-1}}\,dX' \\ &= \frac{m2^{m+1}\sigma_{m-1}y}{|P|^{m+1}}\int_0^{5|P|}\frac{\psi(t)}{t}\,dt \\ &\leqslant (5m2^m\sigma_{m-1})\frac{y}{|P|^m}, \end{aligned}$$

since $\psi(t) \leqslant t/2$.

On the other hand if $5|P| \leqslant |X'| \leqslant 1$ then $|P - X'| \geqslant 2|P|(1 + |X'|^2)^{\frac{1}{2}}$ so that

$$\frac{1}{|P - X'|^m} \leqslant \frac{2^{-m}}{|P|^m(1 + |X'|^2)^{m/2}}$$

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and hence, from (4·6),

$$U_3(P) \leqslant (2^{-m}-1)\frac{y}{|P|^m}\int_{E_3}\frac{\psi(|X'|)}{|X'|^m(1+|X'|^2)^{m/2}}dX'$$
$$\leqslant -\left(\frac{\sigma_{m-1}}{2^m}\right)\frac{y}{|P|^m}\int_{5|P|}^1\frac{\psi(t)}{t^2}dt.$$

Combining these estimates gives (4·9).

To prove (4·10) note that if $y \geqslant |X|$ then $y \geqslant |P|/\sqrt{2}$ so that $|P-X'| \geqslant |P|/\sqrt{2}$ also. Thus, by (4·8),

$$U_1(P)+U_2(P) \leqslant \frac{2my}{|P|}\int_{|X'|\leqslant 5|P|}\frac{\psi(|X'|)}{|X'|^{m-1}|P-X'|^m}dX'$$
$$\leqslant \frac{m2^{m+1}y}{|P|^{m+1}}\int_{|X'|\leqslant 5|P|}\frac{\psi(|X'|)}{|X'|^{m-1}}dX'$$
$$\leqslant (5m2^m\sigma_{m-1})\frac{y}{|P|^m},$$

as before. This completes the proof of the lemma.

We can now prove Theorem 3. From (4·9) it follows that if $|P| \leqslant c_1(m)$ and $y \leqslant \phi(|X|)$ then

$$U(P) \leqslant C_3(m)\frac{\phi(|X|)}{|X|^m}$$
$$= C_3(m)\lambda(\phi(|X|)),$$

by (4·3), and so

$$U(P) \leqslant C_3(m)\lambda(y), \quad (|P| \leqslant c_1(m), \quad y \leqslant \phi(|X|)). \tag{4·11}$$

Next suppose that $|P| \leqslant c_1(m)$ and $\phi(|X|) \leqslant y \leqslant |X|$. Then $\sqrt{2}|X| \geqslant |P|$ so that, by (4·9),

$$U(P) \leqslant \frac{y}{|P|^m}\left(C_4(m)-c_2(m)\int_{5|P|}^1\frac{\psi(t)}{t^2}dt\right).$$

Thus, by (4·4), there exists $c_1 > 0$ so small that

$$U(P) \leqslant 0 \quad (|P| \leqslant c_1, \quad \phi(|X|) \leqslant y \leqslant |X|). \tag{4·12}$$

Finally we deduce from (4·10) that there exists $c_2 > 0$ so small that

$$U(P) \leqslant -c_3(m)\frac{y}{|P|^m}\int_{5|P|}^1\frac{\psi(t)}{t^2}dt \quad (|P| \leqslant c_2, \quad y \geqslant |X|), \tag{4·13}$$

and in particular that

$$y^{m-1}U(O,y) \to -\infty \quad (y \to 0).$$

Also we see from (4·11), (4·12) and (4·13) that

$$U(P) \leqslant C_3(m)\lambda(y) \quad (|P| \leqslant \min[c_1(m), c_1, c_2]).$$

Outside this neighbourhood of the origin U is bounded above, say by C, $0 < C < \infty$. Since $\lambda(y) \geqslant 1$ for $y > 0$ the function

$$u(P) = U(P)/\max[C_3(m), C],$$

satisfies (4·1) and (4·2), and vanishes at P_1. This completes the proof of Theorem 3.

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It is worth pointing out that the function $-u+1$ is positive and harmonic in $\{(X,y): |X| < 1, y > \phi(|X|)\}$ and so, in view of (4·2), it provides a counterexample for Theorem 1.

5. *Concluding remarks*

It is clear that versions of Theorem 2 could be given for domains other than $B(1)$. A particularly simple case would be a half-space, but the proof could be modified to work in a Dini domain (see (3) for definitions).

It is more interesting to ask whether there are unit ball versions of the other results about harmonic functions proved by Hayman and Korenblum in (1). For example, if Γ is a cone in R^m with vertex at the origin and $E = \partial B(1) \cap \Gamma$, what conditions on the function k will guarantee that, if u is harmonic in $B(1)$ and

$$u(P) \leqslant k(|P|) \quad (|P| < 1),$$

then

$$\lim_{r \to 1} \int_E u(rQ)\, d\sigma(Q)$$

exists? Here $d\sigma$ denotes surface area on $\partial B(1)$. It seems likely that the right condition for this problem is still

$$\int_0^1 \left(\frac{k(r)}{1-r}\right)^{\frac{1}{2}} dr < \infty,$$

when $m > 2$.

Finally, I should like to thank the referee for a number of helpful comments.

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