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Vortex–wave interactions: long-wavelength streaks and spatial localization in natural convection

Philip Hall†

Department of Mathematics, Imperial College, London SW7 2BZ, UK

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The unidirectional shear flow driven by buoyancy effects in a vertical channel when a temperature difference is maintained between the walls of the channel is considered. The flow is unstable to waves which interact to reinforce the original flow and make it 'streaky'. Such 'vortex–wave' interactions have been the subject of much recent research but little is yet known about what happens when the wavelength of the roll/streak flow becomes large. An asymptotic structure for long-wavelength interactions is derived and the tendency of the fluid to resist this state and the flow to become localized is revealed. Here the high-Grashof-number limit is considered and it is shown how a self-sustained process can occur with vortices interacting with a wave system in a manner similar to that discussed by Hall & Smith (*J. Fluid Mech.*, vol. 227, 1991, pp. 641–666) and Hall & Sherwin (*J. Fluid Mech.*, vol. 661, 2010, pp. 178–205). The work is closely related to numerical simulations of self-sustained processes in for example Couette flow but the fact that the basic flow here is generated by buoyancy effects enables us to make analytical progress. It is shown that the wave part of the interaction process has a flat critical layer and its wavelength is twice that of the streaky flow which supports it. Such subharmonic vortex–wave/self-sustained process interactions have not been previously identified.

Key words: buoyancy-driven instability, nonlinear instability

1. Introduction

Recently there has been immense interest in vortex–wave interactions/self-sustained processes in shear flows. Sometimes these are referred to as exact coherent structures and are typified by wave systems driving rolls which then drive streaks which are themselves neutrally stable to the wave. In a recent paper Hall & Sherwin (2010) showed that vortex–wave interaction theory as defined in a series of papers by Hall and Smith; see for example Hall & Smith (1991), is simply the high-Reynolds-number description of what subsequently became known as self-sustained processes: see for example Waleffe (2003) and Wang, Gibson & Waleffe (2007). We concentrate on natural convection flows because here the interactions can occur in the weakly nonlinear regime so that it is easier to understand the structures of interest. In this paper we elucidate the structure of subharmonic states which turn out to satisfy

† Email address for correspondence: philhall@ic.ac.uk

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symmetries not previously discussed and we find a mechanism which allows wide streaky flows to break down and exhibit localized structures.

The vortex–wave interaction theory of Hall & Smith (1991) and Hall & Sherwin (2010), hereafter referred to as HS1, HS2 respectively, and the self-sustained-process work are both linked to earlier ideas by Benney (1984) on 'mean flow' interactions. Subsequently Hall (2011), hereafter referred to as H1, derived the interaction equations for natural convection and found a weakly nonlinear solution for waves and streaks of $O(1)$ wavelengths. Here we concentrate on the natural convection problem when the roll/streak flow has a large wavelength. This limit was beyond the scope of the Couette flow problem of HS1 and the small-amplitude limit of H1. Here we focus directly on the long-wavelength limit and consider larger amplitudes than those found in H1, and derive a theory that suggests a tendency for long-wavelength structures in natural convection to localize.

We consider the flow of a Boussinesq fluid in a channel with walls located at $y = \pm 1$ and of infinite extent in the y, z-directions with gravity acting in the x-direction. The channel walls are maintained at different temperatures so a linear temperature gradient is established across the channel and through buoyancy effects this drives a unidirectional flow in the x-direction. The velocity profile is antisymmetric about the centre of the channel and has an inflection point so is unstable to waves propagating in the x-direction. However since the inflection point occurs where the fluid is at rest the inviscid limit of the wave turns out to correspond to a stationary wave. The instability of this flow to transverse rolls was investigated by Gotoh & Ikeda (1972) and Ruth (1979). Transverse rolls are periodic in the x-direction and so correspond to what we refer to as a wave here. We shall henceforth refer to disturbances periodic in the x-direction as waves since this is consistent with terminology of shear-flow instability theory. Gotoh & Ikeda (1972) and Ruth (1979) calculated the critical Grashof number for various values of the Prandtl number and found that a finite band of unstable wavenumbers exists in the limit of large Grashof number. The large-Grashof-number limit, with Prandtl number held fixed, corresponds to the inviscid limit described above, so there will be a finite wavenumber where the wave is neutral and smaller wavenumbers will correspond to inviscidly unstable waves. Here we shall concentrate on the latter neutral wave. At finite Grashof numbers and high values of the Prandtl number travelling waves are possible. The inviscid neutral wave at high Grashof numbers will be allowed in what follows to be modulated in the spanwise, i.e. z, direction. Vortex–wave interaction theory shows that the wave then generates a flow in the plane perpendicular to the unperturbed flow direction. In shear-flow instability theory the latter flow is referred to as a roll and we adopt that terminology here; see figure 1. The roll then modifies the flow in the x-direction to produce a streaky flow which is then unstable to the wave and the interaction is closed. The relevance of vortex–wave interactions/self-sustained processes to turbulence is well-known and we therefore expect that the analysis here might well describe structures related to turbulent natural convection at high Grashof numbers.

For the sake of brevity we will quote the appropriate governing equations from for example Bergholz (1978). With respect to Cartesian coordinates (x, y, z) and the usual notation for the convective derivative, the Laplacian and the gradient operator, the required momentum, continuity and energy equations take the form:

$$\frac{\mathrm{D}\widehat{\boldsymbol{u}}}{\mathrm{D}t} = -\boldsymbol{\nabla}\widehat{p} + \frac{1}{G}\nabla^2\widehat{\boldsymbol{u}} + \frac{\widehat{\Theta}}{G}(1, 0, 0), \tag{1.1}$$

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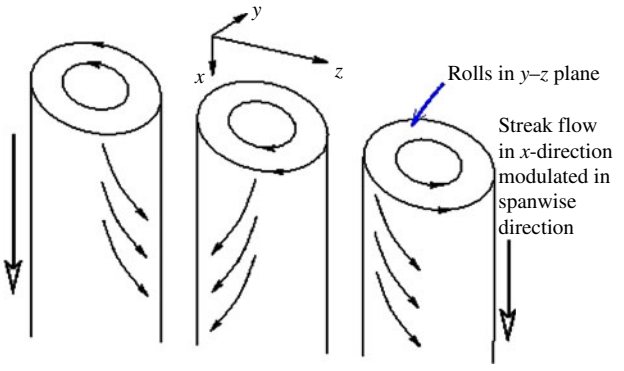


FIGURE 1. (Colour online available at journals.cambridge.org/flm) Sketch of the roll/streak flow, with the roll corresponding to flow in a plane at a fixed value of x and the velocity component in the x-direction being the streak.

$$\nabla \cdot \widehat{\boldsymbol{u}} = 0, \tag{1.2}$$

$$\frac{\mathrm{D}\widehat{\Theta}}{\mathrm{D}t} = \frac{1}{\sigma G}\nabla^2\widehat{\Theta}. \tag{1.3}$$

Here $\widehat{\boldsymbol{u}} = (\widehat{u}, \widehat{v}, \widehat{w})$ represents the fluid velocity whilst $\widehat{\Theta}$ is the temperature, G the Grashof number, σ the Prandtl number and $\widehat{p}$ the pressure. For convenience we have multiplied the scaled pressure introduced by Bergholz by a factor of G. The channel is bounded by $y = \pm 1$ where the appropriate boundary conditions are therefore given by

$$\widehat{\boldsymbol{u}} = (0, 0, 0), \quad \widehat{\Theta} = \pm 1, y = \pm 1. \tag{1.4}$$

Thus the flow satisfies the no-slip conditions at the channel walls and a constant temperature difference is maintained between the walls. In the absence of any disturbance there exists a linear temperature profile $\widehat{\Theta} = \theta_b(y) = y$ and a unidirectional flow $(u_b(y), 0, 0)$ in the x-direction where

$$u_b = \frac{y - y^3}{6}. \tag{1.5}$$

We note that since gravity is perpendicular to the temperature gradient the system does not support Rayleigh–Bénard convection. However a further important point to note here is that this flow is asymmetric about the centre of the channel and has an inflection point at the origin and is therefore unstable to travelling waves, and the neutral mode will have zero wave speed and its critical layer will be at the centreplane. Our formulation is valid when a pressure gradient is applied in the x-direction or when the walls move. In that case the wave will propagate so we shall continue to refer to a disturbance as a wave even though it does not propagate. Following H1 we note that the above equations admit an exact solution corresponding to a roll/streak solution of the type discussed in HS2 evolving on the longer time scale $\tau = t/G$ with $\widehat{\boldsymbol{u}} = (u(y, z, \tau), v(y, z, \tau)/G, w(y, z, \tau)/G), \widehat{p} = p(y, z, \tau)/G^2$. An examination of the energy equation shows that the temperature field associated with this roll/streak flow takes the form $\widehat{\Theta} = \theta(y, z, \tau)$. The flow is taken to be periodic in the spanwise direction with wavelength $2\pi/k$. We now follow the analysis of H1 and assume that the Grashof number is large; in that case we perturb the steady form of the roll/streak

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flow by writing:

$$\widehat{\boldsymbol{u}} = \left(u(y,z,\tau), \frac{v(y,z,\tau)}{G}, \frac{w(y,z,\tau)}{G}\right) + G^{-7/6}\boldsymbol{U}(y,z,\tau)E + \text{c.c.} + \cdots, \quad (1.6)$$

$$\widehat{p} = \frac{p(y,z,\tau)}{G^2} + G^{-7/6}P(y,z,\tau)E + \text{c.c.} + \cdots, \quad (1.7)$$

$$\widehat{\Theta} = \theta(y,z,\tau) + G^{-7/6}T(y,z,\tau)E + \text{c.c.} + \cdots, \quad E = \exp\left\{\text{i}\alpha(x - G\int c(\tau)\,\text{d}\tau)\right\}, \quad (1.8)$$

so that the wave propagates in the x-direction with instantaneous speed c and has wavenumber α. Here we shall concentrate on waves with zero wave speed but for completeness we shall initially show the form of the solution for travelling waves which would be relevant to the case when walls move or a pressure gradient in the x direction is present. The size of the wave, $O(G^{-7/6})$, is fixed as in HS1, HS2, H1 in such a way that in the critical layer associated with the wave $O(1)$ jumps in the roll stresses are induced across the critical layer. The roll/streak and associated temperature field then satisfy:

$$u_\tau + vu_y + wu_z = u_{yy} + u_{zz} + \theta, \quad (1.9)$$

$$v_\tau + vv_y + wv_z + p_y = v_{yy} + v_{zz}, \quad (1.10)$$

$$w_\tau + vw_y + ww_z + p_z = w_{yy} + w_{zz}, \quad (1.11)$$

$$\theta_\tau + v\theta_y + w\theta_z = \frac{1}{\sigma}[\theta_{yy} + \theta_{zz}], \quad (1.12)$$

$$v_y + w_z = 0, \quad (1.13)$$

subject to

$$u = (0,0,0), \quad \theta = \pm 1, y = \pm 1 \quad (1.14)$$

Away from the critical layer the wave satisfies the linearized system

$$\text{i}\alpha(u-c)U + Vu_y + Wu_z = -\text{i}\alpha P, \quad (1.15)$$

$$\text{i}\alpha(u-c)V = -P_y, \quad (1.16)$$

$$\text{i}\alpha(u-c)W = -P_z, \quad (1.17)$$

$$\text{i}\alpha U + V_y + W_z = 0, \quad (1.18)$$

$$\text{i}\alpha(u-c)T + V\theta_y + W\theta_z = 0. \quad (1.19)$$

We see that at high Grashof numbers the temperature perturbation decouples from the velocity–pressure perturbations which means that the wave depends only on the streak velocity field and, following Hall & Horseman (1991), elimination of the wave velocity field leads to the wave pressure eigenvalue problem:.

$$P_{yy} + P_{zz} - \alpha^2 P - \frac{2u_y P_y}{u-c} - \frac{2u_z P_z}{u-c} = 0, \quad P_y = 0, y = \pm 1. \quad (1.20)$$

In the spanwise direction we require that P is periodic. Throughout the interaction the wave is taken to be instantaneously neutral so that c is real. The coupling between the wave and roll/streak flow is completed in the critical layer which occupies a region of width $G^{-1/3}$ around $y = f(z,\tau)$ where $u = c$. Here we will restrict our attention to the case when the wave speed is zero and we will see that the special symmetries of the solution we construct lead to a flat critical layer with $f = 0$. In the critical

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layer (1.15)–(1.19) show that the wave velocity components and temperature increase by $O(G^{1/3})$ and $O(G^{2/3})$ respectively. Within the thin critical layer the wave interacts with itself to drive the roll flow. When the roll flow inside the critical layer is matched to the outer flow we see that the result of the wave driving the roll is to produce jumps in the stresses associated with the roll for the flow outside the critical layer. The velocity and pressure fields associated with the wave, roll and streak within the critical layer are found in exactly the same manner as given in HS1 and HS2 so we do not repeat it here. The temperature field independent of x develops no new structure at leading order in the critical layer so it and its first derivatives are unchanged across the layer. Likewise the roll velocity components, v, w together with u, u_y are continuous across the critical layer. However, following HS1, HS2 we find that the Reynolds stresses associated with the wave lead to the following jumps for the roll flow across the layer:

$$[w_y]_-^+ = \frac{[v_y]_-^+}{f_z} = J(z,\tau), \tag{1.21}$$

$$J = \left(\frac{2}{3}\right)^{2/3} \Gamma\left(\frac{1}{3}\right) \frac{2\pi}{a^{5/3}\Delta^5} \left\{ \left(-\frac{7\Delta_z}{2\Delta} - \frac{5a_z}{3a} \right) |P_z|^2 + \frac{\partial}{\partial z}|P_z|^2 \right\}, \tag{1.22}$$

$$[p]_-^+ = K = -\left(\frac{2}{3}\right)^{2/3} \Gamma\left(\frac{1}{3}\right) \frac{2\pi}{a^{5/3}\Delta^5} f_{zz}|P_z|^2, \tag{1.23}$$

where $\Delta = 1 + f_z^2$ and $a = \mu\alpha/\Delta$, where μ is u_y evaluated at the critical layer, and the $\pm$ signs denote values evaluated to the right and left of the critical layer and all z derivatives are to be evaluated along the critical layer $y = f(z,\tau)$. Also the wave pressure in the above equations is to be evaluated on the critical layer. As explained in HS1 the easiest way to understand the origin of the jumps is by thinking of the flow in the critical layer as a steady streaming problem. Thus in the critical layer the wavefield becomes relatively large and its interaction with itself generates Reynolds stresses which cause the roll velocity component parallel to the critical layer to have a jump in its normal derivative whilst the momentum equation normal to the layer produces a jump in the pressure. Thus the Reynolds stresses basically produce jumps in the roll stresses normal and parallel to the critical layer. Note that the non-wave part of the temperature field, θ, is passive in this process and so θ, θ_y are continuous across the layer. Finally in this section we note that the wave has a viscous layer at the walls of the same thickness as the critical layer but no jumps are produced at leading order there because the streamwise momentum equation shows that the wave pressure is zero there.

2. Long-wavelength solutions of the vortex–wave interaction equations

It is useful at this stage to summarize the task associated with the solution of the interaction equations. Firstly we suppose that a streaky flow u exists and that at any instant in time it is neutrally stable to an inviscid wave disturbance. The inviscid wave satisfies the pressure equation (1.20) and interacts with itself in the critical layer to generate discontinuities in the stresses associated with the roll flow (v, w). The roll flow then drives the streak directly through the convective terms in the Navier–Stokes equations and indirectly by modifying the temperature field which itself then modifies the streaky flow. Thus the interaction is slightly more complex than the situation of HS1, HS2 where temperature effects do not enter and there is only the direct roll/streak/wave process taking place. Now let us suppose that the flow is periodic

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in the spanwise direction with wavenumber k which we take to be small. In HS1, HS2 and H1 no results were found for this limit and the possibility of wide rolls has apparently not been addressed in the context of self-sustained processes. We take the wave pressure P to be of size δ and assume that the critical layer displacement from the origin can be neglected so it follows from the jump conditions that the roll will then have $v \backsim \delta^2 k^4$, $w \backsim \delta^2 k^3$. This implies that perturbations to u and θ are of size $\delta^2 k^4$. If we then consider the terms in the pressure equation corresponding to diffusion in the spanwise direction and the deviation of u from its unperturbed state we see that a balance occurs when $\delta^2 k^4 \backsim k^2$ and we arrive at the distinguished limit $\delta \backsim k^{-1}$. In this case $v \backsim k^2$, $w \backsim k$, and $p \backsim O(1)$ and the perturbations to the 'vortex' part of the temperature and streak fields are both of size k^2. This scaling for the pressure means that at leading order the pressure jump induced by the Reynolds stresses of the wave is zero across the critical layer since the wave leads only to a jump of size $O(f)$ there.

Now let us find the reduced form of the interaction equations in this limit bearing in mind that we can only take k to be so small that the previous limit involving the Grashof number remains valid. In fact the wide vortex limit of the interaction equations corresponds to a lubrication theory description of the roll flow driven by jumps in the stress across the fictitious interface $y = 0$. We confine our attention to the equilibrium case and henceforth drop the τ dependence of the interaction equations. We rescale the spanwise dependence of the flow by making the transformation $z \to kz$ in the interaction equations. The roll, streak, pressure and temperature fields associated with the vortex flow expand as

$$\left(\frac{u-u_b}{k^2}, \frac{v}{k^2}, \frac{w}{k}, p\right) = (u_0, v_0, w_0, p_0) + O(k), \quad \frac{\theta - y}{k^2} = \theta_0 + O(k). \tag{2.1}$$

We note that since u is perturbed by an amount $O(k^2)$ the critical layer displacement is at most of the same size and therefore the leading-order contribution to J is found by setting $f = 0$. If we substitute into the streak, roll and temperature equations we find that the leading-order terms in these equations satisfy:

$$u_{0yy} = v_0 u_{by} - \theta_0, \quad p_{0y} = 0, \quad w_{0yy} = p_{0z}, \quad v_{0y} + w_{0z} = 0, \quad \theta_{0yy} = \sigma v_0. \tag{2.2}$$

These equations must be solved subject to the no-slip condition at $y = \pm 1$ and $\theta = 0, y = \pm 1$. Before we can write down further conditions associated with the jumps in stresses we must expand the wave pressure perturbation in the form

$$P = k^{-1} P_0 + k P_1 + \cdots. \tag{2.3}$$

The function P_0 therefore corresponds to the neutral disturbance of the undisturbed shear flow u_b. That flow is unstable for wavenumbers in the range $0 < \alpha < \alpha_n$ with c purely imaginary and calculations show $\alpha_n \backsimeq 2.28$. We thus take $c = 0$ in what follows and perturb the wavenumber away from its neutral value by writing

$$\alpha = \alpha_n + k^2 \frac{\alpha_1}{2\alpha_n} + \cdots, \tag{2.4}$$

where the factor $2\alpha_n$ has been introduced for convenience. Since z does not appear in the equation for P_0 we can write $P_0 = A(z) q_0(y) + \cdots$, where

$$q_{0yy} - \alpha_0^2 q_0 - \frac{2 u_{by} q_{0y}}{u_b} = 0, \quad q_{0y} = 0, y = \pm 1, \tag{2.5}$$

and $A(z)$ is an unknown amplitude function to be found at higher order. Having found the form of the wave pressure we can complete the problem for $u_0, v_0, w_0, p_0, \theta_0$ by

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writing down the leading-order approximation to the jump conditions:

$$[w_y]^+_- = J_0 \frac{\mathrm{d}}{\mathrm{d}z}|A_z|^2, \quad [p_0]^+_- = 0. \tag{2.6}$$

$$J_0 = \left(\frac{2}{3}\right)^{2/3} \Gamma\left(\frac{1}{3}\right) 2\pi \Big/ \left(\frac{1}{6}\alpha_n\right)^{5/3} q_0^2(0). \tag{2.7}$$

We see then that the pressure jump induced by the wave is negligible at leading order. The equations for $u_0, v_0, w_0, p_0, \theta_0$ are then solved by first noting that p_0 is a function of z only. We then denote u_0, v_0, w_0, θ_0 for $y > 0, y < 0$ by $u_0^{\pm}, v_0^{\pm} w_0^{\pm}, \theta_0^{\pm}$ respectively. The spanwise momentum equation can then be integrated to give $w_0^{\pm}$ subject to the no slip condition being satisfied at the walls. We can then integrate the continuity equation in each layer to find $v_0^{\pm}$ again subject to the no-slip condition being satisfied at the walls. The continuity of v_0, w_0 and the imposition of the required jump in w_{oy} then fixes the unknown constants and yields the Reynolds lubrication equation for the pressure; this gives $p_{0z} = -(3/4)J_0(\mathrm{d}/\mathrm{d}z)\,|A_z|^2$. The complete solution is found to be:

$$u_0^+ = \frac{J_0}{48}\frac{\mathrm{d}^2}{\mathrm{d}z^2}|A_z|^2\left[-\frac{\sigma e_1(y)}{84} + e_2(y)\right], \quad v_0^+ = \frac{J_0}{8}\frac{\mathrm{d}^2}{\mathrm{d}z^2}|A_z|^2 y\,(y-1)^2, \tag{2.8}$$

$$w_0^+ = \frac{J_0}{8}\frac{\mathrm{d}}{\mathrm{d}z}|A_z|^2(y-1)(1-3y), \tag{2.9}$$

$$\theta_0^+ = \frac{\sigma J_0}{16}\frac{\mathrm{d}^2}{\mathrm{d}z^2}|A_z|^2\left(\frac{y^5}{10} - \frac{y^4}{3} + \frac{y^3}{3} - \frac{y}{10}\right), \tag{2.10}$$

with u_0, w_0, θ_0 odd in y and v_0 even in y and $e_1 = 3y^7 - 14y^6 + 21y^5 - 21y^3 + 11y$, $e_2 = -(y^7/14) + (y^6/5) - (y^5/10) - (y^4/6) + (y^3/6) - (y/35)$. It remains for us to determine $A(z)$ which specifies the spanwise structure of the wave pressure. At order k^2 we find that

$$P_{1yy} - \alpha_n^2 P_1 - \frac{2u_{by}P_{1y}}{\overline{u} - c} = u_b^2\left[-A_{zz}\frac{q_0}{u_b^2} + \frac{2q_0 A\alpha_n\alpha_1}{u_b^2} + 2\left(\frac{u_0^+}{u_b}\right)'\frac{P_{0y}}{u_b^2}\right]. \tag{2.11}$$

where a prime denotes $\mathrm{d}/\mathrm{d}t$. After substituting into the above equation for u_0 and applying the condition that the above equation should have a solution subject to P_{1y} vanishing at the walls we obtain an equation to determine $A(z)$. After a little manipulation the solvability condition yields:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}z^2} + \alpha_1 A = \lambda A \frac{\mathrm{d}^2}{\mathrm{d}z^2}|A_z|^2, \tag{2.12}$$

where the constant λ is given by:

$$\lambda = -\int_0^1 v_0^2 \frac{[u_{0yy}^+ u_b - u_0^+ u_{byy}]}{u_b^2}\,\mathrm{d}y \Big/ \int_0^1 v_0^2\,\mathrm{d}y. \tag{2.13}$$

Here v_0 is the normal velocity component of the wave pressure corresponding to q_0 and α_1 can be positive or negative depending on whether α corresponds to a stable or an unstable inviscid wave perturbation to u_b. The constant λ is found to be always positive so that after rescaling we can sensibly write the equation for A as

$$\frac{\mathrm{d}^2 A}{\mathrm{d}z^2} + \alpha_1 A = A\frac{\mathrm{d}^2}{\mathrm{d}z^2}|A_z|^2. \tag{2.14}$$

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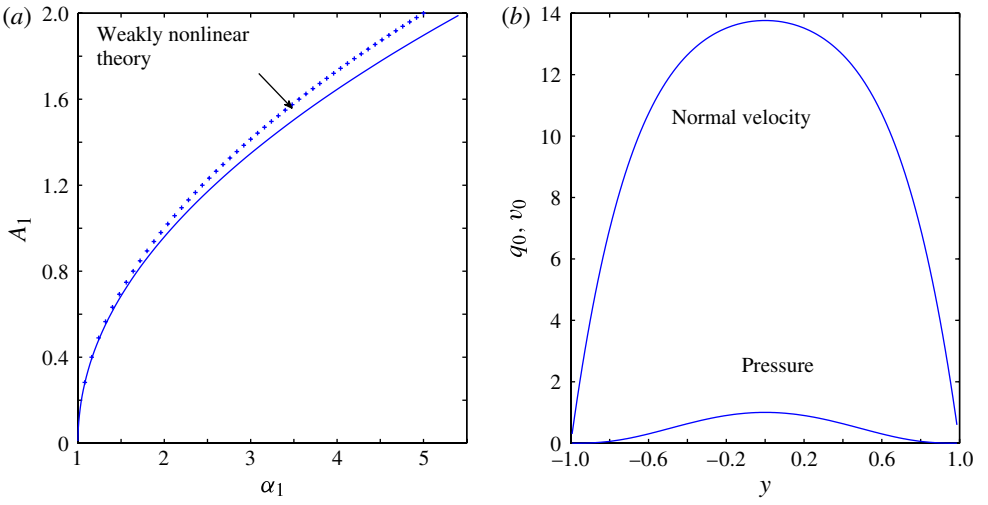


FIGURE 2. (Colour online) (*a*) Dependence of amplitude $A(z)$ on the perturbed streamwise wavenumber; the continuous curve represents the full solution and the other curve the weakly nonlinear solution. The norm of the amplitude used as a measure is A_1. (*b*) The normal velocity component and imaginary part of the pressure associated with the wave.

This is apparently a new amplitude equation in hydrodynamic instability theory and is at first sight unusual in that the highest derivative appears in the nonlinear term. This is because the equation governs the interaction of two types of disturbance namely a wave and roll/streak flow which operate on different spatial scales. For an evolution equation involving one disturbance or indeed sets of disturbances operating on the same length scale the structure of the Navier–Stokes equations means that the highest derivative will be linear. We will solve the amplitude equation subject to $A(z+2\pi)=A(z)$; the possibility of localized solutions exists but our concern is with the periodic ones.

It is easy to see that the linearized form of the amplitude equations has periodic solutions associated with the points $\alpha_1 = 1, 4, 9, \ldots$ so we can perform a weakly nonlinear analysis around these points to find the initial nature of the bifurcating solution. Thus if we write $\alpha_1 = 1 + \chi^2 + \cdots$ then we can show that for small χ, $A(z)$ can be approximated by $\chi \sin z$. In fact this corresponds to a subcritical bifurcation since when $\alpha_1 > 0$ the flow is stable according to linear theory. Away from the bifurcation points the solution may only be found numerically and this was achieved by expanding A in the form

$$A = \sum_{n=1}^{n=\infty} A_n \sin nz, \tag{2.15}$$

substituting into the amplitude equation and then using Newton iteration to solve the resultant set of nonlinear equations for A_n. The number of terms used to obtain converged results depended on the amplitude of the disturbance and results shown correspond to taking sufficient number of terms to obtain graphical accuracy. We restricted our calculations to the case when the amplitude A is real. Figure 2(*a*) shows the amplitude of the first bifurcating solution as a function of α; the measure of the amplitude is taken to be A_1. Figure 2(*b*) shows the imaginary part of v_0

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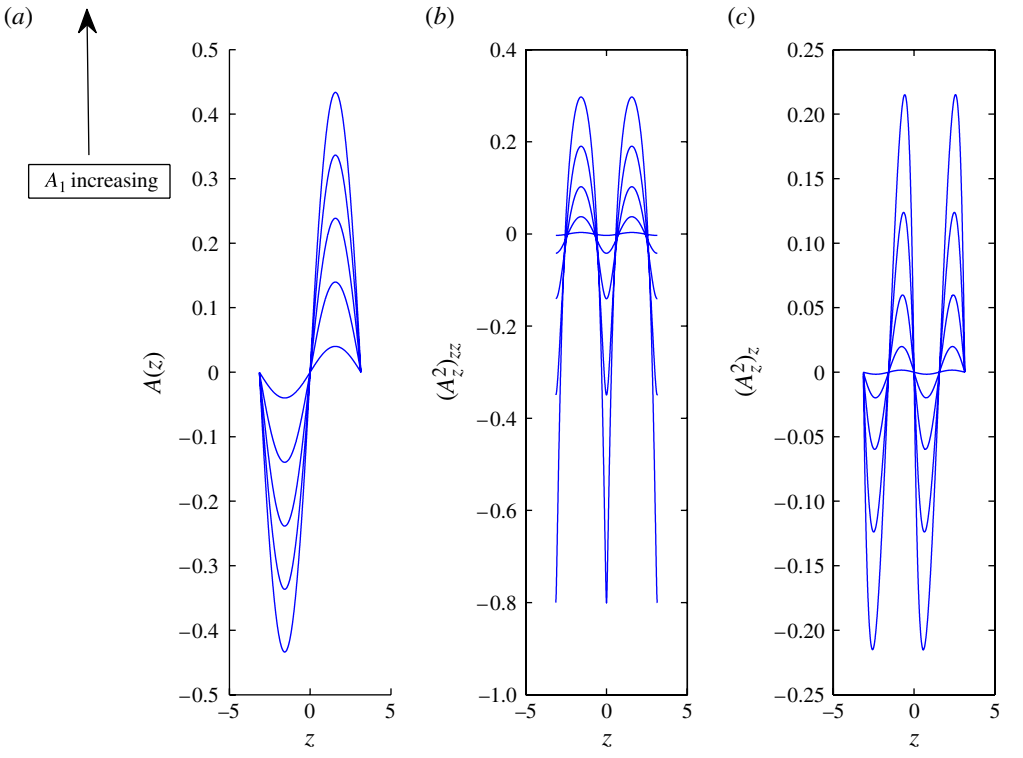


FIGURE 3. (Colour online) (*a*) Spanwise amplitude $A(z)$ of the pressure as a function of z for different values of the perturbed streamwise wavenumber; the solutions shown correspond to $A_1 = 0.05, 0.15, 0.25, 0.35, 0.45$. (*b*) $[(A_z)^2]_{zz}$ as a function of z for the same values of A_1. (*c*) $[(A_z)^2]_z$ as a function of z for the same values of A_1.

and q_0. Note that the leading-order wave pressure is real whilst the normal velocity component is purely imaginary. Also shown in figure 2(*a*) is the weakly nonlinear solution. The solution converges rapidly at small amplitudes and naturally needs more terms as the amplitude increases. For values of $A_1 > 0.6$ the solutions develop small-amplitude oscillations typical of the Gibbs phenomenon. The reason for this will become apparent below after we discuss the results for $A(z)$ in a little more detail.

In figure 3(*a–c*) we show plots of A, $(A_z^2)_z$, and $(A_z^2)_{zz}$ for various values of A_1. The curves for A are quite smooth and differ little from the sinusoidal small-amplitude bifurcating solution. However we see that the results for $(A_z^2)_{zz}$ indicate a variation on a shorter scale developing near the points $z = 0, \pi, 2\pi, \ldots$. This shorter scale variation causes the Fourier series approximation of this function to exhibit Gibbs phenomenon and our procedure loses accuracy. In fact the short-scale behaviour can be anticipated from the amplitude equation directly and corresponds to a large α_1 limit. In that case the dominant balance in the equation ignores the linear second-derivative term over most of the interval for z. At the points $z = 0, \pi, 2\pi, \ldots$ thin layers of thickness $\alpha_1^{1/2}$ develop. We do not pursue the subsequent development of the amplitude in this limit here but its consequences are of significance. This is because it suggests the development of spanwise localized small-scale structures in the streaky flow as the pressure wave amplitude increases. This means that in an experiment the streaky flow

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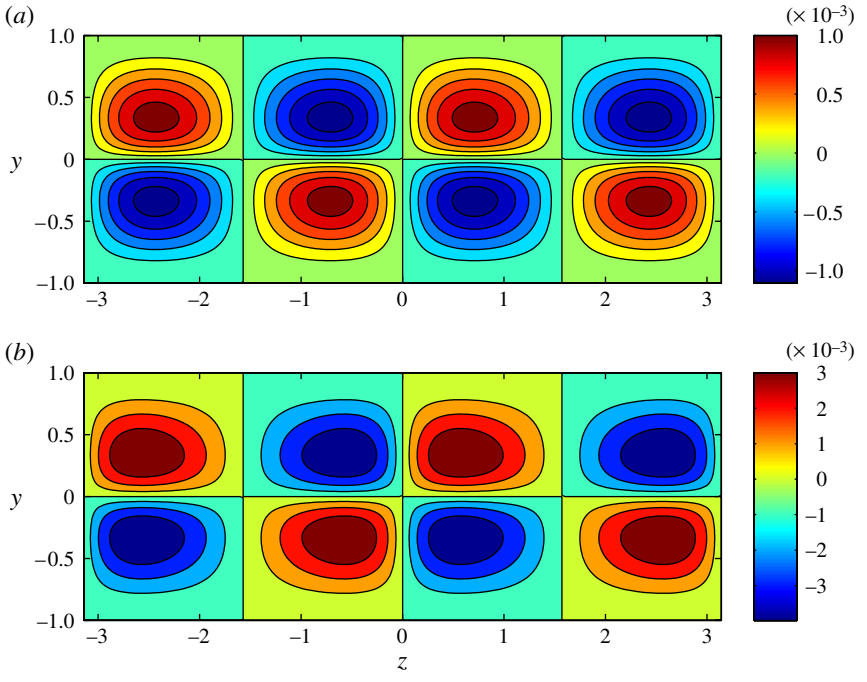


FIGURE 4. Streamfunction associated with the roll flow for $A_1 = 0.25$ (a) and $A_1 = 0.45$ (b).

would be apparently confined to widely separated narrow regions in the spanwise direction. However we note that the function $(A'^2)_{zz}$ which represents the spatial amplitude of the mean flow correction is in fact non-zero everywhere but it simply changes faster in the layers where the new structure develops. We further note that the development of the instability once the new structure develops requires us to return to the full equations and retain spanwise derivatives at leading order. In figure 4 we show plots of the streamfunction associated with the roll flow at increasing values of the wave amplitude. We see that the localized structure of $(A'^2)_{zz}$ has not yet caused a localized structure in the roll flow to develop. By way of a contrast we see in figure 5 that when the wave amplitude increases the mean flow correction does indeed vary on a shorter length scale at particular spanwise locations.

Numerical simulations of natural convection have for the most part concentrated on the two-dimensional case, i.e. independent of z in our notation, and are therefore not relevant to the present calculation. Some work has done on the three-dimensional case but the emphasis has been on elucidating the different patterns which emerge after the flow becomes unstable. However Bratsun, Zyuzgin & Putin (2003) do report finding what in our terminology correspond to travelling waves with a spanwise modulation but do not mention any associated roll/streak flow. At high Grashof numbers where the flow is turbulent there has apparently been no attempt to look for coherent structures embedded in the flows as has become routine in shear flows. Several numerical investigations of turbulent natural convection in channels or adjacent to a single vertical wall have reported the generation of streaks; see for example Fedorovich & Shapiro (2009). In pipe flow and plane Couette flow the lower-branch states of the type we discuss here are visited regularly when the flow is turbulent and define edge states between turbulent and laminar flows: see for Gibson, Halcrow & Cvitanovic (2008) and Willis & Kerswell (2009). In turbulent natural convection

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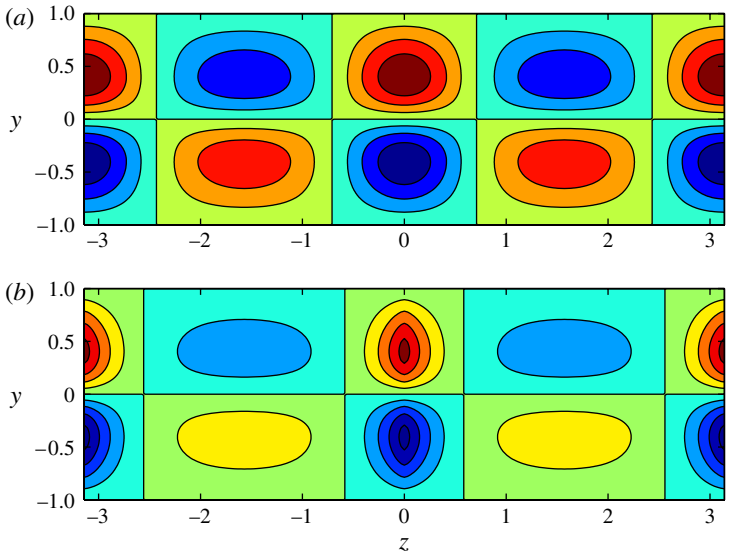


FIGURE 5. The 'streaky' part of the streamwise velocity component u_0^+ for (*a*) $A_1 = 0.25$ and (*b*) $A_1 = 0.45$.

numerical simulations of the turbulent regimes have not yet focused on such issues and therefore the relevance of the lower-branch states described here is an open question.

We have shown then that the vortex–wave interaction theory applied to natural convection leads to a subharmonic interaction between waves and roll/streaks. At small amplitudes the waves are purely sinusoidal but at higher amplitudes there is a tendency for localized structures to occur. The development of these structures is beyond the scope of this paper but we can tentatively conclude that wide rolls exhibit a tendency to lose coherences and develop a localized behaviour. The nature of the subharmonic solutions found suggests that similar structures might be found in Couette flow and that possibility is now being investigated.

We have seen above that the long-wavelength vortex–wave interaction takes the form of a subharmonic type of interaction with the wave pressure having twice the wavelength of the roll/streak flow. This type of interaction survives if we allow the walls to move with equal and opposite speeds so it is likely that the solutions connect with similar structures in Couette flow. A further interesting property of the interaction is that to the order of approximation we considered the critical layer is flat. Indeed a careful examination of the full vortex–wave interaction equations shows that they allow solutions with a roll/streak flow having the streamwise and spanwise velocity components that are odd functions of y so that the critical layer is flat at all orders. The wave pressure associated with that flow has twice the spanwise wavelength as the roll/streak flow and can be either odd or even in z. No such structures have been found for Couette flow but the above calculation suggests strongly that they exist.

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